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Restriction of Laplace operator on one-forms: from \mathbb{R}^{n+2} and \mathbb{R}^{n+1} ambient spaces to embedded (A)dS_n submanifolds

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The Laplace-de Rham operator acting on a one-form a : $\square a$, in \mathbb{R}^{n+2} or \mathbb{R}^{n+1} spaces is restricted to n -dimensional pseudo-spheres. This includes, in particular, the n -dimensional de Sitter and Anti-de Sitter space-times. The restriction is designed to extract the corresponding n -dimensional Laplace-de Rham operator acting on the corresponding n -dimensional one-form on pseudo-spheres. Explicit formulas relating these two operators are given in each situation. The converse problem, of extending an n -dimensional operator composed of the sum of the Laplace-de Rham operator and additional terms to ambient spaces Laplace-de Rham operator, is also studied. We show that for any additional term this operator on the embedded space is the restriction of Laplace-de Rham operator on the embedding space. These results are translated to the Laplace-Beltrami operator thanks to the Weitzenböck formula, for which a proof is also given.

I. INTRODUCTION

This paper deals with the restriction from different embedding spaces (\mathbb{R}^{n+2} and \mathbb{R}^{n+1}) to the n -dimensional de Sitter (dS) and Anti-de Sitter (AdS) space-times of the Laplace-de Rham operator, hereafter generically denoted by $\square := -(d\delta + \delta d)$, acting either on

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one-form or on scalar fields.

Embedding curved space-times in flat higher dimensional space is an old and still pursued topic (see [1] for a wealth of references and [2–5] for current endeavors), often with an interest in the description of geodesics and determination of Hawking temperature. Here, on the contrary, the present work is focused on the restriction of differential operators from the embedding space to the space-time (the embedded submanifold). Such an approach has been used in the study of quantum field theories from an ambient space point of view [6–23]. It has also been already used in the case of quantum mechanics constrained to a surface in which the restricted (scalar) laplacian shows additional terms due to the embedding [24–27].

The lowering of dimensionality, between embedding and embedded spaces, leads to constraints or additional degrees of freedom for fields on which operators act, depending on the situation. For instance, the restriction of $\square A$, A being a vector field, to a lower dimensional embedded manifold, implies that the number of components of A has to be reduced. For the restricted operator, these “additional” components become “additional” degrees of freedom, they may be used to impose conditions such as, for instance, transversality or homogeneity on the field. This kind of method is of common use when the restricted operator is already known. See, in particular, the work initiated by Dirac [6, 7], extended by Fronsdal [8–13] and continued by others [14–23], where one builds in the embedding space a tensor calculus isomorphic to that of the embedded space.

Here, we take a different approach, no conditions are imposed on the one-form or scalar fields in the embedding space (neither transversality nor homogeneity), and the restricted operator retains its most general form. To summarize, an explicit relation between the Laplace-de Rham operator on the embedding space and that on the embedded space is proved for both $n + 1$ and $n + 2$ dimensional cases. Precisely, the restricted operator is decomposed into two parts, the former being the Laplace-de Rham operator of the considered pseudo-sphere, the latter contains terms involving intrinsic geometrical structures (normal derivative to the manifold, Lie derivative along dilation vectors, . . .) and the field over the embedding space (that is unrestricted to the manifold). This last term, that we shall denote by AT (for “Additional Term”) in the sequel, is thus determined through

$$(\square_d \alpha)_\Sigma = \square_\Sigma \alpha_\Sigma + \text{AT},$$

where \square_d is the Laplace-de Rham operator on the d -dimensional embedding space, Σ is the

embedded space, \square_Σ is the Laplace-de Rham operator on Σ , and $(\square_d\alpha)_\Sigma$, α_Σ , are restrictions to the embedded submanifold, in a sense to be precised later, of respectively $\square_d\alpha$ and α .

The generic form of AT encompass many known situations as, in particular, the one where the restricted operator corresponds to a one-form (or scalar) field which belongs to an unitary irreducible representations of the (A)dS group [28]. Moreover, this form allows us to draw general conclusions on the freedom left in the interplay between operators in embedding and embedded spaces. Precisely, the geometry of the embedding being given, the additional term AT appearing in the restriction from the embedding space to the embedded pseudo-sphere is completely determined by the field. Conversely, if one considers on the embedded pseudo-sphere an operator $\square_\Sigma\beta + \chi(\beta)$ where β is a scalar field or a 1-form on Σ , and χ is any expression depending (or not) on β we will show that there always exists an extension $\varpi(\beta)$ of the field β to the embedding space for which AT is χ , explicitly:

$$(\square_d\varpi(\beta))_\Sigma = \square_\Sigma\beta + \chi(\beta).$$

There is therefore no constraint on the additional term which can be any desired expression.

The considered embedding spaces \mathbb{R}^{n+2} (respectively \mathbb{R}^{n+1}) are the $n + 2$ (respectively $n + 1$) dimensional real vector spaces endowed with the metric η left invariant under the conformal group $SO(2, n)$ (respectively the isometrical groups $SO(1, n)$ or $SO(2, n-1)$ depending on whether we consider dS or AdS space-time). Note that, we consider the embedding space \mathbb{R}^{n+2} as our main goal to allow us further generalizations (see the conclusion Sec. VII). In this respect, the embedding space \mathbb{R}^{n+1} appears as an intermediate step.

The paper is organized as follows. In Sec. II we summarize the steps and the methods used to derive the expression of the restricted operators, Sec. III gives the expression of the Laplace-de Rham operators on one-forms and scalars in terms of general frames and anholonomy coefficients. The restriction from $n + 1$ -dimensional space is considered in Sec. IV, it is generalized to the restriction from $n + 2$ -dimensional space in Sec. V. The relation between restriction and extension is examined in Sec. VI. In Sec. VII we conclude and consider the possibility of the extension of this work to other space-times. In appendix A 1 a summary of differential geometry and its most used formulas, relevant to this article, is given. The Weitzenböck formula is provided, with a proof in our notations, enabling us to extend our results from the Laplace-de Rham operator to the Laplace-Beltrami operator. The calculation of the co-differential of a co-frame $\delta(e^a)$, used in particular in Sec. III, is

detailed in appendix A2. The calculations of the formulas stated in the paper are detailed in A3–A7.

Throughout the paper except otherwise specified a, b, c denote abstract indexes, while $\mu, \nu, \dots = 0, \dots, n - 1$, and $A, B, \dots = 0, \dots, n$ and, finally, $\alpha, \beta, \dots = 0, \dots, n + 1$ are related, respectively, to Σ_n , \mathbb{R}^{n+1} , and \mathbb{R}^{n+2} . The canonical coordinates of a point y of \mathbb{R}^{n+2} (respectively \mathbb{R}^{n+1}) are denoted $\{y^\alpha\}$ (respectively $\{y^A\}$) and their associated Cartesian ortho-normal basis is denoted $\{\partial_\alpha\}$ (respectively $\{\partial_A\}$).

II. GEOMETRIC CONTEXT AND METHOD

We proceed in three main steps. We first express the Laplace-de Rham operator on a d -dimensional pseudo-Riemannian oriented manifold (M, g) , in terms of any ortho-normalized (local) field of frames and their associated anholonomy coefficients.

The $n + 1$ -dimensional case is then tackled. The metric manifold (M, g) is here the embedding space \mathbb{R}^{n+1} endowed with a pseudo-metric η . We define specific frames of \mathbb{R}^{n+1} adapted to any submanifold of M . They are built from orthonormal frames of M completed by its outgoing normal, and extended to the embedding space. Anholonomic coefficients of these frames are then computed for the pseudo-sphere. The expressions of, both scalar and one-form, Laplace-de Rham operator obtained before are then particularized to these frames and coefficients and then restricted to the n -dimensional pseudo-sphere. The resulting expressions are rewritten using intrinsic, frame independent, quantities.

The restriction of the Laplace-de Rham operator to n -dimensional (A)dS space from the embedding \mathbb{R}^{n+2} space is finally considered. The (A)dS space is obtained as the intersection of the null cone of \mathbb{R}^{n+2} and an $n + 1$ -dimensional hyper-plane. In that hyper-plane, the (A)dS space is a pseudo-sphere and we recover the geometry of the \mathbb{R}^{n+1} case. The frames designed for the restriction in the \mathbb{R}^{n+1} situation are extended to frames adapted to the \mathbb{R}^{n+2} situation. The restriction of operators is performed in complete analogy to the \mathbb{R}^{n+1} case. Here again, the restricted operators are re-expressed under an intrinsic form.

III. EXPRESSION OF $\square a$ AND $\square \phi$ IN ORTHO-NORMAL FRAMES

In order to restrict the Laplace-de Rham operator \square we first need an expression of it which we can work with. This is achieved by expressing it in an orthonormal basis with its anholonomy coefficients.

Proposition 1. *Let (M, g) a d -dimensional pseudo-Riemannian manifold endowed (at least locally) by a (pseudo-) ortho-normalized frame e_a , $a = 1, \dots, d$, with structure coefficients c_{bc}^a which satisfy*

$$[e_a, e_b] = c_{ab}^c e_c \text{ and } de^a = -\frac{1}{2} c_{bc}^a e^b \wedge e^c.$$

Then, on a one-form or a scalar field, a and ϕ respectively, the Laplace-de Rham operator reads

$$\begin{aligned} \square a = & [\eta^{bc} e_c(e_b(a_a)) + c_{ab}^c e_c(a^b) + \eta^{bc} c_{ac}^d e_b(a_d) - \eta^{bd} \eta_{na} c_{cd}^n e_b(a^c) \\ & + \eta^{bc} c_{pc}^p e_b(a_a) - \eta^{bd} a_c e_b(c_{da}^c) - \eta^{bm} a_c c_{ba}^c c_{pm}^p + a^b e_a(c_{pb}^p) \\ & - \frac{1}{2} a_c \eta^{mf} \eta^{bd} \eta_{na} c_{mb}^c c_{fd}^n] e^a, \end{aligned} \quad (1)$$

and

$$\square \phi = \eta^{ab} [e_a e_b(\phi) + e_a(\phi) c_{pb}^p]. \quad (2)$$

Proof. Eq. (2) is obtained through a straightforward calculation detailed in appendix A3. A lengthier calculation detailed in appendix A4, making use of Eq. (2), establishes Eq. (1). \square

IV. RESTRICTION FROM \mathbb{R}^{n+1}

A. Adapted frames

Let M_n be any n -dimensional submanifold isometrically embedded in some oriented d -dimensional pseudo-euclidian space E . In this situation, for our purpose, it will be convenient to use a so-called local *adapted frame*, that is to say a field of frames of E adapted to this embedding. This is defined as follows.

Definition 1. Let Q be a point of M_n and U_Q a E -neighborhood of Q . A field of positively oriented ortho-normal frames of E : e_A , $A = 0, \dots, d-1$ on U_Q is said to be *adapted to M_n* iff at each point of $V_Q := M_n \cap U_Q$, e_μ , $\mu = 0, \dots, n-1$ is a local field of ortho-normal frames of M_n .

Remark. From this definition, necessarily e_n, \dots, e_{d-1} are orthogonal to M_n .

In this section, we restrict M_n to be the n -dimensional pseudo-sphere denoted Σ_n in \mathbb{R}^{n+1} and defined through

$$y^2 := y^A y_A = -\epsilon H^{-2},$$

H being a positive constant and $\epsilon = \pm 1$. We assume that \mathbb{R}^{n+1} is oriented and that Σ_n is oriented thanks to its outer normal. In the following, we focus on the de Sitter space obtained with $\epsilon = 1$, the signature of E being $(1, n)$, and the Anti-de Sitter space obtained with $\epsilon = -1$, the signature of E being $(2, n - 1)$. Nevertheless, the final results of this section, Eq. (6) and Eq. (7), concerning the \mathbb{R}^{n+1} embedding remain valid in other cases and, in particular, for the euclidean sphere S_n obtained with $\epsilon = -1$, the signature of E being $(n + 1, 0)$. This is no more the case concerning the \mathbb{R}^{n+2} embedding.

Adapted frame for Σ_n in \mathbb{R}^{n+1} . A set of adapted frames of Σ_n can be constructed on \mathbb{R}^{n+1} as follows. In a neighborhood (in the sense of Σ_n) V_Q of some point Q of Σ_n , let e_μ be a field of direct ortho-normal frames of Σ_n defined at each point of V_Q . This field of frames is then extended to a field on Σ_n of frames of \mathbb{R}^{n+1} by adding the unit outer normal to Σ_n : e_n which is, from the very definition of M_n , non isotropic. Then, this whole field of frames e_A is extended to each point of the cone $U_Q = \mathcal{C}(V_Q)$ of \mathbb{R}^{n+1} , which is the union of all the open half-lines coming from the origin of \mathbb{R}^{n+1} and crossing V_Q . This is done by moving each frame identically to itself along the half-lines that intercept the origin of the frame. To this end, each component, in the canonical basis of \mathbb{R}^{n+1} , of a vector that belongs to the frame of U_Q is required to be homogeneous of degree zero. Namely, the e_A^B , defined through $e_A = e_A^B \partial_B$, are homogeneous of degree zero and, more explicitly, since for $y \in U_Q$, we have $y/\sqrt{|y|^2} \in V_Q$ we define $e_A^B(y)$ through $e_A^B(y) = e_A^B(y/\sqrt{|y|^2})$. We note that the unit outer normal explicitly reads

$$e_n = \frac{1}{\sqrt{|y|^2}} D,$$

where D is the dilation vector $D = y^A \partial_A$. We finally remark, that these choices lead to $\eta_{nn} = \eta^{nn} = \text{sgn}(y^2) = -\epsilon$ (in a neighborhood of Q). The adapted frame just built will be used in the following sections.

B. Coefficients of anholonomy

We now compute the coefficients of anholonomy in the field of adapted frames defined Sec. IV A using the general relation $[e_b, e_c] = c_{bc}^a e_a$.

We first note that since $\{e_\mu\}$ is a frame of Σ_n , a sub-manifold of \mathbb{R}^{n+1} , one has $[e_\mu, e_\nu] = c_{\mu\nu}^\lambda e_\lambda$. A first consequence is that the coefficients with Greek-index only do not change when restricted to Σ_n , they will appear inside the Laplace-de Rham operator on Σ_n only and there is no need to compute them explicitly, a second consequence is that $c_{\mu\nu}^n = 0$. Then the anholonomy coefficients we have to calculate are $c_{\mu n}^\nu$ and $c_{\mu n}^n$.

Proposition 2.

$$c_{\mu n}^\nu = \frac{1}{\sqrt{|y^2|}} \delta_\mu^\nu \text{ and } c_{AB}^n = 0.$$

Proof. To obtain $c_{\mu n}^\nu$ we recall from Sec. IV A that the coefficients e_μ^ν are homogeneous functions of degree zero, then on an arbitrary homogeneous function φ of degree r one has:

$$\begin{aligned} [e_n, e_\mu]\varphi &= \frac{1}{\sqrt{|y^2|}} D e_\mu \varphi - e_\mu \frac{1}{\sqrt{|y^2|}} D \varphi \\ &= \frac{r-1}{\sqrt{|y^2|}} e_\mu \varphi - e_\mu \left(\frac{1}{\sqrt{|y^2|}} \right) r \varphi - \frac{1}{\sqrt{|y^2|}} r e_\mu \varphi \\ &= -\frac{e_\mu}{\sqrt{|y^2|}} \varphi, \end{aligned}$$

where, in the r.h.s of the second line, we used the fact that $e_\mu(y^2) = 0$ since $e_\mu(y^2) = \langle dy^2, e_\mu \rangle$. The anholonomy coefficients do not depend on the choice of φ , then the last line of the above calculation shows that

$$c_{\mu n}^\nu = \frac{1}{\sqrt{|y^2|}} \delta_\mu^\nu \text{ and } c_{AB}^n = 0,$$

as stated. \square

Remark. This expression leads to $c_{An}^A = n/\sqrt{|y^2|}$ which corresponds to the result of Eq. (A 2).

Finally, the derivatives of the coefficients of anholonomy are obtained thanks to their homogeneity: since e_A^B is homogeneous of degree zero, $e_A = e_A^B \partial_B$, $\{\partial_B\}$ being the canonical basis of \mathbb{R}^{n+1} , is homogeneous of degree -1 , it follows from the general relation $[e_b, e_c] = c_{bc}^a e_a$, that c_{BC}^A are homogeneous of degree -1 , consequently

$$e_n(c_{BC}^A) = -\frac{1}{\sqrt{|y^2|}} c_{BC}^A \text{ and } e_\lambda(c_{\mu n}^\nu) = 0.$$

C. Restriction to the pseudo-sphere Σ_n

1. Restriction of the one-form operator

We now restrict the one-form $\square_{n+1}a$ to the sphere Σ_n . This is obtained thanks to the pullback l^* of the canonical injection $l : \Sigma_n \rightarrow \mathbb{R}^{n+1}$.

Notation. If α is a p -form on \mathbb{R}^{n+1} we set $\alpha_\Sigma := l^*\alpha$.

From the definition of an adapted frame Sec. IV A one has $l^*e^\mu = e^\mu$ and $l^*e^n = 0$, then for α a one-form:

$$\alpha_\Sigma = l^*\alpha = (l^*\alpha_A)l^*(e^A) = (\alpha_\mu)_\Sigma e^\mu.$$

Proposition 3. *The restricted Laplace-de Rham operator on the one form: $(\square_{n+1}a)_\Sigma = \square_\Sigma a_\Sigma + {}^nB_\Sigma$, reads*

$$(\square_{n+1}a)_\Sigma = \square_\Sigma a_\Sigma - \epsilon H^2 [D^2(a_\mu)_\Sigma e^\mu + (n-1)D(a_\mu)_\Sigma e^\mu + 2d_\Sigma(i_D a) + (n-2)a_\Sigma], \quad (3)$$

where $d_\Sigma = l^*d$.

Proof. In the restriction of $\square_{n+1}a$ from Eq. (1) only the μ -component remains, the contribution to e^n being mapped to zero, that is: $(\square_{n+1}a)_\Sigma = [(\square_{n+1}a)_\mu]_\Sigma e^\mu =: [B_\mu e^\mu]_\Sigma$. The r.h.s. of this last expression divides into two parts: one that contains terms with Greek indexes only, another, that we shall denote nB , contains terms indexed by n . The former leads after restriction to the Laplace-de Rham operator on Σ_n acting on a_Σ : $\square_\Sigma(a_\Sigma)$, the latter is calculated in appendix A 5, one finds

$${}^nB = -\epsilon \frac{1}{|y^2|} [D^2(a_\mu)e^\mu + (n-1)D(a_\mu)e^\mu + 2e_\mu(i_D a)e^\mu + (n-2)a_\mu e^\mu]. \quad (4)$$

Adding this latter term to the former, with $|y^2| = H^{-2}$, proves the formula. \square

Remark. For a transverse one-form ($y^A a_A = 0$) the term $d_\Sigma(i_D a)$ vanishes. Also, for a homogeneous of degree r (thus a_μ homogeneous of degree $r-1$) the above expression reduces to

$$(\square_{n+1}a)_\Sigma = \square_\Sigma a_\Sigma - \epsilon H^2 [(r+1)(r+n-2)a_\Sigma + 2d_\Sigma(i_D a)]. \quad (5)$$

Eq. (3) might be recast in an alternate form containing the Lie derivative along D which provides us the following key formula.

Theorem 1. *The Laplace-de Rham operator acting upon a one-form a restricted from \mathbb{R}^{n+1} to Σ_n reads:*

$$(\square_{n+1}a)_\Sigma = \square_\Sigma a_\Sigma - \epsilon H^2[\mathcal{L}_D^2(a) + (n-3)\mathcal{L}_D(a) + 2d(i_D a)]_\Sigma, \quad (6)$$

where no reference frame is involved.

Proof. Since $\mathcal{L}_D e^A = e^A$, thus implying $D(a_A)e^A = \mathcal{L}_D(a) - a$ and $D^2(a_A)e^A = \mathcal{L}_D^2(a) - 2\mathcal{L}_D(a) + a$, then making these replacements in the term ${}^n B_\Sigma$ recasts Eq. (3) into the intrinsic form Eq. (6). \square

Remark (1). Eq. (6) with $\epsilon = -1$ is still valid in the case of the euclidean sphere ($\Sigma_n = S_n$) embedded in the euclidean space \mathbb{R}^{n+1} . This can be proved through a straightforward adaptation of our proof.

Remark (2). Thanks to the Weitzenböck formula, see appendix A 1, Eq. (6) can be adapted to the Laplace-Beltrami operator:

$$(\Delta_{n+1}a)_\Sigma = \Delta_\Sigma a_\Sigma - \epsilon H^2[\mathcal{L}_D^2(a) + (n-3)\mathcal{L}_D(a) + 2d(i_D a)]_\Sigma - r(\sharp a_\Sigma, \text{Id}),$$

where r is the Ricci tensor and $r(\sharp a_\Sigma, \text{Id})$ is the 1-form such that $r(\sharp a_\Sigma, \text{Id})(u) = r(\sharp a_\Sigma, u)$.

2. Restriction of the scalar operator

For completeness, we note that the scalar operator can be restricted along the same lines as the one-form. The splitting between Greek-indexed and n -indexed term in Eq. (2) leads again to a first term which restricts in $\square_\Sigma \phi_\Sigma$ and a second term which, keeping the notation of Sec. IVC 1 for simplicity, reads

$${}^n B := \eta^{nn}[e_n^2(\phi) + e_n(\phi)c_{An}^A].$$

Thanks to the expressions of η^{nn} , e_n and c_{An}^A Sec. IV A-IV B one obtains

$${}^n B := -\frac{\epsilon}{|y|^2} (D^2 + (n-1)D) \phi.$$

Using the above formula then proves the following.

Theorem 2. *The restriction of the Laplace-de Rham operator on a scalar field ϕ reads*

$$(\square_{n+1}\phi)_\Sigma = \square_\Sigma \phi_\Sigma - \epsilon H^2[D^2(\phi) + (n-1)D(\phi)]_\Sigma, \quad (7)$$

independently of any frame.

Once again this formula, with $\epsilon = -1$ is still valid in the case of the euclidean sphere ($\Sigma_n = S_n$) embedded in the euclidean space \mathbb{R}^{n+1} .

Remark. If ϕ is homogeneous of degree r the above expression reduces to

$$(\square_{n+1}\phi)_\Sigma = \square_\Sigma\phi_\Sigma - \epsilon H^2 r(r+n-1)\phi_\Sigma.$$

V. RESTRICTION FROM \mathbb{R}^{n+2}

A. Adapted frames of \mathbb{R}^{n+2}

Hereafter, \mathbb{R}^{n+2} stands for the oriented pseudo-euclidean space, the signature of the metrics being $(2, n)$.

Notation. In \mathbb{R}^{n+2} , we consider the $n+1$ -dimensional plane P defined by the equation $f(y) = 1$ where f is a homogeneous polynomial of degree one. We assume that the normal vector field of P is nowhere isotropic and we note $F = \sharp df$ this normal vector, and also $F^2 = \epsilon H^2$ where $H > 0$.

Proposition 4. *The intersection Σ_n of the null cone \mathcal{C} of \mathbb{R}^{n+2} and P , endowed with the metrics induced from that of the ambient space, is the de Sitter or Anti-de Sitter space, whether $\epsilon = 1$ or $\epsilon = -1$, with radius H . The position of F , relative to the null cone of \mathbb{R}^{n+2} , is “time-like” ($F^2 > 0$) for de Sitter space and “space-like” ($F^2 < 0$) for Anti-de Sitter space. In this construction P is identified with the \mathbb{R}^{n+1} of the previous section $P \simeq \mathbb{R}^{n+1}$.*

Proof. We can, without loss of generality, consider that the canonical Cartesian coordinates basis of \mathbb{R}^{n+2} is such that

$$f(y) = Hy^{n+1}, \quad H > 0, \quad (8)$$

and that the metric reads $\eta = \text{diag}(+, -, \dots, -, -\epsilon, \epsilon)$. See [17] for another choice to control the zero curvature limit. This implies that

$$F = \sharp df = \epsilon H \partial_{n+1}. \quad (9)$$

In effect, written in this coordinate system, the equations for the intersection of \mathcal{C} and P

$$\begin{cases} C(y) = y^\mu y_\mu - \epsilon(y^n)^2 + \epsilon(y^{n+1})^2 = 0, \\ f(y) = Hy^{n+1} = 1, \end{cases}$$

define Σ_n as the de Sitter, or Anti-de Sitter space for, respectively, $\epsilon = +1$, and $\epsilon = -1$. \square

Note, for future use, that

$$D = y^\alpha \partial_\alpha = D_P + D_{n+1}, \quad (10)$$

where $D_P := y^A \partial_A$ is the dilation operator of P , that appears in the \mathbb{R}^{n+1} reduction Sec. IV, and $D_{n+1} := y^{n+1} \partial_{n+1}$.

Adapted frame for Σ_n in \mathbb{R}^{n+2} . Now, a field of frames adapted to Σ_n in \mathbb{R}^{n+2} is built as follows. We first build the adapted frames to Σ_n in $U_Q \subset P$: $\{e_\mu, e_n\}$, as described in Sec. IV A. Then, one supplements each frame $\{e_\mu, e_n\}$ by the normal to P : ∂_{n+1} , located at the same point of P . Finally, we define the cylinder W_Q which is the union of all the “vertical” lines directed by ∂_{n+1} and crossing U_Q . We then extend the frames $\{e_\mu, e_n, \partial_{n+1}\}$ of P to $W_Q \subset \mathbb{R}^{n+2}$ by translating, without change, along the vertical lines. Namely, for $y \in W_Q$ we note y_P the orthogonal projection of y on P and define $e_A^B(y)$ through the formula $e_A^B(y) = e_A^B(y_P) = e_A^B(y^0, \dots, y^n)$ and, writing for simplicity $e_{n+1} = \partial_{n+1}$, $e_{n+1}^A = e_A^{n+1} = 0$ and $e_{n+1}^{n+1} = 1$. We thus obtained an adapted frame $e_\alpha, \alpha = 0, \dots, n+1$ fulfilling the important properties $D(e_\alpha^\beta) = F(e_\alpha^\beta) = 0$, that is: the coefficients are homogeneous of degree 0 and do not depend on y^{n+1} . This implies that $[e_\mu, \partial_{n+1}] = [e_n, \partial_{n+1}] = 0$. That is, all anholonomy coefficients containing the index $n+1$ are equal to zero. The adapted frame just built will be used in the following sections.

B. Restriction to (A)dS spaces

1. Restriction of the one-form operator

In the present paragraph we derive the restriction of the one form $\square_{n+2} a$ to the submanifold Σ_n , the (A)dS space. We consider Eq. (1) as the starting point, we apply on it the pullback m^* of the canonical injection map $m : \Sigma_n \rightarrow \mathbb{R}^{n+2}$.

Notation. If α is a p -form on \mathbb{R}^{n+2} we set $\alpha_\Sigma := m^* \alpha$.

From the definition of an adapted frame (see Sec. IV A) one has $m^* e^\mu = e^\mu$ and $m^* e^n = m^* e^{n+1} = 0$, then, if α is a 1-form,

$$\alpha_\Sigma = m^* \alpha = (m^* \alpha_\alpha) m^*(e^\alpha) = (\alpha_\mu)_\Sigma e^\mu.$$

Proposition 5. *The restricted Laplace-de Rham operator on the one form $(\square_{n+2}a)_\Sigma$ reads*

$$m^*(\square_{n+2}a) = \square_\Sigma a_\Sigma + m^* \left[\epsilon(\partial_{n+1}^2 a_\mu) e^\mu - \epsilon H^2 \left(\mathcal{L}_{D_P}^2(a) + (n-3)\mathcal{L}_{D_P}(a) + 2d(i_{D_P}a) \right) \right]. \quad (11)$$

Proof. The calculation parallels that of Sec. [IV C 1](#). The r.h.s. of the pullback by m of Eq. (1) divides now into three parts: the first one, containing only Greek indexes μ, ν, \dots , leads after restriction to the Laplace-de Rham operator on Σ_n acting on a_Σ : $\square_\Sigma(a_\Sigma)$, the second one denoted nB as in Sec. [IV C 1](#), contains terms indexed by n but not by $n+1$, the last one contains terms indexed by $n+1$, and since anholonomy coefficients involving $n+1$ are equal to zero, it reduces to $m^*\epsilon(\partial_v^2 a_\mu) e^\mu$. The calculation of nB is the same as in the \mathbb{R}^{n+1} embedding space Sec. [IV C 1](#), one obtains thus Eq. (4) with the replacements $D \rightarrow D_P$ and $y^2 \rightarrow y_P^2 := y^A y_A$, that accounts for the embedding space \mathbb{R}^{n+2} in place of \mathbb{R}^{n+1} . \square

Finally, the second term of the r.h.s. of this equation can be recast under an intrinsic form, that is without reference to specific frames.

Theorem 3. *The Laplace-de Rham operator acting upon the one-form a restricted from \mathbb{R}^{n+2} to Σ_n reads:*

$$\begin{aligned} m^*(\square_{n+2}a) &= \square_\Sigma a_\Sigma + m^* \left[-\epsilon H^2 \left(\mathcal{L}_D^2 + (n-3)\mathcal{L}_D + 2di_D \right) + 2\mathcal{L}_F \mathcal{L}_D \right. \\ &\quad \left. + (n-4)\mathcal{L}_F + 2di_F \right] (a) \\ &= \square_\Sigma a_\Sigma + m^* \left[-\epsilon H^2 \left(\mathcal{L}_D^2 + (n-3)\mathcal{L}_D + 2di_D \right) + 2\mathcal{L}_D \mathcal{L}_F \right. \\ &\quad \left. + n\mathcal{L}_F + 2di_F \right] (a), \end{aligned} \quad (12)$$

Proof. The details of the calculation can be found in appendix [A 6](#). Switching between both expressions makes use of $[F, D] = 2F$, due to the homogeneity of f . \square

Remark. Thanks to the Weitzenböck formula, see appendix [A 1](#), Eq. (12) can be adapted to the Laplace-Beltrami operator:

$$\begin{aligned} (\Delta_{n+2}a)_\Sigma &= \Delta_\Sigma a_\Sigma + m^* \left[-\epsilon H^2 \left(\mathcal{L}_D^2 + (n-3)\mathcal{L}_D + 2di_D \right) + 2\mathcal{L}_F \mathcal{L}_D \right. \\ &\quad \left. + (n-4)\mathcal{L}_F + 2di_F \right] (a) - r(\sharp a_\Sigma, \text{Id}), \end{aligned}$$

where r is the Ricci tensor and $r(\sharp a_\Sigma, \text{Id})$ is the 1-form such that $r(\sharp a_\Sigma, \text{Id})(u) = r(\sharp a_\Sigma, u)$.

2. Restriction of the scalar operator

Let us consider again the scalar case. Eq. (2) for a, not necessarily homogeneous, scalar field ϕ splits, as in the one-form case, into three terms corresponding to those containing Greek indexes only, those involving the index n , and those indexed by $n+1$. The part relating to Greek indexes only, leads, upon reduction, to the scalar Laplace-de Rham operator on Σ_n , the (A)dS space, the others are calculated in a straightforward way in appendix A7. This proves the following formula.

Theorem 4. *The Laplace-de Rham operator acting upon a scalar field ϕ restricted from \mathbb{R}^{n+2} to Σ_n reads:*

$$\begin{aligned} m^*(\square_{n+2}\phi) &= \square_{\Sigma}\phi_{\Sigma} - \epsilon H^2 [D^2\phi + (n-1)D\phi]_{\Sigma} + [2FD\phi + (n-2)F\phi]_{\Sigma} \\ &= \square_{\Sigma}\phi_{\Sigma} - \epsilon H^2 [D^2\phi + (n-1)D\phi]_{\Sigma} + [2DF\phi + (n+2)F\phi]_{\Sigma}. \end{aligned} \quad (13)$$

Remark. For ϕ homogeneous of degree r , the above expression reduces to

$$(\square_{n+2}\phi)_{\Sigma} = \square_{\Sigma}\phi_{\Sigma} - \epsilon H^2 r(r+n-1)\phi_{\Sigma} + (2r+n-2)F(\phi)_{\Sigma}.$$

Note that for $n=4$ we retrieve our previous result (Eq. (9) of [22]) as a particular case.

VI. THE ADDITIONAL TERMS

We observe that all restricted operators we obtained in previous sections Eqs. (6, 7, 12, 13), share the common structure:

$$(\square_d\alpha)_{\Sigma} = \square_{\Sigma}\alpha_{\Sigma} + \text{AT}(\alpha), \quad (14)$$

where α is a one-form or a scalar field and AT stands for ‘‘Additional Term’’.

A natural question is whether the Additional Term has some universal characteristic or can be anything. In order to address this question, we are interested in the converse situation in which one considers as a starting point the expression

$$\square_{\Sigma}\beta + \chi(\beta), \quad (15)$$

where β and χ live on Σ_n and are respectively a one-form or scalar field, and an arbitrary smooth operator. We will show that it is possible to choose an extension, to the embedding

space, of β such that the Additional Term corresponding to this extension is equal to χ . Such an extension of β will be denoted $\varpi(\beta)$ where ϖ is a right inverse (sometimes called a *section*) of m^* : $m^* \circ \varpi = \text{Id}$. The answer to this question is given by the following theorem.

Theorem 5. *Let β be a 1-form or a scalar field on Σ_n and χ a smooth operator, then there exists an extension ϖ_χ such that*

$$(\square_{n+2}\varpi_\chi(\beta))_\Sigma = \square_\Sigma\beta + \chi(\beta).$$

In other words, any additional term χ on the embedded pseudo-sphere Σ_n can be obtained by the choice of an appropriate extension of the one-form or scalar field. One can see easily that proving this theorem is equivalent to finding an extension ϖ_χ such that $\text{AT} \circ \varpi_\chi = \chi$. The following subsections are devoted to building explicitly these extensions.

Note that the same result holds in \mathbb{R}^{n+1} context and, for technical reasons, we begin with addressing this case.

A. Additional terms in the \mathbb{R}^{n+1} context

We recall that l is the canonical injection of Σ_n into \mathbb{R}^{n+1} and its pullback l^* the corresponding restriction for scalar and one-forms.

Definition 2. Let $y \in \mathbb{R}^{n+1}$ be a point such that $\text{sgn}(y^A y_A) = -\epsilon$, let ρ be a map over Σ_n , and let $r \in \mathbb{R}$. Then, ρ is *extended by homogeneity* from Σ_n to \mathbb{R}^{n+1} by setting

$$\rho^{(r)}(y) = \rho \left(\frac{y}{H\sqrt{|y_A y^A|}} \right) \left(H\sqrt{|y_A y^A|} \right)^r. \quad (16)$$

Remark. The condition $\text{sgn}(y^A y_A) = -\epsilon$ ensures that the half-line emanating from the center of \mathbb{R}^{n+1} and containing y intercepts Σ_n .

Since $y/(H\sqrt{|y_A y^A|}) \in \Sigma_n$, then $\rho^{(r)}$ is homogeneous of degree r and $(\rho^{(r)})_\Sigma = \rho$ by construction. In regards to the one-forms, one can define the extension by homogeneity of order s for any one-form $h = h_\mu e^\mu$ on Σ_n through

$$h^{(s)} = h_\mu^{(s-1)} e^\mu.$$

Remark that, since D is along the (outer) normal to Σ_n , any one-forms $k = k_\mu e^\mu$ on E satisfy the transversality condition $i_D k = 0$. This is the case for $h^{(s)}$ which is thus an

extension of h , transverse and homogeneous of degree s . From Eq. (6) we see that AT, in the present situation, involves the Lie derivative \mathcal{L}_D . Let us determine its action on $h^{(s)}$, one has successively

$$\begin{aligned}
\mathcal{L}_D(h^{(s)}) &= i_D dh_\mu^{(s-1)} e^\mu + d i_D h^{(s)} \\
&= D(h_\mu^{(s-1)}) e^\mu + h_\mu^{(s-1)} i_D de^\mu \\
&= (s-1) h_\mu^{(s-1)} e^\mu - \frac{1}{2} h_\mu^{(s-1)} \sqrt{|y^2|} i_n c_{AB}^\mu e^A \wedge e^B \\
&= (s-1) h^{(s)} - \frac{1}{2} h_\mu^{(s-1)} \sqrt{|y^2|} c_{AB}^\mu (\delta_n^A e^B - \delta_n^B e^A) \\
&= (s-1) h^{(s)} - h_\mu^{(s-1)} \sqrt{|y^2|} c_{nA}^\mu e^A \\
&= s h^{(s)},
\end{aligned}$$

where transversality has been used to eliminate the second term on the r.h.s. in the first line.

Proposition 6. *Let $b := \beta$ in Eq. (15) be a one-form field on Σ_n , then a section ϖ_χ such that $AT \circ \varpi_\chi(b) = \chi(b)$ is given by*

$$\varpi_\chi(b) = b^{(0)} - \epsilon H^{-2} (\chi^{(s)} - \chi^{(0)}),$$

with s a solution of $S(s) = 1$ where $S(X) = X^2 + (n-3)X$.

Proof. Note first, that applying l^* to both members of the above relation shows that ϖ is a section of l^* . Then, starting from Eq. (6), we obtain:

$$AT = -\epsilon H^2 l^* \circ [S(\mathcal{L}_D) + 2d(i_D)],$$

and

$$\begin{aligned}
AT(\varpi_\chi(b)) &= -\epsilon H^2 l^* S(\mathcal{L}_D) \varpi_\chi(b) \\
&= -\epsilon H^2 l^* S(\mathcal{L}_D) [b^{(0)} - \epsilon H^{-2} (\chi^{(s)}(b) - \chi^{(0)}(b))] \\
&= 0 + l^* S(\mathcal{L}_D) (\chi^{(s)}(b)) \\
&= S(s) l^* (\chi^{(s)}(b)) \\
&= \chi(b),
\end{aligned}$$

where the transversality of $\chi^{(s)}(b)$ has been used in the first and the third line. \square

The scalar case can be treated in a similar way and we can prove the following proposition in a similar manner.

Proposition 7. *Let $\psi := \beta$ be a scalar field on Σ_n , and let ϖ_χ be defined through*

$$\varpi_\chi(\psi) = \psi^{(0)} - \epsilon H^{-2}(\chi^{(s)}(\psi) - \chi^{(0)}(\psi)),$$

where s is a solution of the equation $Q(s) := s^2 + (n-1)s = 1$. Then ϖ_χ is a section of l^* , that is it fulfills $l^* \circ \varpi_\chi = Id$, which satisfies $AT \circ \varpi_\chi = \chi$.

B. Additional terms in the \mathbb{R}^{n+2} context

As in the previous section, we have to find ϖ'_χ a section of m^* such that $AT' \circ \varpi'_\chi = \chi$ where AT' is the additional term in \mathbb{R}^{n+2} context, see Eq. (12), namely

$$AT' = m^* [-\epsilon H^2(S(\mathcal{L}_D) + 2di_D) + 2\mathcal{L}_D\mathcal{L}_F + n\mathcal{L}_F + 2di_F].$$

Let p be the orthogonal projection of \mathbb{R}^{n+2} onto $P \sim \mathbb{R}^{n+1}$ (remember that the normal vector of P is non-isotropic), one can verify that $l = p \circ m$ and, as a consequence, $p^* \circ \varpi_\chi$ is a section of m^* .

Proposition 8. *Let p be the orthogonal projection of \mathbb{R}^{n+2} onto $P \sim \mathbb{R}^{n+1}$, as above, let ϖ_χ be a section in \mathbb{R}^{n+1} , as in Proposition 6 or 7. Then $\varpi'_\chi := p^* \circ \varpi_\chi$ fulfills $AT' \circ \varpi'_\chi = \chi$.*

Before proving that, remarks regarding p^* are in order. Let $h = h_A e^A$ be a one-form on $P \sim \mathbb{R}^{n+1}$, then we have

$$(p^*h)(y^0, \dots, y^n, y^{n+1}) = h_A(y^0, \dots, y^n)e^A.$$

This proves several facts. First $\mathcal{L}_F \circ p^* = i_F \circ p^* = 0$, second p^*h is homogeneous of degree s as soon as h is, and finally $i_D p^*h = 0$ as soon as $i_D h = 0$.

Proof. Taking into account the properties of p^* , in the case of the one-form field we have

$$\begin{aligned} AT' \circ \varpi'_\chi &= m^* [-\epsilon H^2(S(\mathcal{L}_D) + 2di_D) + 2\mathcal{L}_D\mathcal{L}_F + n\mathcal{L}_F + 2di_F] \circ p^* \circ \varpi_\chi \\ &= -\epsilon H^2 m^* S(\mathcal{L}_D) \circ p^* \circ \varpi_\chi \\ &= -\epsilon H^2 m^* S(s) p^* \circ \varpi_\chi \\ &= -\epsilon H^2 l^* \circ \varpi_\chi \\ &= \chi. \end{aligned}$$

The scalar case can be treated in the same way: replacing ϖ_χ with $p^* \circ \varpi_\chi$ leads to the same result. \square

We again see that any operator $\square_\Sigma b + \chi(b)$ on Σ_n can be reached as a restriction of the operator $\square_{n+2} a$ where a is an appropriate extension of b .

VII. CONCLUSION

The keystone of our construction is the existence of an adapted frame for which the commutation relations, leading to anholonomic coefficients, are easily calculated. This adapted frame is deeply related to the pseudo-sphere and the maximal symmetry of (A)dS. Other space-times are no longer maximally symmetric. Nevertheless, many of them share the same conformal group $SO(2, n)$, including Friedmann-Lemaître-Robertson-Walker spaces. This should allow us, in a future work, to generalize to these spaces the results obtained here.

Appendix A: Details on calculations

1. Overview of the main notions of differential geometry used in this work

We gather and comment here some definitions and formulas from differential geometry frequently used in this paper (the notations are those of [29]). We also relate, using our notations, the Laplace-de Rham to the Laplace-Beltrami operators through the Weitzenböck formula.

We consider an oriented pseudo-Riemannian d -dimensional manifold (M, g) and the set $\Lambda(M) = \bigoplus_p \Lambda^p(M)$ of differential forms on M . We note $\langle \alpha, v \rangle$ the natural pairing between a 1-form $\alpha \in \Lambda^1(M)$ and a vector $v \in T(M)$. We let, as usual, $\sharp\alpha$ be the vector such that $g(\sharp\alpha, v) = \langle \alpha, v \rangle$ and $\flat v$ be the 1-form such that $\langle \flat v, u \rangle = g(v, u)$ for any vector u . We often use the notation \sim instead of both \sharp and \flat . We note also $\hat{\eta}$ the linear operator of $\Lambda(M)$ defined through: $\hat{\eta}\alpha = (-1)^p \alpha$ iff $\alpha \in \Lambda^p(M)$.

Beside the well-known interior product i_v we define the corresponding creator j_v through $j_v\alpha = \tilde{v} \wedge \alpha$ where $v \in T(M)$. The basic relationship between them is

$$i_u j_v + j_v i_u = g(u, v).$$

The interior product relates to the Lie derivatives through the Cartan formula:

$$\mathcal{L}_v = i_v d + d i_v.$$

In this paper we make intensive use of positively oriented orthonormal frames e_a verifying $g(e_a, e_b) = \eta_{ab}$ and their dual basis e^a . In such a frame, the volume form reads $\omega = e^1 \wedge \dots \wedge e^d$. We note $i^a = \eta^{ab} i_{e_b}$ and $j^a = \eta^{ab} j_{e_b}$ and obtain the very useful algebra:

$$\eta^{ab} = i^a j^b + j^b i^a,$$

$$i^a i^b = -i^b i^a,$$

$$j^a j^b = -j^b j^a.$$

In order to introduce the Hodge operator, we begin with the inner product of 1-forms $(\alpha, \beta)_g = g(\tilde{\alpha}, \tilde{\beta})$ that we extend to the Gram inner product of two p -forms $\alpha_1 \wedge \dots \wedge \alpha_p$ and $\beta_1 \wedge \dots \wedge \beta_p$, where α_i and β_i are 1-forms, through

$$(\alpha_1 \wedge \dots \wedge \alpha_p, \beta_1 \wedge \dots \wedge \beta_p)_g = \det M,$$

where M is the $p \times p$ matrix whose entries are $M_{ij} = (\alpha_i, \beta_j)_g$.

We can now introduce the Hodge operator $\alpha \mapsto *\alpha$, $\Lambda^p(M) \rightarrow \Lambda^{d-p}(M)$ such that, for any p -form β ,

$$\beta \wedge *\alpha = (\beta, \alpha)_g \omega.$$

This operator fulfills the elementary properties

$$*^{-1} = \text{sgn}(g) * \hat{\eta}^{d+1},$$

$$*\omega = \text{sgn}(g),$$

$$*1 = \omega,$$

and interacts with the i and j operators in the following way

$$*i_v = -j_v * \hat{\eta},$$

$$*j_v = i_v * \hat{\eta},$$

from which we obtain the useful formulas

$$*(e^a \wedge \cdots \wedge e^b) = i^b \dots i^a \omega,$$

$$*i^a \dots i^b \omega = \text{sgn}(g) \hat{\eta}^{d+1} e^b \wedge \cdots \wedge e^a.$$

Moreover, the $*$ -operator allows us to define the co-differential operator δ on $\Lambda(M)$ through

$$\delta = *^{-1} d * \hat{\eta},$$

which, in the case of 1-forms, relates to the usual divergence through $\delta\alpha = -\text{div}(\tilde{\alpha})$. We can now define the Laplace-de Rham operator \square , which we use intensively in this paper, through

$$\square = -(d\delta + \delta d).$$

This operator is related to the usual Laplace-Beltrami operator. Indeed, if ∇ is the Levi-Civita connection associated to g then the second covariant derivative $\nabla_{uv}^2 = \nabla_u \nabla_v - \nabla_{\nabla_u v}$, where u and v are tangent vectors, brings about the Laplace-Beltrami operator through its trace, namely:

$$\Delta = \text{tr} \nabla^2 = g^{ab} (\nabla_a \nabla_b - \nabla_{\nabla_a e_b}).$$

The link between the Laplace-de Rham \square and the Laplace-Beltrami Δ operators on p -forms is given by the Weitzenböck formula, which we recall here with a short proof written in our notations.

Theorem 6 (Weitzenböck formula). *Let α a p -form and e_a any frame, then*

$$\square\alpha = \Delta\alpha + j^a i^b R(e_a, e_b)\alpha, \quad (\text{A1})$$

where $R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u, v]}$ is the curvature operator and ∇ the Levi-Civita connection.

Proof. We first recall the link between the operators d , δ and the Levi-Civita connection ∇ :

$$\begin{aligned} \delta &= -i^a \nabla_a, \\ d &= j^a \nabla_a, \end{aligned}$$

in any frame e_a . We now choose a point p of the manifold and a normal (geodesic) frame centered at p . As a consequence, in this frame, $\Gamma_{ab}^c(p) = 0$ and $g_{ab}(p) = \eta_{ab}$. We can now work at this point and in this frame and obtain

$$\begin{aligned} d\delta\alpha &= -j^a \nabla_a i^b \nabla_b \alpha \\ &= -j^a i^b \nabla_a \nabla_b \alpha, \end{aligned}$$

and

$$\begin{aligned} \delta d\alpha &= -i^b \nabla_b j^a \nabla_a \alpha \\ &= -i^b j^a \nabla_b \nabla_a \alpha \\ &= +j^a i^b \nabla_b \nabla_a \alpha - \eta^{ab} \nabla_b \nabla_a \alpha. \end{aligned}$$

This proves that, in this frame at the point p ,

$$\square\alpha = \eta^{ab} \nabla_b \nabla_a \alpha + j^a i^b (\nabla_a \nabla_b - \nabla_b \nabla_a)\alpha.$$

Moreover, in this frame, $\Delta = \eta^{ab} \nabla_a \nabla_b$ and $R(e_a, e_b) = (\nabla_a \nabla_b - \nabla_b \nabla_a)$, and we obtain that the formula A1 is correct in this frame and at this point. However R is tensorial in each entries and then the formula is independent of the frame. \square

This formula takes a simpler form in two special cases.

Scalar case ($p = 0$). Since R vanishes when applied on scalar field we get

$$\square\alpha = \Delta\alpha. \quad (\text{A2})$$

1-form case ($p = 1$). Since $[\nabla, \sharp] = 0$ we have

$$\begin{aligned}
j^a i^b R(e_a, e_b) \alpha &= j^a i^b \sharp \sharp R(e_a, e_b) \alpha_c e^c \\
&= j^a i^b \sharp R(e_a, e_b) \sharp \alpha_c e^c \\
&= j^a i^b \alpha^c \sharp R(e_a, e_b) e_c \\
&= j^a i^b \alpha^c \sharp e_d R^d_{cab} \\
&= -\alpha^c j^a i^b \eta_{de} e^e R^d_{cba} \\
&= -\alpha^c j^a \eta^{be} \eta_{de} R^d_{cba} \\
&= -\alpha^c j^a \delta_d^b R^d_{cba} \\
&= -\alpha^c j^a R^b_{cba} \\
&= -\alpha^c j^a r_{ca} \\
&= -j^a r(\sharp \alpha, e_a),
\end{aligned}$$

with r the Ricci tensor. In this case the Weitzenböck formula becomes

$$\square \alpha = \Delta \alpha - j^a r(\sharp \alpha, e_a),$$

in other words

$$\square \alpha(u) = \Delta \alpha(u) - r(\sharp \alpha, u).$$

2. Calculation of $\delta(e^a)$

$$\begin{aligned}
\delta(e^a) &= -\operatorname{sgn}(g) * d * e^a \\
&= -\operatorname{sgn}(g) * di^a \omega \\
&= -\operatorname{sgn}(g) \eta^{aa} * d(-1)^{a-1} e^1 \wedge \cdots \wedge e^a \wedge \cdots \wedge e^d \\
&= \operatorname{sgn}(g) (-1)^a \eta^{aa} * J,
\end{aligned}$$

with $J = d(e^1 \wedge \cdots \wedge e^a \wedge \cdots \wedge e^d)$. Thus, using exceptionally the symbol \sum to indicate summations, one has successively

$$\begin{aligned}
J &= \sum_{p < a} (-1)^{p-1} e^1 \wedge \cdots \wedge de^p \wedge \cdots \wedge e^a \wedge \cdots \wedge e^d \\
&\quad + \sum_{p > a} (-1)^p e^1 \wedge \cdots \wedge e^a \wedge \cdots \wedge de^p \wedge \cdots \wedge e^d \\
&= \sum_{p < a} (-1)^p e^1 \wedge \cdots \wedge \left(\sum_{b < c} c_{bc}^p e^b \wedge e^c \right) \wedge \cdots \wedge e^a \wedge \cdots \wedge e^d \\
&\quad + \sum_{p > a} (-1)^{p+1} e^1 \wedge \cdots \wedge e^a \wedge \cdots \wedge \left(\sum_{b < c} c_{bc}^p e^b \wedge e^c \right) \wedge \cdots \wedge e^d \\
&= \sum_{p < a} (-1)^p e^1 \wedge \cdots \wedge (c_{pa}^p e^p \wedge e^a) \wedge \cdots \wedge e^a \wedge \cdots \wedge e^d \\
&\quad + \sum_{p > a} (-1)^{p+1} e^1 \wedge \cdots \wedge e^a \wedge \cdots \wedge (c_{ap}^p e^a \wedge e^p) \wedge \cdots \wedge e^d \\
&= \sum_{p < a} (-1)^p (-1)^{p-a+1} c_{pa}^p \omega + \sum_{p > a} (-1)^{p+1} (-1)^{a+1-p} c_{ap}^p \omega \\
&= (-1)^{a+1} \omega \sum_{p \neq a} c_{pa}^p.
\end{aligned}$$

Finally, returning to the Einstein summation convention, one gets

$$\delta(e^a) = -\eta^{ab} c_{pb}^p.$$

3. Derivation of Eq. (2)

The expression Eq. (2) for the scalar field is obtained through the following steps:

$$\begin{aligned}
\Box\phi &= -\delta d\phi \\
&= -\delta e_a(\phi)e^a \\
&= - *^{-1} d * \hat{\eta} e_a(\phi)e^a \\
&= -\text{sgn}(g) * \hat{\eta}^{d+1} d * (-1)e_a(\phi)e^a \\
&= \text{sgn}(g) * (-1)^{d(d+1)} d e_a(\phi) i^a \omega \\
&= \text{sgn}(g) * (d(e_a(\phi)) \wedge i^a \omega + e_a(\phi) d i^a \omega) \\
&= \text{sgn}(g) * (e_b(e_a(\phi)) e^b \wedge i^a \omega + e_a(\phi) \mathcal{L}_{\tilde{e}^a}(\omega)) \\
&= \text{sgn}(g) * (e_b(e_a(\phi)) j^b i^a \omega + e_a(\phi) \mathcal{L}_{\tilde{e}^a}(\omega)) \\
&= \text{sgn}(g) * (\eta^{ab} e_b(e_a(\phi)) \omega - e_a(\phi) \delta(e^a) \omega) \\
&= \eta^{ab} (e_b(e_a(\phi)) - e_a(\phi) \delta(e^a)) \\
&= \eta^{ab} [e_a e_b(\phi) + e_a(\phi) c_{pb}^p],
\end{aligned}$$

where the expression of the term $\delta(e^a)$, derived in appendix A 2, has been used in the last line.

We note for future reference that $\delta\alpha$, α being a one-form, is obtained by the replacement of $e_a(\phi)$ by α_a in the above calculation leading to

$$\delta(\alpha) = -\eta^{ab} e_a(\alpha_b) + \alpha_a \delta(e^a). \quad (\text{A3})$$

4. Derivation of Eq. (1)

In order to express $\Box a$ in an ortho-normal basis. We first compute the Maxwell operator:

$$-\delta d(a_a e^a) = \underbrace{-\delta d(a_a)}_I \wedge e^a + \underbrace{-\delta a_c}_{II} d e^c.$$

Calculation of I :

$$\begin{aligned}
I &= -\delta e_b(a_a) e^b \wedge e^a \\
&= (-1)^d \operatorname{sgn}(g) * d * e_b(a_a) e^b \wedge e^a \\
&= (-1)^d \operatorname{sgn}(g) * d e_b(a_a) i^a i^b \omega \\
&= (-1)^d \operatorname{sgn}(g) \left(\underbrace{*d(e_b(a_a)) i^a i^b \omega}_{I_1} + \underbrace{*e_b(a_a) d i^a i^b \omega}_{I_2} \right).
\end{aligned}$$

For the term I_1 one has successively:

$$\begin{aligned}
I_1 &= *d(e_b(a_a)) i^a i^b \omega \\
&= *e_c((e_b(a_a))) j^c i^a i^b \omega \\
&= *e_c((e_b(a_a))) (\eta^{ca} - i^a j^c) i^b \omega \\
&= e_a((e_b(a_a))) * i^b \omega - e_c((e_b(a_a))) * i^a \eta^{cb} \omega \\
&= \operatorname{sgn}(g) (-1)^{d+1} (e_a((e_b(a_a))) e^b - \eta^{cb} e_c((e_b(a_a))) e^a) \\
&= \operatorname{sgn}(g) (-1)^{d+1} (e_a e_b(a_a) e^b - \eta^{cb} e_c e_b(a_a) e^a).
\end{aligned}$$

For the term I_2 one has successively:

$$\begin{aligned}
I_2 &= *e_b(a_a) d i^a i^b \omega \\
&= *e_b(a_a) (\mathcal{L}_{\tilde{e}^a} - i^a d) i^b \omega \\
&= *e_b(a_a) \left[\mathcal{L}_{\tilde{e}^a}(\tilde{e}^b] \omega - i^a d i^b \omega \right] \\
&= *e_b(a_a) \left[\mathcal{L}_{\tilde{e}^a}(\tilde{e}^b] \omega + \tilde{e}^b] \mathcal{L}_{\tilde{e}^a}(\omega) - i^a \mathcal{L}_{\tilde{e}^b} \omega \right] \\
&= *e_b(a_a) \left[[\tilde{e}^a, \tilde{e}^b] \omega - \delta(e^a) i^b \omega + \delta(e^b) i^a \omega \right] \\
&= *e_b(a_a) \left[\eta^{ac} \eta^{bm} c_{cm}^l \eta_{lf} i^f \omega - \delta(e^a) i^b \omega + \delta(e^b) i^a \omega \right] \\
&= (-1)^{d+1} \operatorname{sgn}(g) e_b(a_a) \left[\delta(e^b) e^a + \eta^{ac} \eta^{bm} \eta_{lf} c_{cm}^l e^f - \delta(e^a) e^b \right] \\
&= (-1)^{d+1} \operatorname{sgn}(g) e_b(a_a) \left[-\eta^{bd} c_{pd}^p e^a + \eta^{ac} \eta^{bm} \eta_{lf} c_{cm}^l e^f + \eta^{ad} c_{pd}^p e^b \right] \\
&= (-1)^{d+1} \operatorname{sgn}(g) \left[-\eta^{bd} c_{pd}^p e_b(a_a) + \eta^{bm} \eta_{na} c_{cm}^n e_b(a^c) + c_{pb}^p e_a(a^b) \right] e^a,
\end{aligned}$$

where Eq. (A 2) for $\delta(e^a)$ as been used.

We now compute II . We first remark that II rewrites

$$\begin{aligned} II &= -\delta a_c d e^c \\ &= -\delta a_c \left(-\frac{1}{2}\right) c_{ab}^c e^a \wedge e^b \\ &= -\delta \frac{1}{2} a_c c_{ab}^c e^b \wedge e^a, \end{aligned}$$

which is nothing but I with $e_b(a_a)$ in place of $\frac{1}{2}a_c c_{ab}^c$. Thus, one can recast II under the form $II = (-1)^d \text{sgn}(g)(II_1 + II_2)$, where the terms II_1 et II_2 are evaluated as follow:

$$\begin{aligned} II_1 &= \frac{1}{2} \text{sgn}(g)(-1)^{d+1} [\eta^{aa} e_a(a_c c_{ab}^c) e^b - \eta^{cb} e_c(a_D c_{ab}^d) e^a] \\ &= \frac{1}{2} \text{sgn}(g)(-1)^{d+1} [\eta^{ad} e_a(a_c c_{db}^c) e^b - \eta^{cb} e_c(a_D c_{ab}^d) e^a] \\ &= \frac{1}{2} \text{sgn}(g)(-1)^{d+1} [\eta^{ad} e_a(a_c c_{db}^c) e^b + \eta^{ad} e_a(a_c c_{db}^c) e^b] \\ &= \text{sgn}(g)(-1)^{d+1} [\eta^{ad} e_a(a_c c_{db}^c) e^b] \\ &= \text{sgn}(g)(-1)^{d+1} [\eta^{ad} e_a(a_c) c_{db}^c e^b + \eta^{ad} a_c e_a(c_{db}^c) e^b], \end{aligned}$$

and

$$\begin{aligned} II_2 &= \frac{1}{2} \text{sgn}(g)(-1)^{d+1} a_c c_{ab}^c [\delta(e^b) e^a + \eta^{ac} \eta^{bm} \eta_{lf} c_{cm}^l e^f - \delta(e^a) e^b] \\ &= \text{sgn}(g)(-1)^{d+1} \left[\eta^{bm} a_c c_{ba}^c c_{pm}^p + \frac{1}{2} a_c \eta^{mn} \eta^{pq} \eta_{ab} c_{mp}^c c_{nq}^b \right] e^a. \end{aligned}$$

The Maxwell part of \square being determined one focuses on the second part, namely: $-d\delta(a)$.

Thanks to Eq. (A3) one obtains

$$\begin{aligned} III &= -d\delta(a) \\ &= -d[-e_a(a^a) + a_a \delta(e^a)] \\ &= e_b[e_a(a^a) + a^a c_{pa}^p] e^b \\ &= e_b(e_a(a^a)) e^b + e_b(a^a) c_{pa}^p e^b + a^a e_b(c_{pa}^p) e^b. \end{aligned}$$

Finally

$$\square a = (-1)^d \text{sgn}(g)(I_1 + I_2 + II_1 + II_2) + III,$$

which is Eq. (1).

5. Expression of the term nB of Eq. (4)

From Eq. (1) the terms containing the n index denoted nB reads

$${}^nB = \sum_{m=1}^9 {}^nB^m,$$

where ${}^nB^m$ is the m -th term of the r.h.s. of Eq. (1):

$${}^nB^1 = \eta^{nn} e_n(e_n(a_\mu))e^\mu ; {}^nB^2 = (c_{\mu\nu}^n e_n(a^\nu) + c_{\mu n}^\nu e_\nu(a^n) + c_{\mu n}^n e_n(a^n))e^\mu \text{ etc.}$$

Thanks to the relation $a_n = i_{e_n} a = i_{D} a / \sqrt{|y^2|}$, and the fact that $e_\mu(y^2) = 0$, one obtains

$$\begin{aligned} {}^nB^1 &= -\epsilon \frac{1}{|y^2|} (D^2(a_\mu) - D(a_\mu))e^\mu ; {}^nB^2 = -\epsilon \frac{1}{|y^2|} e_\mu(i_D a)e^\mu ; {}^nB^3 = -\epsilon \frac{1}{|y^2|} D(a_\mu)e^\mu ; \\ {}^nB^4 &= \epsilon \frac{1}{|y^2|} (D(a_\mu)e^\mu - e_\mu(i_D a)e^\mu) ; {}^nB^5 = -\epsilon \frac{n}{|y^2|} D(a_\mu)e^\mu ; {}^nB^6 = \epsilon \frac{1}{|y^2|} a_\mu e^\mu ; \\ {}^nB^7 &= -\epsilon \frac{n}{|y^2|} a_\mu e^\mu ; {}^nB^8 = 0 ; {}^nB^9 = \epsilon \frac{1}{|y^2|} a_\mu e^\mu. \end{aligned}$$

Gathering these terms gives Eq. (4).

6. Intrinsic form for Eq. (11)

We now rewrite the second term of the r.h.s. of Eq. (11). We now show that

$$m^* ((\partial_{n+1}^2 a_\mu)e^\mu) = m^* (H^{-2} \mathcal{L}_F^2(a)). \quad (\text{A4})$$

To begin with, we have $\mathcal{L}_{\partial_{n+1}}(e^\mu) = 0$, since

$$\begin{aligned} \mathcal{L}_{\partial_{n+1}}(e^\mu) &= di_{n+1}e^\mu + i_{n+1}de^\mu \\ &= 0 - \frac{1}{2} i_{n+1} c_{\alpha\beta}^\mu e^\alpha \wedge \beta \\ &= 0, \end{aligned}$$

in which we used $c_{\alpha n+1}^\mu = 0$. As a consequence, $\mathcal{L}_{n+1}^2(a_\mu)e^\mu = \mathcal{L}_{n+1}^2(a_\mu e^\mu)$ and the result follows using Eq. (9).

Thanks to Eqs. (10)–(9) and $y^{n+1} = H^{-1}f$ one has $D_P = D - \epsilon H^{-2}fF$, which allows us to eliminate D_P from Eq. (11). This is achieved as follows.

One first rewrites $\mathcal{L}_{D_{n+1}}$ thanks to the Cartan formula under the form:

$$\mathcal{L}_{\phi_V} = \phi \mathcal{L}_V + d\phi \wedge i_V,$$

ϕ being a scalar and V a vector. Then

$$\mathcal{L}_{D_{n+1}} = y^{n+1}\mathcal{L}_{\partial_{n+1}} + dy^{n+1} \wedge i_{\partial_{n+1}}. \quad (\text{A5})$$

using Eq. (9) and $y^{n+1} = H^{-1}f$ this expression rewrites $\mathcal{L}_{D_{n+1}} = \epsilon H^{-2}(f\mathcal{L}_F + df \wedge i_F)$, noting that $m^*f = 1$, $m^*df = 0$, one obtains

$$m^*\mathcal{L}_{D_{n+1}} = m^*(\epsilon H^{-2}\mathcal{L}_F), \quad (\text{A6})$$

where, here and after, we note, for any operator A , m^*A instead of $m^* \circ A$. This allows us to recast $m^*\mathcal{L}_{D_P}$ under the form

$$m^*\mathcal{L}_{D_P} = m^*(\mathcal{L}_D - \epsilon H^{-2}\mathcal{L}_F). \quad (\text{A7})$$

We now consider $m^*\mathcal{L}_{D_P}^2 = m^*(\mathcal{L}_D^2 + \mathcal{L}_{D_{n+1}}^2 - [\mathcal{L}_D, \mathcal{L}_{D_{n+1}}]_+)$, and first compute $m^*\mathcal{L}_{D_{n+1}}^2$. From Eq. (A5), taking into account $m^*dy^{n+1} = 0$ and $m^*y^{n+1} = H^{-1}$, one has

$$\begin{aligned} m^*\mathcal{L}_{D_{n+1}}^2 &= m^*(y^{n+1}\mathcal{L}_{\partial_{n+1}} + dy^{n+1} \wedge i_{\partial_{n+1}})(y^{n+1}\mathcal{L}_{\partial_{n+1}} + dy^{n+1} \wedge i_{\partial_{n+1}}) \\ &= m^*y^{n+1}\mathcal{L}_{\partial_{n+1}}y^{n+1}\mathcal{L}_{\partial_{n+1}} + m^*y^{n+1}\mathcal{L}_{\partial_{n+1}}dy^{n+1} \wedge i_{\partial_{n+1}} \\ &= m^*y^{n+1}(\mathcal{L}_{\partial_{n+1}}(y^{n+1})\mathcal{L}_{\partial_{n+1}} + y^{n+1}\mathcal{L}_{\partial_{n+1}}^2) + m^*y^{n+1}\mathcal{L}_{\partial_{n+1}}(dy^{n+1}) \wedge i_{\partial_{n+1}} \\ &= m^*(y^{n+1}\mathcal{L}_{\partial_{n+1}} + (y^{n+1})^2\mathcal{L}_{\partial_{n+1}}^2). \end{aligned}$$

Then we determine the term $m^*[\mathcal{L}_D, \mathcal{L}_{D_{n+1}}]_+$. Noting that $[D, D_{n+1}] = 0$ and thus $[\mathcal{L}_D, \mathcal{L}_{D_{n+1}}] = 0$ one has

$$\begin{aligned} m^*[\mathcal{L}_D, \mathcal{L}_{D_{n+1}}]_+ &= m^*(2\mathcal{L}_{D_{n+1}}\mathcal{L}_D) \\ &= m^*(2(y^{n+1}\mathcal{L}_{\partial_{n+1}} + dy^{n+1} \wedge i_{\partial_{n+1}})\mathcal{L}_D) \\ &= m^*2(y^{n+1}\mathcal{L}_{\partial_{n+1}}\mathcal{L}_D), \end{aligned}$$

where $m^*dy^{n+1} = 0$ has been used.

Gathering the above results and using Eq. (9) and $y^{n+1} = H^{-1}f$ gives

$$m^*\mathcal{L}_{D_P}^2 = m^*(\mathcal{L}_D^2 + H^{-4}\mathcal{L}_F^2 + \epsilon H^{-4}\mathcal{L}_F^2(1 - 2\mathcal{L}_D)). \quad (\text{A8})$$

The last term of Eq. (11) containing D_P , rewrites as

$$m^*di_{D_P} = m^*(di_D - \epsilon H^{-2}di_F). \quad (\text{A9})$$

Finally, using the above relations Eqs. (A4), (A7), (A8) and (A9), the second term of the r.h.s. of Eq. (11):

$$X := m^* \left[\epsilon (\partial_{n+1}^2 a_\mu) e^\mu - \epsilon H^2 \left(\mathcal{L}_{D_P}^2(a) + (n-3)\mathcal{L}_{D_P}(a) + 2d(i_{D_P}a) \right) \right],$$

writes

$$X = m^* \left[-\epsilon H^2 (\mathcal{L}_D^2 + (n-3)\mathcal{L}_D + 2di_D) + 2\mathcal{L}_F \mathcal{L}_D + (n-4)\mathcal{L}_F + 2di_F \right] (a).$$

7. Derivation of Eq. (13)

The Eq. (2), as for the one-form, splits into three terms:

$$\begin{aligned} {}^r C &= \eta^{\mu\nu} [e_\mu e_\nu(\phi) + e_\mu(\phi) c_{\alpha\mu}^\alpha], \\ {}^n C &= \eta^{nn} [e_n e_n(\phi) + e_n(\phi) c_{\alpha n}^\alpha], \\ {}^{n+1} C &= \eta^{n+1n+1} [\partial_{n+1} \partial_{n+1}(\phi) + \partial_{n+1}(\phi) c_{\alpha n+1}^\alpha]. \end{aligned}$$

As already said $m^* {}^r C = \square_\Sigma \phi_\Sigma$. Now, using the notations of Sec. VA, and $y_P^2 := y^A y_A$, the two remaining terms read

$$\begin{aligned} {}^n C &= \eta^{nn} [e_n e_n(\phi) + e_n(\phi) c_{\alpha n}^\alpha] \\ &= -\epsilon \left[\frac{1}{\sqrt{|y_P^2|}} (D_P) \frac{1}{\sqrt{|y_P^2|}} (D_P) \phi + \frac{n}{|y_P^2|} (D_P) \phi \right] \\ &= -\frac{\epsilon}{|y_P^2|} [(D_P)^2 \phi - (D_P) \phi + n(D_P) \phi] \\ &= -\frac{\epsilon}{|y_P^2|} [D^2 \phi - 2D_{n+1} D \phi + (y^{n+1})^2 \partial_{n+1}^2 \phi + D_{n+1} \phi + (n-1)(D - D_{n+1}) \phi], \end{aligned}$$

and ${}^{n+1} C = \epsilon \partial_{n+1}^2 \phi$.

Using $y^{n+1} = H^{-1} f$ one remarks that the term ${}^{n+1} C$ simplify with a term in ${}^n C$:

$$m^* \left(-\frac{\epsilon}{|y_P^2|} (y^{n+1})^2 \partial_{n+1}^2 \phi + \epsilon \partial_{n+1}^2 \phi \right) = -\frac{\epsilon}{H^{-2}} H^{-2} \partial_{n+1}^2 \phi + \epsilon \partial_{n+1}^2 \phi = 0,$$

and one thus has successively

$$\begin{aligned} m^*(\square_{n+2}\phi) &= \square_\Sigma \phi_\Sigma - \epsilon H^2 [D^2 \phi - 2D_{n+1} D \phi + D_{n+1} \phi + (n-1)(D \phi - D_{n+1}) \phi]_\Sigma \\ &= \square_\Sigma \phi_\Sigma - \epsilon H^2 [D^2 \phi - 2D_{n+1} D \phi + D_{n+1} \phi + (n-1)(D_P) \phi]_\Sigma \\ &= \square_\Sigma \phi_\Sigma - \epsilon H^2 [D^2 \phi + (n-1)D \phi - 2H^{-1} \partial_{n+1} D \phi + (2-n)H^{-1} \partial_{n+1} \phi]_\Sigma \\ &= \square_\Sigma \phi_\Sigma - \epsilon H^2 [D^2 \phi + (n-1)D \phi - 2\epsilon H^{-2} F D \phi + \epsilon(2-n)H^{-2} F \phi]_\Sigma \\ &= \square_\Sigma \phi_\Sigma - \epsilon H^2 [D^2 \phi + (n-1)D \phi]_\Sigma + [2F D \phi + (n-2)F \phi]_\Sigma, \end{aligned}$$

which is Eq. (13).

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