



**HAL**  
open science

# On proper 2-labellings distinguishing by sums, multisets or products

Julien Bensmail, Foivos Fioravantes

► **To cite this version:**

Julien Bensmail, Foivos Fioravantes. On proper 2-labellings distinguishing by sums, multisets or products. [Research Report] Université Côte d'Azur. 2022. hal-03536425v1

**HAL Id: hal-03536425**

**<https://hal.science/hal-03536425v1>**

Submitted on 19 Jan 2022 (v1), last revised 16 Aug 2022 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On proper 2-labellings distinguishing by sums, multisets or products

Julien Bensmail<sup>a</sup>, Foivos Fioravantes<sup>a</sup>

<sup>a</sup>Université Côte d'Azur, CNRS, Inria, I3S, France

---

## Abstract

Given a graph  $G$ , a  $k$ -labelling  $\ell$  of  $G$  is an assignment  $\ell : E(G) \rightarrow \{1, \dots, k\}$  of labels from  $\{1, \dots, k\}$  to the edges. We say that  $\ell$  is *s-proper*, *m-proper* or *p-proper*, if no two adjacent vertices of  $G$  are incident to the same sum, multiset or product, respectively, of labels.

Proper labellings are part of the field of distinguishing labellings, and have been receiving quite some attention over the last decades, in particular in the context of the well-known 1-2-3 Conjecture. In recent years, quite some progress was made towards the main questions of the field, with, notably, the analogues of the 1-2-3 Conjecture for *m-proper* and *p-proper* labellings being solved. This followed mainly from a better global understanding of these types of labellings.

In this note, we focus on a question raised by Paramaguru and Sampathkumar, who asked whether graphs with *m-proper* 2-labellings always admit *s-proper* 2-labellings. We give a negative answer to this question, showing that recognising graphs with *m-proper* 2-labellings but no *s-proper* 2-labellings is an NP-hard problem. We also prove a similar result for *m-proper* 2-labellings and *p-proper* 2-labellings, and raise a few directions for further work on the topic.

*Keywords:* proper labelling; sum of labels; multiset of labels; product of labels.

---

## 1. Introduction

Let  $G$  be a graph. By a *labelling*  $\ell$  of  $G$ , we mean an assignment of *labels* (numbers) from a given set to the edges of  $G$ . For a set  $S$  of labels,  $\ell$  is called an *S-labelling* if it assigns labels from  $S$ , while, if  $S = \{1, \dots, k\}$  for some  $k \geq 1$ , then we call  $\ell$  a *k-labelling*. In so-called **distinguishing labellings**, we are interested in designing labellings that permit to distinguish certain pairs of vertices accordingly to some function inferred by the assigned labels. As attested by the survey [12] by Gallian, this general definition is very flexible, and it gave birth, throughout the years, to a tremendous number of such distinguishing labelling notions.

In this work, we are interested in a subset of distinguishing labellings, called **proper labellings**. In proper labellings, the pairs of vertices that are required to be distinguished are the pairs of adjacent vertices. Regarding the distinguishing function, there are again many possibilities. We are here interested in three such functions, being the *sums*, *multisets* and *products* of labels incident to the vertices. Formally, let us consider a graph  $G$ , together with a labelling  $\ell$ . For any vertex  $v$  of  $G$ , we denote by  $\sigma(v)$  the sum of labels assigned by  $\ell$  to the edges incident to  $v$ . Similarly, we denote by  $\mu(v)$  and  $\rho(v)$  the multiset and product, respectively, of these incident labels. Now, we say that  $\ell$  is *s-proper* if no two adjacent vertices  $u$  and  $v$  of  $G$  are incident to the same sum of labels, i.e., if  $\sigma(u) \neq \sigma(v)$  for every edge  $uv \in E(G)$ . Similarly, we say that  $\ell$  is *m-proper* and *p-proper* if every two adjacent vertices are distinguished through the functions  $\mu$  and  $\rho$ , respectively.

In proper labellings, the goal is generally to design labellings that are not only proper, but also  $k$ -labellings for  $k$  as small as possible. This leads to the definition of three parameters, denoted  $\chi_S(G)$ ,  $\chi_M(G)$  and  $\chi_P(G)$  for a given graph  $G$ , which denote the smallest  $k \geq 1$  such that *s-proper*, *m-proper* and *p-proper*, respectively,  $k$ -labellings of  $G$  exist. It is worth pointing out now that these three parameters are not defined for all graphs  $G$ , as it can easily be observed that  $K_2$ , the complete graph on 2 vertices, admits no proper labellings at all. However, greedy arguments can be employed to show that this is the only pathological connected graph. Consequently, the parameters  $\chi_S$ ,  $\chi_M$  and  $\chi_P$  are more precisely studied in the context of *nice graphs*, which are those graphs which do not have  $K_2$  as a connected component.

The notion of  $s$ -proper labellings emerged in the literature as a local version of the *irregularity strength* of graphs, which was initially introduced, back in 1988, by Chartrand *et al.* [10]. Since then,  $s$ -proper labellings have been studied for their own interest, and they have actually been attracting a lot of attention due to the intriguing **1-2-3 Conjecture**:

**1-2-3 Conjecture** (Karoński, Łuczak, Thomason [14]). *If  $G$  is a nice graph, then  $\chi_S(G) \leq 3$ .*

For details on the 1-2-3 Conjecture, we refer the interested reader to [17]. Let us just mention, for now, that this conjecture has been proven to be quite challenging. To date, the best result towards the conjecture is that  $\chi_S(G) \leq 5$  for every nice graph  $G$  (see [13]). The conjecture was proven to hold for 3-colourable graphs [14], and simple graph classes such as complete graphs [9], for which, already, all of the labels 1, 2, 3 are required.

The general hardness behind the 1-2-3 Conjecture is one of the main reasons that led people to consider  $m$ -proper labellings instead, as it can be observed that an  $s$ -proper labelling is always  $m$ -proper. This observation led the authors of [1] to consider a multiset version of the 1-2-3 Conjecture (asking, naturally, whether  $\chi_M(G) \leq 3$  for every nice graph  $G$ ). Right away, this presumption that distinguishing vertices via multisets is easier than distinguishing via sums was proven correct, as the authors showed that  $\chi_M(G) \leq 4$  for all nice graphs  $G$ . More recently, this was definitely confirmed, with Vučković proving, in [20], the multiset version of the 1-2-3 Conjecture.

In parallel of these results, a product version of the 1-2-3 Conjecture, stating that  $\chi_P(G) \leq 3$  for every nice graph  $G$ , was introduced by Skowronek-Kaziów in [18]. An interesting fact is that, due to 2 and 3 being coprime,  $m$ -proper 3-labellings and  $p$ -proper 3-labellings are actually not so distant objects (see [6] for more details), which sort of gave the impression that the product version of the 1-2-3 Conjecture might be of intermediate difficulty, between the sum and multiset versions, yet closer to the latter one. Again, this was confirmed recently in [7], with Bensmail, Hocquard, Lajou and Sopena proving the product version of the 1-2-3 Conjecture in full.

We have thus reached a point of time where quite some progress towards the 1-2-3 Conjecture has been recently made, through the upper bound, 5, being very close to what is conjectured exactly, and two related conjectures, which for quite some time seemed of close hardness, being proved. What made this recent progress possible, is definitely a better understanding over the inherent behaviour of  $s$ -proper,  $m$ -proper and  $p$ -proper 3-labellings, which on themselves, gave birth to interesting side investigations (see [4] and the references there for examples). As a matter of fact, even if the 1-2-3 Conjecture turned out to be proven soon, there would still remain lots of interesting questions to answer towards fully understanding proper labellings.

One of these questions, which is precisely the one we investigate throughout this note, is about the real difference between  $s$ -proper,  $m$ -proper and  $p$ -proper labellings. As partly mentioned earlier,  $s$ -properness and  $p$ -properness both imply  $m$ -properness, that is  $\chi_M(G) \leq \min\{\chi_S(G), \chi_P(G)\}$  for every nice graph  $G$ . This led, in particular, to the following question:

**Question 1.1** (Paramaguru, Sampathkumar [16]). *Does every nice graph  $G$  verify  $\chi_M(G) = \chi_S(G)$ ?*

Note that  $\chi_S(G) = 1$  for a graph  $G$  if and only if  $G$  is *locally irregular*, i.e., does not have adjacent vertices with the same degree [3], in which context also  $\chi_M(G) = 1$ . Now, since the multiset version of the 1-2-3 Conjecture holds [20], we have  $\chi_M(G) \leq 3$  for every nice graph  $G$ . So the natural question to ask in order to advance towards answering Question 1.1, is whether there exist graphs  $G$  such that  $2 \leq \chi_M(G) < \chi_S(G)$ . Now, if we assume that the 1-2-3 Conjecture holds as well, then this question reduces to:

**Question 1.2.** *Are there graphs  $G$  with  $\chi_M(G) = 2 < 3 = \chi_S(G)$ ?*

Surprisingly, it turns out that the previous results on the topic, in particular the graphs for which the 1-2-3 Conjecture was shown to hold, do not allow to come up with examples of graphs having the properties described in Question 1.2, and neither does the examination of small graphs.

In this work, we answer Question 1.2 (and thus Question 1.1) by showing that there exist infinitely many graphs verifying the conditions of the statement. Even stronger, we prove that determining whether  $\chi_S(G) = 3$  for a given graph  $G$  with  $\chi_M(G) = 2$  is NP-hard. Thus, recognising graphs  $G$  with  $\chi_M(G) = 2 < 3 = \chi_S(G)$  cannot be done in polynomial time, unless  $P=NP$ . We

also prove that such a result holds for m-proper and p-proper labellings, although the answer to a question equivalent to Question 1.2 can be deduced from previous works (see later Section 5).

This note is organised as follows. We start, in Section 2, by recalling a few facts on s-proper, m-proper and p-proper labellings. In Section 3, we give a first general result on m-proper and p-proper labellings, from which we provide a first insight into our proof arguments. From these, we then give our main result in Section 4, answering positively to Question 1.2. We conclude in Section 5 with a few more remarks and questions related to our investigations.

## 2. General tools and previous results

We start by recalling a few facts on proper labellings, which were already observed in previous works (such as [9]). We begin with the following, which is evident.

**Observation 2.1.** *Let  $G$  be a nice graph, and  $\ell$  be a  $k$ -labelling of  $G$ . If  $uv \in E(G)$  is an edge with  $d(u) = 1$  (and  $d(v) > 1$  since  $G$  is nice), then  $\sigma(u) \neq \sigma(v)$ .*

Another obvious observation, is the fact that p-proper labellings cannot assign label 0. This is because if we have  $\ell(uv) = 0$  for an edge  $uv$  of a graph  $G$  by a labelling  $\ell$ , then  $\rho(u) = \rho(v) = 0$ . Due to this property, whenever dealing with p-proper  $S$ -labelling throughout this note (in particular in upcoming Observation 2.3 and Corollary 2.4), we implicitly assume that  $0 \notin S$ .

**Observation 2.2.** *P-proper labellings cannot assign label 0.*

The next result is another elementary one of the field. Yet, it has several interesting consequences. Particularly, it gives contexts in which s-proper, m-proper and p-proper 2-labellings stand as equivalent objects.

**Observation 2.3.** *Let  $G$  be a nice graph, and  $\ell$  be an  $\{a, b\}$ -labelling of  $G$  for any two distinct real numbers  $a, b$ . For  $\sigma(u) \neq \sigma(v)$  to hold for some  $uv \in E(G)$  with  $d(u) = d(v)$ , the number of edges labelled  $a$  incident to  $u$  must be different from the number of edges labelled  $a$  incident to  $v$  (and similarly for the numbers of edges incident to  $u$  and  $v$  labelled  $b$ ). Consequently, if  $\sigma(u) \neq \sigma(v)$ , then also both  $\mu(u) \neq \mu(v)$  and  $\rho(u) \neq \rho(v)$ .*

*Proof.* If we denote by  $n_x(u)$  and  $n_x(v)$  the number of edges incident to  $u$  and  $v$ , respectively, assigned some label  $x$  by  $\ell$ , then the facts that  $d(u) = d(v)$  and  $n_a(u) = n_a(v)$  imply that also  $n_b(u) = n_b(v)$ , and from that we deduce that  $\sigma(u) = \sigma(v)$ ,  $\mu(u) = \mu(v)$  and  $\rho(u) = \rho(v)$  (assuming  $0 \notin \{a, b\}$  in this last case, recall Observation 2.2). This implies the claim directly.  $\square$

**Corollary 2.4.** *In regular graphs, finding an s-proper, m-proper or p-proper  $\{a, b\}$ -labelling for some distinct real numbers  $a, b$ , is equivalent to finding an s-proper, m-proper or p-proper  $\{a', b'\}$ -labelling for any distinct  $a', b'$ .*

In the next sections, the main results we prove relate to the following decision problems:

S-PROPER 2-LABELLING

**Input:** A graph  $G$ .

**Question:** Do we have  $\chi_S(G) \leq 2$ , i.e., does  $G$  admit s-proper 2-labellings?

P-PROPER 2-LABELLING

**Input:** A graph  $G$ .

**Question:** Do we have  $\chi_P(G) \leq 2$ , i.e., does  $G$  admit p-proper 2-labellings?

Note that these problems are clearly in NP. They were also showed to be NP-hard in general, first by Dudek and Ważj in [11] for general graphs. Later on, Ahadi, Dehghan and Sadeghi proved in [2] that these problems remain NP-hard when restricted to cubic graphs. From Corollary 2.4, this implies the following:

**Theorem 2.5** (Ahadi, Dehghan, Sadeghi [2]). *For any two distinct  $a, b$ , deciding whether a (cubic) graph admits an s-proper, m-proper or p-proper  $\{a, b\}$ -labelling is NP-hard.*

### 3. Multisets versus products

The main result to be established in this section, relies on the following simple fact, already established e.g. in [15].

**Observation 3.1.** *Let  $G$  be a nice graph with a vertex  $v$ , and let  $H$  be obtained from  $G$  by attaching a pending path  $(v, a, b, c, d)$  of length 4 at  $v$ . If  $\ell$  is a  $p$ -proper 2-labelling of  $H$ , then the restriction of  $\ell$  to  $G$  is also  $p$ -proper. Conversely, any  $p$ -proper 2-labelling of  $G$  can be extended to one of  $H$ .*

*Proof.* Assume  $\ell$  is a  $p$ -proper 2-labelling of  $H$ . So that  $\rho(c) \neq \rho(d)$ , note that we must have  $\ell(bc) = 2$ . From this, so that  $\rho(a) \neq \rho(b)$ , we must have  $\ell(va) = 1$ , meaning that  $\ell(va)$  does not contribute to  $\rho(v)$ . We thus deduce that, for  $\ell$  to be  $p$ -proper in  $H$ , the restriction of  $\ell$  to the edges of  $G$  must also be  $p$ -proper in  $G$ .

Regarding the last part of the statement, consider a  $p$ -proper 2-labelling  $\ell$  of  $G$ . We extend this labelling to a  $p$ -proper 2-labelling of  $H$ , by first setting  $\ell(va) = 1$  and  $\ell(bc) = 2$ , and then setting either  $\ell(ab) = 1$  and  $\ell(cd) = 2$  (if  $\rho(v) \neq 1$ ) or  $\ell(ab) = 2$  and  $\ell(cd) = 1$  (otherwise). It can be checked that this raises no conflict in both cases.  $\square$

We now prove that P-PROPER 2-LABELLING is NP-hard even when restricted to graphs  $G$  with  $\chi_M(G) = 2$ . So, assuming that  $P \neq NP$ , not only does our result imply that there exist infinitely many graphs  $G$  with  $\chi_M(G) = 2 < 3 = \chi_P(G)$ , but also that recognising the graphs  $G$  that verify  $\chi_M(G) = \chi_P(G) = 2$  cannot be done in polynomial time.

**Theorem 3.2.** *P-PROPER 2-LABELLING is NP-hard for graphs  $G$  with  $\chi_M(G) = 2$ .*

*Proof.* The proof is by reduction from the P-PROPER 2-LABELLING problem in cubic graphs, which is NP-hard (recall Theorem 2.5). Let  $G$  be a cubic graph, an instance of P-PROPER 2-LABELLING. We can assume that  $G$  is connected. We can also assume that  $G$  is not  $K_4$  (since  $\chi_P(K_4) = 3$ , see [9]), thus that its chromatic number is at most 3 by Brooks' Theorem [8]. In other words, we can assume we also have a proper vertex-colouring  $\phi : V(G) \rightarrow \{0, 1, 2\}$  of  $G$ .

We construct a graph  $H$  verifying  $\chi_M(H) = 2$ , and such that  $\chi_P(H) = 2$  if and only if  $\chi_P(G) = 2$ . For that, we start from  $G$ , consider every vertex  $v \in V(G)$  in turn, and attach at  $v$  exactly  $\phi(v)$  new disjoint pending paths of length 4. Note that the construction of  $H$  is clearly achieved in polynomial time.

The fact that  $\chi_P(G) = 2$  if and only if  $\chi_P(H) = 2$  follows mainly from Observation 3.1. That is, a  $p$ -proper 2-labelling of  $H$  directly infers one of  $G$ , due to the fact that the paths of length 4 do not contribute to the products of the vertices in  $V(H) \cap V(G)$ . Regarding the other direction, a  $p$ -proper 2-labelling of  $G$  can be extended to the pending paths we have added to form  $H$ , by repeated applications of the arguments in the proof of Observation 3.1. Thus, we have the desired equivalence.

To see now that we always have  $\chi_M(H) = 2$  (regardless of  $G$ ), it suffices to note that, for any two vertices  $u$  and  $v$ , by any labelling of a graph, we have  $\mu(u) \neq \mu(v)$  whenever  $d(u) \neq d(v)$ . Now, for any vertex  $v \in V(H) \cap V(G)$ , note that  $d_H(v) = d_G(v) + \phi(v)$ . In particular, because  $\phi$  is a proper  $\{0, 1, 2\}$ -vertex-colouring of  $G$ , for every two adjacent vertices  $u, v \in V(H) \cap V(G)$ , we have  $d_H(u) \neq d_H(v)$ , for  $d_H(u), d_H(v) \in \{3, 4, 5\}$ , and thus  $\mu(u) \neq \mu(v)$  by any labelling of  $H$ . Similarly, if a vertex  $v \in V(H) \cap V(G)$  is incident to a pending path (added when constructing  $H$  from  $G$ ), and is thus adjacent to a degree-2 vertex  $u$ , then we have  $d_H(v) \geq 3 > 2 = d_H(u)$ , and again  $\mu(u) \neq \mu(v)$  by any labelling. Thus, designing an  $m$ -proper 2-labelling of  $H$  falls down to 2-labelling the pending paths in an  $m$ -proper way. From this, we deduce that assigning label 1 to all edges of  $E(H) \cap E(G)$ , and, then, for every pending path  $(v, a, b, c, d)$ , where  $v \in V(H) \cap V(G)$  (and thus  $d_H(d) = 1$ ), assigning label 2 to  $dc$  and  $cb$  and label 1 to  $ba$  and  $av$ , results in an  $m$ -proper 2-labelling of  $H$ .  $\square$

### 4. Multisets versus sums

Before getting to proving our main result, Theorem 4.3, we first need to introduce a few gadgets and constructions, and to point out some of their properties.

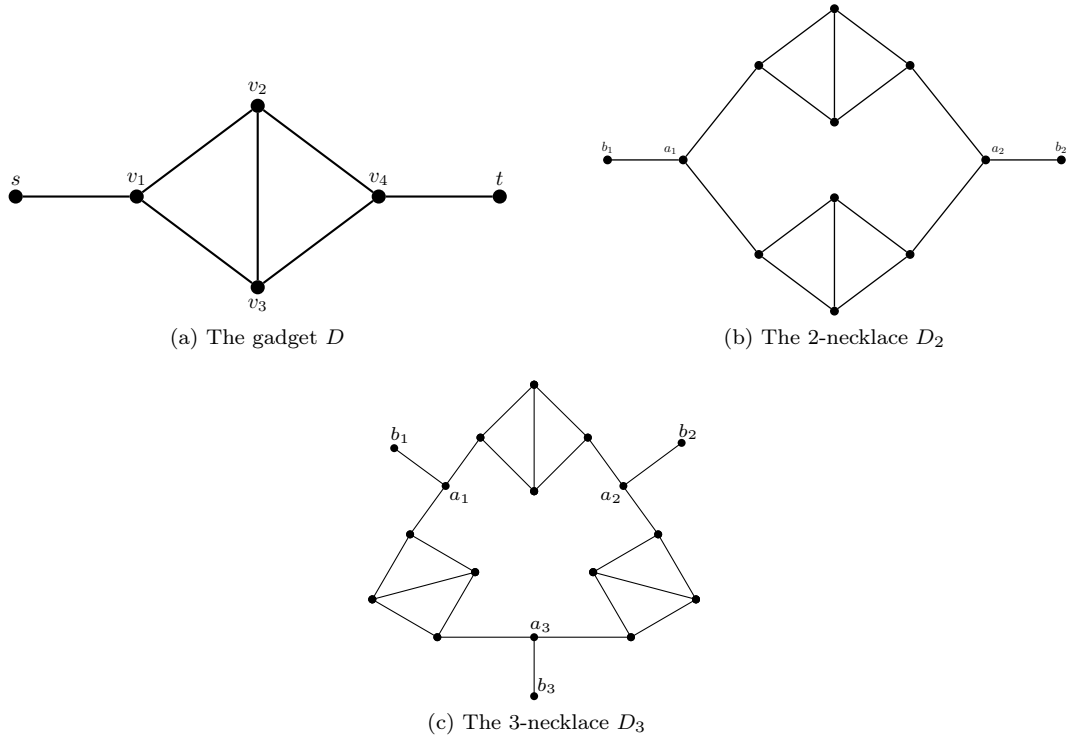


Figure 1: Gadgets used to prove Theorem 4.3.

The gadget  $D$  is depicted in Figure 1(a). Throughout this section, we deal with the vertices and edges of  $D$  following the notation given in Figure 1. We call  $s$  and  $t$  the two *ends* of  $D$ , while we say that edges  $sv_1$  and  $tv_4$  are its two *outputs*. This gadget  $D$  was already used in [2], where it was noticed it has the following labelling properties:

**Theorem 4.1.**  $D$  fulfils the following properties:

1.  $D$  admits  $s$ -proper 2-labellings  $\ell$  where  $\ell(sv_1) = 1$ ;
2.  $D$  admits  $s$ -proper 2-labellings  $\ell$  where  $\ell(sv_1) = 2$ ;
3. if  $\ell$  is any  $s$ -proper 2-labelling of  $D$ , then:
  - (a)  $\ell(sv_1) = \ell(tv_4)$ ;
  - (b)  $\sigma(v_1) = \sigma(v_4) = 4$  if  $\ell(sv_1) = 1$ , and  $\sigma(v_1) = \sigma(v_4) = 5$  otherwise.

*Proof.* The vertices of  $D$  being of degree 1 and 3 only, it can be observed (through Observations 2.1 and 2.3) that if we have an  $s$ -proper 2-labelling of  $D$  and reverse all labels (1's become 2's, and *vice versa*) then what results is an  $s$ -proper 2-labelling of  $D$ . From this, we get that if Item 1. holds then Item 2. holds. Similarly, if the first part of Item 3.(b) holds, then the second part holds.

Let  $\ell$  be an  $s$ -proper 2-labelling of  $D$ . By the previous arguments, we can assume that  $\ell(v_2v_3) = 1$ . Assume first that  $\ell(v_2v_1) = \ell(v_2v_4) = 1$ . Then,  $\sigma(v_2) = 3$ . So that  $\sigma(v_2) \neq \sigma(v_3)$ , at least one of  $\ell(v_3v_1)$  and  $\ell(v_3v_4)$  must be 2. There are two cases:

- Assume first that  $\ell(v_3v_1) \neq \ell(v_3v_4)$ , say  $\ell(v_3v_1) = 1$  and  $\ell(v_3v_4) = 2$ . Thus,  $\sigma(v_4) = 4$ . Now observe that, regardless of  $\ell(v_1s)$ , we must have  $\sigma(v_1) \in \{3, 4\} = \{\sigma(v_2), \sigma(v_3)\}$ , resulting in a conflict, thus a contradiction.
- Assume now that  $\ell(v_3v_1) = \ell(v_3v_4) = 2$ . Then,  $\sigma(v_3) = 5$ . So that  $\sigma(v_1), \sigma(v_4) \notin \{3, 5\} = \{\sigma(v_2), \sigma(v_3)\}$ , note that we must have  $\ell(v_1s) = \ell(v_4t) = 1$ , which indeed yields  $\sigma(v_1) = \sigma(v_4) = 4$ , as claimed.

If  $\ell(v_2v_1) = \ell(v_2v_4) = 2$ , then, so that  $\sigma(v_2) \neq \sigma(v_3)$ , either  $\ell(v_3v_1) = \ell(v_3v_4) = 1$ , in which case we fall into a previous case (up to renaming the vertices), or  $\ell(v_3v_1) \neq \ell(v_3v_4)$ . In that last case, if we have, say,  $\ell(v_3v_1) = 1$  and  $\ell(v_3v_4) = 2$ , then  $\{\sigma(v_2), \sigma(v_3)\} = \{4, 5\}$ ; note that, again, regardless of  $\ell(v_1s)$ , we must have  $\sigma(v_1) \in \{4, 5\} = \{\sigma(v_2), \sigma(v_3)\}$ , thus a conflict.

Lastly, if  $\ell(v_2v_1) \neq \ell(v_2v_4)$ , then, so that  $\sigma(v_2) \neq \sigma(v_3)$ , we must have  $\ell(v_3v_1) = \ell(v_3v_4)$ . Up to renaming the vertices of  $D$ , this is a case we have covered earlier.  $\square$

Using copies of the gadget  $D$ , we can construct bigger gadgets. For any  $k \geq 1$ , the  $k$ -necklace  $D_k$  is obtained in the following way (see Figures 1(b) and (c) for examples):

- Start from  $k$  disjoint copies  $G_0, \dots, G_{k-1}$  of the gadget  $D$ . For every  $i \in \{0, \dots, k-1\}$ , we denote by  $x_i$  and  $y_i$  the ends of  $G_i$ .
- For every  $i \in \{0, \dots, k-1\}$ , identify  $y_i$  and  $x_{(i+1) \bmod k}$  to connect the  $G_i$ 's in a cyclic fashion. Note that all vertices have degree 3, except for exactly  $k$  of them, which we denote by  $a_1, \dots, a_k$ , which result from the identifications.
- For every  $i \in \{1, \dots, k\}$ , attach a pending vertex  $b_i$  at  $a_i$ .

We call the  $b_i$ 's the *ends* of  $D_k$ , while we call the  $a_i b_i$ 's its *outputs*. Labelling properties of the gadget  $D$  actually infer labelling properties for necklaces; in particular:

**Theorem 4.2.** *For any  $k \geq 1$ , the  $k$ -necklace  $D_k$  fulfils the following properties:*

1. *by any  $s$ -proper 2-labelling of  $D_k$ , all outputs must be assigned the same label;*
2.  *$D_k$  admits  $s$ -proper 2-labellings assigning label 1 to the outputs;*
3.  *$D_k$  admits  $s$ -proper 2-labellings assigning label 2 to the outputs;*
4. *if  $\ell$  is any  $s$ -proper 2-labelling of  $D_k$ , then  $\sigma(a_1) = \dots = \sigma(a_k) = 3$  if  $\ell(a_1 b_1) = 1$ , and  $\sigma(a_1) = \dots = \sigma(a_k) = 6$  otherwise.*

*Proof.*  $D_k$  having vertices of degree 1 and 3 only, by Observations 2.1 and 2.3 we get that, upon reversing labels by  $s$ -proper 2-labellings, Item 3. holds as soon as Items 1. and 2. hold.

Let  $\ell$  be an  $s$ -proper 2-labelling of  $D_k$ . First off, we claim that, for each of the  $k$  copies  $G_0, \dots, G_{k-1}$  of the gadget  $D$  forming  $D_k$ , its two outputs must be assigned the same label by  $\ell$ . In other words, we claim that the  $2k$  outputs of  $G_0, \dots, G_{k-1}$  must be assigned the same label. Indeed, assume this is wrong. Recall that, by Item 3.(a) of Theorem 4.1, for any copy of  $D$  in  $D_k$ , its two outputs must be assigned the same label by  $\ell$ . Now, by our hypothesis, there must be a vertex  $a_i$  in  $D_k$  resulting from the identification of two ends from two copies  $G_{i-1}$  and  $G_i$  of  $D$  such that, say, the outputs of  $G_{i-1}$  are assigned label 1 by  $\ell$  while the outputs of  $G_i$  are assigned label 2. By Item 3.(b) of Theorem 4.1, this means that  $a_i$  is adjacent to a vertex  $v$  of  $G_{i-1}$  that has sum 4 by  $\ell$ , and adjacent to a vertex  $u$  of  $G_i$  that has sum 5. Then, there is a conflict either between  $a_i$  and  $v$  (if  $\ell(a_i b_i) = 1$ ) or between  $a_i$  and  $u$  (if  $\ell(a_i b_i) = 2$ ).

Thus, the  $2k$  outputs of the copies of  $D$  in  $D_k$  must be assigned the same label by  $\ell$ . Assume this label 1, and focus on any vertex  $a_i$  resulting from the identification of two ends of  $G_{i-1}$  and  $G_i$ , being copies of  $D$ . Then the two neighbours of  $a_i$  in  $G_{i-1}$  and  $G_i$ , by Item 3.(b) of Theorem 4.1, have sum 4 by  $\ell$ . So that there is no conflict between  $a_i$  and these vertices, we must have  $\ell(a_i b_i) = 1$  so that  $\sigma(a_i) = 3$ . Similar arguments show that we must have  $\ell(a_i b_i) = 2$  in case the other two edges incident to  $a_i$  are assigned label 2, in which case  $\sigma(a_i) = 6$ .  $\square$

We are now ready to prove our main result. Again, this result implies that there exist infinitely many graphs  $G$  verifying  $\chi_M(G) = 2 < 3 = \chi_S(G)$ . Even worse, recognising such graphs cannot be done in polynomial time, unless  $P=NP$ .

**Theorem 4.3.** *S-PROPER 2-LABELLING is NP-hard for graphs  $G$  with  $\chi_M(G) = 2$ .*

*Proof.* The proof starts similarly as that of Theorem 3.2, and is done by a reduction from the S-PROPER 2-LABELLING problem in cubic graphs. Let  $G$  be a connected cubic graph given together with a proper vertex-colouring  $\phi : V(G) \rightarrow \{0, 2, 4\}$ , where  $G$  is an instance of S-PROPER 2-LABELLING. We construct, in polynomial time, a graph  $H$  with  $\chi_M(H) = 2$ , and such that  $\chi_S(G) = 2$  if and only if  $\chi_S(H) = 2$ .

The construction of  $H$  is mainly achieved through using an  $\alpha$ -necklace with sufficiently many outputs, connected to the rest of the graph in some fashion to guarantee that 1) any s-proper 2-labelling must propagate in a very particular way, and that 2) some adjacent vertices have distinct degrees so that they cannot be an obstacle to m-properness. We voluntarily avoid specifying the value of  $\alpha$  for now, and just assume it is sufficiently large so that we always have unused outputs in hands whenever we need some. Let us just mention that  $\alpha$  will be a linear function of  $|V(G)|$ .

Before describing the heart of the reduction, let us make some preparations. We start from  $A$ , an  $\alpha$ -necklace with  $\alpha$  outputs. Recall that by any s-proper 2-labelling, all outputs of  $A$  must be assigned the same label, and that this label can be 1 or 2 (recall Theorem 4.2). We first modify  $A$ , to force this label on the outputs to be 1. For that, we first need to point out the following easy observation:

**Claim 1.** *Let  $uv$  be an output of  $A$ , where  $v$  is the degree-1 vertex. Attach a pending path  $(v, x, y)$  of length 2 at  $v$ , resulting in a new graph  $A'$ . Then:*

1. every s-proper 2-labelling of  $A$  can be extended to one of  $A'$ ;
2. if  $\ell$  is any s-proper 2-labelling of  $A'$ , then:
  - (a)  $\ell(uv) \neq \ell(xy)$ ;
  - (b)  $\sigma(x) = 3$  if  $\ell(uv) = 1$ , and  $\sigma(x)$  can be anything in  $\{2, 3\}$  otherwise.

*Proof of the claim.* Item 2.(a) is necessary, as it is the only reason that guarantees  $\sigma(v) \neq \sigma(x)$ . To see now that Item 1. holds, consider an s-proper 2-labelling of  $A$ , and consider extending it to the edges  $vx$  and  $xy$  of  $A'$ . If  $\ell(uv) = 1$ , then recall that  $\sigma(u) = 3$  by Theorem 4.2. In that case, we must set  $\ell(vx) = 1$  and  $\ell(xy) = 2$ , which yields  $\sigma(v) = 2$ ,  $\sigma(x) = 3$  and  $\sigma(y) = 2$ , thus no conflicts. Now, if  $\ell(uv) = 2$ , then, by Theorem 4.2, recall that  $\sigma(u) = 6$ . In that case, we can set e.g.  $\ell(vx) = 1$  (or  $\ell(vx) = 2$ ) and  $\ell(xy) = 1$ , which yields  $\sigma(v) = 3$  (or  $\sigma(v) = 4$ ),  $\sigma(x) = 2$  (or  $\sigma(x) = 3$ ) and  $\sigma(y) = 1$ , raising no conflicts.  $\diamond$

In what is to come, we will need to “extend” some outputs of  $A$  as described in Claim 1. That is, given an output  $uv$  of  $A$  where  $v$  is the degree-1 vertex, by *extending*  $uv$  we mean attaching a pending path  $(v, x, y)$  of length 2 at  $v$ . We regard the pending edge  $xy$  as another output of the resulting graph, its end being  $y$ . Conversely,  $uv$  is no longer regarded as an output of the resulting graph. So, from  $A$ , by extending some outputs we get another graph  $A'$  with  $\alpha$  outputs which can now be of two possible types: an output of  $A'$  either is an *extended output*, resulting precisely from an output extension, or is one of the initial outputs of  $A$  that was not extended, which we call *regular outputs* from now on to avoid any confusion. Due to Claim 1, the main difference, in brief, between regular and extended outputs, is that regular outputs should all be assigned the same label by an s-proper 2-labelling, while all extended outputs must be assigned the other label.

We are now ready to modify  $A$  to force the label of its outputs by any s-proper 2-labelling. For that, we proceed as follows. Take one output  $e$  with end  $u$  of  $A$ , and two more outputs  $f_1, f_2$ . Start by extending  $f_1, f_2$ , resulting in two new extended outputs  $f'_1, f'_2$ . Calling  $v_1$  and  $v_2$ , respectively, their ends, we modify  $A$  by first identifying  $v_1$  and  $v_2$  to a new vertex  $v$ , and adding an edge between  $u$  and  $v$ . This results in a new graph  $A'$ , in which we no longer regard  $e, f'_1, f'_2$  as outputs, and which thus has  $\alpha - 3$  outputs (all of which are regular).  $A'$  also has the following properties:

**Claim 2.**  *$A'$  fulfils the following properties:*

1. by any s-proper 2-labellings of  $A'$ , all (regular) outputs must be assigned label 1;
2.  $A'$  admits s-proper 2-labellings.



*Proof of the claim.* We start by proving the first item. Assume Item 1. is wrong, and that  $A'$  admits s-proper 2-labellings  $\ell$  where all outputs are assigned label 2. By Theorem 4.2, following the terminology above, by the construction of  $A'$  from  $A$ , the edge  $e$  must be assigned label 2 by  $\ell$ , while  $f'_1, f'_2$ , being extensions of  $f_1, f_2$ , must be assigned label 1 (by Claim 1). This implies that we must have  $\sigma(u) = \sigma(v)$  regardless of  $\ell(uv)$ , a contradiction.

Assume now that  $\ell$  is an s-proper 1-labelling of  $A$  where all outputs are assigned label 1 (which exists by Theorem 4.2). We claim it can be extended to  $A'$ , thereby proving the second item of the claim. By Claim 1,  $\ell$  extends from  $f_1, f_2$  to  $f'_1, f'_2$ , and it must be that  $\ell(f'_1) = \ell(f'_2) = 2$ , while the ends of  $f'_1, f'_2$  other than  $v$  have sum 3. By setting  $\ell(uv) = 1$ , we obtain  $\sigma(u) = 2$  and  $\sigma(v) = 5$ , thus raising no conflicts. In particular, recall that the neighbour of  $u$  different from  $v$  must have sum 3 by Theorem 4.2.  $\diamond$

At this point, we know that  $A'$  admits s-proper 2-labellings, all of which must assign label 1 to all of the  $\alpha - 3$  (regular) outputs. We now describe how to construct  $H$  from  $G$  and  $A'$ . Start from  $G$  and  $A'$ . Consider then every vertex  $v$  of  $G$  in turn, and pick  $x = \phi(v)$  new regular outputs  $e_1, \dots, e_x$  of  $A'$ , as well as  $y = \frac{4-\phi(v)}{2}$  other new regular outputs  $f_1, \dots, f_y$ . Start by extending each of  $f_1, \dots, f_y$ , resulting in  $y$  extended outputs  $f'_1, \dots, f'_y$ . Note that  $x + y \in \{2, 3, 4\}$ . Now, just identify  $v$  and each of the  $x + y$  ends of  $e_1, \dots, e_x, f'_1, \dots, f'_y$ . Once all vertices  $v$  of  $G$  have been treated that way, the resulting graph is our  $H$ .

Before proceeding with proving the equivalence between  $G$  and  $H$ , let us first comment on  $\alpha$ , the number of outputs that  $A$  must have for the whole construction to be achieved. For a vertex  $v \in V(H) \cap V(G)$  with  $\phi(v) = 0$ , we have attached two extended outputs at  $v$  (and no regular outputs). If  $\phi(v) = 2$ , then we have attached two regular outputs and one extended output at  $v$ . Last, if  $\phi(v) = 4$ , then we have attached four regular outputs (and no extended outputs). Thus, the number of attached outputs (regardless of their type) of  $A'$  is at most  $4|V(G)|$ , and remember that three additional outputs were necessary to go from  $A$  to  $A'$ . Thus it is sufficient to have  $\alpha$  linear in  $|V(G)|$ , and the construction of  $H$  from  $G$  and  $A'$  is clearly achieved in polynomial time.

We now prove that we have the desired equivalence between  $G$  and  $H$ .

- Assume first that  $H$  admits an s-proper 2-labelling  $\ell$ . Recall that all regular outputs of  $A'$  must be assigned label 1 by  $\ell$  due to Claim 2, and that all extended outputs must be assigned label 2 by Claim 1. By how  $H$  was constructed, in particular with respect to  $\phi$ , note that, for every vertex  $v \in V(H) \cap V(G)$ , the edges of  $E(A')$  incident to  $v$  have their assigned labels summing up to exactly 4. In particular, for every edge  $uv$  in  $E(H) \cap E(G)$ , we have  $\sigma(u) \neq \sigma(v)$ , and, thus,  $\sigma(u) - 4 \neq \sigma(v) - 4$ . This implies that the restriction of  $\ell$  to the edges of  $G$  is also s-proper.
- Conversely, assume  $G$  admits an s-proper 2-labelling  $\ell$ . We claim that  $\ell$  can be extended to an s-proper 2-labelling of  $H$ . To see this is true, start from  $\ell$  being a partial labelling of  $H$  (following the labelling  $\ell$  of  $G$ ), and simply extend this  $\ell$  to the edges of  $A'$  by considering an s-proper 2-labelling of  $A'$  (such a labelling exists, by Claim 2). Let us denote by  $\ell'$  the resulting 2-labelling of  $H$ . We claim  $\ell'$  is s-proper. First off, for every vertex  $v \in V(H) \cap V(G)$ , as mentioned earlier by  $\ell'$  we have  $\sigma(v) = x + 4$ , where  $x$  is the value of  $\sigma(v)$  by  $\ell$ . Since  $\ell$  is s-proper in  $G$ , this implies that, for every  $uv \in E(H) \cap E(G)$ , by  $\ell'$  we have  $\sigma(u) \neq \sigma(v)$ . By how  $\ell'$  was obtained, also  $\sigma(u) \neq \sigma(v)$  by  $\ell'$  for every  $uv \in E(H) \cap E(A')$  such that  $u, v \notin V(A') \cap V(G)$ . It remains to prove that we also have  $\sigma(u) \neq \sigma(v)$  by  $\ell'$  for every two adjacent vertices  $u, v$  with  $u \in V(H) \cap V(G)$  and  $v \in V(H) \cap V(A')$ . Due to the outputs of  $A'$  attached at  $u$ , which, by  $\ell'$ , bring exactly 4 to  $\sigma(u)$ , and  $d_G(u) = 3$ , note that  $\sigma(u) \geq 7$  by  $\ell'$ . On the other hand,  $v$  is either of degree 3 (case of a regular output  $uv$  attached at  $u$ ) or 2 (otherwise, case of an extended output  $uv$ ), meaning that  $\sigma(v) \leq 6$  by  $\ell'$ . Thus, we always have  $\sigma(u) > \sigma(v)$ , and  $\ell'$  is thus s-proper.

We conclude the proof by showing that  $\chi_M(H) = 2$ , regardless of  $G$ . This is by the following arguments. For every vertex  $v \in V(H) \cap V(G)$ , note that we have  $d_H(v) = d_G(v) + \phi(v) + \frac{4-\phi(v)}{2}$ . Since  $G$  is cubic, we thus have  $d_H(v) = 7$  if  $\phi(v) = 4$ ,  $d_H(v) = 6$  if  $\phi(v) = 2$ , and  $d_H(v) = 5$  otherwise, i.e., if  $\phi(v) = 0$ . Since  $\phi$  is proper, this means that, for every  $uv \in E(H) \cap E(G)$ , we have  $d_H(u) \neq d_H(v)$ , implying that  $\mu(u) \neq \mu(v)$  by any 2-labelling of  $H$ . Similarly, for every vertex

$v \in V(H) \cap V(A')$ , we have  $d_H(v) \leq 3$ , implying that  $d_H(u) \neq d_H(v)$  for every  $u \in V(H) \cap V(G)$ , thus  $\mu(u) \neq \mu(v)$  again by any 2-labelling of  $H$ . All these arguments imply that if we just start from an s-proper 2-labelling of  $A'$  (which exists by Claim 2), and extend it to the whole of  $H$  by assigning arbitrary labels to the edges of  $E(H) \cap E(G)$ , then what results is an m-proper 2-labelling of  $H$ . Thus,  $\chi_M(H) = 2$ .  $\square$

## 5. Discussion

Through Theorem 4.3, we have provided a positive answer to Question 1.2, thus a negative answer to Question 1.1. A general way to pursue the investigations on this topic, could be by wondering about classes of graphs for which Question 1.1 is true or not, and similarly for its counterpart for p-proper labellings. For instance:

- In the case of nice trees  $T$ , it has been known for long that  $\chi_S(T) \leq 2$  always holds (see [9]), which implies that Question 1.1 is true for trees. This is a neat difference with p-proper labellings, as Szabo Lyngsie proved in [15] that there exist infinitely many trees  $T$  with  $\chi_P(T) = 3$ , which, fortunately, can be recognised in polynomial time. Thus, for trees  $T$ , we sometimes have  $\chi_M(T) = 2 < 3 = \chi_P(T)$ , but such situations can be recognised easily.
- Regarding bipartite graphs, it was proved by Thomassen, Wu and Zhang [19] that bipartite graphs  $G$  with  $\chi_S(G) = 3$  form exactly the class of so-called *odd multi-cacti*, which can be recognised in polynomial time. Looking at the structure of odd multi-cacti, it is not too complicated to prove that each  $G$  of them also verifies  $\chi_M(G) = 3$  (refer e.g. to [5] for more insight in the structure of these graphs). In other words, the answer to Question 1.1 is also yes for bipartite graphs. Regarding p-proper labellings, we have already mentioned earlier that trees form a context in which we sometimes do not have equality between the parameters  $\chi_M$  and  $\chi_P$ . Regarding the complexity of deciding whether this is the case or not for a given bipartite graph, it is still unclear, as we still do not know whether bipartite graphs  $G$  with  $\chi_P(G) \leq 2$  can be recognised in polynomial time (see [15] for partial results).
- Regarding graphs with bounded maximum degree, we note that our reduction in the proof of Theorem 4.3 builds graphs of maximum degree at most 7. A question could thus be whether Theorem 4.3 also holds for graphs with maximum degree less than 6.

Note that graphs  $G$  with maximum degree 2 are paths and cycles, for which it is not complicated to prove, through Observations 2.1 and 2.3, that we always have  $\chi_M(G) = \chi_S(G) = 2$ . We note also that for any  $\Delta \in \{3, 4, 5, 6\}$ , we can construct graphs with maximum degree  $\Delta$  that are counterexamples to Question 1.1. Let us give an example for  $\Delta = 3$ , which generalises easily for any  $\Delta \in \{4, 5, 6\}$ . Reusing the terminology from the proof of Theorem 4.3, start from a 6-necklace  $D_6$ , with outputs  $e_1, \dots, e_6$ . Now extend  $e_4, e_5, e_6$  to new extended outputs  $e'_4, e'_5, e'_6$ . Denote by  $v_1, \dots, v_6$  the ends of  $e_1, e_2, e_3, e'_4, e'_5, e'_6$ , respectively. Now identify  $v_4$  and  $v_5$  to a new vertex  $x$ , and similarly identify  $v_2$  and  $v_3$  to a new vertex  $y$ , and just add the edges  $v_1x$  and  $v_6y$ , to obtain a graph  $G$ . It is not too hard to check that  $\Delta(G) = 3$ , and that  $\chi_M(G) = 2 < 3 = \chi_S(G)$ , by similar arguments as in the proof of Theorem 4.3. In particular, note that, just as in the proof with  $A'$ , the edge  $v_1x$  forces all outputs of  $D_6$  to be assigned label 1 by an s-proper 2-labelling of  $G$ , while the edge  $v_6y$ , by similar arguments, forces all outputs of  $D_6$  to be assigned label 2. It follows that  $\chi_S(G) > 2$ . On the other hand, due to the number of outputs of  $D_6$  we have identified, note that  $v_1$  and  $x$ , and similarly  $v_6$  and  $y$ , have different degrees and thus cannot be in conflict when considering multisets.

Regarding the same concerns for p-proper labellings, note that the reduction in the proof of Theorem 3.2 provides graphs with maximum degree 5. Thus, a similar question as above concerns the complexity of the same problem for graphs with maximum degree 3 or 4, given, again, that some of these graphs  $G$  sometimes verify  $\chi_M(G) = 2 < 3 = \chi_P(G)$  (which can be proved reusing the same ideas as above, but with the labelling ideas from the proof of Theorem 3.2).

Another question could concern the order of the smallest graphs for which Question 1.1 does not hold. The main reason why the answer to that question was not so obvious, is that, perhaps, graphs  $G$  with  $\chi_M(G) = 2 < 3 = \chi_S(G)$  are always large. We note that the example we have constructed above from the 6-necklace  $D_6$  has order 40, and we wonder about smaller similar graphs. Regarding p-products, we note that the path  $P_6$  on 6 vertices verifies  $\chi_P(P_6) = 3$ , while  $\chi_M(P_6) = \chi_S(P_6) = 2$ .

## References

- [1] L. Addario-Berry, R.E.L. Aldred, K. Dalal, B.A. Reed. Vertex colouring edge partitions. *Journal of Combinatorial Theory, Series B*, 94(2):237-244, 2005.
- [2] A. Ahadi, A. Dehghan, M.-R. Sadeghi. Algorithmic complexity of proper labeling problems. *Theoretical Computer Science*, 495:25-36, 2013.
- [3] O. Baudon, J. Bensmail, J. Przybyło, M. Woźniak. On decomposing regular graphs into locally irregular subgraphs. *European Journal of Combinatorics*, 49:90-104, 2015.
- [4] J. Bensmail, F. Fioravantes, F. Mc Inerney. On the Role of 3s for the 1-2-3 Conjecture. *Theoretical Computer Science*, 892:238-257, 2021.
- [5] J. Bensmail, F. Fioravantes, F. Mc Inerney, N. Nisse. Further Results on an Equitable 1-2-3 Conjecture. *Discrete Applied Mathematics*, 297:1-20, 2021.
- [6] J. Bensmail, H. Hocquard, D. Lajou, É. Sopena. Further Evidence Towards the Multiplicative 1-2-3 Conjecture. *Discrete Applied Mathematics*, 307:135-144, 2022.
- [7] J. Bensmail, H. Hocquard, D. Lajou, É. Sopena. A proof of the Multiplicative 1-2-3 Conjecture. Preprint, 2021. Available online at <https://arxiv.org/abs/2108.10554>.
- [8] R.L. Brooks. On colouring the nodes of a network. In *Proceedings Cambridge Philosophical Society, Mathematical and Physical Sciences*, 37:194–197, 1941.
- [9] G.J. Chang, C. Lu, J. Wu, Q. Yu. Vertex-coloring edge-weightings of graphs. *Taiwanese Journal of Mathematics*, 15(4):1807-1813, 2011.
- [10] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba. Irregular networks. *Congressus Numerantium*, 64:197–210, 1988.
- [11] A. Dudek, D. Wajc. On the complexity of vertex-coloring edge-weightings. *Discrete Mathematics and Theoretical Computer Science*, 13(3):45-50, 2011.
- [12] J.A. Gallian. A dynamic survey of graph labeling. *Electronic Journal of Combinatorics*, 6, 1998.
- [13] M. Kalkowski, M. Karoński, F. Pfender. Vertex-coloring edge-weightings: towards the 1-2-3 Conjecture. *Journal of Combinatorial Theory, Series B*, 100:347-349, 2010.
- [14] M. Karoński, T. Łuczak, A. Thomason. Edge weights and vertex colours. *Journal of Combinatorial Theory, Series B*, 91:151–157, 2004.
- [15] K. Szabo Lyngsie. On neighbour sum-distinguishing  $\{0, 1\}$ -edge-weightings of bipartite graphs. *Discrete Mathematics and Theoretical Computer Science*, 20(1), 2018.
- [16] N. Paramaguru, R. Sampathkumar. Graphs with vertex-coloring and detectable 2-edge-weighting. *AKCE International Journal of Graphs and Combinatorics*, 13:146-156, 2016.
- [17] B. Seamone. The 1-2-3 Conjecture and related problems: a survey. Preprint, 2012. Available online at <http://arxiv.org/abs/1211.5122>.
- [18] J. Skowronek-Kaziów. Multiplicative vertex-colouring weightings of graphs. *Information Processing Letters*, 112(5):191-194, 2012.

- [19] C. Thomassen, Y. Wu, C.-Q. Zhang. The 3-flow conjecture, factors modulo  $k$ , and the 1-2-3-conjecture. *Journal of Combinatorial Theory, Series B*, 121:308-325, 2016.
- [20] B. Vučković. Multi-set neighbor distinguishing 3-edge coloring. *Discrete Mathematics*, 341:820-824, 2018.