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Large N limit of Yang–Mills partition function and Wilson loops on compact surfaces

Antoine Dahlqvist* Thibaut Lemoine†

Abstract

We compute the large N limit of several objects related to the two-dimensional Euclidean Yang–Mills measure on compact connected orientable surfaces Σ of genus $g \geq 1$ with a structure group taken among the classical groups of order N . We first generalise to all classical groups the convergence of partitions functions obtained by the second author in [29], then we apply this result to show that all Wilson loop observables for loops included within a topological disc of Σ converge towards the limit of the Wilson loop where the latter disc is embedded in the plane in place of Σ . This solves a conjecture of B. Hall [25] for closed surfaces. Besides, using similar arguments, we show that Wilson loops vanish for all non-contractible simple loops.

1 Introduction

The purpose of this article is to study a model of random matrices originating from 2D Euclidean quantum field theory, known as the two-dimensional Yang–Mills measure. This model is defined for a classical compact Lie group and a compact surface endowed with an area measure and can be understood as a gauge theory. Each loop on the surface defines an observable called Wilson loop. We are interested here in Wilson loops when the size of the matrices N goes to infinity, while the loop and the surface are fixed.

Let us attempt to give an account of the origin and state of this problem. The question seems to have first appeared in a mathematical paper in [50]. Therein, the candidate limit of Wilson loops on an arbitrary surface Σ is called master field. This paper was inspired by the large N limits considered in quantum gauge theories [12, 27, 42, 39, 22, 21], which started after the landmark work of t’Hooft [52]. Since then, the Yang–Mills measure in two dimensions has been rigorously defined¹ [23, 17, 48, 30], and the latter question has received the attention of several mathematicians² [54, 49, 2, 34, 19, 18, 11, 25]. It is now known that the limit indeed exists when the

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¹See [37] for a historical review on the motivations of these approaches.

²See [35] for a recent reviews paper on the Yang–Mills measure and the master fields in two dimensions. See also [6] for a study of large N limits of Wilson loops, for discrete Yang–Mills measure in dimension higher than 2.

surface is the plane [2, 33] or the sphere [11], giving rise to two different objects. Simultaneously to and independently from [11], conditional convergence results were obtained in [25] for loops within a topological disc embedded in a compact surface, with different set of assumptions for simple loops. Both works [11, 25] relied on the so-called Makeenko–Migdal equations, which allow to tackle the problem recursively in the complexity of loops. These equations first appeared in physics in [39, 27] and were proved rigorously on the plane in [33, 8, 19] and on surfaces in [19, 18].

Our main result, Theorem 2.14, shows that for all loops included within a topological disc of Σ , the limit Wilson loop converges and its limit is given by the value of the *master field on the plane* evaluated at the loop obtained by embedding the same disc in the plane instead of Σ , in a way that preserves the area measure. In some loose sense, the behaviour of the Yang–Mills measure within a topological disc is not affected asymptotically by the topological constraint imposed by the surface. Though it was probably known by physicists, we were unable to find this phenomenon mentioned explicitly in the physics literature.

Our second result Theorem 2.15 shows that for any simple non-contractible loop the Wilson loop of any iteration vanishes asymptotically. This leaves the case of loops which are neither embedded in a topological disc nor a power of a simple loop. We investigate this question in the next paper in the series [10].

Our argument does not rely here on the Makeenko–Migdal equations but mainly on the convergence of another quantity: the partition function. We prove the convergence of the partition function using harmonic analysis on the classical compact groups, generalising a result of [24] and of the second author [29].

The rest of the paper is divided in three sections. The first one recalls the setup of the problem and presents the main results. The second section recalls the necessary notions from representation theory of compact groups and the result of [24, 29] on partition functions which we generalise to all group series of compact matrix Lie groups and all area parameter. The last one gives the proof of our main results on Wilson loops.

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2 Setup and statement of results

2.1 Heat kernel on compact Lie groups

2.1.1 Heat kernel

In this text, G will denote a compact Lie group endowed with a bi-invariant inner product. The heat kernel on G is the family of smooth functions $(p_t, t \in (0, \infty))$ on G satisfying

$$\frac{d}{dt}p_t(g) = \frac{1}{2}\Delta(p_t)(g) \text{ for } t > 0 \text{ and } g \in G$$

and for any continuous function f ,

$$\lim_{t \rightarrow 0} \int_G f(g)p_t(g)dg = f(1).$$

We denote here by Δ and dg the Laplace–Beltrami operator and the Haar measure on G , while we write 1 for the identity element of G . The functions p_t are central for all $t > 0$ and form a semigroup for the convolution product on G , that is for $t > 0$,

$$p_t(xgx^{-1}) = p_t(g), \forall x, g \in G$$

and

$$p_t * p_s = p_{t+s}, \forall s > 0.$$

2.1.2 Classical compact Lie groups

In the sequel, we shall say that G_N is a *classical group of size N^3* if it is equal to one of the following matrix Lie groups:

1. The unitary group $U(N) = \{U \in \text{GL}_N(\mathbb{C}) : UU^* = I_N\}$,
2. The special unitary group $SU(N) = \{U \in U(N) : \det(U) = 1\}$,
3. The special orthogonal group $SO(N) = \{O \in \text{GL}_N(\mathbb{R}) : OO^t = I_N\}$,

³Although $\text{Sp}(N)$ is a group of complex matrices of size $2N$, the denomination “size N ” is not misleading, because it can be also considered as a subgroup of $\text{GL}_N(\mathbb{H})$. We do not exploit this property, and choose such naming convention only for the sake of simplicity.

4. The compact symplectic group $\mathrm{Sp}(N) = U(2N) \cap \{S \in \mathrm{GL}_{2N}(\mathbb{C}) : S^t J S = J\}$ where $J \in \mathrm{GL}_{2N}(\mathbb{C})$ is defined by

$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

We fix the bi-invariant metric as follows. Assume that G_N is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ (so that $n = N$ for $U(N)$, $SU(N)$, $SO(N)$ and $n = 2N$ for $\mathrm{Sp}(N)$). We set an integer parameter⁴ β which is equal to 1 if $G_N = SO(N)$, 2 if $G_N = U(N)$, $SU(N)$, and 4 if $G_N = \mathrm{Sp}(N)$. We endow the Lie algebra \mathfrak{g}_N of G_N with the scalar product

$$\langle X, Y \rangle = \frac{\beta n}{2} \mathrm{Tr}(X^* Y), \forall X, Y \in \mathfrak{g}_N. \quad (1)$$

It is conjugation-invariant in the sense that

$$\langle gXg^{-1}, gYg^{-1} \rangle = \langle X, Y \rangle, \text{ for all } X, Y \in \mathfrak{g}_N \text{ and } g \in G_N,$$

and defines therefore a bi-invariant metric on G_N . Here Tr denotes the trace of the above matrix, that is the non-normalised sum of diagonal coefficients.

Remark. Except in case 1., there is up to constant a unique invariant inner-product. The above choice of scaling is standard in random matrices. For instance, the gaussian vector on Hermitian matrices obtained by composition of the above scalar product with the multiplication by i is the classical Gaussian Unitary Ensemble.

2.2 2D-maps and multiplicative functions

Assume that Σ is a two dimensional compact Riemannian manifold, with Riemannian distance d_Σ and genus $g \geq 0$. It is homeomorphic to a $4g$ -gon whose sides are, counterclockwise, $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$, such that each two edges with same letter are glued together, while respecting the orientation. The polygon is called the *fundamental domain* of the surface. If Σ has a nonempty boundary $\partial\Sigma$, the latter can be described as the union of n connected components $(C_i)_{1 \leq i \leq n}$, each homeomorphic to the unit circle S^1 . We introduce here some notations used throughout this text.

Denote by $\mathrm{P}(\Sigma)$ the set of continuous maps $[0, 1] \rightarrow \Sigma$ with positive and finite length, up to Lipschitz reparametrisation. When $\gamma \in \mathrm{P}(\Sigma)$, $\mathcal{L}(\gamma)$ denotes its length, $\gamma^{-1} \in \mathrm{P}(\Sigma)$ its reverse, $\underline{\gamma} = \tilde{\gamma}(0)$ its starting point and $\bar{\gamma} = \tilde{\gamma}(1)$ its endpoint, where $\tilde{\gamma}$ is some parametrisation. When $\bar{\gamma} = \underline{\gamma}$, we say that γ is a *loop* of Σ and write

$$\mathrm{L}(\Sigma) = \{\gamma \in \mathrm{P}(\Sigma) : \underline{\gamma} = \bar{\gamma}\}.$$

When $\gamma_1, \gamma_2 \in \mathrm{P}(\Sigma)$ with $\underline{\gamma_2} = \bar{\gamma_1}$, $\gamma_1 \gamma_2 \in \mathrm{P}(\Sigma)$ stands for their concatenation. A distance on $\mathrm{P}(\Sigma)$ is defined [33] setting for all $\gamma_1, \gamma_2 \in \mathrm{P}(\Sigma)$,

$$d(\gamma_1, \gamma_2) = |\mathcal{L}(\gamma_1) - \mathcal{L}(\gamma_2)| + \inf_{t \in [0, 1]} \inf_{\tilde{\gamma}_2(t)} \{d_\Sigma(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t))\}$$

⁴This parameter corresponds to the Dyson index in random matrix theory. For details over its significance, see for instance Section 4.1 of [1]. We introduce this parameter so that standard Brownian motions on these Lie algebras all converge to the same process when $N \rightarrow \infty$, see for instance [9, Section 2] for an explanation.

where the first infimum is taken over all Lipschitz parametrisations $\tilde{\gamma}_1, \tilde{\gamma}_2$ of respectively γ_1 and γ_2 . In the following paragraphs, we will consider subsets of $P(\Sigma)$ corresponding to loops traced in embedded graphs.

2.2.1 Topological maps on compact surfaces

We follow here conventions of [28, Section 1.3.2]. A *graph* \mathcal{G} is a triple $(\mathbb{V}, \mathbb{E}, I)$ consisting of a set \mathbb{V} of *vertices*, a set \mathbb{E} of *edges* and an *incidence relation* I such that an edge is incident to either one vertex or two distinct vertices, called *endpoints*. It can be given an *orientation* by setting, for any $e \in \mathbb{E}$, a source $\underline{e} \in \mathbb{V}$ and a target $\bar{e} \in \mathbb{V}$. An oriented graph can be then represented by a quadruple $(\mathbb{V}, \mathbb{E}, s, t)$ where \mathbb{V} is the set of vertices, \mathbb{E} the set of oriented edges, and $s, t : \mathbb{E} \rightarrow \mathbb{V}$ are the functions that map respectively an edge e to its source and target. An *isomorphism* between two graphs $\mathcal{G}_1 = (\mathbb{V}_1, \mathbb{E}_1, s_1, t_1)$ and $\mathcal{G}_2 = (\mathbb{V}_2, \mathbb{E}_2, s_2, t_2)$ is a bijection $\phi : \mathbb{V}_1 \cup \mathbb{E}_1 \rightarrow \mathbb{V}_2 \cup \mathbb{E}_2$ that sends \mathbb{V}_1 (resp. \mathbb{E}_1) onto \mathbb{V}_2 (resp. \mathbb{E}_2), and such that

$$s_2(\phi(e)) = \phi(s_1(e)), \quad t_2(\phi(e)) = \phi(t_1(e)), \quad \forall e \in \mathbb{E}.$$

A *topological map* M on Σ is a finite oriented graph $\mathcal{G} = (\mathbb{V}, \mathbb{E}, s, t)$ endowed with an embedding $\theta : \mathcal{G} \rightarrow \Sigma$, called *drawing of the graph*, such that:

- the vertices are drawn as distinct points of Σ ,
- oriented edges are drawn as oriented continuous curves that only intersect at their endpoints,
- For any $e \in \mathbb{E}$, there is an edge $e^{-1} \in \mathbb{E}$ such that $\theta(e^{-1}) = \theta(e)^{-1}$,
- the set $\mathbb{F} = \Sigma \setminus \bigcup_{e \in \mathbb{E}} \{\theta(e)\}$ is given by a union of open discs called *faces* of the map.

In order to avoid too heavy notations, we will always identify vertices and edges with their drawing, so that a map on Σ can be represented by $(\mathbb{V}, \mathbb{E}, \mathbb{F})$ (the applications s and t will remain implicit from now on). Σ is called the *underlying surface* of M . If $v \in \mathbb{V}$ is a vertex in M , we denote by $P(M)$ (resp. $L(M), L_v(M)$) the set of paths (resp. loops, loops with base v) in M obtained by concatenation of edges.

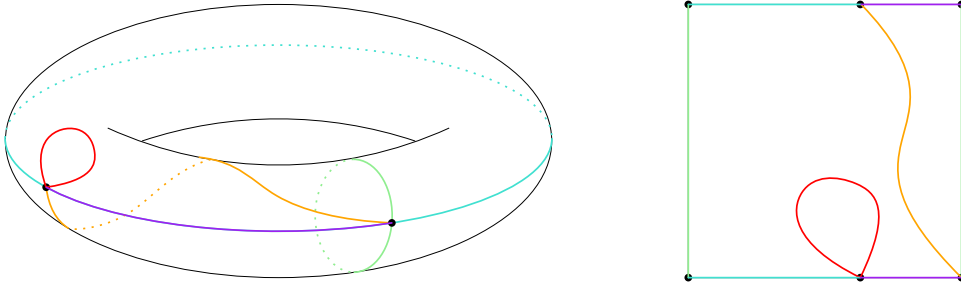


Figure 1: An example of topological map on a torus (on the left), and its representation as an abstract graph (on the right) whose opposite edges of the same colour are identified.

Two maps $M_1 = (\mathbb{V}_1, \mathbb{E}_1, \mathbb{F}_1)$ and $M_2 = (\mathbb{V}_2, \mathbb{E}_2, \mathbb{F}_2)$, respectively on Σ_1 and Σ_2 and with associated graphs \mathcal{G}_1 and \mathcal{G}_2 , are *equivalent* if there is an orientation-preserving homeomorphism $\phi : \Sigma_1 \rightarrow \Sigma_2$ such that the restriction of ϕ to M_1 induces

an isomorphism between \mathcal{G}_1 and \mathcal{G}_2 . The *genus* of a topological map in Σ is the genus of Σ , and it does not depend on its equivalence class. Note that Dehn twists, that generate the mapping class group of Σ (see [26]), preserve the equivalence class of a map on Σ .

The careful reader may wonder how to tackle the presence of a nonempty boundary; according to the following result, which is a particular case of Proposition 1.3.10 in [33], it is a simple detail.

Proposition 2.1. *Let M be a topological map on a compact surface Σ such that $\partial\Sigma$ is the union of elements of $L(\Sigma)$ ⁵. Any connected component C of $\partial\Sigma$ is an element of $L(M)$.*

2.2.2 Topological maps on \mathbb{R}^2 or an open disc

A topological map M on the *plane* \mathbb{R}^2 or an *open disc* is a topological map on the sphere \mathbb{S}^2 together with a marked face f_∞ . The latter is called the infinite face of M , and it contains the point at infinity when one makes the identification $\mathbb{S}^2 \simeq \mathbb{R}^2 \cup \{\infty\}$. From now on, for any map M , be it on the plane, a disc or a compact surface, \mathbb{F} will denote by default the set of *bounded* faces, and $\hat{\mathbb{F}}$ will denote $\mathbb{F} \cup \{f_\infty\}$ whenever it makes sense.

2.2.3 Multiplicative functions

For any $P \subset P(\Sigma)$ and any compact group G , a *multiplicative function* $h : P \rightarrow G$ is a function that satisfies $h_{\gamma^{-1}} = h_\gamma^{-1}$ for any $\gamma \in P$ such that $\gamma^{-1} \in P$, and $h_{\gamma_1\gamma_2} = h_{\gamma_2}h_{\gamma_1}$ for any $\gamma_1, \gamma_2 \in P$ such that $\gamma_1\gamma_2 \in P$. We denote by $\mathcal{M}(P, G)$ the space of multiplicative functions from P to G . The gauge group $\Gamma = \mathcal{C}^\infty(\Sigma, G)$ acts on it by *gauge transformations*: for any $j \in \Gamma$ and $h \in \mathcal{M}(P, G)$,

$$(j \cdot h)_\gamma = j(\bar{\gamma})^{-1}h_\gamma j(\underline{\gamma}), \quad \forall \gamma \in P.$$

One of the numerous advantages of considering topological maps is that the gauge-invariance of multiplicative functions is equivalent to a G -invariance, as stated by the following lemma which is a particular instance of [33, Lemma 2.1.5].

Lemma 2.2. *Let M be a topological map in Σ . For any $v \in \mathbb{V}$, and $\gamma_1, \dots, \gamma_n \in P(M)$, there exist $\ell_1, \dots, \ell_n \in L_v(M)$ such that for any $f : G^n \rightarrow \mathbb{C}$ such that $h \mapsto f(h_{\gamma_1}, \dots, h_{\gamma_n})$ is gauge-invariant on $\mathcal{M}(P(M), G)$, there exists a function $\tilde{f} : G^n \rightarrow \mathbb{C}$ that is invariant by the diagonal action of G by conjugation such that*

$$f(h_{\gamma_1}, \dots, h_{\gamma_n}) = \tilde{f}(h_{\ell_1}, \dots, h_{\ell_n}), \quad \forall h \in \mathcal{M}(P(M), G).$$

Two sigma-fields may be put on $\mathcal{M}(P(M), G)$:

- The smallest sigma-field \mathcal{C} such that for any $\gamma \in P(M)$ the evaluation function⁶

$$H_\gamma : \begin{cases} \mathcal{M}(P(M), G) & \rightarrow G \\ h & \mapsto h_\gamma \end{cases}$$

is measurable: it is the *cylindrical sigma-field*.

⁵In other terms, we ask that the boundary of Σ has positive and finite length!

⁶It is also called *holonomy* in the literature.

- The smallest sigma-field \mathcal{J} that makes

$$h \mapsto f(h_{\ell_1}, \dots, h_{\ell_n})$$

measurable, for all $v \in \mathbb{V}$, $n \in \mathbb{N}$, $\ell_1, \dots, \ell_n \in L_v(M)$ and $f : G^n \rightarrow \mathbb{C}$ G -invariant: it is the *invariant sigma-field*.

We will mainly work with \mathcal{C} but some results will only hold on \mathcal{J} , therefore we will precise which sigma-field we consider.

2.2.4 Wilson loops

If $\chi : G \rightarrow \mathbb{C}$ is a central function, for any $\ell \in L(M)$, $\chi(h_\ell)$ does not depend on the choice of based point. The function

$$\begin{aligned} \mathcal{M}(L(M), G) &\longrightarrow \mathbb{C} \\ h &\longmapsto \chi(h_\ell) \end{aligned}$$

is then called a Wilson loop. When G belongs to one of the four series given above, we will be interested in the Wilson loops

$$W_\ell : \begin{cases} \mathcal{M}(L(M), G) &\longrightarrow \mathbb{C} \\ h &\longmapsto \text{tr}(h_\ell), \end{cases} \quad (2)$$

where ℓ is a loop of M , while for any $M \in M_d(\mathbb{C})$, $\text{tr}(M) = \frac{1}{d} \sum_{i=1}^d M_{i,i}$.

Remark (Gauge equivalence). For most compact Lie groups G , it can be shown that the family of Wilson loops separate points of $\mathcal{M}(P(M), G)/\Gamma$ endowed with the quotient topology. When G belongs to one of the four series of the previous section, it can further be shown [46, 31] that it is enough to consider the family $\{W_\ell, \ell \in L(M)\}$.

2.2.5 Area-weighted maps

An *area-weighted map* is a topological map M together with a function $a : \mathbb{F} \rightarrow \mathbb{R}_+$: the marked faces are excluded because their area is either considered be infinite (in the case of \mathbb{R}^2) or to be undefined (in the case of a surface with boundary). Two maps (M, a) and (M', a') are *equivalent* if M and M' are equivalent as maps and the associated homomorphism ϕ of Σ defines a bijection $\mathfrak{F} : \mathbb{F} \rightarrow \mathbb{F}'$ with

$$a' \circ \mathfrak{F} = a.$$

For any $T > 0$ we set

$$\Delta_M(T) = \{a : \mathbb{F} \rightarrow \mathbb{R}_+ : \sum_{f \in \mathbb{F}} a_f = T\}$$

when M is a map of a closed surface and when (M, a) is an area-weighted map we write $\|a\|_1 = \sum_{f \in \mathbb{F}} a_f$. If the surface is endowed with a Riemannian volume vol , then it induces in particular an area function $\text{vol} : \mathbb{F} \rightarrow \mathbb{R}_+$.

2.3 Two dimensional Yang–Mills measure

In this section we define the Yang–Mills measure in several steps:

1. Given a topological map M , we define a uniform measure $U_{M,C,G}$ on $\mathcal{M}(\mathbb{P}(M), G)$ with and without constraints,
2. We define the discrete Yang–Mills measure $YM_{M,C,a,G}$ on an area-weighted topological map (M, a) as an absolutely continuous measure with respect to $U_{M,G,C}$,
3. We define the Yang–Mills holonomy field $(H_\ell)_{\ell \in \mathbb{P}(\Sigma)}$ on any compact measured surface (Σ, vol) , whose distribution $YM_{\Sigma,C,G}$ is the continuous version of $YM_{M,C,a,G}$

2.3.1 Uniform measure on $\mathcal{M}(\mathbb{P}(M), G)$

For any topological map M , the space $\mathcal{M}(\mathbb{P}(M), G)$ is a compact Lie group when endowed with the pointwise multiplication and has a unique Haar measure μ_M , on both sigma-fields \mathcal{C} and \mathcal{J} . Fixing a set \mathbb{E}^+ of positively oriented edges (so that for any $e \in \mathbb{E}$, only e or e^{-1} belongs to \mathbb{E}^+), $\mathcal{M}(\mathbb{P}(M), G)$ can be identified with $G^{\mathbb{E}^+}$ and μ_M is the push-forward of the direct product of Haar measures on G . Let us call it the *unconstrained uniform measure* and denote it by $U_{M,G}$. In contrast, we need to define a constrained version to put boundary conditions when necessary. If Σ is a compact surface with boundary, denote by $\mathcal{B}(\Sigma)$ the set of connected components of $\partial\Sigma$, each taken twice, one for each orientation. There is a natural action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{B}(\Sigma)$ by orientation-reversal.

A *set of boundary conditions* on a compact surface Σ is a $\mathbb{Z}/2\mathbb{Z}$ -equivariant map

$$C : \mathcal{B}(\Sigma) \rightarrow G/\text{Ad},$$

where G/Ad is the set of conjugation classes of G where $\mathbb{Z}/2\mathbb{Z}$ acts by inversion. The fact that boundary conditions take values in G/Ad rather than simply G will be explained later. If (M, a) is an area-weighted map on Σ , then the boundary conditions also apply to M , as $\mathcal{B}(\Sigma) \subset \mathcal{M}(\mathbb{P}(M), G)$. Let us denote by L_1, \dots, L_p the elements of $\mathcal{B}(\Sigma)$ oriented positively, then there is a bijection between $(G/\text{Ad})^{\mathcal{B}(\Sigma)}/(\mathbb{Z}/2\mathbb{Z})$ and $(G/\text{Ad})^p$, so that any set C of boundary conditions can be written $C = (t_1, \dots, t_p)$ with $t_1, \dots, t_p \in G/\text{Ad}$.

For any $t \in G/\text{Ad}$ and $n \geq 1$, the set $t(n) = \{(x_1, \dots, x_n) \in G^n : x_1 \cdots x_n \in t\}$ is a homogeneous space for the G^n -action

$$(g_1, \dots, g_n) \cdot (x_1, \dots, x_n) = (g_1 x_1 g_2^{-1}, \dots, g_n x_n g_1^{-1}).$$

We denote by $\delta_{t(n)}$ the extension to G^n of the unique G^n -invariant probability measure on $t(n) \subset G^n$. It can be thought of as the *conditional Haar measure* on G^n , under the condition $x_1 \cdots x_n \in t$.

A drawback of this conditional measure is that it is not permutation-invariant anymore: for instance, whereas $dg_1 \otimes dg_2 = dg_2 \otimes dg_1$, we have $\delta_{\mathcal{O}(2)}(dg_1 dg_2) \neq \delta_{\mathcal{O}(2)}(dg_2 dg_1)$. Let us choose a specific labelling of \mathbb{E} in order to avoid too heavy notations. If we denote again by L_1, \dots, L_p the connected components of $\partial\Sigma$, we choose the labels of the oriented edges $e \in \mathbb{E}^+$ such that $L_i = e_{i,1} \cdots e_{i,n_i}$ for every i . Denote finally by e_1, \dots, e_m the remaining edges of \mathbb{E}^+ . The *uniform measure* on

$\mathcal{M}(\mathbb{P}(M), G)$ with boundary conditions $C = (t_1, \dots, t_p)$ is the measure $U_{M,C,G}$ on $(\mathcal{M}(\mathbb{P}(M), G), \mathcal{C})$ defined by

$$U_{M,(t_1, \dots, t_p), G}(dh) = dg_1 \otimes \dots \otimes dg_m \otimes \bigotimes_{i=1}^p \delta_{t_i(n_i)}(dg_{i,1} \dots dg_{i,n_i}).$$

As we announced it earlier, here is an explanation of why it is better to consider G/Ad -valued boundary conditions. A gauge transformation $j \in \Gamma$ acts on the boundary conditions by

$$j \cdot (t_1, \dots, t_p) = (j(\bar{\ell})^{-1} t_1 j(\bar{\ell}), \dots, j(\bar{\ell})^{-1} t_p j(\bar{\ell})).$$

Note that any $\ell \in \mathcal{B}(\Sigma)$ is a loop, thus its source and target are identical, so that the action above is an action by conjugation. It induces an action of $j \in \Gamma$ on $U_{M,G,C}$ by

$$j \cdot U_{M,(t_1, \dots, t_p), G} = U_{M, j \cdot (t_1, \dots, t_p), G},$$

and it appears that the measure $U_{M,G,C}$ is gauge-invariant, for any topological map M and set of boundary conditions C , by construction of the measures $\delta_{t_i(n_i)}$.

2.3.2 Discrete Yang–Mills measure

Consider a compact orientable surface Σ with genus $g \geq 0$, and an area-weighted map (M, a) with graph \mathcal{G} embedded in Σ . For each face $f \in \mathbb{F}$, its boundary ∂f is the loop of \mathcal{G} given by the concatenation of edges bordering f . It can be endowed with two possible orientations. The *Yang–Mills measure* on (M, a) with structure group G and boundary conditions⁷ $C = (t_1, \dots, t_p)$ is the probability measure $\text{YM}_{M,C,a,G}$ on $(\mathcal{M}(\mathbb{P}(M), G), \mathcal{C})$ with density⁸

$$\frac{1}{Z_{M,a,G}(t_1, \dots, t_p)} \prod_{f \in \mathbb{F}} p_{a_f}(h_{\partial f}) \quad (3)$$

with respect to $U_{M,C,G}$, where $Z_{M,a,G}(t_1, \dots, t_p) > 0$ is a positive constant. Since the heat kernel defines central functions on G , each term of the above product is a well defined Wilson loop. Using the semigroup property leads to the following elementary but remarkable lemma.

Lemma 2.3 ([47, 33]). *The constant*

$$Z_{M,a,G}(t_1, \dots, t_p) = \int_{\mathcal{M}(\mathbb{P}(M), G)} \prod_{f \in \mathbb{F}} p_{a_f}(h_{\partial f}) U_{M,t_1, \dots, t_p, G}(dh)$$

depends only on the genus g of Σ , the total area $T = \sum_{a \in \mathbb{F}} a_f$ and the boundary conditions C .

⁷It is understood that $C = \emptyset$ if there is no boundary condition. If C is not empty, the measure is also called the *conditional Yang–Mills measure*: in fact, conditions can be put on any set of disjoint simple loops in the map, not only the ones representing the boundary of the underlying surface. For more details, see for instance §1.5.2 of [30].

⁸This measure seems to have been introduced first in the physics literature in [42], see also [41] and [53] where it was used to compute symplectic volumes of flat connexions. Weighting with the heat kernel in gauge theories or continuous spin systems also bares the name of *Villain action*, see for instance [43].

This constant is called the *partition function*. We shall denote it by $Z_{g,T,G}(t_1, \dots, t_p)$ or $Z_{g,T}(t_1, \dots, t_p)$ if there is no ambiguity on the group. We shall give an explicit expression using representation theory, in section 3.1.

When (M', a') is another map with genus g and volume T , we say that (M', a') is finer than (M, a) if any face f of M is a disjoint union of faces f_1, \dots, f_k of M' with $a_f = a'_{f_1} + \dots + a'_{f_k}$. In this case, the restriction of $h \in \mathcal{M}(\mathbb{P}(M'), G)$ to the edges of M defines an element $\mathcal{R}_M^{M'}(h)$ of $\mathcal{M}(\mathbb{P}(M), G)$. We have the following lemma.

Lemma 2.4 ([47, 33]). *When (M', a') is finer than (M, a) , then*

$$(\mathcal{R}_M^{M'})_*(\text{YM}_{M',t_1,\dots,t_p,a',G}) = \text{YM}_{M,t_1,\dots,t_p,a,G}.$$

In particular if $\ell \in \mathbb{L}(M)$, the random variable W_ℓ has same law under $\text{YM}_{M',t_1,\dots,t_p,a',G}$ and $\text{YM}_{M,t_1,\dots,t_p,a,G}$.

Let us finally discuss the link between the conditional and unconditional Yang–Mills measures:

Proposition 2.5 ([33, 30]). *Let (M, a) be an area-weighted map embedded in a compact surface Σ with genus $g \geq 0$. Let ℓ be a loop in M , and denote by $\text{YM}_{M,\ell,t,a,G}$ the Yang–Mills measure on (M, a) with the condition that $[H_\ell] = t \in G/\text{Ad}$. We have*

$$Z_{M,a,G} \text{YM}_{M,a,G} = \int_{G/\text{Ad}} \text{YM}_{M,\ell,t,a,G} Z_{M,\ell,t,a,G} dt. \quad (4)$$

2.4 Yang–Mills holonomy field

The compatibility relation of Lemma 2.4 suggests that the measures considered can be obtained as the image of a single measure on a larger probability space. This has indeed been achieved in [30] and was then generalised in [33], leading to the notion of (continuous) Markovian holonomy fields, allowing to consider a very large family of loops at once. Though it is not crucial to our argument, we recall here their definition, as it allows to reformulate our main results in a unified continuous model. A different rigorous continuous approach has been given earlier by [17, 45, 47]. It relies on stochastic analysis of the white noise on the plane and the formula (3) is obtained as a consequence of the construction and it is called the *Driver–Sengupta formula*. The random holonomy field can be understood as the parallel transport of a random connection with curvature given by a white noise. Recently, yet another continuous construction, defining a random connection one form has been given in [7] recovering the latter formula.

Let Σ be a compact connected orientable surface, an open disc of \mathbb{R}^2 or \mathbb{R}^2 itself, endowed with an area measure vol. For any $\ell \in \mathbb{L}(\Sigma)$, we use the same notation as in (2), for the random variable

$$W_\ell = \text{tr}(H_\ell).$$

Furthermore, whenever $h \in \mathcal{M}(\mathbb{P}(\Sigma), G)$, the restriction of h to a topological map M on Σ defines an element $\mathcal{R}_M(h)$ of $\mathcal{M}(\mathbb{P}(M), G)$. Let us finally mention that the space $\mathcal{M}(\mathbb{P}(\Sigma), G)$ can be endowed with two sigma-fields \mathcal{C} and \mathcal{J} defined as in the discrete case. The continuous extension to the discrete Yang–Mills measure is provided by the following result.

Theorem 2.6 ([33]). *Assume that Σ is a compact connected surface, an open disc of \mathbb{R}^2 or \mathbb{R}^2 , endowed with an area measure vol . For any set $C = (t_1, \dots, t_p)$ of boundary conditions, there exists a unique probability measure $\text{YM}_{\Sigma, C, G}$ on $(\mathcal{M}(\text{P}(\Sigma), G), \mathcal{C})$ such that*

1. *For any topological map M on Σ ,*

$$(\mathcal{R}_M)_*(\text{YM}_{\Sigma, C, G}) = \text{YM}_{M, C, \text{vol}, G}.$$

2. *If $(\gamma_n)_{n \geq 0}$ is a sequence of $\text{P}(\Sigma)$ and $\gamma \in \text{P}(\Sigma)$ with $\gamma_n = \gamma, \overline{\gamma_n} = \overline{\gamma}, \forall n \geq 0$ and $d(\gamma_n, \gamma) \rightarrow 0$ as $n \rightarrow \infty$, then, under $\text{YM}_{\Sigma, C, G}$, $(H_{\gamma_n})_{n \geq 0}$ converges in distribution towards H_γ .*

Moreover, if Ψ is a diffeomorphism of Σ preserving vol and $\psi(\ell) \in \text{L}(\Sigma)$ is the loop of Σ obtained by composition a parametrisation of ℓ with Ψ , then W_ℓ has same law as $W_{\Psi(\ell)}$.

The random process $(H_\gamma, \gamma \in \text{P}(\Sigma))$ with distribution given by $\text{YM}_{\Sigma, C, G}$ is called the *Yang–Mills holonomy field* on Σ . We are primarily interested in the random variables W_ℓ , for $\ell \in \text{L}(\Sigma)$, with structure group a classical group G_N of size N .

2.5 Master fields, conjectures and main results

Following the physics literature, I. Singer raised in [50] the following question, that we will reformulate here in a slightly⁹ modified form as a conjecture.

Conjecture 2.7. *Let Σ be a fixed two dimensional compact Riemannian manifold or \mathbb{R}^2 or a disc of \mathbb{R}^2 . For any loop $\ell \in \text{L}(\Sigma)$, there is a constant $\tau(\ell)$ such that, for any (G_N) such that G_N is a classical group of size N and any set $C = (t_1, \dots, t_p)$ of boundary conditions, under $\text{YM}_{\Sigma, C, G}$,*

$$W_\ell \rightarrow \Phi_\Sigma(\ell) \text{ in probability as } N \rightarrow \infty. \quad (5)$$

If Ψ is a diffeomorphism of Σ preserving its volume form and $\psi(\ell) \in \text{L}(\Sigma)$ is the loop of Σ obtained by composition a parametrisation of ℓ with Ψ ,

$$\Phi_\Sigma(\Psi(\ell)) = \Phi_\Sigma(\ell).$$

The limit function

$$\Phi_\Sigma : \text{L}(\Sigma) \rightarrow \mathbb{C}$$

is called a *master field*. This conjecture has been partly proved for the plane, working with a smaller class of loops in [54, 2], for unitary groups. In [34], it was simultaneously proved for the plane and for all above group series. Recently another argument using Makeenko–Migdal equations was also given in [25].

Remark. For any loop, $\ell \in \text{L}(\Sigma)$, since H_ℓ is a unitary matrix, $|\text{tr}(H_\ell)| \leq 1$ and the convergence in probability (5) is equivalent to

$$\mathbb{E}[|\text{tr}(H_\ell) - \tau(\ell)|] \rightarrow 0 \quad (*)$$

as $N \rightarrow \infty$. When we say that (5) holds uniformly in the set of area vectors a , it is equivalent to the uniformity of the convergence (*) in a . To show (*), it is sufficient to show

$$\mathbb{E}[\text{tr}(H_\ell)] \rightarrow \tau(\ell) \text{ and } \text{Var}(\text{tr}(H_\ell)) = \mathbb{E}[\text{tr}(H_\ell)\overline{\text{tr}(H_\ell)}] - |\mathbb{E}[\text{tr}(H_\ell)]|^2 \rightarrow 0.$$

⁹the specific family of loops was not specified in [50].

Remark. The linear extension of a master field Φ_Σ to $\mathbb{C}[\mathbb{L}(\Sigma)]$ comes automatically with a structure of non-commutative probability space [34, 11]. In the case of the plane, this non-commutative distribution can be characterised using free probability ([34] and [5]). In that case, one can recover the distribution of a free Brownian motion from the master field ([3, 34, 5]).

Theorem 2.8 ([54, 2, 34] and [11]). *The conjecture 2.7 holds true when Σ is an open disc of the plane or a sphere of total area $T > 0$. If ℓ is a fixed combinatorial loop, for any area-weighted map (M, a) with fixed total area T in which ℓ is drawn, under $\text{YM}_{M,a}$,*

$$W_\ell \rightarrow \Phi_{M,a}(\ell) \text{ in probability,}$$

where the right-hand side is deterministic and depends continuously on a , over $a \in \mathbb{R}_+^{\mathbb{F}}$ in the case of the plane and $a \in \Delta_M(T)$ in the case of the sphere.

The work [34] was the first to show rigorously that the master field $\Phi_{\mathbb{R}^2}$ satisfies a set of differential equations named after Makeenko–Migdal equations, that appeared earlier in the physics papers [39, 27]. In [11], the conjecture was proved for unitary groups, in the case of the sphere.

Example 2.9 ([54, 2, 34, 11]). *Assume that $\ell \in \mathbb{L}(\mathbb{R}^2)$ is simple and encloses an area t . For $n \geq 1$,*

$$\mu_t(n) := \Phi_{\mathbb{R}^2}(\ell^n) = \frac{e^{-\frac{nt}{2}}}{n} \sum_{m=0}^{n-1} \frac{(-nt)^m}{m!} \binom{n}{m+1}.$$

Denote by \mathbb{S}_T^2 the two-dimensional Euclidean sphere with total volume T . When¹⁰ $T \leq \pi^2$ and $\ell \in \mathbb{L}(\mathbb{S}_T^2)$ is simple, cutting the sphere into two domains of area t and $T - t$,

$$\mu_{t,T}(n) := \Phi_{\mathbb{S}_T^2}(\ell^n) = J_1(2n\sigma) = \int_{-2}^2 \exp(in\sigma x) \frac{\sqrt{4-x^2} dx}{\pi},$$

where $\sigma = \sqrt{\frac{t(T-t)}{T}}$ and

$$J_1(x) = \sum_{m \geq 0} \frac{(-1)^m}{m!(m+1)!} \left(\frac{x}{2}\right)^{2m}$$

is a Bessel function of the first kind.

- Remark.*
1. Note that when $n \geq 1$ and $T \leq \pi^2$ are fixed, $\mu_t(n)$ and $\mu_{t,T}(n)$ are different functions of t . This is also true [11] when $T > \pi^2$, though the expression of $\mu_{t,T}$ is different.¹¹ Therefore, when ℓ is a simple loop enclosing a disc of area t , the expression of the master field is not the same when the surface in which ℓ is drawn is the plane or the sphere.
 2. Let us highlight nonetheless two relations between the master field on the sphere and on the plane. On the one hand, it can be shown [11] that

$$\lim_{T \rightarrow \infty} \mu_{t,T}(n) = \mu_t(n), \forall n \geq 1. \quad (6)$$

¹⁰An the expression for any $T > 0$ was proved in [11, Thm 2.4].

¹¹In the physics literature, the regimes $T \leq \pi^2$ and $T > \pi^2$ are respectively called the weak and the strong regimes [15].

On the other hand, it follows from dominated convergence that for all $t \geq 0$

$$\lim_{k \rightarrow \infty} \mu_{\frac{t}{k^2}}(nk) = J_1(2nt).$$

Therefore, for all $0 \leq t \leq T \leq \pi^2$,

$$\lim_{k \rightarrow \infty} \mu_{\frac{\sigma}{k^2}}(nk) = \mu_{t,T}(n), \forall n \geq 1. \quad (7)$$

3. The sequences $\mu_t(n)$ and $\mu_{t,T}(n)$, $n \geq 1$ are moment sequences of measures μ_t and $\mu_{t,T}$ on the unit circle, associated to a time marginal of the free Brownian motion [3] and of the free brownian bridge [11]. Since both μ_t and $\mu_{t,T}$ are invariant by complex conjugation, (6) and (7) imply the weak convergences

$$\mu_{t,T} \rightarrow \mu_t \text{ as } T \rightarrow \infty$$

and for any $t \leq T \leq \pi^2$,

$$\mu_{\frac{\sigma}{k^2}}^k \rightarrow \mu_{t,T} \text{ as } k \rightarrow \infty,$$

where for any measure ν on the unit circle, ν^k denotes the push forward of ν by $z \mapsto z^k$.

The conjecture 2.7 remains open for general surfaces. Though, using the Makeenko–Migdal equations on surfaces proved in [18], it was realised in [11, 25] that it is sometimes enough to show the convergence for a restricted family of loops. This idea was exploited for general surfaces in [25] yielding the following theorem¹². Let us say that a loop ℓ of a map M is included in a disc if it is included in an open contractible set U of Σ .

Theorem 2.10 ([25]). *Consider $G_N = U(N)$. Assume that whenever $\ell = s^n$ with $n \geq 0$ and s is a simple loop included in a disc, of an area-weighted topological map (M, a) , under YM_{M,a,G_N} , the random variable W_ℓ converges in probability towards a constant as $N \rightarrow \infty$. Then this also holds true for W_ℓ for any combinatorial loop ℓ included in a disc.*

B. Hall conjectured further that the above assumption can be removed.

Conjecture 2.11 ([25]). *Consider $G_N = U(N)$. Whenever $\ell = s^n$ with $n \geq 0$ and s is a simple loop included in a disc of an area-weighted topological map (M, a) , W_ℓ converges in probability towards a constant as $N \rightarrow \infty$, under YM_{M,a,G_N} .*

Our main result is that this conjecture holds true for any closed surface Σ and any group series of classical groups. We simultaneously prove theorem 2.10 without using the Makeenko–Migdal equations. Instead we shall use the convergence of the partition of function; a result proved by the first author in [11] for unitary groups that we recall and generalise to other group series in section 3. Remarkably the limit in the above conjecture is given in terms of the master field on the plane as follows. Assume that ℓ is a loop of an area weighted map (M, a) included in a disc U . Considering only vertices, edges and area-weighted faces of M included in U and replacing all faces intersecting $\Sigma \setminus U$ by a single marked face yields an area-weighted map (\tilde{M}_U, a_U) of \mathbb{R}^2 . We then denote by the same symbol the loop of \tilde{M}_U obtained by concatenating the edges of ℓ .

¹²Another similar result is obtained in [25], where only an assumption on simple loops is made. For $g \geq 1$, the conclusion is then weaker and holds for loops with constrained area vector.

Theorem 2.12. Consider an area-weighted topological map (M, a) with genus $g \geq 1$ and total volume T . Assume that ℓ is a topologically trivial loop of (M, a) included in a disc U of M . Then for any classical group G_N of size N , under YM_{M,a,G_N} ,

$$W_\ell \rightarrow \Phi_{\tilde{M}_U, a_U}(\ell) \text{ in probability as } N \rightarrow \infty,$$

uniformly in $a \in \Delta_M(T)$.

Example 2.13. In particular, if $\ell = s^n$ where s is a simple contractible loop enclosing an area $t \in (0, T]$, then

$$W_\ell \rightarrow \frac{e^{-\frac{nt}{2}}}{n} \sum_{m=0}^{n-1} \frac{(-nt)^m}{m!} \binom{n}{m+1} \text{ in probability as } N \rightarrow \infty.$$

This result can be generalised as follows.

Theorem 2.14. Assume that Σ is a closed compact Riemann surface with Riemannian volume vol , $\Psi : U \rightarrow D_R = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < R\}$ is a diffeomorphism, where $\pi R^2 < T$, U is an open set of Σ and $\Psi_*(\text{vol}|_U)$ is the Lebesgue measure. Then, for any $\ell \in L(U)$, and any classical group G_N of size N , under YM_{Σ, G_N} ,

$$W_\ell \rightarrow \Phi_{\mathbb{R}^2}(\Psi(\ell)) \text{ in probability as } N \rightarrow \infty.$$

Assuming furthermore that $\Psi^{-1} : D_R \rightarrow U$ can be extended continuously to \overline{D}_R , with piecewise continuous derivatives, for any loop $\ell \in L(\overline{D}_R)$,

$$W_{\Psi^{-1}(\ell)} \rightarrow \Phi_{\mathbb{R}^2}(\ell) \text{ in probability as } N \rightarrow \infty.$$

Note that a simple loop is included in a disc if and only if it is contractible. Our second result below is concerned with simple loops ℓ which are not contractible.

Theorem 2.15. Assume that Σ is a closed compact Riemann surface with Riemannian volume vol and genus $g \geq 1$, and ℓ is a simple non-contractible loop on Σ . Then, for any $k \in \mathbb{Z}^*$ and any classical group G_N of size N , under YM_{Σ, G_N} ,

$$W_{\ell^k} \rightarrow 0 \text{ in probability as } N \rightarrow \infty.$$

Our result covers all simple loops on orientable closed surfaces. We further believe it holds true for non-orientable surfaces. In a sequel to the current work [10], we investigate the master field question for all loops with self-intersections.

3 Asymptotics of partition functions

Our argument relies on the convergence of partition functions of closed orientable surfaces, for classical groups, generalising results of [24] and of the second author [29]. This result was mentioned without proof in [44] for $g > 1$ and in [16] for $g = 1$ and $G = \text{U}(N)$ or $\text{SU}(N)$. For $g \geq 2$, for all group series except the unitary group series, it is a simple consequence of [24]. For $g = 1$ it is proved in [29] for $\text{U}(N)$ and $\text{SU}(N)$. For all classical groups series, the limit involves the Jacobi theta function¹³ and the Euler phi function, both defined for $q \in \mathbb{C}$ such that $|q| < 1$, with

$$\theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2} \quad \text{and} \quad \phi(q) = \prod_{m=1}^{\infty} (1 - q^m).$$

¹³The Jacobi theta function only appears for the unitary group series.

Let $r \geq 1$ be an integer, we denote respectively by $\tilde{A}_r, A_r, B_r, C_r, D_r$ the type of $U(r), SU(r+1), SO(2r+1), Sp(r)$ or $SO(2r)$, in reference to the type of their root system (see [4] for instance). Note that we will alternatively use r as the rank of the root system or N as the size of the classical group, and their relations will be implicit: for instance, if we consider $SU(N)$, it will be of type A_r with $r = N - 1$. Conversely, if we consider the classical group of type D_r , it will be $SO(2r)$. This change of index may be confusing at first, but it will be helpful in order to deal simultaneously with all considered root systems. In any case, considering limits when $r \rightarrow \infty$ and when $N \rightarrow \infty$ is strictly equivalent in the end.

Theorem 3.1. *For any $T > 0, (r, g) \in (\mathbb{N}^*)^2$ and any type X_r , with $X \in \{B, C, D\}$, let us denote by Z_{g,T,X_r} the Yang–Mills partition function on an orientable compact surface of genus g and area T with structure group of type X_r (we will also write $Z_{g,T,r}$ when the type of the structure group is unambiguous). Set $q_T = e^{-\frac{T}{2}}$.*

1. *For all $g \geq 1$ and $T > 0$, $\lim_{r \rightarrow \infty} Z_{g,T,r}$ exists and is given by the table (8) below.*
2. *Moreover if $g = 2$, for all $X \in \{A, B, C, D\}$, $Z_{0,g,X_r} = \lim_{T \rightarrow 0} Z_{T,g,X_r}$ is well defined and*

$$\lim_{r \rightarrow \infty} Z_{g,0,r} = 1.$$

Type	\tilde{A}_r	A_r	B_r, C_r	D_r	
$g = 1$	$\frac{\theta(q_T)}{\phi(q_T)^2}$	$\frac{1}{\phi(q_T)^2}$	$\frac{1}{\phi(q_T)}$	$\frac{1+q_T}{(1-q_T)^2 \phi(q_T)}$	(8)
$g \geq 2$	$\theta(q_T)$	1	1	1	

As we can see, the limits for odd and even orthogonal groups are different when $g = 1$, which justify our choice to make the distinction between the root systems rather than the groups themselves.

Point 2 was proved for all types except \tilde{A} in [24], whereas 1 was proved in [29] for group series A, \tilde{A} , proving independently point 2 for types \tilde{A} . Theorem 3.1 will be proved in Section 3. Let us also mention that, whereas the partition function on the plane or a disc in the plane is equal to 1 for any group, the case of the sphere behaves very differently, as the partition function goes to zero exponentially fast, at a speed of order r^2 ([13]). It further displays a phase transition that was discovered by and named after Douglas and Kazakov [15] and proved (in the case of unitary groups) by Lévy and Maïda [36].

Theorem 3.2 (Douglas–Kazakov phase transition). *For any $T \geq 0$, set $Z_{T,N}$ as the Yang–Mills partition function on the sphere of area T with structure group $U(N)$. Then the quantity*

$$F(T) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_{T,N}$$

exists. It defines a function $F \in \mathcal{C}^2(\mathbb{R}_+^) \cap \mathcal{C}^\infty(\mathbb{R}_+^* \setminus \{\pi^2\})$ which admits a third-order jump at the area π^2 .*

We will prove Theorem 3.1 in section 3.2. Beforehand, we recall in the next section a well known expression of the partition function using representation theory of compact groups.

3.1 Character decomposition of the partition function

We shall express the partition functions Z_{g,T,X_r} for any $X \in \{\tilde{A}, A, B, C, D\}$ as a sum over non-increasing sequences of integers. The result of this section are well known [47, 30]. This will follow from standard representation theory of compact groups, and we will prove it succinctly for the sake of completeness. The main result we want to prove is Prop. 3.7, and the remaining of the section may be skipped by anyone familiar with representation theory. Most of the results we will present can be found in [4] and [20].

Definition 3.1. Let G be a compact Lie group.

1. A *complex representation* of G is a couple (φ, V) , where V is a complex vector space and $\varphi : G \rightarrow \text{GL}(V)$ is a smooth group morphism.
2. A representation (φ, V) is *irreducible* if V is the only nontrivial subspace left invariant by φ .
3. The *dimension* d_φ of a representation (φ, V) is defined as the dimension of the vector space V .
4. The *character* of a representation (φ, V) is the function $\chi_\varphi : G \rightarrow \mathbb{C}$ defined by

$$\chi_\varphi(g) = \text{Tr}(\varphi(g)).$$

In the following, we will only consider finite-dimensional representations, unless stated otherwise. The two main results of the representation theory that we will be using are the celebrated *Schur's lemma* and *Plancherel's theorem*.

Lemma 3.3 (Schur's lemma). 1. Let (φ_1, V_1) and (φ_2, V_2) be two irreducible representations of G . If $A : V_1 \rightarrow V_2$ is a nonzero linear map such that

$$A\varphi_1(g) = \varphi_2(g)A, \quad \forall g \in G, \quad (9)$$

then A is an isomorphism.

2. Let (φ, V) be an irreducible representation of G . If $A \in \text{End}(V)$ satisfies

$$A\varphi(g) = \varphi(g)A, \quad \forall g \in G, \quad (10)$$

then there exists $\alpha \in \mathbb{C}$ such that $A = \alpha I$, where I denotes the identity of V .

Among the numerous consequences of this lemma, one can state that the relation (9) defines an equivalence relation between irreducible representations of G , and it enables to define the set \widehat{G} of equivalence classes of irreducible representations. This set, sometimes called *dual space* of G , is countable whenever G is compact, and it is also a group when G is abelian. Another consequence of Schur's lemma is that two irreducible representations within the same class $\lambda \in \widehat{G}$ have same dimension d_λ and same character χ_λ .

Theorem 3.4 (Plancherel's theorem). For any continuous function $f \in C^\infty(G)$, the following sum absolutely converges

$$f(g) = \sum_{\lambda \in \widehat{G}} d_\lambda (f * \chi_\lambda)(g), \quad \forall g \in G, \quad (11)$$

where for any $\chi \in C(G)$, $f * \chi(h) = \int_G f(h)\chi(h^{-1}g)dh, \forall g \in G$.

From these results one can prove the following, which is a particular instance of Thm. 4.2 in [38].

Theorem 3.5. *Let $(p_t)_{t>0}$ be the heat kernel on G . Then for all $t > 0$, $p_t \in C^\infty(G)$ and the following sum*

$$p_t(g) = \sum_{\lambda \in \widehat{G}} e^{-\frac{t}{2}c_\lambda} d_\lambda \chi_\lambda(g), \quad \forall g \in G, \quad (12)$$

absolutely converges, where $c_\lambda \geq 0$ is the non-negative real number such that

$$\Delta_G \chi_\lambda = -c_\lambda \chi_\lambda.$$

Before we state the Fourier decomposition of Yang–Mills partition function, let us also mention an easy but useful result about the irreducible characters of compact groups.

Proposition 3.6. *Let $\lambda \in \widehat{G}$ be an equivalence class of irreducible representations of a compact group G . We have*

$$\int_G \chi_\lambda(x[y, z]) dy = d_\lambda^{-1} \chi_\lambda(xz^{-1}) \chi_\lambda(z), \quad \forall x, z \in G \quad (13)$$

and

$$\int_{G^2} \chi_\lambda(x[y, z]) dy dz = \frac{\chi_\lambda(x)}{d_\lambda^2}, \quad \forall x \in G. \quad (14)$$

Proof. Consider an element (φ, V) of the equivalence class λ and set

$$A_z = \int_G \varphi(yzy^{-1}) dy, \quad \forall z \in G.$$

Since the Haar measure by multiplication, A_z satisfies (10) of Schur’s lemma. Therefore, $A_z = \alpha_z I$ for some scalars $\alpha_z \in \mathbb{C}$ with

$$\alpha_z d_\lambda = \text{Tr}(A_z) = \int_G \chi_\lambda(yzy^{-1}) dy = \chi_\lambda(z).$$

We can now write the left-hand side of (13) as

$$\text{Tr}(\varphi(x) A_z \varphi(z^{-1})) = \frac{\chi_\lambda(z)}{d_\lambda} \text{Tr}(\varphi(z^{-1}x)) = d_\lambda^{-1} \chi_\lambda(z) \chi_\lambda(z^{-1}x), \quad \forall x, z \in G,$$

while using Plancherel theorem, the left-hand side of (14) reads

$$d_\lambda^{-1} \int_G \chi_\lambda(z) \chi_\lambda(z^{-1}x) dz = \frac{1}{d_\lambda} \chi_\lambda * \chi_\lambda(x) = \frac{\chi_\lambda(x)}{d_\lambda^2}, \quad \forall x \in G.$$

□

Proposition 3.7. *Let Σ be a connected orientable surface of genus $g \geq 1$. If $\partial\Sigma$ is connected, then the Yang–Mills partition function on Σ with structure group of type X_r and boundary condition $t \in G/\text{Ad}$ is given by*

$$Z_{g,T,X_r}(t) = \sum_{\lambda \in \widehat{G}} e^{-\frac{T}{2}c_\lambda} d_\lambda^{1-2g} \chi_\lambda(t). \quad (15)$$

If Σ has no boundary, then the Yang–Mills partition function on Σ with structure group G is given by

$$Z_{g,T,X_r} = \sum_{\lambda \in \widehat{G}} e^{-\frac{T}{2}c_\lambda} d_\lambda^{2-2g}. \quad (16)$$

Proof. Let us start with the simplest case, which is (16). Consider the area-weighted map (M, a) of genus g with 1 vertex, $2g$ edges $(a_1, b_1, \dots, a_g, b_g)$ and 1 face f with area $T > 0$ and boundary

$$\partial f = b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1.$$

Then

$$Z_{g,T,X_r} = Z_{M,a,G} = \int_{G^{2g}} p_T([x_1, y_1] \cdots [x_g, y_g]) dx_1 dy_1 \cdots dx_g dy_g.$$

Using the Fourier decomposition of the heat kernel (12), we get

$$Z_{g,T,X_r} = \sum_{\lambda \in \widehat{G(N)}} e^{-\frac{T}{2}c_\lambda} d_\lambda \int_{G^{2g}} \chi_\lambda([x_1, y_1] \cdots [x_g, y_g]) dx_1 dy_1 \cdots dx_g dy_g.$$

We now integrate out all commutators using (14), which yields

$$\int_{G^{2g}} \chi_\lambda([x_1, y_1] \cdots [x_g, y_g]) dx_1 dy_1 \cdots dx_g dy_g = \frac{1}{d_\lambda^{2g-1}}.$$

The results follows. The proof of (15) is similar, using an area-weighted map with one face whose boundary is $\partial f = b_g^{-1} a_g^{-1} b_g a_g \cdots b_1^{-1} a_1^{-1} b_1 a_1 \ell$, where ℓ is the simple loop corresponding to $\partial \Sigma$, oriented positively. \square

Now we will specify (16) to all compact classical groups: to do this, we only need to know the dual space \widehat{G} , and the numbers c_λ and d_λ for every $\lambda \in \widehat{G}$. This can be done thanks to their root systems. Most definitions and results are borrowed from [4], and can also be recovered with much clarity from Sections 2.2 and 2.3 in [40].

Definition 3.2. Let G be a compact connected Lie group.

1. A *weight* of a representation (π, V) of G is a group morphism $\omega : T \rightarrow \mathrm{U}(1)$ where T is a maximal torus of G , such that the space $V^\omega = \{v \in V : \rho(t)v = \omega(t)v, \forall t \in T\}$ is not trivial. The weights form a lattice $I_r = \mathbb{Z}\Omega = \bigoplus_{i=1}^r \mathbb{Z}\omega_i$, where (ω_i) is a distinguished basis of the lattice and r is the rank of the weight lattice. The normaliser of T in G acts by conjugation on T yielding an action on I_r . The vector space $\mathcal{V}_r = \mathbb{R}\Omega = I_r \otimes_{\mathbb{Z}} \mathbb{R}$ can be endowed with an invariant inner product $\langle \cdot, \cdot \rangle$. In fact, if \mathfrak{t} is the Lie algebra of the maximal torus $T \subset G$, we have $\mathbb{R}\Omega \simeq \mathfrak{t}^*$ and $\langle \cdot, \cdot \rangle$ can be taken as the dual of the inner product (1).
2. A *root* of G is a non-zero weight of the adjoint representation of G . The *root system* Φ can be split into $\Phi_+ \sqcup \Phi_-$ with $\Phi_- = -\Phi_+$. The elements of Φ_+ are called *positive roots*.
3. The *Weyl chamber* \mathcal{C}_r is the set

$$\mathcal{C}_r = \{x \in \mathcal{V}_r : \langle x, \alpha \rangle > 0 \forall \alpha \in \Phi_+\}.$$

It is an open cone in \mathcal{V}_r .

4. A *dominant weight* is a weight that belongs to the closure $\overline{\mathcal{C}_r}$ of the Weyl chamber . We denote by Λ_r the set of dominant weights.
5. The element

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$$

has the property that for any $\omega \in I_r$

$$\omega \in \overline{\mathcal{C}_r} \text{ if and only if } \rho + \omega \in \mathcal{C}_r. \quad (17)$$

Below are listed the root systems corresponding to the classical groups depending on their root systems. Note that we do not treat the root systems of exceptional Lie algebras E_6, E_7, E_8, F_4 and G_2 , as we focus in this paper on unitary, special unitary, orthogonal and symplectic groups.

Group	Type	Φ
$U(r)$	A_r	$\{e_i - e_j, 1 \leq i, j \leq r, i \neq j\}$
$SU(r+1)$	A_r	$\{e_i - e_j, 1 \leq i, j \leq r+1, i \neq j\}$
$SO(2r+1)$	B_r	$\{\pm e_i \pm e_j, 1 \leq i < j \leq r\} \cup \{\pm e_i, 1 \leq i \leq N\}$
$Sp(r)$	C_r	$\{\pm e_i \pm e_j, 1 \leq i < j \leq r\} \cup \{\pm 2e_i, 1 \leq i \leq N\}$
$SO(2r)$	D_r	$\{\pm e_i \pm e_j, 1 \leq i < j \leq r\}$

For $SU(r+1)$, we identify \mathcal{V}_r with $\mathcal{V}_r = \mathbb{R}^{r+1}/\mathbb{R}(1, \dots, 1)$ endowed with the norm

$$\|[x]\|^2 = \frac{1}{N+1} \sum_{i=1}^{N+1} \left(x_i - \frac{1}{N+1} \sum_{j=1}^{N+1} x_j\right)^2, \quad \forall [x] \in \mathbb{R}^{N+1}/\mathbb{R}(1, \dots, 1).$$

For any classical group G associated with the type X_r , with $X \in \{\tilde{A}, A, B, C, D\}$, there is a bijection $\widehat{G} \simeq \Lambda_r$: this is a consequence of the highest-weight theory (see [4] for instance). From now on, λ will indistinctly represent an equivalence class of irreducible representations or an element of Λ_r .

Proposition 3.8. *Let G_N be a classical group associated with one of the types $\tilde{A}_r, A_r, B_r, C_r, D_r$ and $\lambda \in \widehat{G}_N \simeq \Lambda_r$ be an equivalence class of irreducible representations. We have*

$$c_\lambda = \langle \lambda + 2\rho, \lambda \rangle \quad (18)$$

and

$$d_\lambda = \frac{\prod_{\alpha \in \Phi_+} \langle \lambda + \rho, \alpha \rangle}{\prod_{\alpha \in \Phi_+} \langle \rho, \alpha \rangle}. \quad (19)$$

We list below some of the objects defined previously, specified for the corresponding classical groups. Most of the explicit computations can be found in [4].

- $U(r)$: type \tilde{A}_r

$$\begin{aligned}
\Phi_+ &= \{e_i - e_j, 1 \leq i < j \leq r\} \\
\mathcal{C}_r &= \{x \in \mathbb{R}^r : x_1 \geq \cdots \geq x_r\} \\
\Lambda_r &= \{\lambda \in \mathbb{Z}^r : \lambda_1 \geq \cdots \geq \lambda_r\} \\
\rho &= \left(\frac{r-1}{2}, \frac{r-3}{2}, \dots, \frac{3-r}{2}, \frac{1-r}{2}\right) \\
c_\lambda &= \frac{1}{r} \sum_{i=1}^r \lambda_i (\lambda_i + r + 1 - 2i), \quad \forall \lambda \in \Lambda_r \\
d_\lambda &= \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i}, \quad \forall \lambda \in \Lambda_r
\end{aligned}$$

- $SU(r+1)$: type A_r

$$\begin{aligned}
\Phi_+ &= \{[e_i - e_j], 1 \leq i < j \leq r+1\} \\
\mathcal{C}_r &= \{[x] \in \mathbb{R}^{r+1} / \sim : x_1 \geq \cdots \geq x_{r+1}\} \\
\Lambda_r &= \{[\lambda] \in \mathbb{Z}^{r+1} / \mathbb{R}(1, \dots, 1) : \lambda_1 \geq \cdots \geq \lambda_{r+1}\} \\
\Lambda_r &\simeq \{\lambda \in \mathbb{Z}^{r+1} : \lambda_1 \geq \cdots \geq \lambda_{r+1} = 0\} \\
\rho &= \left[\left(\frac{r}{2}, \frac{r}{2} - 1, \dots, 1 - \frac{r}{2}, -\frac{r}{2}\right)\right] \\
c_\lambda &= \frac{1}{r+1} \sum_{i=1}^{r+1} \lambda_i (\lambda_i + r + 2 - 2i) - \frac{1}{(r+1)^2} \left(\sum_{i=1}^{r+1} \lambda_i\right)^2, \quad \forall [\lambda] \in \Lambda_r \\
d_\lambda &= \prod_{1 \leq i < j \leq r+1} \frac{\lambda_i - \lambda_j + j - i}{j - i}, \quad \forall [\lambda] \in \Lambda_r
\end{aligned}$$

- $SO(2r+1)$: type B_r

$$\begin{aligned}
\Phi_+ &= \{e_i \pm e_j, 1 \leq i < j \leq r\} \\
\mathcal{C}_r &= \{x \in \mathbb{R}^r : x_1 \geq \cdots \geq x_r \geq 0\} \\
\Lambda_r &= \{\lambda \in \mathbb{N}^r : \lambda_1 \geq \cdots \geq \lambda_r\} \\
\rho &= \left(r - \frac{1}{2}, r - \frac{3}{2}, \dots, \frac{1}{2}\right) \\
c_\lambda &= \frac{1}{2r+1} \sum_{i=1}^r \lambda_i (\lambda_i + 2r + 1 - 2i), \quad \forall \lambda \in \Lambda_r \\
d_\lambda &= \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq r} \frac{\lambda_i + \lambda_j + 2r + 1 - i - j}{2r + 1 - i - j}, \quad \forall \lambda \in \Lambda_r
\end{aligned}$$

- $\text{Sp}(r)$: type C_r

$$\begin{aligned}\Phi_+ &= \{e_i \pm e_j, 1 \leq i < j \leq r\} \cup \{2e_i, 1 \leq i \leq r\} \\ \mathcal{C}_r &= \{x \in \mathbb{R}^r : x_1 \geq \cdots \geq x_r \geq 0\} \\ \Lambda_r &= \{\lambda \in \mathbb{N}^r : \lambda_1 \geq \cdots \geq \lambda_r\} \\ \rho &= (r, r-1, \dots, 1) \\ c_\lambda &= \frac{1}{2^r} \sum_{i=1}^r \lambda_i (\lambda_i + 2r + 2 - 2i), \quad \forall \lambda \in \Lambda_r \\ d_\lambda &= \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq r} \frac{\lambda_i + \lambda_j + 2r + 2 - i - j}{2r + 2 - i - j}, \quad \forall \lambda \in \Lambda_r\end{aligned}$$

- $\text{SO}(2r)$: type D_r

$$\begin{aligned}\Phi_+ &= \{e_i \pm e_j, 1 \leq i < j \leq r\} \cup \{e_i, 1 \leq i \leq r\} \\ \mathcal{C}_r &= \{x \in \mathbb{R}^r : x_1 \geq x_2 \geq \cdots \geq |x_r|\} \\ \Lambda_r &= \{\lambda \in \mathbb{Z}^r : \lambda_1 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r|\} \\ \rho &= (r-1, r-2, \dots, 0) \\ c_\lambda &= \frac{1}{2^r} \sum_{i=1}^r \lambda_i (\lambda_i + 2r - 2i), \quad \forall \lambda \in \Lambda_r \\ d_\lambda &= \prod_{1 \leq i < j \leq r} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2r - i - j)}{(j - i)(2r - i - j)}, \quad \forall \lambda \in \Lambda_r\end{aligned}$$

We will prove Theorem 3.1 using an asymptotic estimation of (16).

3.2 Proof of convergence of partition functions

We first recall the result of [24] and give an alternative proof based on the result of [29] for A_r series.

3.2.1 Witten zeta function

When $g \geq 2$, for the root systems A_r, B_r, C_r and D_r , a proof relies on an asymptotic estimation of the *Witten zeta function*

$$\zeta_{X_r}(s) = \sum_{\lambda \in \Lambda_r} \frac{1}{d_\lambda^s}, \quad \forall s > 1, \quad \forall X \in \{A, B, C, D\}. \quad (20)$$

The first claim we can do is that, for any root system X_r with $X \in \{A, B, C, D\}$, any $T \in \mathbb{R}_+$ and any $g \geq 2$, we have

$$1 \leq \sum_{\lambda \in \Lambda_r} e^{-\frac{T}{2} c_\lambda} d_\lambda^{2-2g} \leq \zeta_{X_r}(2). \quad (21)$$

Indeed, the sum in the middle contains only nonnegative terms, therefore is bounded from below by the term corresponding to $\lambda = (0, \dots, 0) \in \Lambda_r$, which is equal to 1. The upper bound comes from the fact that for any $\lambda \in \Lambda_r$ and $g \geq 2$, the number

c_λ is nonnegative, and $d_\lambda^{2g-2} \geq d_\lambda^2$. The consequence of (21) is that we only need to prove

$$\lim_{r \rightarrow \infty} \zeta_{X_r}(2) = 1 \quad (22)$$

for $X \in \{A, B, C, D\}$ in order to prove Theorem 3.1 for $g \geq 2$.

Proposition 3.9 ([24]). *For any real $s > 1$ and $X \in \{A, B, C, D\}$, one has*

$$\lim_{r \rightarrow \infty} \zeta_{X_r}(s) = 1.$$

Proof. The case of A_r is detailed in [29], we argue here that it implies the other cases. We will compare the dimensions and the sets of dominant weights for different root systems, thus we will denote by Λ_{X_r} the set of dominant weights of type X_r and $d_{X_r, \lambda}$ the dimension of an irreducible representation of the compact connected group of type X_r with highest weight λ . First, let us remark that the sets of dominant weights are the same for A_r, B_r and C_r . Let $X \in \{B, C\}$ and $\lambda \in \Lambda_{X_r}$ a dominant weight different from $(0, \dots, 0)$. As we have $\lambda_i \geq 0$ for any i , it is clear that

$$d_{X_r, \lambda} \geq \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i} = d_{A_r, \lambda}.$$

It follows that, for any $s > 1$:

$$0 \leq \sum_{\substack{\lambda \in \Lambda_{X_r} \\ \lambda \neq (0, \dots, 0)}} \frac{1}{(d_{X_r, \lambda})^s} \leq \sum_{\substack{\lambda \in \Lambda_{A_r} \\ \lambda \neq (0, \dots, 0)}} \frac{1}{(d_{A_r, \lambda})^s}.$$

We get the right limit by letting r go to infinity.

In the case of D_r , some dominant weights have a negative coefficient, therefore an extra care must be taken. Let $\lambda \in \Lambda_{D_r}$ and $1 \leq i < j \leq r-1$, we have $\lambda_i + \lambda_j \geq 0$ as the sum of nonnegative integers. Now, if $1 \leq i \leq r-1$, we also know that $\lambda_i \leq \lambda_{r-1} \leq |\lambda_r|$ therefore $\lambda_i - \lambda_r \geq 0$ as well. In any case, we deduce that

$$d_{D_r, \lambda} \geq \prod_{1 \leq i < j \leq r} \frac{\lambda_i - \lambda_j + j - i}{j - i} = d_{A_r, \tilde{\lambda}},$$

where $\tilde{\lambda} = (\lambda_1, \dots, \lambda_{r-1}, |\lambda_r|)$. Any dominant weight λ of type A_r corresponds to at most two different dominant weights of type D_r : the same weight, and the one obtained by changing the sign of λ_r . Hence, we have

$$0 \leq \sum_{\substack{\lambda \in \Lambda_{D_r} \\ \lambda \neq (0, \dots, 0)}} \frac{1}{(d_{D_r, \lambda})^s} \leq 2 \sum_{\substack{\lambda \in \Lambda_{A_r} \\ \lambda \neq (0, \dots, 0)}} \frac{1}{(d_{A_r, \lambda})^s}.$$

The right-hand side converges to 0 when $r \rightarrow \infty$ therefore we obtain the expected limit for D_r . \square

Proof of Theorem 3.1 with $g \geq 2$. According to Prop. 3.9, for any $g \geq 2$ and $X \in \{A, B, C, D\}$ we have

$$\lim_{r \rightarrow \infty} \zeta_{X_r}(2g-2) = 1.$$

Together with (21), we get

$$\lim_{r \rightarrow \infty} Z_{g, T, X_r} = 1$$

\square

3.2.2 Discrete Gaussian random variables in a cone

In this section we prove that the convergence of Z_{1,T,X_r} as $r \rightarrow \infty$ and $T > 0$ is fixed for types $\{B, C, D\}$.

Definition 3.3. Let $r \geq 1$ be an integer and Λ_r be the set of dominant elements of type X_r , with $X \in \{\tilde{A}, A, B, C, D\}$. A random variable μ on Λ_r is *Gaussian* with parameter t if for any $\Lambda \subset \Lambda_r$

$$\mathbb{P}(\mu \in \Lambda) = \frac{1}{Z_{r,t}} \sum_{\lambda \in \Lambda} e^{-\frac{t}{2}c_\lambda} \quad (23)$$

and the associated *partition function* is

$$Z_{r,t} = \sum_{\lambda \in \Lambda_r} e^{-\frac{t}{2}c_\lambda}. \quad (24)$$

The denomination ‘Gaussian’ may not be obvious at the first glance, but it can become more explicit. Indeed, recall that for any classical group $G_N \subset \mathrm{GL}_n(\mathbb{C})$ of type X_r and any $\lambda \in \widehat{G}_N$,

$$c_\lambda = \frac{1}{n} \langle \lambda + \rho, \lambda \rangle = \frac{1}{n} (\|\lambda + \rho\|^2 - \|\rho\|^2).$$

It follows that for any $\lambda \in \Lambda_r$,

$$\mathbb{P}(\mu = \lambda) \propto e^{-\frac{t}{2n}\|\lambda + \rho\|^2},$$

which thanks to (17) defines, after a shift by ρ , a discrete Gaussian distribution conditioned to belong to $\rho + (I_r \cap \overline{\mathcal{C}_r}) = (\rho + I_r) \cap \mathcal{C}_r$. Note that if G_N is a classical group of type X_r , the partition function $Z_{r,T}$ of the Gaussian distribution on Λ_r with parameter T is equal to the Yang–Mills partition function Z_{1,T,X_r} on a torus of area T with structure group G_N . Before we prove Theorem 3.1 for $g = 1$, let us give a few notations that we will use. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a sequence of non-negative real numbers, we denote by $|\lambda| = \sum_{i \geq 1} \lambda_i$ its total sum. We also denote by

$$\mathcal{P} = \{(\alpha_1, \alpha_2, \dots) \in \mathbb{N}^{\mathbb{N}^*} : \alpha_1 \geq \alpha_2 \geq \dots\}$$

the set of integer partitions. For any $\alpha \in \mathcal{P}$ we write $\ell(\alpha)$ its number of non-zero parts. It is well known that the generating function of partitions is given by the inverse of Euler function:

$$\sum_{\alpha \in \mathcal{P}} q^{|\alpha|} = \frac{1}{\phi(q)}, \quad \forall q \in \mathbb{C} \text{ s.t. } |q| < 1. \quad (25)$$

Proof of Thm 3.1 with $g = 1$. Types B and C. Let us define an increasing sequence setting $\check{\rho} = (r - \rho_1, \dots, r - \rho_r)$ and write

$$\langle \lambda + 2\rho, \lambda \rangle = \|\lambda\|^2 + 2 \sum_{i=1}^r \lambda_i \rho_i = 2r|\lambda| + \|\lambda\|^2 - 2 \sum_{i=1}^{\ell(\lambda)} \lambda_i \check{\rho}_i.$$

For the types B_r and C_r we have $\Lambda_r \simeq \{\lambda \in \mathcal{P} : \ell(\lambda) \leq r\}$ therefore we have

$$Z_{r,T} = \sum_{\lambda \in \mathcal{P} : \ell(\lambda) \leq r} e^{-\frac{rT}{n}|\lambda|} e^{-\frac{T}{2n}(\|\lambda\|^2 - 2 \sum_{i=1}^{\ell(\lambda)} \lambda_i \check{\rho}_i)}.$$

Since $\check{\rho}$ is increasing and λ is non-increasing while both are non-negative,

$$\sum_{1 \leq i, j \leq \ell(\lambda)} (\lambda_i - \lambda_j)(\check{\rho}_i - \check{\rho}_j) \leq 0.$$

It follows that

$$0 \leq 2 \sum_{i=1}^{\ell(\lambda)} \lambda_i \check{\rho}_i \leq \frac{2}{\ell(\lambda)} \sum_{i=1}^{\ell(\lambda)} \lambda_i \sum_{i=1}^{\ell(\lambda)} \check{\rho}_i \leq |\lambda| \ell(\lambda),$$

where we used for the last inequality that in B_r and C_r cases, $\sum_{i=1}^{\ell(\lambda)} \check{\rho}_i$ is either $\frac{\ell(\lambda)^2}{2}$ or $\frac{\ell(\lambda)(\ell(\lambda)-1)}{2}$. We obtain

$$\sum_{\lambda \in \mathcal{P}: \ell(\lambda) \leq r} e^{-\frac{rT}{n} |\lambda| - \frac{T}{2n} \|\lambda\|^2} \leq Z_{r,T} \leq \sum_{\lambda \in \mathcal{P}: \ell(\lambda) \leq r} e^{-(\frac{rT}{n} - \frac{\ell(\lambda)T}{2n}) |\lambda| - \frac{T}{2n} \|\lambda\|^2}.$$

Recalling that n is $2r+1$ or $2r$, both sides converge to $\phi(q_T)^{-1}$ when $r \rightarrow \infty$ by dominated convergence, and we therefore obtain the expected limit.

Type D. In this case, let us introduce two notations. For $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $\lambda_1 \geq \dots \geq \lambda_r$ and $m \in \mathbb{Z}$, we set $\tilde{\lambda} = (\lambda_1, \dots, \lambda_{r-1})$ and $\lambda + m = (\lambda_1 + m, \dots, \lambda_r + m)$. We have

$$\langle \lambda + 2\rho, \rho \rangle = \|\lambda\|^2 + 2 \sum_{i=1}^{r-1} \lambda_i (r-i) = 2r|\tilde{\lambda}| + \lambda_r^2 + \|\tilde{\lambda}\|^2 - 2 \sum_{i=1}^{r-1} i \tilde{\lambda}_i.$$

We find then

$$\begin{aligned} Z_{r,T} &= \sum_{\substack{\lambda \in \mathcal{P}, k \in \mathbb{Z} \\ \ell(\lambda) \leq r-1, |k| \leq \lambda_{r-1}}} e^{-\frac{rT}{n} |\lambda| - \frac{T}{2n} (k^2 + \|\lambda\|^2 - 2 \sum_{i=1}^{r-1} i \lambda_i)} \\ &= \sum_{m \geq 0} \sum_{\substack{\lambda \in \mathcal{P}, k \in \mathbb{Z} \\ \ell(\lambda) \leq r-1, \lambda_{r-1} = m, |k| \leq m}} e^{-\frac{rT}{n} |\lambda| - \frac{T}{2n} (k^2 + \|\lambda\|^2 - 2 \sum_{i=1}^{r-1} i \lambda_i)} \\ &= \sum_{m \geq 0} \left(\sum_{|k| \leq m} e^{-\frac{Tk^2}{2n}} \right) q_T^m \sum_{\substack{\mu \in \mathcal{P} \\ \ell(\mu) \leq r-2}} q_T^{|\mu|} e^{-\frac{T}{2n} (\|\mu+m\|^2 - 2 \sum_{i=1}^{r-2} i(\mu_i+m))}. \end{aligned}$$

Using the same argument as we did for the types B and C , we have

$$2 \sum_{i=1}^{r-1} i(\mu_i + m) \leq \ell(\mu) |\mu + m|,$$

it follows that

$$Z_{r,T} \leq \sum_{m \geq 0} \left(\sum_{|k| \leq m} e^{-\frac{Tk^2}{2n}} \right) q_T^m \sum_{\substack{\mu \in \mathcal{P} \\ \ell(\mu) \leq r-2}} q_T^{|\mu|} e^{-\frac{T}{2n} \|\mu+m\|^2}$$

and

$$Z_{r,T} \geq \sum_{m \geq 0} \left(\sum_{|k| \leq m} e^{-\frac{Tk^2}{2n}} \right) q_T^m \sum_{\substack{\mu \in \mathcal{P} \\ \ell(\mu) \leq r-2}} q_T^{|\mu|} e^{-\frac{T}{2n} (\|\mu+m\|^2 - \ell(\mu) |\mu+m|)}.$$

By dominated convergence both bounds converge to

$$\sum_{m \geq 0} (2m+1) q_T^m \sum_{\lambda \in \mathcal{P}} q_T^{|\lambda|} = \frac{1+q_T}{(1-q_T)^2 \phi(q_T)},$$

hence we deduce the expected limit for $Z_{r,T}$. \square

Remark. For any $T > 0$, it is also possible to deduce the case $g \geq 2$ from the case $g = 1$ using that when G is of type X_r , for any non-trivial representation $\lambda \in \widehat{G}$, $d_\lambda \geq r$.

Remark. A proof for the groups $U(N)$ and $SU(N+1)$ using highest weights can be produced similarly to the other cases yielding a different proof of the result of [29], the latter being formulated in terms of Young diagrams.

4 Proof of Wilson loops convergence

In this section, we will prove theorems 2.12 and 2.15. The proof relies on an absolute continuity relation and the boundedness of the Yang–Mills partition function¹⁴, except for a class of loops considered in 4.2.2, which requires an independent argument. In the following sections, G_N will denote a classical group of size N .

4.1 Loops within in a disc

Assume that (M, a) is an area-weighted map on Σ of genus $g \geq 1$ and area $\|a\|_1 = T$, that U is an open connected set of Σ obtained by union of faces of M , with area $\|a_U\|_1$, such that $u = T - \|a_U\|_1 > 0$. The boundary of U is given by a loop $\partial U \in L(\tilde{M}_U)$.

Lemma 4.1. *The measure $(\mathcal{R}_{\tilde{M}_U}^M)_*(\text{YM}_{M,a,G_N})$ has density*

$$\frac{Z_{g,u,G_N}([h_{\partial U}^{-1}])}{Z_{g,T,G_N}}$$

with respect to $\text{YM}_{\tilde{M}_U,a_U,G_N}$, where $[h] \in G_N/\text{Ad}$ is the conjugation class of $h \in G_N$.

Proof. Thanks to Lemma 2.4, we can assume that M has same number of faces as \tilde{M}_U , that is, that exactly one face of M is not included in U . Denote by k be the number of edges of ∂U . Let M' be the map with $k+1$ vertices, $k+2g+1$ edges $u_1, \dots, u_k, e, a_1, b_1, \dots, a_g, b_g$ and two faces U' and V_g , such that $\partial U' = u_k \cdots u_1$ and $\partial V_g = e^{-1} [b_g^{-1}, a_g^{-1}] \cdots [b_1^{-1}, a_1^{-1}] e u_1^{-1} \cdots u_k^{-1}$. Let us assume that M is obtained by gluing M' with \tilde{M}_U , identifying ∂U with $\partial U'$ so that the base of ∂U is sent to an endpoint of e . For any $h \in \mathcal{M}(\text{P}(M), G_N)$, let us write $x_l = h_{a_l}, y_l = h_{b_l}$ for all $1 \leq l \leq g$ and $z = h_e$. Denote the set of non-marked faces of \tilde{M}_U by \mathbb{F}_U . Then $\text{YM}_{M,a,G_N}(dh)$ can be written as

$$\frac{1}{Z_{g,T,G_N}} p_u(h_{\partial U}^{-1} z [x_1, y_1] \cdots [x_g, y_g] z^{-1}) dz \prod_{i=1}^g dx_i dy_i \prod_{f \in \mathbb{F}_U} p_{a_f}(h_{\partial f}) \text{U}_{\tilde{M}_U, G_N}(dh),$$

¹⁴which follows from the convergence discussed in the Section 3.

It follows that under YM_{M,a,G_N} , the law of $\mathcal{R}_{\tilde{M}_U}^M$ is absolutely continuous with respect to U_{M,G_N} with density

$$\frac{Z_{g,u,G_N}([h_{\partial U}^{-1}])}{Z_{g,T,X_r}}, \quad \forall h \in \mathcal{M}(\text{P}(M), G_N),$$

which is the expected result. \square

Proof of Thm. 2.12. Let us first assume that $\|a_U\| < T = \|a\|$. For any $h \in G_N$ and $\lambda \in \widehat{G}_N$, since the representation is unitary, $\|\chi_\lambda\|_\infty = \chi_\lambda(1) = d_\lambda$. For any $s > 0$, if we set $\psi_{g,u,G_N} : G_N \rightarrow \mathbb{C}$, $g \mapsto Z_{g,u,G_N}([g])$, we deduce from (15) that

$$\|\psi_{g,u,G_N}\|_\infty = \sum_{\lambda \in \widehat{G}_N} d_\lambda^{2-2g} e^{-\frac{u}{2}c_\lambda} = Z_{g,u,G_N}. \quad (26)$$

Together with the previous lemma, $\mathcal{R}_{\tilde{M}_U}^M(\text{YM}_{M,a,G_N})$ has a density with respect to $\text{YM}_{\tilde{M}_U,a_U,G_N}$ which is bounded from above by

$$\frac{Z_{g,\|a\|_1-\|a_U\|_1,G_N}}{Z_{g,\|a\|_1,G_N}}.$$

According to Theorem 3.1, the right-hand side is uniformly bounded in N . In particular for any $\varepsilon > 0$ and any loop $\ell \in \text{L}(\tilde{M}_U)$,

$$\mathbb{P}_{\text{YM}_{M,a}}(|W_\ell - \Phi_{\tilde{M}_U,a_U}(\ell)| > \varepsilon) \leq \frac{Z_{g,\|a\|_1-\|a_U\|_1,G_N}}{Z_{g,\|a\|_1,G_N}} \mathbb{P}_{\text{YM}_{\tilde{M}_U,a_U}}(|W_\ell - \Phi_{\tilde{M}_U,a_U}(\ell)| > \varepsilon).$$

Together with Theorem 2.8 for the plane, we conclude that the right-hand side converges towards zero as $N \rightarrow \infty$.

To conclude it remains to prove the uniformity of the convergence in the set A of $a \in \Delta_M(T)$ with $a(f) > 0$ for any face f and $\|a_U\| < T$. For this, it is enough to show that for any adjacent faces f_1, f_2 of M , there is a constant $K > 0$ such that for all $a, a' \in A$, such that $a(f) = a'(f)$ for all faces distinct from f_1, f_2 and $\|a - a'\| < \frac{T}{2}$,

$$|\mathbb{E}_{\text{YM}_{M,a}}[|W_\ell - \Phi_{\tilde{M}_U,a_U}(\ell)|] - \mathbb{E}_{\text{YM}_{M,a'}}[|W_\ell - \Phi_{\tilde{M}_U,a'_U}(\ell)|]| \leq K\|a - a'\|^{1/2}. \quad (27)$$

Now when f_1, f_2 and a, a' are as above, there is a simple loop s enclosing an area $\|a - a'\|$ and paths α, β with $\ell = \alpha\beta$, such that under $\text{YM}_{M,a}$, $W_{\alpha\beta}$ has same law as W_ℓ under $\text{YM}_{M,a'}$. Using this identity in law, the left-hand-side of (27) equals

$$|\mathbb{E}_{\text{YM}_{M,a}}[|W_{\alpha\beta} - \Phi_{\tilde{M}_U,a_U}(\alpha\beta)|] - |W_{\alpha\beta} - \Phi_{\tilde{M}_U,a'_U}(\alpha\beta)|]| \leq \mathbb{E}_{\text{YM}_{M,a}}[|W_{\alpha\beta} - W_{\alpha\beta}|].$$

The right-hand-side is bounded by

$$\begin{aligned} \mathbb{E}_{\text{YM}_{M,a}}[|W_{\alpha\beta} - W_{\alpha\beta}|] &\leq \mathbb{E}_{\text{YM}_{M,a}}[|\text{tr}(H_{\alpha\beta} - H_{\alpha\beta})|] = \mathbb{E}_{\text{YM}_{M,a}}[|\text{tr}(H_{\beta\alpha} - H_{\beta\alpha}H_s)|] \\ &\leq \mathbb{E}_{\text{YM}_{M,a}}[\text{tr}[(H_{\beta\alpha} - H_sH_{\beta\alpha})(H_{\beta\alpha} - H_sH_{\beta\alpha})^*]]^{1/2} \\ &\leq \sqrt{2}\mathbb{E}_{\text{YM}_{M,a}}[1 - \Re(\text{tr}(H_s))]^{1/2} \\ &\leq \frac{\sqrt{2}Z_{g,T-\|a-a'\|}}{Z_{g,T}} \mathbb{E}_{\text{YM}_{\tilde{M}_U,a_U}}[1 - \Re(\text{tr}(H_s))]^{1/2} \\ &\leq \frac{\sqrt{2}Z_{g,T/2}}{Z_{g,T}} \mathbb{E}_{\text{YM}_{\tilde{M}_U,a_U}}[1 - \Re(\text{tr}(H_s))]^{1/2}, \end{aligned}$$

where we used the same argument as in the first part of the proof for the penultimate inequality. Now under $\text{YM}_{\tilde{M}_U, a_U}$, H_s has same law as a Brownian motion at time $\|a - a'\|$. It follows from [32] or [9] that there is a constant $c > 0$ with

$$\mathbb{E}_{\text{YM}_{\tilde{M}_U, a_U}} [1 - \Re(\text{tr}(H_s))] \leq 1 - e^{-c\|a - a'\|} \leq c\|a - a'\|,$$

leading to the required bound (27). \square

Proof of Theorem 2.14. We only sketch here the additional argument and refer to [33] for more details. For any multiplicative function $h \in \mathcal{M}(\text{P}(\Sigma), G_N)$, the composition of its restriction to $\text{P}(U)$ with Ψ defines an element $\Psi(h)$ of $\mathcal{M}(\text{P}(D_R), G_N)$. According to the splitting property of Markovian holonomy fields (property (A3) of [33, Def. 3.1.2]) and Lemma 4.1, $\Psi_*(\text{YM}_{\Sigma, G_N})$ has density

$$\frac{\psi_{g, T - \text{vol}(U)}(H_{\partial D_R})}{Z_{g, T, G_N}}$$

with respect to YM_{D_R, G_N} . The claim then follows with the same argument as for the proof of Theorem 2.12. \square

4.2 Simple non-contractible loops

Consider a simple non-contractible loop ℓ on a compact surface Σ of genus $g \geq 1$. Two possibilities occur when cutting Σ along ℓ . If the surface is cut into two surfaces with exactly one connected boundary component, each with genus larger than 1¹⁵, then the loop is *separating*. Otherwise, the new surface has one connected component and two boundary components, and the loop is called *non-separating*. Both cases are illustrated in Fig. 2 below. We refer to [51] for details on these loops. It is now important to understand how the Yang–Mills measure interacts with surgery: we therefore begin with a few results that will help us in the next section.

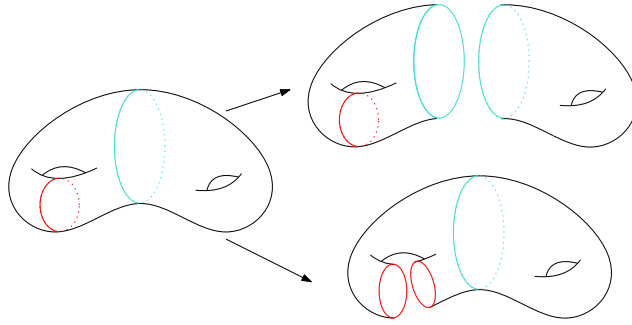


Figure 2: The surface on the left can be seen as the result of a binary gluing of two surfaces along a separating loop (top right), or of one surface along a nonseparating loop (bottom right).

¹⁵Indeed, if one of the boundary components has genus 0, it implies that ℓ is contractible.

4.2.1 Separating loops

Consider first two compact connected orientable surfaces Σ_1 and Σ_2 such that $\partial\Sigma_1$ and $\partial\Sigma_2$ are connected. Denote by L_1 and L_2 the corresponding loops, oriented positively. Let $\psi : L_1 \rightarrow L_2$ be an orientation-reversing diffeomorphism, and Σ be the gluing of Σ_1 and Σ_2 along ψ . Let L be the corresponding loop in the surface Σ . For $i \in \{1, 2\}$, denote by $\tilde{\mathcal{C}}_i$ the sigma-field generated by $(H_{\gamma_1}, \dots, H_{\gamma_n}, n \geq 1, \gamma_i \in \mathcal{P}(\Sigma_i))$. For any i , a function $f : \mathcal{M}(\mathcal{P}(\Sigma), G_N) \rightarrow \mathbb{C}$ is $\tilde{\mathcal{C}}_i$ -measurable if and only if $f \circ \mathcal{R}_{\Sigma_i} : \mathcal{M}(\mathcal{P}(\Sigma_i), G_N) \rightarrow \mathbb{C}$ is \mathcal{C}_i -measurable. The next theorem is a particular case of [30, Thm. 5.1.1].

Theorem 4.2. *The sigma-fields $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ are independent on $\mathcal{M}(\mathcal{P}(\Sigma), G_N)$ under YM_{Σ, G_N} conditionnally to the random variable H_L . Moreover, for any $\tilde{\mathcal{C}}_i$ -measurable function $f_i : \mathcal{M}(\mathcal{P}(\Sigma), G_N) \rightarrow \mathbb{C}$, for $i \in \{1, 2\}$, the product $f_1 f_2$ is measurable with respect to \mathcal{C} , and for any $t \in G_N/\text{Ad}$, the following equality holds true:*

$$\int f_1(h) f_2(h) \text{YM}_{\Sigma, L, t, G_N}(dh) = \int f_1 \circ \mathcal{R}_{\Sigma_1}(h) \text{YM}_{\Sigma_1, t, G_N}(dh) \int f_2 \circ \mathcal{R}_{\Sigma_2}(h) \text{YM}_{\Sigma_2, t^{-1}, G_N}(dh). \quad (28)$$

This theorem provides a sort of spacial Markov property, which is a consequence of the semigroup property of the heat kernel. It has the following application.

Corollary 4.3. *Let ℓ be a separating loop in an area-weighted map (M, a) of genus $g \geq 2$, splitting M into two respective maps M_1 and M_2 of genus g_1, g_2 and total area T_1 and T_2 . Let $f : \mathcal{M}(\mathcal{P}(M), G_N) \rightarrow \mathbb{C}$ be a bounded $\tilde{\mathcal{C}}_1$ -measurable function. We have*

$$\int_{\mathcal{M}(\mathcal{P}(M), G_N)} f(h) \text{YM}_{M, a, G_N}(dh) = \int_{\mathcal{M}(\mathcal{P}(M_1), G_N)} f \circ \mathcal{R}_{M_1}^M(h) I(h_\ell^{-1}) \text{YM}_{M_1, a, G_N}(dh), \quad (29)$$

where for any $x \in G_N$,

$$I(x) = \frac{Z_{g_1, T_1, G_N} Z_{g_2, T_2, G_N}([x])}{Z_{g, T, G_N}}.$$

Proof. For the sake of clarity, we will drop the subscripts G_N in the measures, as the structure group remains fixed. First of all, using (4) we have

$$\int_{\mathcal{M}(\mathcal{P}(M), G_N)} f(h) \text{YM}_{M, a}(dh) = \frac{1}{Z_{g, T}} \int_{G/\text{Ad}} \int_{\mathcal{M}(\mathcal{P}(M), G_N)} f(h) \tilde{Z}_{g, T}(t) \text{YM}_{M, a, t}(dh) dt.$$

Here $\tilde{Z}_{g, T}(t)$ is not given by Prop. 3.7, because here the condition is not on a boundary but on an inner loop. We can still compute it directly. Using the invariance property of Yang–Mills measure by subdivision, one can assume without loss of generality that for $i \in \{1, 2\}$ M_i possesses 1 vertex $v^{(i)}$, $2g_i + 1$ edges $a_1^{(i)}, b_1^{(i)}, \dots, a_g^{(i)}, b_g^{(i)}, e^{(i)}$ all with $v^{(i)}$ as source and target, and 1 face whose boundary is given by $e^{(i)\varepsilon_i} [a_1^{(i)}, b_1^{(i)}] \cdots [a_{g_i}^{(i)}, b_{g_i}^{(i)}]$ where $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. The map M can then be defined as the quotient of $M_1 \cup M_2$ by the relations $v^{(1)} \sim v^{(2)}$ and

$(e^{(1)})^{-1} \sim e^{(2)}$. We obtain

$$\begin{aligned} Z_{g,T}(t) &= \int_{G_N^{2g+1}} p_{T_1}(x[y_1, z_1] \cdots [y_{g_1}, z_{g_1}]) \\ &\quad \times p_{T_2}(x^{-1}[y'_1, z'_1] \cdots [y'_{g_2}, z'_{g_2}]) dx \prod_{\substack{1 \leq i \leq g_1 \\ 1 \leq j \leq g_2}} dy_i dz_i dy'_j dz'_j \\ &= Z_{g_1, T_1}(t) Z_{g_2, T_2}(t^{-1}), \end{aligned}$$

where this time the conditional partition functions are actually given by Prop. 3.7. From this result and from Theorem 4.2 we deduce

$$\begin{aligned} &\int_{\mathcal{M}(\mathbb{P}(M), G_N)} f(h) \text{YM}_{M,a}(dh) \\ &= \frac{1}{Z_{g,T}} \int_{G/\text{Ad}} \int_{\mathcal{M}(\mathbb{P}(M_1), G_N)} f \circ \mathcal{R}_{M_1}^M(h) Z_{g_1, T_1}(t) \text{YM}_{M_1, a, t}(dh) \\ &\quad \times \int_{\mathcal{M}(\mathbb{P}(M_2), G_N)} Z_{g_2, T_2}(t^{-1}) \text{YM}_{M_2, a, t^{-1}}(dh) dt \end{aligned}$$

The integral over $\mathcal{M}(\mathbb{P}(M_2), G_N)$ is simply equal to $Z_{g_2, T_2}(t^{-1})$; furthermore, under $\text{YM}_{M_1, a}(t)$, $t = [H_\ell]$ so that we have

$$\begin{aligned} &\int_{\mathcal{M}(\mathbb{P}(M), G_N)} f(h) \text{YM}_{M,a}(dh) \\ &= \int_{G/\text{Ad}} \int_{\mathcal{M}(\mathbb{P}(M_1), G_N)} f \circ \mathcal{R}_{M_1}^M(h) \frac{Z_{g_2, T_2}([h_\ell^{-1}])}{Z_{g,T}} Z_{g_1, T_1}(t) \text{YM}_{M_1, a}(t)(dh) dt. \end{aligned}$$

Finally, using again (4) gives the expected equality. \square

To prove Theorem 2.15, we shall need the following similar but simpler Lemma.

Lemma 4.4. *Let ℓ be a separating loop in an area-weighted map (M, a) of genus $g \geq 2$, splitting M into two respective maps M_1 and M_2 of genus g_1, g_2 and total area T_1 and T_2 . Under YM_{M, a, G_N} , the random variable H_ℓ has density*

$$\frac{\psi_{g_1, T_1, G_N}(h^{-1}) \psi_{g_2, T_2, G_N}(h)}{Z_{g, T, G_N}}$$

with respect to the Haar measure on G_N , where for all $g \geq 1, s > 0$, $\psi_{g,s}$ is defined by $\psi_{g,s, G_N} : h \mapsto Z_{g,s, G_N}([h])$.

Proof. Let us use the same M, M_1 and M_2 as in the previous proof. For any central function $f : G_N \rightarrow \mathbb{C}$ measurable and bounded, we have

$$\begin{aligned} \mathbb{E}_{\text{YM}_{M, a, G_N}}[f(H_\ell)] &= \frac{1}{Z_{g, T, G_N}} \int_{G_N^{2g+1}} f(x) p_{T_1}(x^{-1}[y_1, z_1] \cdots [y_{g_1}, z_{g_1}]) \\ &\quad \times p_{T_2}(x[y'_1, z'_1] \cdots [y'_{g_2}, z'_{g_2}]) dx \prod_{\substack{1 \leq i \leq g_1 \\ 1 \leq j \leq g_2}} dy_i dz_i dy'_j dz'_j, \end{aligned}$$

thus by integrating over all variables but x we obtain the expected result. \square

Proof of 1. of Theorem 2.15. Using Lemma 4.4 and the bound (26), we find that under $\text{YM}_{M,a}$, the density of h_ℓ with respect to the Haar measure is uniformly bounded by

$$\frac{Z_{g_1, T_1, N} Z_{g_2, T_2, N}}{Z_{g, T, N}}.$$

Now for a Haar distributed random variable H on G belonging to X_N , according to [14], whenever $n \neq 0$, $\text{tr}(H^n)$ converges in probability towards 0, as $N \rightarrow \infty$. The claim now follows by absolute continuity similarly to the proof of Theorem 2.12. \square

4.2.2 Non-separating loops

Let Σ_1 be a compact connected orientable surface such that $\partial\Sigma_1$ has two connected components; denote the corresponding loops, oriented positively, by L_1 and L_2 . Let $\psi : L_1 \rightarrow L_2$ be an orientation-reversing diffeomorphism, and Σ be the gluing of Σ_1 along ψ . Let L be the corresponding loop in Σ . We have the following, which is a particular case of [30, Thm. 5.4.1]. We use the same notation $\tilde{\mathcal{C}}_1$ for the sigma-field on $\mathcal{M}(\text{P}(\Sigma), G)$ generated by the paths in Σ_1 .

Theorem 4.5. *Let $f : \mathcal{M}(\text{P}(\Sigma), G) \rightarrow \mathbb{C}$ be a $\tilde{\mathcal{C}}_1$ -measurable function. For any $t \in G/\text{Ad}$, the following equality holds true:*

$$\int f(h) \text{YM}_{\Sigma, G}(t)(dh) = \int f \circ \mathcal{R}_{\Sigma_1}(h) \text{YM}_{\Sigma_1, G}(t, t^{-1})(dh). \quad (30)$$

The consequence in terms of absolute continuity is the following.

Corollary 4.6. *Let ℓ be a non-separating loop in an area-weighted map (M, a) of genus $g \geq 1$. Under $\text{YM}_{M, a, G}$, the random variable H_ℓ has density given by*

$$\frac{\varphi_{\|a\|_1, g}(h)}{Z_{g, \|a\|_1, N}}, \forall h \in G$$

with respect to the Haar measure, where for all $g \geq 1, T > 0$,

$$\varphi_{T, g}(h) = \sum_{\lambda \in \hat{G}} d_\lambda^{2-2g} |\chi_\lambda(h)|^2 e^{-\frac{T}{2} c_\lambda}, \forall h \in G. \quad (31)$$

Proof. Choosing M with 1 vertex, 1 face and $2g$ edges, one of which is ℓ , and writing $u = h_\ell$,

$$\text{YM}_{M, a}(dh) = p_T([u, v][x_2, y_2] \cdots [x_g, y_g]) dudv \prod_{i=2}^g dx_i dy_i. \quad (32)$$

It follows that the law of h_ℓ under $\text{YM}_{M, a}$ has density

$$\varphi_{T, g}(u) = Z_{g, T, N}^{-1} \int_{G^{2g-1}} p_T([u, v][x_2, y_2] \cdots [x_g, y_g]) dv \prod_{i=2}^g dx_i dy_i, \forall u \in G,$$

with respect to the Haar measure. Expanding in characters and using (14) yields

$$\varphi_{T, g}(h) = \sum_{\lambda \in \hat{G}} e^{-\frac{T}{2} c_\lambda} I_\lambda(u)$$

with

$$I_\lambda(u) = d_\lambda^{1-2(g-1)} \int_G \chi_\lambda([u, v]) dv.$$

Now (13) yields $I_\lambda(u) = d_\lambda^{2-2g} \chi_\lambda(u) \chi_\lambda(u^{-1})$, $\forall u \in G$ and the claim. \square

Proof of 2. of Theorem 2.15 for $g \geq 2$. Then, using that $\|\chi_\lambda\|_\infty = \chi_\lambda(1)$ for all $\lambda \in \widehat{G}$,

$$\|\varphi_{T,g}\|_\infty = \varphi_{T,g}(1) = Z_{g-1,T,N}, \forall g \geq 1, T > 0.$$

It follows that under $\text{YM}_{M,a}$, the density of h_ℓ with respect to the Haar measure is bounded by

$$\frac{Z_{g-1,\|a\|,N}}{Z_{g,\|a\|,N}}.$$

When $g \geq 2$, it follows from 3.1 that this sequence is uniformly bounded when $N \rightarrow \infty$ and the argument is then identical to the proof of point 1. \square

When $g = 1$, the above argument of absolute continuity argument fails. Indeed for a total area T , the maximum of the density is then given by $\frac{Z_{0,T,N}}{Z_{1,T,N}}$. For \tilde{A}_N , thanks to Theorem 3.2, and more precisely formula (28) of [36],

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log(Z_{0,T,N}) = F(T) = \frac{T}{24} + \frac{3}{4} - \frac{1}{2} \log(T), \forall 0 < T \leq \pi^2.$$

Since $F(T) \geq F(\pi^2) > 0$ for all $T \in (0, \pi^2]$, $\lim_{N \rightarrow \infty} Z_{0,T,N} = +\infty$. Similarly to the strategy of [34, 11, 25] we shall consider the expectation and the variance of the random variable W_ℓ .

Remark. For any z in the center of G , under $\text{YM}_{M,a}$, zh_ℓ has same law as h_ℓ . This can be checked changing variable in formula (32). Alternatively, Schur's lemma implies that z acts by a unitary scalar in any irreducible representation and (32) implies that for any $\varphi_{T,g}(zh) = \varphi_{T,g}(h)$ for any $h \in G$. It follows whenever $D_z \in G$ is a diagonal matrix of multiplication by a scalar $z \in \mathbb{C}$, under $\text{YM}_{M,a}$, $\text{tr}(h_\ell^n) = \text{tr}(h_\ell^n)$ has same law as $z^n \text{tr}(h_\ell^n)$. For $X_N = \{A_N, \tilde{A}_N\}$, the center is given by $\{\zeta \text{Id}_{N+1} : \zeta^{N+1} = 1\}$ and $\{z \text{Id}_N : z \in \mathbb{C}, |z| = 1\}$. It follows that for $X = \tilde{A}$, $\mathbb{E}[W_\ell^n] = 0$ for any $n \neq 0$, whereas for $X = A$, $\mathbb{E}[W_\ell^n] = 0$ for any $n \neq 0$ modulo $N+1$. In particular, when $X \in \{A, \tilde{A}\}$, $\lim_{N \rightarrow \infty} \mathbb{E}[W_\ell^n] = 0$.

Unfortunately, the argument given in the above remark does not apply when $X = \{B, C, D\}$. Also, it does not give any information about $\text{var}(W_\ell) = \mathbb{E}[|W_\ell|^2] - |\mathbb{E}[W_\ell]|^2$. Instead, we shall use the expansion in characters of the heat kernel and the following lemma, which is a consequence of the expression of characters as a ratio of alternated functions.

Lemma 4.7. *Let $G_N \subset \text{GL}_n(\mathbb{C})$ be a classical group of size N and type X_r , for $X \in \{B, C, D\}$. For any $k \neq 0$ and $\lambda \in \widehat{G}_N$,*

$$\text{Tr}(g^k) \chi_\lambda(g) = \sum_{\mu \in \widehat{G}_N} c_{\lambda,k}^\mu \chi_\mu(g), \quad (33)$$

with

$$c_{\lambda,k}^\mu \in \{-1, 0, 1\}, \forall \mu \in \widehat{G}_N \quad (34)$$

and

$$\sum_{\mu \in \widehat{G}_N} |c_{\lambda, k}^\mu| \leq n \quad (35)$$

Proof. Let us introduce a few common notations for the different cases. Set $\mathbb{Z}_{\text{sym}} = \mathbb{Z}$ if r is odd or $\mathbb{Z} + \frac{1}{2}$ if r is even. The mapping

$$\lambda \in \widehat{G}_N \mapsto \lambda + \rho$$

establishes a bijection between the set of highest weights and $\{\mu \in \mathbb{Z}_{\text{sym}}^r : \mu_1 > \cdots > \mu_r\}$. The symmetric group \mathfrak{S}_r acts in an obvious way on

$$\Delta_r = \{\mu \in \mathbb{Z}_{\text{sym}}^r : \mu_1 > \cdots > \mu_r\} \cong \{\mu \in \mathbb{Z}_{\text{sym}}^r : \mu_i \neq \mu_j, \forall i \neq j\} / \mathfrak{S}_r,$$

where the symmetric group \mathfrak{S}_r acts on μ by permutation of its components. For such μ we denote by $[\mu]$ its decreasing rearrangement and $\sigma_\mu \in \mathfrak{S}_r$ the unique permutation such that $[\mu]_i = \mu_{\sigma(i)}$ for all i . We will prove each case separately using the previous notations. The explicit formulae for the characters χ_λ that we will use can be found in [4] or in [40, Sect. 2.3]. The proof for all cases will rely on the following computation: for any $(z_1, \dots, z_r) \in (\mathbb{C}^*)^r$, any $(m_1, \dots, m_r) \in \mathbb{Z}^r$ and any $k \in \mathbb{N}^*$, we have

$$\sum_{\ell=1}^r (z_\ell^k + z_\ell^{-k}) \det(z_i^{m_j} - z_i^{-m_j}) = \sum_{r=1}^r (\det(z_i^{m_j+k\delta_{jr}}) + \det(z_i^{-m_j-k\delta_{jr}})). \quad (36)$$

Let us prove (36). Using the Leibniz formula for determinant and the invariance of $\sum_{\ell} (z_\ell^k + z_\ell^{-k})$ by the action $\ell \mapsto \sigma(\ell)$ of the symmetric group, we have

$$\begin{aligned} \sum_{\ell=1}^r (z_\ell^k + z_\ell^{-k}) \det(z_i^{m_j} - z_i^{-m_j}) &= \sum_{\sigma \in \mathfrak{S}_r} \varepsilon(\sigma) \sum_{\ell=1}^r (z_\ell^k + z_\ell^{-k}) \prod_{i=1}^r (z_{\sigma(i)}^{m_i} - z_{\sigma(i)}^{-m_i}) \\ &= \sum_{\sigma \in \mathfrak{S}_r} \varepsilon(\sigma) \sum_{\ell=1}^r (z_{\sigma(\ell)}^k + z_{\sigma(\ell)}^{-k}) \prod_{i=1}^r (z_{\sigma(i)}^{m_i} - z_{\sigma(i)}^{-m_i}). \end{aligned}$$

We can then put the sum over \mathfrak{S}_r back in by linearity, and for any σ and any ℓ we have

$$(z_{\sigma(\ell)}^k + z_{\sigma(\ell)}^{-k}) \prod_{i=1}^r (z_{\sigma(i)}^{m_i} - z_{\sigma(i)}^{-m_i}) = (z_{\sigma(\ell)}^{m_i+k} - z_{\sigma(\ell)}^{-m_i+k} + z_{\sigma(\ell)}^{m_i-k} - z_{\sigma(\ell)}^{-m_i-k}) \prod_{i \neq \ell} (z_{\sigma(i)}^{m_i} - z_{\sigma(i)}^{-m_i}).$$

If we reintegrate this result in the previous equation, we obtain (36). We can now prove (33) for each type of group.

Case B_r : let $g \in \text{SO}(2r+1)$ be an element with eigenvalues $(z_1^{\pm 1}, \dots, z_r^{\pm 1}, 1)$. We have

$$\chi_\lambda(g) = \frac{\det(z_i^{\lambda_j + \rho_j} - z_i^{-\lambda_j - \rho_j})}{\det(z_i^{\rho_j} - z_i^{-\rho_j})}.$$

If we substitute $\mu = \lambda + \rho \in \mathbb{Z}_{\text{sym}}^r$ we can write

$$\chi_\lambda(g) = \chi_{\mu - \rho}(g) = \frac{\det(z_i^{\mu_j} - z_i^{-\mu_j})}{\det(z_i^{\rho_j} - z_i^{-\rho_j})},$$

so that

$$\mathrm{Tr}(g^k)\chi_\lambda(g) = \left(1 + \sum_{\ell=1}^r (z_\ell^k + z_\ell^{-k})\right) \frac{\det(z_i^{\mu_j} - z_i^{-\mu_j})}{\det(z_i^{\rho_j} - z_i^{-\rho_j})}$$

and by (36) we obtain that

$$\mathrm{Tr}(g^k)\chi_\lambda(g) = \chi_\lambda(g) + \sum_{\ell=1}^r \frac{\det(z_i^{\mu_j+k\delta_{j\ell}} - z_i^{-(\mu_j+k\delta_{j\ell})}) + \det(z_i^{\mu_j-k\delta_{j\ell}} - z_i^{-(\mu_j-k\delta_{j\ell})})}{\det(z_i^{\rho_j} - z_i^{-\rho_j})}$$

Set $\mu_{k,\ell}^+ = (\mu_1, \dots, \mu_\ell + k, \dots, \mu_r)$ and $\mu_{k,\ell}^- = (\mu_1, \dots, \mu_\ell - k, \dots, \mu_r)$. We have

$$\mathrm{Tr}(g^k)\chi_\lambda(g) = \chi_\lambda(g) + \sum_{\ell=1}^r \left(\varepsilon(\sigma_{\mu_{k,\ell}^+})\chi_{[\mu_{k,\ell}^+] - \rho}(g) + \varepsilon(\sigma_{\mu_{k,\ell}^-})\chi_{[\mu_{k,\ell}^-] - \rho}(g) \right),$$

which yields (33) with $c_{\lambda,k}^\lambda = 1$, $c_{\lambda,k}^{[\mu_{k,\ell}^\pm] - \rho} = \varepsilon(\sigma_{[\mu_{k,\ell}^\pm] - \rho})$ for all $1 \leq \ell \leq r$ and $c_{\lambda,k}^\mu = 0$ for any other $\mu \in \widehat{G}_N$, which also proves (34). Indeed, each $[\mu_{k,\ell}^\pm] - \rho$ corresponds to an element of \widehat{G}_N if and only if $[\mu_{k,\ell}^\pm] \in \Delta_r$, and if it is not the case, then $\chi_{[\mu_{k,\ell}^\pm]}$ vanishes. Finally, there are $2r+1$ nonzero coefficients, whose absolute value is always equal to 1, hence (35) is also satisfied.

Case C_r : let $g \in \mathrm{Sp}(r)$ be an element with eigenvalues $(z_1^{\pm 1}, \dots, z_r^{\pm 1})$. We have again

$$\chi_\lambda(g) = \chi_{\mu - \rho}(g) = \frac{\det(z_i^{\mu_j} - z_i^{-\mu_j})}{\det(z_i^{\rho_j} - z_i^{-\rho_j})}.$$

It follows that

$$\mathrm{Tr}(g^k)\chi_\lambda(g) = \sum_{\ell=1}^r \left(\varepsilon(\sigma_{\mu_{k,\ell}^+})\chi_{[\mu_{k,\ell}^+] - \rho}(g) + \varepsilon(\sigma_{\mu_{k,\ell}^-})\chi_{[\mu_{k,\ell}^-] - \rho}(g) \right),$$

as expected.

Case D_r : let $g \in \mathrm{SO}(2r)$ be an element with eigenvalues $(z_1^{\pm 1}, \dots, z_r^{\pm 1})$. We have

$$\chi_\lambda(g) = \chi_{\mu - \rho}(g) = \frac{\det(z_i^{\mu_j} - z_i^{-\mu_j}) + \det(z_i^{\mu_j} + z_i^{-\mu_j})}{\det(z_i^{\rho_j} + z_i^{-\rho_j})},$$

therefore

$$\begin{aligned} \mathrm{Tr}(g^k)\chi_\lambda(g) &= \left(\sum_{\ell=1}^r (z_\ell^k + z_\ell^{-k}) \right) \frac{\det(z_i^{\mu_j} - z_i^{-\mu_j}) + \det(z_i^{\mu_j} + z_i^{-\mu_j})}{\det(z_i^{\rho_j} + z_i^{-\rho_j})} \\ &= \sum_{\ell=1}^r \left(\varepsilon(\sigma_{\mu_{k,\ell}^+})\chi_{[\mu_{k,\ell}^+] - \rho}(g) + \varepsilon(\sigma_{\mu_{k,\ell}^-})\chi_{[\mu_{k,\ell}^-] - \rho}(g) \right). \end{aligned}$$

□

Proof of 2. of Theorem 2.15 for $g = 1$. Let us set $T = \|a\|_1$. We shall compute the expectation and the variance of W_ℓ under YM_{M,a,G_N} , and show that both converge

to zero as $N \rightarrow \infty$. Assume that G_N of type X_r is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ as¹⁶ in section 2.1.2. Using Corollary 4.4 and expanding in characters yields

$$\begin{aligned} n\mathbb{E}[W_{\ell^k}] &= Z_{1,T,X_r}^{-1} \int_{G_N} \varphi_{T,1}(h) \mathrm{Tr}(h^k) dh \\ &= Z_{1,T,X_r}^{-1} \sum_{\lambda \in \widehat{G}_N} e^{-\frac{T}{2}c_\lambda} I_\lambda, \end{aligned}$$

where for all $\lambda \in \widehat{G}_N$, using the orthogonality of characters,

$$I_\lambda = \int_{G_N} \mathrm{Tr}(h^k) \chi_\lambda(h) \chi_\lambda(h^{-1}) dh = \sum_{\mu \in \widehat{G}_N} c_{\lambda,k}^\mu \int_{G_N} \chi_\mu(h) \chi_\lambda(h^{-1}) dh = c_{\lambda,k}^\lambda.$$

Thanks to (34), it follows that

$$|\mathbb{E}[W_{\ell^n}]| \leq (nZ_{1,T,X_r})^{-1} \sum_{\lambda \in \widehat{G}_N} e^{-\frac{T}{2}c_\lambda} = \frac{1}{n}. \quad (37)$$

Similarly,

$$n^2\mathbb{E}(|W_{\ell^k}|^2) = Z_{1,T,X_r}^{-1} \int_{G_N} |\mathrm{Tr}(h^k)|^2 \varphi_{T,1}(h) dh = Z_{1,T,X_r}^{-1} \sum_{\lambda \in \widehat{G}_N} e^{-\frac{T}{2}c_\lambda} J_\lambda,$$

where

$$\begin{aligned} J_\lambda &= \int_{G_N} \mathrm{Tr}(h^k) \chi_\lambda(h) \mathrm{Tr}(h^{-k}) \chi_\lambda(h^{-1}) dh \\ &= \sum_{\mu, \nu} c_{\lambda,k}^\mu c_{\lambda,k}^\nu \int_G \chi_\mu(h) \chi_\nu(h^{-1}) dh = \sum_{\mu \in \widehat{G}_N} (c_{\lambda,k}^\mu)^2. \end{aligned}$$

Using Lemma 4.7, we conclude that

$$0 \leq J_\lambda = \sum_{\mu \in \widehat{G}_N} |c_{\lambda,k}^\mu| \leq n$$

and

$$\mathbb{E}[|W_{\ell^k}|^2] \leq n^{-2} Z_{1,T,X_r}^{-1} \sum_{\lambda \in \widehat{G}_N} e^{-\frac{T}{2}c_\lambda} J_\lambda = \frac{1}{n}. \quad (38)$$

From (37) and (38), we conclude that the expectation and variance of W_ℓ under YM_{M,a,G_N} both vanish as $N \rightarrow \infty$. Therefore, W_{ℓ^k} converges to 0 in probability. \square

Remark. For the A_r type, for all $k \neq 0$ and $\lambda \in \Lambda_r$, $c_{\lambda,k}^\lambda = 0$, and the first part of the above proof yields another argument for $\mathbb{E}[W_{\ell^k}] = 0$ for all $r \geq 1$.

¹⁶so that n is respectively $r, r+1, 2r+1$ for types \tilde{A}_r, A_r, B_r and $2r$ for types $\{C_r, D_r\}$.

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