HAL open science

Large N limit of the Yang-Mills measure on compact surfaces II: Makeenko-Migdal equations and planar master field

Thibaut Lemoine, Antoine Dahlqvist

To cite this version:

Thibaut Lemoine, Antoine Dahlqvist. Large N limit of the Yang-Mills measure on compact surfaces II: Makeenko-Migdal equations and planar master field. 2022. hal-03536317

HAL Id: hal-03536317

https://hal.science/hal-03536317v1

Preprint submitted on 19 Jan 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:2201.05886v1 [math.PR] 15 Jan 2022

Large N limit of the Yang–Mills measure on compact surfaces II: Makeenko–Migdal equations and planar master field

Antoine Dahlqvist* Thibaut Lemoine†

January 19, 2022

Abstract

This paper considers the large N limit of Wilson loops for the two-dimensional Euclidean Yang–Mills measure on all orientable compact surfaces of genus larger or equal to 1, with a structure group given by a classical compact matrix Lie group. Our main theorem shows the convergence of all Wilson loops in probability, given that it holds true on a restricted class of loops, obtained as a modification of geodesic paths. Combined with the result of [16], a corollary is the convergence of all Wilson loops on the torus. Unlike the sphere case, we show that the limiting object is remarkably expressed thanks to the master field on the plane defined in [3, 32] and we conjecture that this phenomenon is also valid for all surfaces of higher genus. We prove that this conjecture holds true whenever it does for the restricted class of loops of the main theorem. Our result on the torus justifies the introduction of an interpolation between free and classical convolution of probability measures, defined with the free unitary Brownian motion but differing from t-freeness of [5] that was defined in terms the liberation process of Voiculescu [55]. In contrast to [16], our main tool is a fine use of Makeenko–Migdal equations, proving their uniqueness under suitable assumptions, generalising the arguments of [17, 29].

1 Introduction

The two-dimensional Yang–Mills measure is a probability model originating from Euclidean quantum field theory in the setting of pure gauge theory. It describes a generalised random connection on a principle bundle over a two dimensional manifold, with a compact Lie group as structure group, making rigorous the path integral over connections for the so-called Yang–Mills action. Different equivalent mathematical definitions have been given in two dimensions and are due to [27, 19, 47, 28, 1, 2, 35, 11]. The work of [59] brought to light many special features

*University of Sussex, School of Mathematical and Physical Sciences, Pevensey 3 Building, Brighton, UK

†Institut de Recherche Mathématique Avancée, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg, France

of the Yang–Mills measure in two dimensions, including its partial integrability, used as a way to perform exact volume computations for the Atiyah–Bott–Goldman measure [4, 23] on the space of flat connections [39, 8, 48].

When a compact Lie group G and a surface Σ are given, the Yang–Mills measure can be mathematically understood as a random matrix model which assigns to any loop[1] of the surface a random matrix so that concatenation and reversion of loops are compatible with the group operations. In [32], it is shown that it gives rise to a random homomorphism from the group of rectifiable reduced loops of the surface to the chosen group G.

We consider here a compact, orientable surface Σ of genus $g \geq 1$ and a group G belonging to a series of classical compact matrix groups. We are primarily interested in the traces of these matrices, called *Wilson loops,* when the rank of G goes to infinity. We ask whether Wilson loops converge in probability under the Yang-Mills measure, towards a deterministic function.

Let us try to give a brief historical account of this problem. In physics, a motivation for the focus on Wilson loops is due to K. Wilson work [57] related to quarks confinement. The introduction of the large rank regime in gauge theories, known as large N limit, is due to t'Hooft's work [53] on QCD. This lead to many articles in theoretical physics in the 80' studying the question in two dimensions, a partial list being [30, 31, 42, 44, 58, 25, 24, 26]. In mathematics, this problem was advertised by I. Singer in [51] where the candidate limit of Wilson loops was called *master field*, following the physics literature. The case of the plane and the sphere have been proved in [60, 3, 32] and[2] [17]. The case of compact surfaces has been first investigated by [29] where loops contained in topological disc can be considered whenever the convergence holds for simple loops. The study of similar questions in the plane for analogs of the Yang–Mills measure has been investigated in [9]. In higher dimension, an analog[3] of this question for a lattice model has also been considered [10]. Very recently and independently from the current work, it was shown in [41, 40] that under the Atiyah–Bott–Goldman measure, which can be understood as the weak limit of the Yang–Mills measure when the area of the surface vanishes, the expectation of Wilson loops converges and has a $\frac{1}{N}$ expansion when the group belongs to the series of special unitary matrices and the surface is closed, orientable and of genus $g \geq 2$. For further details and references on the motivations of this problem, we refer to [16, Sec. 1] and [37, Sec. 2.5.].

In this article, we give a complete answer in the case of the torus and a conjecture and a partial result for all surfaces with genus $g \geq 2$. It is the sequel of [16] where we have shown the convergence for a large[4] but incomplete class of loops. Let us recall that in the case of the plane, the master field can be described thanks to free probability and more specifically in terms of free unitary Brownian motion [3, 32]. The case of the sphere leads to the introduction of a different non-commutative stochastic process called the free unitary Brownian bridge [17]. In contrast, for the

[1] with enough regularity.

[2] See also [29] where a conditional result was obtained implying the case of the sphere, given the convergence for simple loops.

[3] though in this case, there is at the time of writing, no construction of the continuous Yang–Mills measure in dimension 3 and higher is available.

[4] informally described as all simple loops or iteration of simple loops, and all loops which do not visit one handle of the surface.

torus, we show that after lifting loops to the universal cover, the master field is also described by the planar master field and we conjecture that the same holds for any surface of higher genus. In the torus case, the master field leads to an interpolation parametrised by the total area of the torus, between the free and the classical convolution of two Haar unitaries built with the free unitary Brownian motion, which differs from the t-freeness introduced by [5] using the liberation process of [55].

The aim of the current paper is to investigate the stability of Wilson loops convergence under homotopy equivalences. To do so, we will use a set of recursive equations named after Makeenko and Migdal [42]. When a loop is deformed in a specific way – that we call a Makeenko–Migdal deformation – these equations relate the differential of the expected Wilson loops with the expectation of a product of Wilson loops having a smaller number of intersection points. These equations can be understood as a remarkable analog of Schwinger–Dyson equations used in random matrix theory and were first inferred heuristically in [42] as an integration by part for the path integral over the space of connections. A first rigorous proof was given in the case of the plane in[5] [32] and was later remarkably simplified and generalised in [22, 21] in a local way that applies to any surface. Makeenko–Migdal equations were crucial to [17, 29] leading to an induction argument on the number of intersection points that reduced the convergence of all Wilson loops on the sphere to the case of simple loops. In the case of other surfaces, the very same strategy fails a priori, as some loops cannot be deformed to simpler loops without raising the number of intersection points, while some homotopy classes do not contain any loop for which the convergence is known to hold. We show here that the first hurdle can be overcome, allowing to reduce the problem, completely in the torus case and partially when $g \geq 2$, to the class of loops considered in [16]. We leave the completion of this program for all compact surfaces to a future work.

The paper is organised as follows. The first four following sections of the introduction give respectively an informal definition of the Yang–Mills measure and of the main results, a discussion on the relation with the Atiyah–Bott–Goldman measure and the work [40, 41], a consequence of the result on the torus in non-commutative probability, and lastly, a sketch of the strategy of the main proofs. Section 2 recalls and adapts some combinatorial notions of discrete homotopy of loops in embedded graph instrumental to the proof. Section 3 gives the definition of the Yang–Mills measure, a statement of the Makeenko–Migdal equations and states the main results of the article. Section 4 consists in the proof of our main technical result, which is Proposition 3.18. Section 5 describes the behaviour of Wilson loops when one performs surgery on the underlying surface. Section 6 is finally discussing how the master field on the torus is a new candidate for the interpolation between classical and free convolution, different from Voiculescu's liberation process. In an appendix, for the sake of completeness, we recall and prove several results on Makeenko–Migdal equations, that are quite standard in the literature for unitary groups but not necessarily for all classical groups.

[5] See also [14, sect. 7] for a variation of this proof and [32, section 0] for the heuristics of the original proof of [42] based on an integration by part in infinite dimension. See also [20] for a proof closer in spirits to the original argument of [42].

Contents

1 Introduction **1**
- 1.1 Yang–Mills measure and master field, statement of results 4
- 1.2 Atiyah–Bott–Goldman measure 8
- 1.3 Non-commutative distribution and master field on the torus: an interpolation between free and classical convolution 10
  - 1.3.1 Non-commutative probability and free independence 10
  - 1.3.2 Free Unitary Brownian motion and t-freeness 12
  - 1.3.3 A matrix approximation for another interpolation from classical to free convolution 13
- 1.4 Strategy of proof via Makeenko–Migdal deformations 13

2 Homology and homotopy on embedded graphs **18**
- 2.1 Four equivalence relations on paths and loops on maps 18
- 2.2 Discrete homology, winding function and Makeenko–Migdal vectors . 20
- 2.3 Regular polygon tilings of the fundamental cover, tiling-length of a tame loop and geodesic loops 25
- 2.4 Shortening homotopy sequence 26
- 2.5 Nested and marked loops 31
- 2.6 Pull and twist moves 32
- 2.7 Vertex desingularisation 34

3 Yang–Mills measure and Makeenko–Migdal equations **35**
- 3.1 Metric and heat kernel on classical groups 35
- 3.2 Area weighted maps, Yang–Mills measure and area continuity 36
- 3.3 Planar master field, main results and conjecture 41
- 3.4 Invariance in law and Wilson loop expectation 44
- 3.5 Makeenko–Migdal equations, existence and uniqueness problem . . . 46

4 Proof of the main result, stability of convergence under deformation **50**
- 4.1 Non-null homology loops 50
- 4.2 Null homology loops 52

5 Proof of convergence after surgery **58**

6 Liberation and master field on the torus **60**

7 Appendix **62**
- 7.1 Makeenko–Migdal equations 62
- 7.2 Makeenko–Migdal equations 63

1.1 Yang–Mills measure and master field, statement of results

We shall first give an heuristic definition of the Yang–Mills measure in its geometric setting and state informally the main results of the current article. Proper definitions and statements are respectively given in sections 3.2 and 3.3.

Let Σ be either a compact, connected, closed orientable surface of the genus $g \geq 1$ endowed with a Riemannian metric, or the Euclidean plane $\mathbb{R}^2$ with its standard inner product. Let G_N be a classical compact matrix Lie group of size N, i.e. viewed compact subgroup of $\mathrm{GL}_N(\mathbb{C})$. We assume that the Lie algebra $\mathfrak{g}_N$ of G_N is endowed with an Ad-invariant inner product $\langle \cdot, \cdot \rangle$, as in section 3.1.

Given a G_N-principal bundle (P, π, Σ), a connection is a one form ω on M valued in adjoint fibre bundle $\mathrm{ad}(P)$, its curvature is the two-form $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ valued in $\mathrm{ad}(P)$.

The *Yang–Mills action* of a connection ω on a G_N-principal bundle (P, π, Σ) is defined by

$$S_{\mathrm{YM}}(\omega) = \frac{1}{2} \int_\Sigma \langle \Omega \wedge \star \Omega \rangle, \tag{1}$$

where $\star$ denotes the Hodge operator. An important feature of dimension 2 is that whenever Ψ is a diffeomorphism of Σ preserving its volume form,

$$S_{\mathrm{YM}}(\Psi_* \omega) = S_{\mathrm{YM}}(\omega). \tag{2}$$

The *Euclidean Yang–Mills measure* is the formal Gibbs measure

$$d\mu_{\mathrm{YM}}(\omega) \text{“} = \text{”} \frac{1}{Z} e^{-S_{\mathrm{YM}}(\omega)} \mathcal{D}\omega, \tag{3}$$

where $\mathcal{D}\omega$ plays the role of a formal Lebesgue measure on the space of connections over an arbitrary principal bundle[6] and Z is a normalisation constant supposed to ensure the total mass to be 1. We choose here not to include a parameter in front of the action, as it can be included in the volume form of Σ.

The space $\mathcal{A}(P)$ being infinite-dimensional, the latter equation has no mathematical meaning. Though at first stance, as the Yang–Mills action of ω can be seen as the L^2-norm of the curvature Ω, an analogy with Gaussian measures can be hoped. Though, when G_N is not abelian, Ω depends non-linearly on ω which prevents any direct construction of μ_{YM} using a Gaussian measure. In two dimensions, this non-linearity can be compensated by the so-called gauge symmetry of S_{YM} which allows to bypass this problem. This lead to the constructions of [27, 19, 47] based on stochastic calculus. See also [11] for an approach defining further a random, distribution valued, connection on trivial bundles over the two dimensional torus. We follow here instead the approach of [35] which focuses on the holonomy of a connection, whose law can be directly defined using the heat kernel on G_N. The definition we are using is recalled in section 3.2, it agrees with the construction of [27, 19, 47] thanks to the so-called Driver–Sengupta formula. An important feature of this measure is suggested by (2). For any two dimensional Riemannian manifold Σ' diffeomorphic to Σ, and for any diffeomorphism $\Psi : \Sigma \to \Sigma'$, there is an induced pushed forward measure $\Psi_*(\mathrm{YM}_\Sigma)$ on connections of $(P, \Psi \circ \pi, \Sigma')$. Whenever Ψ maps the volume form of Σ to the one of Σ',

$$\Psi_*(\mathrm{YM}_\Sigma) = \mathrm{YM}_{\Sigma'}.$$

[6] There is here an apparant additional issue with this vague definition. A slightly less dubious state space could be obtained by fixing a representant of each principle bundle equivalence class over Σ and by considering instead the set of pairs of a principal bundle belonging to this family together with a connection on it. When Σ is a contractible space or if G_N is simply connected, there is only one equivalence class of G_N-principal bundles over Σ and this issue disappears. We shall not discuss further the question of the type of the principal bundle under the Yang–Mills in this text. For more details and rigorous results we refer to [33].

We shall call this property the *area-invariance* of the Yang–Mills measure. Moreover, for any relatively compact, contractible, open subset U of Σ, the restriction to U induces a measure $\mathcal{R}_*^U(\mathrm{YM}_\Sigma)$ on connections of $(\pi^{-1}(U), \pi, U)$. When Σ is the Euclidean plane $\mathbb{R}^2$ or the hyperbolic Poincaré unit disc $\mathbb{D}_\mathfrak{h}$, with its usual metric, it satisfies[7] $\mathcal{R}_*^U(\mathrm{YM}_\Sigma) = \mathrm{YM}_U$, where U is endowed with the metric of Σ.

If ω is a connection on a G_N-principal bundle (P, π, Σ), and U is an open subset of Σ where $\pi : \pi^{-1}(U) \to U$ can be[8] trivialised. When such a trivialisation has been fixed, its *holonomy* is a function $\gamma \mapsto \mathrm{hol}(\omega, \gamma)$ mapping paths[9] $\gamma : [0, 1] \to U$ to elements of the group G_N such that

$$\mathrm{hol}(\omega, \gamma_1\gamma_2) = \mathrm{hol}(\omega, \gamma_2)\mathrm{hol}(\omega, \gamma_1)$$

for any paths γ_1 and γ_2 such that the endpoint of γ_1 coincides with the starting point of γ_2, while for any path γ,

$$\mathrm{hol}(\omega, \gamma^{-1}) = \mathrm{hol}(\omega, \gamma)^{-1},$$

where $\gamma_1\gamma_2$ and γ^{-1} denote the concatenation and reversion of the paths.

When G_N is a group of matrices of size N and ℓ is a loop of U, the *Wilson loop* associated to ℓ is the function

$$W_\ell(\omega) = \mathrm{tr}(\mathrm{hol}(\omega, \ell)),$$

where $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$, with Tr the usual trace of matrices. This function can be shown to be independent of the choice of local trivialisation of (P, π, Σ) and is therefore only a function of ω and ℓ.

Our primary source of interest is the study of the random variables $W_\ell := W_\ell(\omega)$, for loops of Σ, when ω is sampled according to YM_Σ. We are interested in the large N limit of W_ℓ, when the scalar product $\langle \cdot, \cdot \rangle$ is chosen as in section 3.1 and the volume form of the surface is fixed. The paper [51] seems to be the first mathematical article focusing on this question and motivates the following conjecture, also suggested by [36, 21, 29].

Conjecture 1.1. *Let G_N be a classical compact matrix Lie group of size N, endowed with the metric of section 3.1 and denote by Σ a two-dimensional compact Riemannian manifold, the Euclidean plane $\mathbb{R}^2$ or the hyperbolic Poincaré disc $\mathbb{D}_\mathfrak{h}$. For any loop $\ell \in L(\Sigma)$, there is a constant $\Phi_\Sigma(\ell)$ such that under YM_Σ*

$$W_\ell \to \Phi_\Sigma(\ell) \text{ in probability as } N \to \infty. \tag{4}$$

The functional Φ_Σ is called the master field on Σ.

The works [60, 3] answered to this question in the plane for $G_N = \mathrm{U}(N)$. In [36], the above statement was proved simultaneously to [3] for all groups mentioned, and for a large family of loops given by loops of finite length. Moreover, motivated by the physics articles [42, 44, 31], [36] proved recursion relations giving a way to compute explicitly $\Phi_{\mathbb{R}^2}$ for all loops with finitely many intersections.

[7] Compact surfaces do not have this property but there is still absolute continuity in place of continuity. This was instrumental in [16].

[8] the tubular neighbourhood of a smooth loop or of an embedded graph could be such an open set.

[9] In this section the space of paths is not specified and could be taken as the space of piecewise smooth paths with constant speed and transverse intersections.

By area invariance and restriction property, the result on the hyperbolic plane can be deduced directly from these latter works as follows. According to a theorem of Moser [46], any relatively compact open disc U of $\mathbb{D}_{\mathfrak{h}}$ with hyperbolic volume t can be mapped to the open Euclidean disc D_t of $\mathbb{R}^2$ centered at 0 and of area t, by a diffeomorphism $\Psi : U \to D_t$ sending the restriction of the hyperbolic volume form on U to the restriction of Euclidean volume form on D_t. By area-invariance, $\mathcal{R}_*^U(\mathrm{YM}_{\mathbb{D}_{\mathfrak{h}}}) = \mathrm{YM}_U = \Psi_*^{-1}(\mathrm{YM}_{D_t})$, so that the conjecture holds true for $\mathbb{D}_{\mathfrak{h}}$ with

$$\Phi_{\mathbb{D}_{\mathfrak{h}}}(\ell) = \Phi_{\mathbb{R}^2}(\Psi \circ \ell)$$

for any loop ℓ with range included in U.

The work [17] proved the conjecture for $\Sigma = \mathbb{S}^2$ and all loops of finite length when $G_N = \mathrm{U}(N)$, while [29], written simultaneously, gave a conditional result on $\mathbb{S}^2$ based on an argument similar to [17], as well as a conditional result on other surfaces for loops included in a topological disc, given convergence of for simple loops. In [16] we gave an alternative argument proving a generalisation of the results of [29] on compact surfaces without using the conditions [29], see section 1.4. The current article was written with the aim to strengthen the argument common to [17] and [29] to address the conjecture on all compact manifolds. This lead to the following theorem and conjecture.

Theorem 1.2. *When $\mathbb{T}_T$ is a torus of volume $T > 0$, conjecture 1.1 is valid. Moreover, considering $\mathbb{T}_T$ as the quotient of the Euclidean plane $\mathbb{R}^2$ by $\sqrt{T}.\mathbb{Z}^2$,*

$$\Phi_{\mathbb{T}_T}(\ell) = \begin{cases} \Phi_{\mathbb{R}^2}(\tilde{\ell}) & \text{if } \ell \text{ is contractible,} \\ 0 & \text{otherwise,} \end{cases}$$

where for any continuous loop ℓ in $\mathbb{T}_T$, $\tilde{\ell}$ is a lift of ℓ to $\mathbb{R}^2$, that is a smooth loop of $\mathbb{R}^2$, whose projection to $\mathbb{R}^2/\sqrt{T}.\mathbb{Z}^2$ is ℓ.

We discuss an interpretation of this result in terms of non-commutative probability in section 1.3. For compact surfaces of higher genus, a natural candidate is given as follows. Recall that for any compact surface Σ of volume $T > 0$, there is a covering map $p : \mathbb{D}_{\mathfrak{h}} \to \Sigma$ mapping the hyperbolic metric of $\mathbb{D}_{\mathfrak{h}}$ to the metric of Σ.

Conjecture 1.3. *For any two-dimensional compact manifold of genus $g \geq 2$, with universal cover $p : D_{\mathfrak{h}} \to \Sigma$, the conjecture 1.1 is valid with*

$$\Phi_{\Sigma}(\ell) = \begin{cases} \Phi_{\mathbb{D}_{\mathfrak{h}}}(\tilde{\ell}) & \text{if } \ell \text{ is contractible,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

This conjecture is also justified by the main result of [16] which leads to the following. Recall that a simple loop γ of Σ is non-separating, if the set $\Sigma \setminus \gamma$, where γ denote the range of γ, is connected.

Corollary 1.4. *When $g \geq 2$ and γ is a non-separating loop of Σ, then the convergence (4) holds true with the limit (5), for all loops ℓ that do not intersect γ.*

From this point of view, Theorem 1.2 and Conjecture 1.3 prove or claim the following "asymptotic restriction property": for any closed surface Σ of genus $g \geq 1$, if Σ_b is a surface with boundary embedded in Σ preserving the volume form, for any

loop ℓ included in Σ_b, W_ℓ converges in probability towards a constant both under YM_{Σ_b} and YM_Σ, and the two limits agree, being equal to $\Phi_{\mathbb{R}^2}(\tilde{\ell}), \Phi_{\mathbb{D}_\mathfrak{h}}(\tilde{\ell})$ or 0. See section 3.3 for a proper statement.

We obtained here two conditional results proving stability of the claimed convergence.

Proposition 1.5. *For any two-dimensional compact manifold of genus $g \geq 2$, when G_N is a classical compact matrix group of size N, assume that for any geodesic loop ℓ of Σ with non-zero homology, under* YM_Σ,

$$W_\ell \to 0 \text{ in probability as } N \to \infty. \tag{6}$$

Then (6) *also holds true for all loops with non-zero homology.*

Assume $g \geq 2$ and Γ_g is a discrete subgroup of isometry acting freely, properly on $\mathbb{D}_\mathfrak{h}$ and that $\mathbb{D}_\mathfrak{h}/\Gamma_g$ is a compact surface of genus g with finite total volume $T > 0$. Assume that there is a fundamental domain for this action given by a $4g$ hyperbolic polygon D of volume T, centred at 0.

Let us say that a loop ℓ of Σ is a delayed geodesic if there is a lift $\tilde{\ell}$ of ℓ starting from 0 such that $\tilde{\ell} = \gamma_1\gamma_2$ where γ_2 is a geodesic and γ_1 is smooth path included in $\overline{D}$, intersecting ∂D only once, transversally at its endpoint.

Theorem 1.6. *The conjecture 1.3 holds true if* (6) *is true for all non contractible, delayed geodesic loops.*

Besides, the recent results of [41] are furthermore coherent with the above statement as discussed in the next sub-section.

1.2 Atiyah–Bott–Goldman measure

Another measure on connections is due to [4, 23] when $g \geq 2$. Recently, the limit of Wilson loops under this measure has been investigated by [40, 41], we discuss the relation with our result.

Let G be a compact connected semisimple[10] Lie group G, $\mathfrak{g}$ its Lie algebra, endowed with an invariant inner product, and $Z(G)$ its center. For any $g \geq 2$, let $K_g : G^{2g} \to G$ be the product of commutators:

$$K_g(a_1, b_1, \ldots, a_g, b_g) = [a_1, b_1] \cdots [a_g, b_g].$$

The space

$$\mathcal{M}_g = K_g^{-1}(e)/G$$

is called the *moduli space of flat G-connections* over a compact surface of genus $g \geq 2$, where G acts by diagonal conjugation, as

$$h.(z_1, \ldots, z_{2g}) = (hz_1h^{-1}, \ldots, hz_{2g}h^{-1}), \forall z \in G^{2g}, g \in G.$$

For any $z \in G^{2g}$, its isotropy group is $Z_z = \{h \in G, h.z = z\}$. The set $\mathcal{M}_g^0 = \{z \in G^{2g} : Z_z = Z(G)\}$ can be shown to be a manifold [23, 49] of dimension $2g - 2$, endowed with a symplectic form ω with finite total volume. Besides, using the holonomy map along a suitable $2g-$tuple $\ell_1, \ldots, \ell_{2g}$ of loops, $\mathcal{M}_g^0$ can be identified

[10] Mind that this excludes $\mathrm{U}(N)$.

with a subset of smooth connections ω on a G-principal bundle over Σ such that $S_{\mathrm{YM}}(\omega) = 0$. This subset is a manifold with a symplectic structure [4], equal to the push forward of ω. The Atiyah–Bott–Goldman measure is the volume form on $\mathcal{M}_g^0$ associated to ω, given by

$$\mathrm{vol}_g = \frac{\omega^{\frac{1}{2}\dim \mathcal{M}_g^0}}{(\frac{1}{2}\dim \mathcal{M}_g^0)!}. \tag{7}$$

Let us denote by $\mu_{ABG,g}$ the probability measure on $\mathcal{M}_g^0$ obtained by normalising vol_g. It appeared in [59], that integrating against the Yang–Mills measure on a compact surface of total area T and letting T tend to 0, allows to obtain formulas for integrals against vol_g. This convergence was proved rigorously by Sengupta in [49]. Using the holonomy mapping of the Yang–Mills measure, the convergence can be understood as follows. Consider a heat kernel $(p_t)_{t>0}$ on G, when its Lie algebra $\mathfrak{g}$ is endowed with its Killing form $\langle \cdot, \cdot \rangle$.

Theorem 1.7 (Symplectic limit of Yang–Mills measure)**.** *Let $f : G^{2g} \to \mathbb{C}$ be a continuous G-invariant function, and $\tilde{f} : \mathcal{M}_g^0 \to \mathbb{C}$ be the induced function on the moduli space. Then*

$$\lim_{T\downarrow 0} \int_{G^{2g}} f(x) p_T(K_g(x)) dx = \frac{\mathrm{vol}(G)^{2-2g}}{|Z|} \int_{\mathcal{M}_g^0} \tilde{f} d\mathrm{vol}_g. \tag{8}$$

For any word w in the variables $a_1, \dots, b_g$ and their inverse, setting

$$W_w(z) = \frac{1}{N}\mathrm{Tr}(w(z_1, z_1^{-1}, \dots, z_{2g}, z_{2g}^{-1})), \forall z \in G^{2g}$$

defines also a function on $\mathcal{M}_g^0$. Denoting it also by W_w and considering the loop ℓ_w obtained by the concatenation $w(\ell_1, \ell_1^{-1}, \dots, \ell_{2g}, \ell_{2g}^{-1})$, the last statement can be reformulated as

$$\lim_{T\downarrow 0} \mathbb{E}_{\mathrm{YM}_{\Sigma_T}}[W_{\ell_w}] = \int_{\mathcal{M}_g^0} W_w d\mu_{ABG,g},$$

Consider the surface group

$$\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g | [a_1, b_1] \dots [a_g, b_g] \rangle.$$

Consider the equivalence relation $\sim$ on the set of words with $2g$ letters and their inverse, such that $w \sim w'$ iff $w(a_1, \dots, b_g)$ and $w'(a_1, \dots, b_g)$ are equal in Γ_g. Thanks to the defining relation of $\mathcal{M}_g$, for any word w, the function W_w depends only on the equivalence class of w. When $\gamma \in \Gamma_g$ is the evaluation of w in Γ_g, denote this function by W_γ. In [41], Magee obtained the following analog of asymptotic freeness of Haar unitary random matrices.

Theorem 1.8 ([41] Cor. 1.2)**.** *Consider the group $G = \mathrm{SU}(N)$. For any $\gamma \in \Gamma_g$,*

$$\lim_{N\to\infty} \mathbb{E}_{\mu_{ABG,g}}[W_\gamma] = \begin{cases} 1 & \text{if } \gamma = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since for any word with evaluation $\gamma \in \Gamma_g$, it can be shown that $\gamma = 1$ if and only if the loop ℓ_w is contractible, the above statement can be understood as the $T = 0$ case of the conjecture 1.3, with a weaker convergence given in expectation instead

of in probability. In [40], it is also shown that $\mathbb{E}_{\mu_{ABG,g}}[W_\ell]$ admits an asymptotic expansion in powers of $\frac{1}{N}$.

Our main arguments differ from the one of [41]; the one of [16] relies on the Markovian property of the Yang-Mills measure, while the ones of this paper starts from on the Makeenko–Migdal equations which allow to prove that convergence in probability is stable under a large class of deformations, a problem which does not occur in the zero volume case. Though, a common starting point to [16] and [40] is the convergence of the partition function of the model.

Besides, on the one hand we did not investigate asymptotic expansion in $\frac{1}{N}$ neither here nor in [16]. On the other hand, we prove a convergence in probability instead of in expectation and work also with larger family of matrix groups.

Moreover, when $g = 1$, the construction and results mentioned within this subsection are not valid. Though, we are able to prove a result in this setting when $T > 0$. It gives a matrix approximation result for an interpolation between classical and free convolution.

1.3 Non-commutative distribution and master field on the torus: an interpolation between free and classical convolution

We discuss here the non-commutative distribution associated to the master field on the torus, leading to the corollary 1.11 below, obtained by specialising Theorem 1.2 to projection of loops restrained to the lattice $\sqrt{T}.\mathbb{Z}^2$.

1.3.1 Non-commutative probability and free independence

Let us give an extremely brief account of these notions. We refer to [56, 45] for more details.

A *non-commutative probability space*[11] is the data of a tuple $(\mathcal{A}, *, 1, \tau)$ where $(\mathcal{A}, *, 1)$ is a unital $*$-algebra over $\mathbb{C}$, and τ is a positive, tracial state, that is a linear map $\tau : \mathcal{A} \to \mathbb{C}$ with

$$\tau(aa^*) \geq 0 \text{ and } \tau(ab) = \tau(ba), \forall a, b \in \mathcal{A},$$

with furthermore $\tau(1) = 1$ and $\tau(a^*) = \overline{\tau(a)}, \forall a \in \mathcal{A}$. We shall often leave as implicit the choice of unit and $*$, and denote a non-commutative probability space simply as a pair $(\mathcal{A}, \tau)$.

Example 1.9. *For $N \geq 1$, the tuple $(M_N(\mathbb{C}), *, \mathrm{Id}_N, \mathrm{tr})$, where $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$, gives such a space. Consider the group $\mathrm{U}(N)$ of unitary complex matrices of size N and a group Γ with unit element 1. Let $(\mathbb{C}[\Gamma], *)$ be the group algebra of Γ endowed with the skew-linear idempotent defined by $\gamma^* = \gamma^{-1}, \forall \gamma \in \Gamma$. Then, whenever $\rho : \Gamma \to \mathrm{U}_N(\mathbb{C})$ is a unitary representation of Γ, setting $\tau_\rho = \mathrm{tr} \circ \rho$, the tuple $(\mathbb{C}[\Gamma], *, 1, \tau_\rho)$ is a non-commutative probability space.*

Let $(\mathcal{A}_1, \mathcal{A}_2)$ be unital sub-algebras of a non-commutative probability space $\mathcal{A}_1$.

- They are *classically independent* if $\forall a_1, \ldots, a_n \in \mathcal{A}_1, b_1, \ldots, b_n \in \mathcal{A}_2$,

$$\tau(a_1 b_1 a_2 \ldots a_n b_n) = \tau(a_1 \ldots a_n)\tau(b_1 \ldots b_n).$$

[11] sometimes denoted NCPS

- They are *freely independent* if for any $n \in \mathbb{N}$, for any $\{i_1, \dots, i_n\} \in \{1,2\}^n$ such that $i_1 \neq i_2, \dots, i_{n-1} \neq i_n$ and for any $a_k \in \mathcal{A}_{i_k}$,

$$\tau(a_k) = 0,\ \forall 1 \leq k \leq n \Longrightarrow \tau(a_1 \cdots a_n) = 0.$$

These definitions can be generalised to any number of sub-algebras, and a family of elements $(a_i)_{i\in I}$ of a non-commutative probability space $(\mathcal{A}, \tau)$ is said to be independent (resp. free) if the family $(\mathcal{A}_i)_{i\in I}$ is independent (resp. free), where for all $i \in I$, $\mathcal{A}_i$ is the subalgebra generated by a_i and a_i^*. We shall then say that $(a_i)_{i\in I}$ are resp. independent and free under τ.

When I is an arbitrary set, let us denote by $\mathbb{C}\langle X_i, X_i^*, i \in I\rangle$ the unital $*$-algebra of non-commutative polynomials in the variables $X_i, X_i^*, \in I$, with $*$ mapping X_i to X_i^* for all $i \in I$. When $(\mathcal{A}, *, 1, \tau)$ is a non-commutative probability space and $\mathbf{a} = (a_i)_{i\in I}$ is a family of elements of $\mathcal{A}$, its *non-commutative distribution* is the positive, tracial, state on $\mathbb{C}\langle X_i, X_i^*, i \in I\rangle$ given by

$$\tau_{\mathbf{a}}(P) = \tau(P(a_i, i \in I)), \forall P \in \mathbb{C}\langle X_i, X_i^*, i \in I\rangle,$$

where $P(a_i, i \in I) \in \mathcal{A}$ denotes the evaluation of P replacing X_i and X_i^* by a_i and a_i^*. Likewise, when $\mathcal{A}$ and $\mathcal{B}$ are sub-algebras of a same non-commutative probability space $(\mathcal{C}, \tau)$, we call the state $\tau_{\langle\mathcal{A},\mathcal{B}\rangle}$ on $\mathbb{C}\langle X_a, Y_b, a \in \mathcal{A}, b \in \mathcal{B}\rangle$ given by

$$\tau_{\langle\mathcal{A},\mathcal{B}\rangle}(P(X_a, Y_b; a \in \mathcal{A}, b \in \mathcal{B})) = \tau(P(a, b; a \in \mathcal{A}, b \in \mathcal{B})),$$

the joint distribution of $(\mathcal{A}, \mathcal{B})$ in $(\mathcal{C}, \tau)$.

When a, b are two elements of non-commutative probability spaces with respective non-commutative distribution τ_a and τ_b, there are unique states $\tau_a \star \tau_b$ and $\tau_a *_c \tau_b$ on $\mathbb{C}\langle X, Y, X^*, Y^*\rangle$ such that $\tau_X = \tau_a$ and $\tau_Y = \tau_b$ both under and $\tau_a *_c \tau_b$ and $\tau_a \star \tau_b$, while the joint distribution (X, Y) under $\tau_a \star \tau_b$ and $\tau_a *_c \tau_b$, are respectively freely and classically independent. The states $\tau_a \star \tau_b$ and $\tau_a *_c \tau_b$ are resp. called the *free* and the classical *convolution* of τ_a and τ_b. We define likewise the free and classical convolution of two states on $\tau_{\mathcal{A}}, \tau_{\mathcal{B}}$ of NCPS $(\mathcal{A}, \tau_{\mathcal{A}}), (\mathcal{B}, \tau_{\mathcal{B}})$ as states $\tau_{\mathcal{A}} \star \tau_{\mathcal{B}}$ and $\tau_{\mathcal{A}} *_c \tau_{\mathcal{B}}$ on $\mathbb{C}\langle X_a, Y_b, a \in \mathcal{A}, b \in \mathcal{B}\rangle$.

Let us recall the following result of *asymptotic freeness* due to Voiculescu [54], and for the considered group series by [13], see also [36, Sect. I-3].

Theorem 1.10 ([54, 13, 36])**.** *Let A and B be two deterministic matrices of size N with respective non-commutative distribution satisfying for all fixed $P \in \mathbb{C}\langle X, X^*\rangle$,*

$$\tau_A(P) \to \tau_a(P), \tau_B(P) \to \tau_b(P), \quad \text{as } N \to \infty,$$

for some state τ_a, τ_b on $\mathbb{C}\langle X, X^*\rangle$. Consider U and V two independent Haar unitary matrices on a group G_N and $\rho_N : \mathbb{C}[\mathbb{F}_2] \to G_N$ the associated unitary representation of the free group of rank 2.

Then for any $\gamma \in \mathbb{F}_2$ and $P \in \mathbb{C}\langle X, Y, X^*, Y^*\rangle$, the following limit holds in probability as $N \to \infty$,

$$\tau_{\rho_N}(\gamma) \to \begin{cases} 1 & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma \in \mathbb{F}_2 \setminus \{1\} \end{cases} \tag{9}$$

and

$$\tau_{A,UBU^*}(P) \to \tau_a \star \tau_b(P). \tag{10}$$

On the one hand, the first convergence (9) can be proved to be a special case of (10) when A and B are themselves independent Haar unitary random variables. On the other hand, when A and B are unitary or Hermitian with uniformly bounded spectrum, (10) can be deduced from (9) by functional calculus.

A motivation of the current article was to understand an analog of (9), when (U, V) are sampled according to a different law with correlation, as discussed in section 1.3.3.

Let us first mention a family of states which arises when considering an analog of (10), replacing the Haar measure by a Brownian motion on the group G_N.

1.3.2 Free Unitary Brownian motion and t-freeness

We refer here to [6, 55, 5] for more details. Consider a non-commutative probability space $(\mathcal{A}, \tau, *, 1)$. An element $u \in \mathcal{A}$ is called unitary when $uu^* = u^*u = 1$. It is Haar unitary if for all integer $n > 0$, $\tau(u^n) = \tau((u^*)^n) = 0$.

The *free unitary Brownian motion* on a $*$-probability space $(\mathcal{A}, \tau, *, 1)$ is a family $(u_t)_{t\geq 0}$ of unitary elements of $\mathcal{A}$ such that the increments $u_{t_1}u_0^*, \ldots, u_{t_n}u_{t_{n-1}}^*$ are free for all $0 \leq t_1 \leq \cdots \leq t_n$, and for any $k \in \mathbb{Z}^*$ and $0 < s < t$,

$$\tau((u_t u_s^*)^k) = \tau(u_{t-s}^k)$$

while $\tau(u_t^k) = \nu_t(|k|)$ is C^1 with for all $m \geq 0$,

$$\frac{d}{dt}\nu_t(m) = -\frac{m}{2}\nu_t(m) - \frac{m}{2}\sum_{l=1}^{m}\nu_t(l)\nu_t(m-l), \forall t \geq 0, , \nu_0(m) = 1. \tag{11}$$

Let us set $\nu_t = \tau_{u_t}$. It follows from the above expression that as t tends respectively to 0 and $+\infty$, the distribution ν_t converges pointwise to the one of respectively 1 and a Haar unitary. In view of (10), it is also natural to introduce the following deformation of free convolution.

Definition 1.1 ([55])**.** Let $(\mathcal{A}, \tau_{\mathcal{A}})$ and $(\mathcal{B}, \tau_{\mathcal{B}})$ be two non-commutative probability spaces. Then there is a non-commutative probability space $(\mathcal{C}^{(t)}, \tau_{\mathcal{C}^{(t)}})$ such that

1. $\mathcal{A}$ and $\mathcal{B}$ can be identified with two *independent* sub-algebras of $(\mathcal{C}^{(t)}, \tau_{\mathcal{C}^{(t)}})$ with
$$\tau_{\mathcal{C}^{(t)}}(a) = \tau_{\mathcal{A}}(a) \text{ and } \tau_{\mathcal{C}^{(t)}}(b) = \tau_{\mathcal{B}}(b), \forall (a, b) \in \mathcal{A} \times \mathcal{B}.$$
2. There is a unitary element $u_t \in \mathcal{C}^{(t)}$ free with the sub-algebra of $\mathcal{C}^{(t)}$ generated by $\mathcal{A}$ and $\mathcal{B}$, such that u_t has distribution ν_t.

The *t-free convolution product* of $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ is then the joint distribution $\tau_{\mathcal{A}} \star_t \tau_{\mathcal{B}}$ of $(\mathcal{A}, u_t\mathcal{B}u_t^*)$ in the non-commutative probability space $(\mathcal{C}^{(t)}, \tau_{\mathcal{C}^{(t)}})$. It does not depend on the choice of $(\mathcal{C}^{(t)}, \tau_{\mathcal{C}^{(t)}})$ satisfying 1) and 2). The above construction was introduced more generally [12] by Voiculescu [55] in his study of free entropy and free Fisher information via the liberation process.

For any $t > 0$, two sub-algebras $\mathcal{A}$ and $\mathcal{B}$ of a same non-commutative probability space $(\mathcal{C}, \tau)$ with respective distribution $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are said to be *t-free*, if their joint distribution under τ is given by $\tau_{\mathcal{A}} \star_t \tau_{\mathcal{B}}$. It can be shown ([55, 5]) that the following limits hold pointwise,

$$\lim_{t\downarrow 0} \tau_{\mathcal{A}} \star_t \tau_{\mathcal{B}} = \tau_{\mathcal{A}} *_c \tau_{\mathcal{B}} \text{ and } \lim_{t\to+\infty} \tau_{\mathcal{A}} \star_t \tau_{\mathcal{B}} = \tau_{\mathcal{A}} \star \tau_{\mathcal{B}}.$$

[12] not necessarily with the assumption of classical independence for the initial state.

1.3.3 A matrix approximation for another interpolation from classical to free convolution

Let us present an application of Theorem 1.2. Consider a heat kernel $(p_t)_{t>0}$ on a classical compact matrix Lie group G_N endowed with the metric considered in section 3.1 and for any $T > 0$, define a probability measure setting

$$d\mu_{N,T}(A, B) = Z_T^{-1} p_T([A, B]) dA dB \tag{12}$$

on G_N^2 where $dAdB$ denotes the Haar measure on G_N^2 and $Z_T = \int_{G_N^2} p_T([A, B]) dA dB$. As the limits $\lim_{T\downarrow 0} p_T(U)dU = \delta_{\mathrm{Id}_N}$ and $\lim_{T\to\infty} p_T(U)dU = dU$ hold weakly, we can think about μ_T as a model of random matrices interpolating between commuting and non-commuting settings. In [16, Thm 2.15], we have proved that though A and B are not Haar distributed for N fixed, as $N \to \infty$, they converge individually to Haar unitaries. Moreover, we also saw that under $\mu_{N,T}$, $[A, B]$ converges in non-commutative distribution, with limit given by ν_T, a free unitary Brownian motion at time T. In view of (9), it is then natural to investigate the possible limit of the joint law, hoping for a non-trivial coupling of Haar unitaries. Note that analog models with potentials [13] have been investigated in [12]. A challenge appearing in the setting of [12] is that these general results are limited to weak coupling regimes.[14] A consequence of our work is that $\mu_{N,T}$ has a non-commutative limit for all $T > 0$, leading to an interpolation between independent and free Haar unitaries. Denote by τ_u the distribution of a Haar unitary.

Corollary 1.11. *For any $T > 0$, there is a state Φ_T on $\mathcal{A} = \mathbb{C}\langle X, X^*, Y, Y^*\rangle$, such that for any $P \in \mathcal{A}$, under $\mu_{N,T}$,*

$$\mathrm{tr}(P(A, B)) \to \Phi_T(P) \text{ in probability as } N \to \infty$$

with

$$\lim_{T\downarrow 0} \Phi_T(P) = \tau_u *_c \tau_u(P) \text{ and } \lim_{T\to+\infty} \Phi_T(P) = \tau_u \star \tau_u(P).$$

Besides, for all $T, t > 0$,

$$\Phi_T \neq \tau_u \star_t \tau_u, \tag{13}$$

while

$$\Phi_T((XYX^*Y^*)^n) = \nu_T(n) = \tau_u \star_{\frac{T}{4}} \tau_u((XYX^*Y^*)^n), \forall n \in \mathbb{Z}^*.$$

We prove in section 6 the above corollary together with a few other properties of Φ_T. Let us mention that the interpolation provided by Corollary 1.11 is not the only possible interpolation, even if we exclude the t-free convolution; for instance another interpolation was proposed in [43] using rank one Harish–Chandra–Itzykson–Zuber integrals.

1.4 Strategy of proof via Makeenko–Migdal deformations

An important property formally inferred by integration by part from (3) in [42] and rigorously proved in [36] based on the Driver–Sengupta formula, are a family of

[13] Though the class of potentials considered in [12] do not cover the heat kernel.

[14] meaning that the parameter of the potential responsible for the non-independence of A and B needs to be small enough.

equations almost characterising the function Φ_Σ when Σ is the plane. Other proofs have been given in [14, 22]. The proofs of [22] were much shorter and local, and it was possible to adapt them to all compact surfaces [21]. See also [20] for a different approach using the construction of the Yang–Mills measure using the white noise.

These equations can be described informally as follows. Consider a smooth loop ℓ with a transverse intersection at a point v. Assume that $(\ell_\varepsilon)_\varepsilon$ is a deformation of ℓ in a neighborhood of v such that the areas of the four corners adjacent to v are modified as in Figure 1. Then the Makeenko–Migdal equation at v for a master field

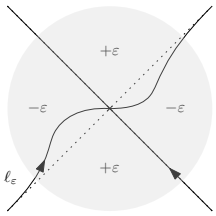

Figure 1: Makeenko–Migdal deformation near an intersection point.

Φ_Σ is given by

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Phi_\Sigma(\ell_\varepsilon) = \Phi_\Sigma(\ell_1)\Phi_\Sigma(\ell_2) \tag{14}$$

where ℓ_1, ℓ_2 are two loops obtained by de-singularising ℓ at v as on figure 2.

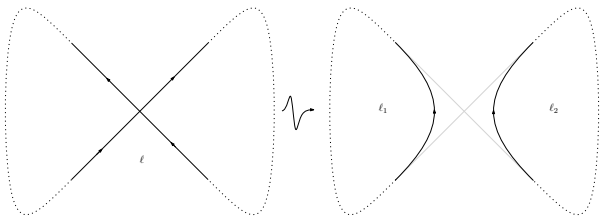

Figure 2: De-singularisation at a simple intersection point.

The works [36, 17, 29] can be understood as a study of existence and uniqueness of variants of the equation (14). Our strategy here is to extendthese results to all compact surfaces of genus $g \geq 1$.

A motivation of [36] for proving these relations was to compute explicitly the planar master field by induction on the number of intersections and to characterise it through differential equations. It was realised there that for the plane there is no uniqueness for the Makeenko–Migdal equations alone, but there is if they are completed by an additional family of equations[15]. In [17, 29], the authors are interested in a perturbation of (14) arising from finite N analogs of (14) in view of proving the convergence of Wilson loops. The same lack of uniqueness occurs but is dealt with differently, adding in some sense boundary conditions, specifying the value of the master field[16] for simple loops. With this boundary condition, both [17, 29] are

[15] associated to each face adjacent to an infinite face.
[16] or the convergence of Wilson loops

able to deduce the convergence of Wilson loops[17] on the sphere by induction on the number of intersection points. To complete the proof of Wilson loops convergence, it is then necessary to prove the convergence for boundary conditions via other means: this was done in [17] using a representation through a discrete[18] β-ensemble.

In [29], the author applied the same argument on all compact surfaces with a boundary condition given by simple loops within a disc and a uniquess or convergence result for loops within a disc. See the introduction of [16] for a more detailed discussion. In [16] we were able, using an independent argument, to prove the same result but without any boundary condition and making a relation with the planar master field.

Theorem 1.12. *Let ℓ be a loop in a compact Riemann surface Σ of genus $g \geq 1$ with area measure* vol.

1. *If ℓ is topologically trivial and included in a disc U such that* $\mathrm{vol}(U) < \mathrm{vol}(\Sigma)$, *then as N tends to infinity, under μ_{YM},*

$$W_\ell \to \tilde{\Phi}(\psi \circ \ell) \text{ in probability,}$$

   where $\tilde{\Phi}$ denotes the master field in the planar disc $\psi(U)$ where $\psi : U \to \psi(U) \subset \mathbb{R}^2$ is an area-preserving diffeomorphism.

2. *If ℓ is simple and non-contractible, then for any $n \in \mathbb{Z}^*$, as N tends to infinity,*

$$W_{\ell^n} \to 0 \text{ in probability.}$$

3. *If γ is a separating loop of Σ and ℓ does not intersect γ, then W_ℓ converges in probability towards a constant.*

A first remark is that evaluating the planar master field at lift of contractible loops to the universal cover of Σ, as in the conjecture 1.3, gives a solution to Makeenko–Migdal equations. Our main focus will therefore be to study uniqueness of the Makeenko-Migdal equations or its deformation arising for finite N.

The general strategy of this article is to use Theorem 1.12 as boundary condition to prove Proposition 1.5 and Theorem 1.6. For the torus, any non-trivial closed geodesic is whether simple or the iteration of a simple closed loop, Proposition 1.5 together with Theorem 1.6 yield Theorem 1.2. For surfaces of genus $g \geq 2$, the result of [16] do not cover delayed geodesic loops and there are then loops whose homotopy class does not include any simple loop, or any loop obtained by iterating a simple loop [7] (moreover most geodesics have intersection points).

Let us now discuss how this strategy is implemented here. When applying the argument of [17] or [29], it is difficult to prove a result better than Theorem 1.12, which, given point 1. of Theorem 1.12, makes the use of Makeenko–Migdal equations pointless. A first obstacle being for instance a loop like in figure 3, where it does not seem possible to apply Makeenko–Migdal equations at any vertex to deform the loop into a simpler loop.

[17] This argument is valid for loops with finitely many transverse intersections. An additional step which is not considered in [29] is to extend it to loops with finite length.

[18] as suggested in [29], another route here could be to relate Wilson loops for simple loops on the sphere to the Dyson Brownian bridge on the unit circle, which has been studied recently at another scale in [38].

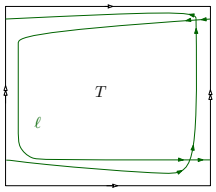

Figure 3: In this example, it is impossible to change the area around any intersection point, respecting the constraint of Makeenko–Migdal given in figure 1, without raising the number of intersection points.

To improve on [29], a first step is to understand exactly in the case of surfaces of genus $g \geq 1$, what family of deformations are allowed to use Makeenko–Migdal equations.Viewing the evaluation at a regular loop of the master field as a function of faces area, we wonder along which deformation of loops, the derivative of the master field is a linear combination of area derivatives such as the one involved in the left-hand-side of (14). This was understood first in the plane by [36]. This is achieved here for surfaces in section 2.2 with the following conclusion. When a loop has non-zero homology, then any reasonable deformation is allowed. When a loop has zero homology, then it is possible to define the winding number and algebraic area of the loop and a deformation is allowed if and only if it preserves the algebraic area.

This observation allows to consider the simpler case of loops with non-zero homology separately. In this case, it is possible to argue as follows by induction, showing at each step that the derivative along a suitable deformation is bounded by induction assumption. First, considering the lift of a loop with non-zero homology to the universal cover, by induction on the number of intersections, it is possible to reduce the problem to loops with non-zero homology such that each strand of the lift going through a fundamental region has[19] no intersection point. Then Proposition 1.5 can be proved by induction on the number of fundamental domains visited. A key remark in this case is that at each intersection point, the two loops obtained by de-singularisation have both non-zero homology and visit strictly less fundamental domains. This programme is carried out in section 4.1.

A second step is to overcome the difficulty met in Figure 3. This loop has vanishing homology. The cause of the obstruction becomes clearer thanks to the first step: it is not possible to decrease the area of the central face as it is a strict maximum of the winding number function. An idea is then to first pull and twist the loop in a face that we want to "inflate" so that the algebraic area remains preserved, as suggested on the following figure.

An apparent issue with this argument is that the number of intersections of the loops involved in the different steps may raise. It is therefore not possible a priori to apply, as in [36, 17, 29], an induction on the number of intersections to prove uniqueness. This suggests to use another form of complexity for loops and the introduction of marked loops, with a main part and a "perturbed part" allowing to keep the algebraic area constant. Despite that the number of intersection points may raise along the deformation, the loops get closer at each step to the loops belonging

[19] We shall call below these loops proper loops

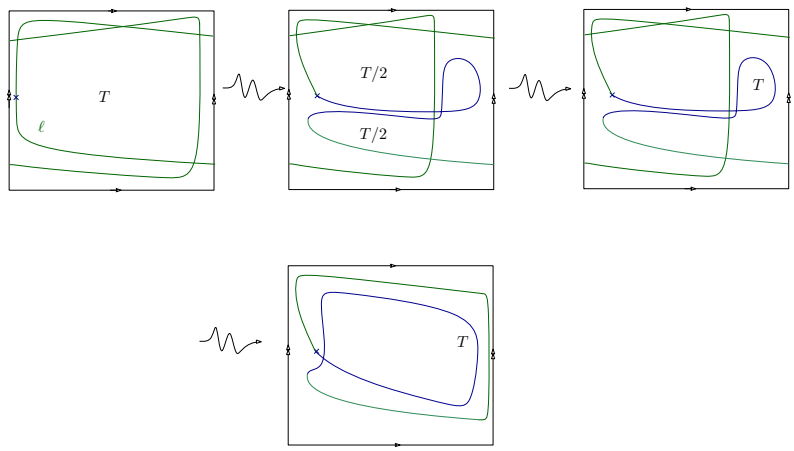

Figure 4: Discrete homotopy towards a loop included in a disc preserving the algebraic area. Faces are labeled by their area. Faces without label have area 0.

to the boundary condition. Defining a convenient notion[20] of "distance" to the loops of the boundary condition is the purpose of section 2.4. We choose a minimal class of loops obtained as "perturbation part" that we introduce in section 2.5 and call *nested loops*.

However, for any choice of complexity, if we allow the number of intersection points to grow, proving the vanishing of the the right-hand side of Makeenko–Migdal equations is not possible by direct induction. This issue also appears in the above example. When going from the second to the third step in Figure 4, it is necessary to apply the Makeenko–Migdal equation at the new intersection point of the blue curve. By area invariance, the right-hand-side of the equation (14) contains a function of the area which is identical to the initial function. To prove uniqueness, we cannot anymore apply directly an induction argument to bound this derivative. Nonetheless this case can be handled as follows. Setting the complexity as the number of crossings of the square defining the torus, all loops obtained by de-singularisation of all other points will have a complexity strictly smaller than the initial one. To conclude an induction step, it is then enough to apply Grönwall's lemma along a parametrisation of the second deformation. This idea is implemented and generalised in Lemma 4.5 and is a key step in our argument.

Using the Lemma 4.5, the main induction for all loops is then carried out in Proposition 4.6, which concludes the main part of the proof for loops with finitely many transverse intersections.

Lastly it remains to extend our convergence result to a wider family of loops. As explained in section 6, to prove corollary 1.11, it is indeed needed to consider all loops within a lattice. We then use a property of uniform continuity for loops with finitely many transverse intersections. Besides, by a more general argument introduced in [9, 17] that built on the construction of [35], it is possible to consider all loops with finite length.[21]

[20] We believe there is a lot of flexibility here in the argument. We choose here a combinatorial approach. It would be interesting to use instead a continuous functional on loops.

[21] This second step is not needed to consider projection of loops on a lattice.

2 Homology and homotopy on embedded graphs

2.1 Four equivalence relations on paths and loops on maps

We recall briefly here standard notions and define some notations of topological discretisation of a surface.

In this text,[22] a map $\mathbb{G}$ of genus g is the data of three finite sets (V, E, F) in bijection with the $0, 1$ and 2 cells of a finite CW complex $\Sigma_{\mathbb{G}}$ which is isomorphic to a connected, closed, compact surface of genus g. Such complex is sometimes called *surface complex*, see [52] for instance. We define a map with boundary as the data of a map (V, E, F) together with a proper subset B of F such that the closure of 2 cells associated to any pair of distinct elements of B do not intersect. We identify E with the set of edges of the graph with vertices V, where a pair $v, w \in V$ is adjacent whenever it forms the boundary of an element of E. We denote by E_o and F_o the set of oriented edges and faces. For any oriented edge $e \in E_o$, we write $\overline{e}$ and $\underline{e}$ for its endpoint and starting point, e^{-1} for the edge with reverse orientation. A *path* in $\mathbb{G}$ is whether a single vertex or a finite string of edges $e_1 \dots e_n$ with $n \geq 1$ such that for all $k \in \{1, \dots, n-2\}$, $\underline{e}_{k+1} = \overline{e}_k$. We say it is constant in the first case and set $|\gamma| = 0$, while in the second, we denote by $\overline{\gamma} = \overline{e}_n$ and $\underline{\gamma} = \underline{e}_1$ its starting point and endpoint and by $|\gamma| = n$ its *length*. A loop of $\mathbb{G}$ is a path γ with $\underline{\gamma} = \overline{\gamma}$. A loop $\mathfrak{l}$ is *based* at a vertex v when $\underline{\mathfrak{l}} = v$. We say it is *simple* when all vertices of $\mathfrak{l}$ occur only once $\mathfrak{l}$ but $\underline{\mathfrak{l}}$ which occurs exactly twice. We write respectively $\mathrm{P}(\mathbb{G})$ and $\mathrm{L}(\mathbb{G})$ for the set of paths and loops of $\mathbb{G}$. The respective sets of paths starting from a vertex $v \in V$ are denoted by $\mathrm{P}_v(\mathbb{G})$ and $\mathrm{L}_v(\mathbb{G})$. Whenever α and β are two paths with $\overline{\alpha} = \underline{\beta}$, $\alpha\beta$ denotes their *concatenation*, while α^{-1} is the path run in reverse direction, with the convention that $\gamma_1\alpha\gamma_2 = \alpha$ when γ_1 and γ_2 are constant paths at $\underline{\alpha}$ and $\overline{\alpha}$. We say that β is a *subpath* of $\delta \in \mathrm{P}(\mathbb{G})$ and write $\beta \prec \delta$, if there are paths α and γ with $\delta = \alpha\beta\gamma$.

Homeomorphic loops: When two maps $\mathbb{G}, \mathbb{G}'$ yields homeomorphic CW complexes, it induces a bijection between cells of same dimension. Denote by $\Phi : E \to E'$ the associated bijection between edges of $\mathbb{G}$ and $\mathbb{G}'$ and the associated bijection between $\mathrm{P}(\mathbb{G})$ and $\mathrm{P}(\mathbb{G}')$. Consider two paths α and β within maps $\mathbb{G}_\alpha$ and $\mathbb{G}_\beta$. We say that α and β are *homeomorphic* and write

$$\alpha \sim_\Sigma \beta$$

if there are maps $\mathbb{G}$ and $\mathbb{G}'$ finer than respectively $\mathbb{G}_\alpha$ and $\mathbb{G}_\beta$ such that $\mathbb{G}$ and $\mathbb{G}'$ have homeomorphic, with induced bijection $\Phi : \mathrm{P}(\mathbb{G}) \to \mathrm{P}(\mathbb{G}')$ such that

$$\Phi(\alpha) = \beta.$$

Cyclically equivalent loops: We say that two loops are *cyclically equivalent* when one can be obtained from the other by cyclically permuting its edges. By convention, two constant loops are cyclically equivalent if they have equal base-point. This defines an equivalence relation $\sim_c$ on $\mathrm{L}(\mathbb{G})$. An element of the quotient $\mathrm{L}_c(\mathbb{G}) = \mathrm{L}(\mathbb{G})/\sim_c$ is called an *unrooted loop*.

Reduced loops: A path γ' is obtained by insertion of an edge in a path γ, if $\gamma = \gamma_1\gamma_2$ and $\gamma' = \gamma_1 e e^{-1} \gamma_2$ with γ_1, γ_2 two subpaths of γ and e an edge, satisfying

[22] We shall not consider here non-orientable surfaces.

$\overline{\gamma_1} = \underline{e} = \underline{\gamma_2}$. Vice-versa, we say in this situation that γ is obtained by erasing of an edge of γ'. Two paths are said to have the same *reduction* if a finite sequence of erasures and insertions of edges transforms one into the other. This defines an equivalence relation $\sim_r$ on $\mathrm{P}(\mathbb{G})$ and we write $\mathrm{RP}(\mathbb{G}) = \mathrm{P}(\mathbb{G})/\sim_r$, $\mathrm{RP}_v(\mathbb{G}) = \mathrm{P}(\mathbb{G})/\sim_r$ and $\mathrm{RL}_v(\mathbb{G}) = \mathrm{L}_v(\mathbb{G})/\sim_r$ for any $v \in V$. The reduction of a path $\gamma \in \mathrm{P}(\mathbb{G})$ is the unique path of minimal length in its $\sim_r$-equivalence class. We say that two loops are $\sim_{r,c}$-equivalent if one can be obtained from the other by iterated cyclic permutations, insertions and erasures of edges.

Lassos and discrete homotopy: For any face $f \in F_o$, its boundary can be identified with an unrooted loop ∂f. When $r \in V$ is a vertex of ∂f, we write $\partial_r f$ for the loop in the $\sim_c$-class of ∂f with $\underline{\partial_r f} = r$. When F_* is a subset of F, a F_*-*lasso* is a loop of the form $\alpha \partial_r f \alpha^{-1}$, where f is an oriented face belonging up to orientation to F_* and $\alpha \in \mathrm{P}(\mathbb{G})$ is a path such that $r = \overline{\alpha}$ is a vertex of ∂f. When $\gamma \in \mathrm{P}(\mathbb{G})$, γ' is obtained by *lasso insertion* from γ if $\gamma = \gamma_1\gamma_2$ for some paths $\gamma_1, \gamma_2 \in \mathrm{P}(\mathbb{G})$ and $\gamma' = \gamma_1 \mathfrak{l} \gamma_2$, where $\mathfrak{l}$ is a lasso with $\overline{\gamma}_1 = \underline{\mathfrak{l}} = \underline{\gamma}_2$. Conversely, γ' is said to be obtained from γ by *lasso erasure*. We say that two paths are *discrete homotopic* if there is a finite sequence of lassos or edge erasures and insertions transforming one into the other. This defines an equivalence relation $\sim_h$ on $\mathrm{P}(\mathbb{G})$ which is also well defined on $\mathrm{RP}(\mathbb{G})$. Moreover, two paths of $\mathbb{G}$ are discrete homotopic if and only if their image in $\Sigma_{\mathbb{G}}$ are homotopic with fixed endpoints. For any $v \in V$, we denote the quotient $\mathrm{P}_v(\mathbb{G})/\sim_h$ and $\mathrm{L}_v(\mathbb{G})/\sim_h$ by $\tilde{V}_v$ and $\pi_{1,v}(\mathbb{G})$. When $F_* \subset F$, we say that two paths of $\mathbb{G}$ are F_*-*homotopic* if there is a finite sequence of F_*-lassos or edge erasures and insertions transforming one into the other. This defines an equivalence relation on $\mathrm{P}(\mathbb{G})$ denoted by $\sim_{F_*}$. When K is a closed, compact, contractible subset of $\Sigma_{\mathbb{G}}$ given by the closure of the union of images of F_*, for any pair of paths $\gamma_1, \gamma_2 \in \mathrm{P}(\mathbb{G})$ whose image in $\Sigma_{\mathbb{G}}$ is included in K and with same endpoints, $\gamma_1 \sim_{F_*} \gamma_2$.

The group of reduced loops and the fundamental group: For any vertex $v \in V$, we define a group by endowing $\mathrm{RL}_v(\mathbb{G})$ with the multiplication given by concatenation and the inverse map given by reversing the orientation of loops. The group $\pi_{1,v}(\mathbb{G})$ is the quotient of $\mathrm{RL}_v(\mathbb{G})$ by the normal subgroup generated by lassos based at v. Since two loops of $\mathbb{G}$ are discrete homotopic if and only if their image in $\Sigma_{\mathbb{G}}$ are homotopic, the group $\pi_{1,v}(\mathbb{G})$ is isomorphic to the fundamental group of the surface $\Sigma_{\mathbb{G}}$. For any group G, let us write $[a,b] = aba^{-1}b^{-1}$, $\forall a, b \in G$. Then $\pi_{1,v}(\mathbb{G})$ is isomorphic to the surface group

$$\Gamma_g = \langle x_1, y_1, \ldots, x_g, y_g | [x_1, y_1] \ldots [x_g, y_g] \rangle.$$

Lemma 2.1 ([35])**.** *For any map $\mathbb{G}$, the following assertions hold:*

1. *The group $\mathrm{RL}_v(\mathbb{G})$ is free of rank $\#E - \#V + 1 = \#F + 2g - 1$.*
2. *Assume that $g \geq 0$ and $\#F = r$. For any $v \in V$, there are lassos $(\mathfrak{l}_i, 1 \leq i \leq r)$ based at v, with faces in bijection with F, and loops $a_1, b_1, \ldots, a_g, b_g \in \mathrm{L}_v(\mathbb{G})$ such that the application*

$$\Gamma_{r,g} = \langle z_1, \ldots, z_r, x_1, y_1, \ldots, x_g, y_g | z_1 \ldots z_r = [x_1, y_1] \ldots [x_g, y_g] \rangle \to \mathrm{RL}_v(\mathbb{G})$$

that sends z_i on $\mathfrak{l}_i$ for all $1 \leq i \leq r$, x_m (resp. y_m) on a_m (resp. b_m) for all

$1 \leq m \leq g$ *is an isomorphism*[23]. *The diagram*

$$\begin{array}{cccc} 1 & \to \Gamma_{r,g} & \to \mathrm{RL}_v(\mathbb{G}) & \to 1 \\ & \downarrow & \downarrow & \\ 1 & \to \Gamma_g & \to \pi_{1,v}(\mathbb{G}) & \to 1 \end{array}$$

is then commutative.

Dual map: When $\mathbb{G} = (V, E, F)$ is a map of genus g with surface $\Sigma_{\mathbb{G}}$, $(V^*, E^*, F^*) = (F, E, V)$ can also be endowed with a map structure $\mathbb{G}^*$ such that $\Sigma_{\mathbb{G}^*} = \Sigma_{\mathbb{G}}$ while two faces of $\mathbb{G}$ are adjacent in (V^*, E^*) if and only if they share an edge.

Refining maps: When $\mathbb{G}' = (V', E', F')$ and $\mathbb{G} = (V, E, F)$ are two maps, $\mathbb{G}'$ is *finer* than $\mathbb{G}$ if (V, E) is a subgraph of (V', E') and $\Sigma_{\mathbb{G}'} = \Sigma_{\mathbb{G}}$, so that we can identify V and E with subsets of respectively V' and $\mathrm{P}(\mathbb{G})$, while any face of $\mathbb{G}$ is the union of faces of $\mathbb{G}'$.

Cutting and gluing maps: When $\mathbb{G} = (V, E, F)$ is a maps and $\mathfrak{l}$ is a simple loop of $\mathbb{G}$, with dual edges $E^*_{\mathfrak{l}}$, we say that $\mathfrak{l}$ is separating if the graph $(F, E^* \backslash E^*_{\mathfrak{l}})$ has exactly two connected components (F_1, E^*_1) and (F_2, E^*_2). Let us denote by E_1, E_2 the edges of $\mathbb{G}$ dual to E_1 and E_2 and by V_1, V_2 the vertices of $\mathbb{G}$ belonging the boundary of E_1 and E_2. Giving the new 2-cell the label $f_{\infty,1}$ and setting $\mathbb{G}_1 = (V_1, E_1, F_1 \sqcup \{f_{1,\infty}\})$. We define likewise $f_{2,\infty}$ and a map $\mathbb{G}_2 = (V_2, E_2, F_2 \sqcup \{f_{2,\infty}\})$. We say that the pair of maps with boundary $(\mathbb{G}_1, \{f_{1,\infty}\}), (\mathbb{G}_2, \{f_{2,\infty}\})$ is the *cut* of $\mathbb{G}$ along $\mathfrak{l}$. We say that the cut is *essential* if $\mathfrak{l}$ is not contractible. A cut is essential if and only if the maps $\mathbb{G}_1$ and $\mathbb{G}_2$ have genus larger or equal to 1.

2.2 Discrete homology, winding function and Makeenko–Migdal vectors

We recall here an elementary definition of discrete homology and discuss its relation to Makeenko–Migdal vectors introduced in [36, 17, 29].

Discrete homology: Consider a map $\mathbb{G} = (V, E, F)$ without double edges, with vertices V and oriented edges E and denote here by R the ring $\mathbb{Z}$ or $\mathbb{R}$. For any oriented edge $e \in E_o$, we denote by $e^{-1} \in E_o$ its reverse, $\underline{e} \in V$ its source and $\overline{e} \in V$ its target. Consider the R-module

$$\Omega^1(\mathbb{G}, R) = \{\omega : E_o \to R : \omega(e^{-1}) = -\omega(e)\}$$

of discrete one-forms. When $e \in E_o$, we write $\omega_e \in \Omega^1(\mathbb{G}, R)$ for the one-form such that $\omega_e(e) = 1, \omega_e(e^{-1}) = -1$ and $\omega_e(e') = 0$ for $e' \notin \{e, e^{-1}\}$. When $\mathfrak{l} = (e_1, \ldots, e_n)$ is a loop of $\mathbb{G}$, we set

$$\omega_{\mathfrak{l}} = \sum_{i=1}^{n} \omega_{e_i}.$$

The discrete exterior derivative is defined by

$$d : \Omega^0(\mathbb{G}, R) = R^V \to \Omega^1(\mathbb{G}, R), f \mapsto df : e \mapsto f(\overline{e}) - f(\underline{e}). \tag{15}$$

[23]denoting here abusively the $\sim_r$ class of a loop by the same symbol as the loop.

Let

$$d^* : \Omega^1(\mathbb{G}, R) \to \Omega^0(\mathbb{G}, R)$$

be the adjoint of d for the non-degenerate bilinear form on $\Omega^0(\mathbb{G}, R)$ and the one on $\Omega^1(\mathbb{G}, R)$ defined by[24]

$$\langle f, g\rangle = \sum_{v\in V} f(v)g(v) \text{ and } \langle \omega, \nu\rangle = \frac{1}{2}\sum_{e\in E} \omega(e)\nu(e),$$

for all $f, g \in \Omega^0(\mathbb{G}, R)$ and $\omega, \nu \in \Omega^1(\mathbb{G}, R)$. Denoting by $\diamondsuit_1$ the module spanned by the one-forms $\omega_{\mathfrak{l}}$ where $\mathfrak{l}$ is a loop of $\mathbb{G}$ and by $\bigstar_1$ the image of d, we have that $\diamondsuit_1 \subset \ker(d^*)$ and the orthogonal of $\diamondsuit_1$ is included in $\bigstar_1$, hence $\diamondsuit_1 = \ker(d^*)$ and the orthogonal decomposition

$$\Omega^1(\mathbb{G}, R) = \bigstar_1 \oplus \diamondsuit_1$$

into exact and co-closed one forms.

Let us define now the space of 2-forms $\Omega^2(\mathbb{G}, R)$ as the space of functions φ on F_o with

$$\varphi(f') = -\varphi(f)$$

whenever f' and f are the same 2-cell with two different orientations. Set

$$\partial : \Omega^2(\mathbb{G}, R) \to \Omega^1(\mathbb{G}, R), \varphi \mapsto \frac{1}{2}\sum_{f\in F} \varphi(f)\omega_{\partial f}.$$

Endowing $\Omega^2(\mathbb{G}, R)$ with the non-degenerate bilinear form

$$\langle \varphi, \psi\rangle = \frac{1}{2}\sum_{f\in F} \varphi(f)\psi(f),$$

the adjoint of ∂ is given by

$$d : \Omega^1(\mathbb{G}) \to \Omega^2(\mathbb{G}), \omega \mapsto \left(f \mapsto \frac{1}{2}\sum_{e\in E} \omega_{\partial f}(e)\omega(e) = \langle \omega_{\partial f}, \omega\rangle \right).$$

By construction

$$\bigstar_1^* := d^*(\Omega^2(\mathbb{G}, R)) \subset \diamondsuit_1 \text{ and } \bigstar_1 \subset \diamondsuit_1^* := \ker(d : \Omega^1 \to \Omega^2)$$

so that so that $d \circ d = 0$, $d^* \circ d^* = 0$ and

$$\Omega^2(\mathbb{G}, R) \xrightarrow{d^*} \Omega^1(\mathbb{G}, R) \xrightarrow{d^*} \Omega^0(\mathbb{G}, R)$$

is a chain complex. The vector space

$$\mathcal{H}_1 = (\bigstar_1^*)^\perp \cap \diamondsuit_1 = \ker(d : \Omega^1 \to \Omega^2) \cap \ker(d^* : \Omega^1 \to \Omega^0)$$

is isomorphic to the *first homology group*

$$H_1(d^*, R) = \ker(d^* : \Omega^1 \to \Omega^0)/d^*(\Omega^2).$$

[24] mind that fixing an orientation of edges, second expression is a sum in R

Changing ring the two constructions are compatible with

$$H_1(d^*, \mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Z}} H_1(d^*, \mathbb{Z}).$$

For any $n \geq 2$, we set

$$H_1(d^*, \mathbb{Z}_n) = \mathbb{Z}_n \otimes_{\mathbb{Z}} H_1(d^*, \mathbb{Z}).$$

Definition 2.1. When $\mathfrak{l}$ is a loop of $\mathbb{G}$, its R-homology $[\mathfrak{l}]_R$ is the image of the element $\omega_{\mathfrak{l}}$ in $H_1(d^*, R)$. For any $n \geq 2$, its $\mathbb{Z}_n$-homology $[\mathfrak{l}]_{\mathbb{Z}_n}$ is the element $1 \otimes [\mathfrak{l}]_{\mathbb{Z}} \in H_1(d^*, \mathbb{Z}_n)$.

Lemma 2.2. *Assume that $\mathbb{G}$ is embedded in an orientable surface of genus g.*

1. *$H_1(d^*, R)$ is free of rank $2g$ and there are $2g$ simple loops $\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{a}_2, \mathfrak{b}_2, \ldots, \mathfrak{a}_g, \mathfrak{b}_g$ of $\mathbb{G}$ such that $[\mathfrak{a}_1]_R, [\mathfrak{b}_1]_R, \ldots, [\mathfrak{a}_g]_R, [\mathfrak{b}_g]_R$ is a free basis of $H_1(d^*, R)$.*
2. *When $g \geq 1$ and $v \in V$, $(\mathfrak{l}_i, 1 \leq i \leq r)$ and $a_1, b_1, \ldots, a_g, b_g \in \mathrm{L}_v(\mathbb{G})$ are as in Lemma 2.1, the map*

$$\Gamma_g = \langle x_1, y_1, \ldots, x_g, y_g | [x_1, y_1] \ldots [x_g, y_g] \rangle \to H_1(d^*, \mathbb{Z})$$

$$x_m \mapsto [a_m]_{\mathbb{Z}}, y_m \mapsto [b_m]_{\mathbb{Z}}, \forall 1 \leq m \leq g$$

is a well defined onto morphism, with kernel given by the commutator group $[\Gamma_g, \Gamma_g]$.

3. *Denoting by $\tilde{\mathcal{H}}$ the R-span of $\omega_{\mathfrak{a}_1}, \ldots, \omega_{\mathfrak{b}_g}$,*

$$\tilde{\mathcal{H}} \oplus \bigstar_1^* = \Omega^1(G, R).$$

For any loop $\mathfrak{l}$ of $\mathbb{G}$, there is therefore a unique pair $(n_{\mathfrak{l}}, \tilde{h})$ with $n_{\mathfrak{l}} \in \Omega^2(\mathbb{G}, R)/R\mu_*$ and $\tilde{h} \in \tilde{\mathcal{H}}$ such that

$$\omega_{\mathfrak{l}} = d^* n_{\mathfrak{l}} + \tilde{h}_{\mathfrak{l}}.$$

We call the two-form $n_{\mathfrak{l}}$ the $\tilde{\mathcal{H}}$-winding function of $\mathfrak{l}$. When $[\mathfrak{l}]_R = 0$, it does not depend on $\tilde{\mathcal{H}}$ and we call it simply the winding function of $\mathfrak{l}$. When furthermore $R = \mathbb{R}$, we shall identify $n_{\mathfrak{l}}$ with an element of $\{\mu_*\}^{\perp}$.

Makeenko–Migdal vectors for a regular loop: Let us first define a notion of loops with transverse simple intersections.

Consider a loop of a topological map $\mathbb{G} = (V, E, F)$ which uses each non-oriented edge at most once and each vertex at most twice. We denote then by $E_{\mathfrak{l}}$ the subset of edges $e \in E$ such that $\mathfrak{l}$ runs through e or e^{-1}.

Whenever a vertex v is visited twice, the four outgoing edges at v visited by $\mathfrak{l}$ can be ordered e_1, e_2, e_3, e_4 respecting the counterclockwise, cyclic ordering of the orientation of the map, so that $\mathfrak{l}$ is cyclically equivalent to a tame loop of the form $\alpha e_1^{-1} e_3 \beta e_2^{-1} e_4 \gamma$, $\alpha e_1^{-1} e_4 \beta e_3^{-1} e_2 \gamma$, $\alpha e_1^{-1} e_4^{-1} \beta e_2^{-1} e_3 \gamma$ or $\alpha e_1^{-1} e_4 \beta e_2^{-1} e_3 \gamma$, these four cases being exclusive. See Figure 6. We say that $\mathfrak{l}$ is a *tame loop* if only the first case occurs. The set $V_{\mathfrak{l}}$ of vertices visited twice by $\mathfrak{l}$ are then called the *(transverse) intersection points* of $\mathfrak{l}$.

The Makeenko–Migdal vector at an intersection $v \in V_{\mathfrak{l}}$ of a tame loop is then the 2-form

$$\mu_v = d(\omega_{e_1}) + d(\omega_{e_3}) = -d(\omega_{e_2}) - d(\omega_{e_4}). \tag{16}$$

The vector space $\mathfrak{m}_{\mathfrak{l}}$ of Makeenko–Migdal vectors is the linear span of Makeenko–Migdal vectors $\mu_v, v \in V_{\mathfrak{l}}$ and of the 2-forms $d\omega_e$ with $e \notin E_{\mathfrak{l}}$.

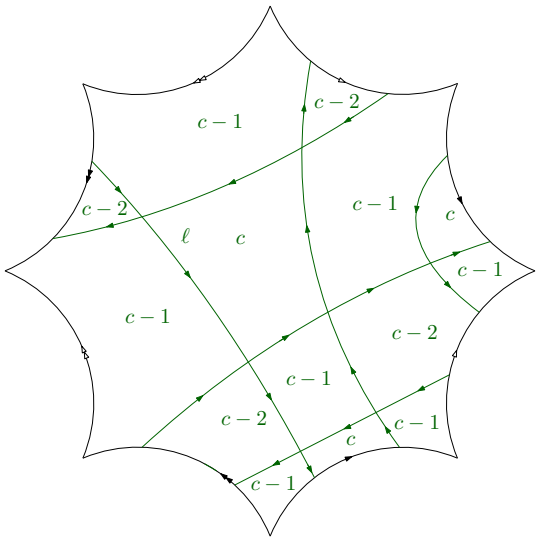

Figure 5: A representant of the winding number function with $c \in R$, for a loop ℓ of null homology, on a map of genus 2. The loop is drawn in green and the value on each positively oriented face is displayed on each 2-cell.

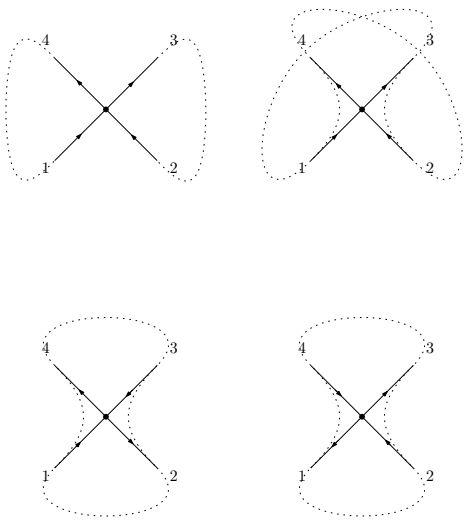

Figure 6: The four types of intersections at a vertex visited twice.

Lemma 2.3. *Let* $\mathfrak{l}$ *be a tame loop of a map* $\mathbb{G}$*. Then*

$$\mathfrak{m}_{\mathfrak{l}} = \begin{cases} \{\alpha \in \Omega^2(\mathbb{G}, \mathbb{R}) : \langle \alpha, \mu_* \rangle = 0\} & \text{if } [\mathfrak{l}] \neq 0, \\ \{\alpha \in \Omega^2(\mathbb{G}, \mathbb{R}) : \langle \alpha, \mu_* \rangle = \langle \alpha, n_{\mathfrak{l}} \rangle = 0\} & \text{if } [\mathfrak{l}] = 0. \end{cases}$$

Proof. Let us first remark that the above construction is invariant by the following appropriate subdivisions. Let us call subdivision of an oriented face f_*, the operation of adding two new vertices on its boundary and adding an edge e connecting them;

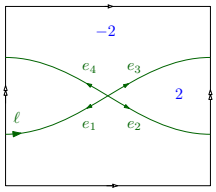

Figure 7: A tame loop in a graph with one vertex and 2 faces. The value of μ_v is displayed on each face in blue.

the new map $\mathbb{G}'$ has 2 new vertices, 1 more edge and 1 more face, with in place of f_*, two faces f_1 and f_2 with the same orientation induced from f_*, while any other face is identified with a face of $\mathbb{G}$. The map $\mathbb{G}'$ being finer than $\mathbb{G}$, $\mathfrak{l}$ can be identified with a tame loop $\mathbb{G}'$ that we denote by the same letter. Denote respectively by $\mathfrak{m}'_{\mathfrak{l}}, \mu'_*$ and, when $[\ell] = 0$, $n'_{\mathfrak{l}}$, the space of Makeenko–Migdal vectors, the constant 2-form and the winding function of $\mathfrak{l}$ on $\mathbb{G}'$. Consider the map $P : \Omega^2(\mathbb{G}') \to \Omega^2(\mathbb{G})$ with $P(\varphi)(f) = \varphi(f')$ whenever a face f of $\mathbb{G}$ is identified with a face of $\mathbb{G}'$ and $\varphi(f_1) + \varphi(f_2)$ when $f = f_*$. On the one hand, $P(d\omega_e) = 0$ and P maps all other vectors of the defining generating family of $\mathfrak{m}'_{\mathfrak{l}}$ to the generating family of $\mathfrak{m}_{\mathfrak{l}}$. Therefore, $P(\mathfrak{m}'_{\mathfrak{l}}) = \mathfrak{m}_{\mathfrak{l}}$. As $P : \{d\omega_e\}^\perp \to \Omega^2(\mathbb{G})$ is an isometry, while $d\omega_e \in \mathfrak{m}'_{\mathfrak{l}} \cap \ker(P)$, $P({\mathfrak{m}'_{\mathfrak{l}}}^\perp) = \mathfrak{m}_{\mathfrak{l}}^\perp$. On the other hand, $P(\mu'_*) = \mu_*$ and when $[\mathfrak{l}] = 0$, $P(n'_{\mathfrak{l}}) = n_{\mathfrak{l}}$. We conclude that it is enough to prove the claim for any subdivision of $\mathbb{G}$.

Therefore, we can w.l.o.g. assume that $\mathfrak{l}$ and the paths $\mathfrak{a}_1, \mathfrak{b}_1, \dots, \mathfrak{a}_g, \mathfrak{b}_g$ of Lemma 2.2 do not share any edge in common. Under this assumption, let us set $\mathscr{S} = \{\mathfrak{l}, \mathfrak{a}_1, \mathfrak{b}_1, \dots, \mathfrak{a}_g, \mathfrak{b}_g\}$, denote by $\tilde{\mathcal{H}}$ as in Lemma 2.2 and by $\mathrm{T}(\mathscr{S})$ the set of oriented edges e such that an element of $\mathscr{S}$ runs through e or e^{-1}. Let η be the permutation of the edges E such that $\eta(e^{-1}) = \eta(e)^{-1}$ for any edge $e \in E$, with $2+4g$ non-trivial cycles associated to elements of $\mathscr{S}$ forgetting the base point. More precisely, for each $\gamma \in \{\mathfrak{l}, \mathfrak{a}_1, \mathfrak{b}_1, \dots, \mathfrak{a}_g, \mathfrak{b}_g\}$ with $\gamma = e_1 \dots e_n$, $(e_1, \dots, e_n)$ and $(e_1^{-1}, \dots, e_n^{-1})$ are cycles of η, whereas $\eta(e) = e$ for any $e \notin \mathrm{T}(\mathscr{S})$. For any $\omega \in \Omega^1(\mathbb{G})$, setting

$$\eta.\omega = \omega \circ \eta^{-1}$$

defines a one-form. We claim that for any oriented edge $e \in \mathrm{T}(\mathscr{S})$,

$$\alpha_e = d\omega_e - d(\eta.\omega_e) \in \mathfrak{m}_{\mathfrak{l}}.$$

Indeed, it is non zero only when $\gamma \in \mathscr{S}$ runs through e or e^{-1}, in which case, it follows from (16) that α_e is a Makeenko–Migdal vector at respectively $\overline{e}$ or $\underline{e}$.

Let us now consider $\beta \in \mathfrak{m}_{\mathfrak{l}}^\perp \cap \{\mu_*\}^\perp$. Then

$$\langle \beta, \alpha_e \rangle = \langle \beta, (d - d \circ \eta)\omega_e \rangle = 0, \forall e \in \mathrm{T}(\mathscr{S}), \text{ whereas } \langle \beta, d\omega_e \rangle = 0, \forall e \notin \mathrm{T}(\mathscr{S}),$$

so that

$$d^*\beta = (d \circ \eta)^*(\beta) = \eta^{-1} \circ d^*\beta \text{ and } \langle d^*\beta, \omega_e \rangle = 0, \forall e \notin \mathrm{T}(\mathscr{S}).$$

It follows that

$$d^*\beta = c\omega_{\mathfrak{l}} + \sum_{i=1}^{g} (a_i \omega_{\mathfrak{a}_i} + b_i \omega_{\mathfrak{b}_i}), \text{ for some } c, a_1, b_1, \dots, a_g, b_g \in \mathbb{R}.$$

Using the decomposition $\diamondsuit_1 = \bigstar_1^* \oplus \tilde{\mathcal{H}}$, we find

$$d^*\beta = cd^*n_{\mathfrak{l}} \text{ and } c\tilde{h}_{\mathfrak{l}} + \sum_{i=1}^{g}(a_i\omega_{\mathfrak{a}_i} + b_i\omega_{\mathfrak{b}_i}) = 0.$$

Since $\beta \in \hat{\mathfrak{m}}_{\mathfrak{l}}^{\perp}$, for any edge e such that e, e^{-1} do not belong to $\mathfrak{l}$, $\langle d^*\beta, \omega_e\rangle = 0$. In particular, $a_i = b_i = 0$ for all i and $d^*\beta = c\omega_{\mathfrak{l}}$. Since $\beta \in \mu_*^{\perp}$, it follows that whether $[\mathfrak{l}] = 0$ and $\beta = cn_{\mathfrak{l}}$ or $c = 0$ and $\beta = 0$. We conclude that whether $[\mathfrak{l}] = 0$ and $\mathfrak{m}_{\mathfrak{l}}^{\perp} \cap \{\mu_*\}^{\perp} = \mathbb{R}.n_{\mathfrak{l}}$, or $[\mathfrak{l}] \neq 0$ and $\mathfrak{m}_{\mathfrak{l}}^{\perp} \cap \{\mu_*\}^{\perp} = \{0\}$. □

2.3 Regular polygon tilings of the fundamental cover, tiling-length of a tame loop and geodesic loops

To simplify the presentation, we shall work only with surfaces of genus g obtained by a standard quotient of $4g$ polygons. We fix here notations and definitions relative to the universal cover of such maps. We refer to [7] for more details.

Regular maps and regular loops: A $2g$-*bouquet map* is a map (V, E, F) with 1 vertex v, 1 face and $2g$ edges, so that for $f \in F_o$, there are $2g$ oriented edges $a_1, b_1, \ldots, a_g, b_g \in E_o$ corresponding to distinct edges, with $\partial_v f = [a_1, b_1] \ldots [a_g, b_g]$. A $2g$-bouquet map can be obtained by labelling the edges of a $4g$-polygon counter-clockwise and gluing the $i+4k$-th edge with the $i+2+4k$ edge for all $0 \leq k \leq g-1$ $i \in \{1, 2\}$. A *regular map* is a pair given by a map $\mathbb{G} = (V, E, F)$ and a $2g$-bouquet map $\mathbb{G}_g$, such that $\mathbb{G}$ is finer than $\mathbb{G}_g$. Each edge of $\mathbb{G}_g$ is uniquely decomposed as a concatenation of edges of $\mathbb{G}$. Let $\partial E \subset E$ be the set of non-oriented edges appearing in these concatenations. We then set ∂V the set of endpoints of edges of ∂E and $\overset{o}{V} = V \setminus \partial V$. When $(\mathbb{G}, \mathbb{G}_g)$ is a regular map, we refine the notion of tame loops defined in the previous section as follows. A *loop* $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$ is *regular* whenever it is tame, none of its edges belong to ∂E and $\underline{\mathfrak{l}} \in \overset{o}{V}$. In particular its intersection points satisfy $V_{\mathfrak{l}} \subset \overset{o}{V}$.

Universal cover of a regular map and its tiling: Let $(\mathbb{G}, \mathbb{G}_g)$ be a regular map with $\mathbb{G} = (V, E, F)$. When $g = 1$, consider the closed square P_1 with vertices coordinates in $\{-\frac{1}{2}, \frac{1}{2}\}$ and the tiling of $\mathbb{R}^2$ by translation of P_1 by $\mathbb{Z}^2$. When $g \geq 2$, consider a tiling of the Poincaré hyperbolic disc $\mathbb{H}$ by a family of closed regular $4g$-polygons of $\mathbb{H}$ whose sides do not intersect 0 and denote by P_1 the polygon among them enclosing 0. The group Γ_g can be identified with $\mathbb{Z}^2$ when $g = 1$ and with a subgroup of Möbius transformations that acts properly by isometry on $\mathbb{H}$ when $g \geq 2$. The group Γ_g acts freely on the set of tiles and for each $h \in \Gamma_g$, there is a unique tile P_h with $g.0$ belonging to the interior of P_h. Let us define $\tilde{\Sigma}_{\mathbb{G}}$ as $\mathbb{R}^2$ when $g = 1$ and $\mathbb{H}$ when $g \geq 2$. The quotient of $\tilde{\Sigma}_{\mathbb{G}}$ by Γ_g is homeomorphic to $\Sigma_{\mathbb{G}}$ and we denote by $p : \tilde{\Sigma}_{\mathbb{G}} \to \Sigma_{\mathbb{G}}$ the quotient mapping. There is a unique CW approximation of $\tilde{\Sigma}_{\mathbb{G}}$ such that the restriction of p to the interior of each cell in $\tilde{\Sigma}_{\mathbb{G}}$ is an homeomorphism onto the interior of a cell of $\Sigma_{\mathbb{G}}$ labeled by an element of V, E or F. We denote a labelling of the cells of this CW complex by $\tilde{\mathbb{G}} = (\tilde{V}, \tilde{E}, \tilde{F})$ and call $\tilde{\mathbb{G}}$ a *universal cover* of $\mathbb{G}$. There is a natural map from $\tilde{V}, \tilde{E}, \tilde{F}$ to respectively V, E and F that we also denote by p. As for maps, the pair $(\tilde{V}, \tilde{E})$ can be identified with a graph, and we denote by $\mathrm{P}(\tilde{\mathbb{G}})$ its set of paths. For each path $\gamma = e_1 \ldots e_n \in \mathrm{P}(\mathbb{G})$ and $\tilde{v} \in p^{-1}(v_0)$, the lift of γ from $\tilde{v}$ is the unique path $\tilde{\gamma} = (\tilde{e}_1, \ldots, \tilde{e}_n) \in \mathrm{P}(\mathbb{G})$ with $\tilde{v}_0 = \tilde{v}$ and $p(\tilde{e}_k) = e_k$

for all $1 \leq k \leq n$. For all $h \in \Gamma_g$, we denote by $D_h \subset \tilde{V}, D_h^* \subset \tilde{F}$ and $\overset{o}{D}_h \subset \tilde{V}$ the subsets of vertices and faces of $\mathbb{G}$, whose image in $\tilde{\Sigma}_{\mathbb{G}}$ is included respectively in P_h and its interior $\overset{o}{P}_h$. The projection $\overset{o}{D}$ of $\overset{o}{D}_h$ does not depend on $h \in \Gamma$. When $U \subset \tilde{F}$ and $E_c \subset \tilde{E}$, we denote by $U \setminus E_c$ the subgraph of the graph of $\tilde{\mathbb{G}}^*$ where all faces from $\tilde{F} \setminus U$ and all edges dual to E_c are removed. Let us consider the oriented graph with vertices Γ_g such that there is an edge between a and b if and only if P_a and P_b share a side. The action of Γ_g on $\mathbb{H}$ induces a free, transitive, isometric action on this graph and we denote by $|h|_{\Gamma_g}$ the *distance* between any $h \in \Gamma_g$ and 1.

For any path $\gamma \in \mathrm{P}(\mathbb{G})$ with lift $\tilde{\gamma}$,

$$|\gamma|_D = |\gamma| - \#\{0 \leq i \leq |\gamma| - 1 : \exists h \in \Gamma_g \text{ with } \{\tilde{\gamma}_i, \tilde{\gamma}_{i+1}\} \subset D_h\}$$

does not depend on the choice of starting point of $\tilde{\gamma}$, and we call it *tiling length* of γ. The *tile decomposition* of $\mathfrak{l}$ are the paths $\gamma_0, \ldots, \gamma_{|\mathfrak{l}|_D}$ of $\mathbb{G}$, with lifts $\tilde{\gamma}_0, \ldots, \tilde{\gamma}_{|\mathfrak{l}|_D}$, such that we can decompose $\tilde{\mathfrak{l}}$ as

$$\tilde{\mathfrak{l}} = \tilde{\gamma}_0 \ldots \tilde{\gamma}_{|\mathfrak{l}|_D} \tag{17}$$

where for all $0 \leq k \leq |\mathfrak{l}|_D$, there are $h_0, h_1, \ldots, h_{|\mathfrak{l}|_{|D|}} \in \Gamma_g$ such that all vertices of $\tilde{\gamma}_k$ belong to D_{h_k}, while $\mathfrak{l}_\Gamma = (h_0, \ldots, h_{|\mathfrak{l}|_D})$ is a path in Γ_g. We call $\mathfrak{l}_D = \gamma_{|\mathfrak{l}|_D}\gamma_0$ the *initial strand* of $\mathfrak{l}$. We call $\mathfrak{l}_\Gamma$ the *tiling path* of $\mathfrak{l}$ and set

$$|\mathfrak{l}|_\Gamma = |h_{|\mathfrak{l}|_D}|_\Gamma.$$

A loop $\mathfrak{l}_1$ of $(\mathbb{G}, \mathbb{G}_g)$ is called an *inner loop* of $\mathfrak{l}$ if $\mathfrak{l}_1$ is regular, included in $\overset{o}{D}$ and $\mathfrak{l}_1 \prec \mathfrak{l}$. We then say that $\mathfrak{l}_1$ is a *contractible intersection point* of $\mathfrak{l}$ and denote by $V_{c,\mathfrak{l}}$ the set of such points. A *proper loop* is a regular loop $\mathfrak{l}$ with $\#V_{c,\mathfrak{l}} = 0$.

A path $\gamma \in \mathrm{P}(\mathbb{G})$ is said to be *geodesic* when the image in $\tilde{\Sigma}_{\mathbb{G}}$ of its lift $\tilde{\gamma} \in \mathrm{P}(\tilde{\mathbb{G}})$ is the restriction of a line (resp. a circle orthogonal to the unit disc), when $g = 1$ (resp. $g \geq 2$). A *path* in Γ_g is *geodesic* if it is the tiling path of a geodesic path of a regular map.

2.4 Shortening homotopy sequence

We define here operations on regular loops allowing to decrease their tiling length.

We say that a sequence $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ is a *shortening homotopy sequence* from $\mathfrak{l}_1$ to $\mathfrak{l}_n$ if $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ are regular loops such that $|\mathfrak{l}_1|_D \geq \ldots \geq |\mathfrak{l}_n|_D$ and for all $1 \leq l < n$,

$$\#V_{c,l} = \#V_{c,l+1} = 0 \text{ or } \#V_{c,l} > \#V_{c,l+1},$$

while there is a regular map (V, E, F) with $\mathfrak{l}_l, \mathfrak{l}_{l+1} \in \mathrm{P}(\mathbb{G})$ and a subset of faces $K_l \underset{\neq}{\subset} F$, with

$$\mathfrak{l}_l \sim_{K_l} \mathfrak{l}_{l+1}.$$

The aim of this section is to prove the following.

Proposition 2.4. *For any proper loop $\mathfrak{l}$, there is a shortening homotopy sequence $\mathfrak{l}_1, \ldots, \mathfrak{l}_m$, a geodesic loop $\mathfrak{l}'$ and a path η within the same map $\mathbb{G} = (V, E, F)$ as $\mathfrak{l}_m$, such that $\mathfrak{l}_m \sim_K \eta\mathfrak{l}'\eta^{-1}$ for some $K \subset F$ with $K \neq F$. The path η can be chosen simple, within a fundamental domain and crossing $\mathfrak{l}_m$ and $\mathfrak{l}'$ only at their endpoints.*

We need two additional notions for this proof.

Bulk of a loop: Consider a regular map $(\mathbb{G}, \mathbb{G}_g)$ with $\mathbb{G} = (V, E, F)$, and a contractible loop $\mathfrak{l}$ of $\mathbb{G}$ whose lift is a loop $\tilde{\mathfrak{l}}$ of $\tilde{\mathbb{G}}$. Let E_c be the set of edges used by $\tilde{\mathfrak{l}}$ and let $O_{\mathfrak{l}}$ be the unbounded component of $\tilde{\mathbb{G}}^* \setminus E_c$. The *bulk* of $\mathfrak{l}$ is then $K_{\mathfrak{l}} = p(\tilde{F} \setminus O_{\mathfrak{l}})$. Since $E_{\mathfrak{l}}$ is connected, the image of $O_{\mathfrak{l}}$ in $\tilde{\Sigma}_{\mathbb{G}}$ is a surface with one boundary and the image $\tilde{X}_{\mathfrak{l}}$ of $\tilde{F} \setminus O_{\mathfrak{l}}$ in $\tilde{\Sigma}_{\mathbb{G}}$ is a contractible set. The image of $\mathfrak{l}$ is then contractible within $X_{\mathfrak{l}} = p(\tilde{X}_{\mathfrak{l}})$ and

$$\mathfrak{l} \sim_{K_{\mathfrak{l}}} \mathfrak{l}_*$$

where $\mathfrak{l}_*$ is the constant loop at $\underline{\mathfrak{l}}$.

Adding a rim to a regular map: When $(\mathbb{G}, \mathbb{G}_g)$ is a regular map, let us define a map $\mathbb{G}_r$ finer than $\mathbb{G}$ in the following way. First add exactly one vertex to each edge of $E \setminus \partial E$ with one endpoint in ∂V and exactly two when both endpoints belong to ∂V. Each new vertex is paired uniquely with a vertex of ∂V and their set inherit the cyclic order of vertices of ∂V. Second add an edge for each consecutive new vertices. We denote by $\mathbb{G}_r$ the new map defined thereby and call the set $\partial_r E$ of edges added in the second step the *rim* of $\mathbb{G}$. Each face of the new map, whose boundary has an edge in ∂E has exactly four adjacent edges with exactly one in ∂E_r. We denote this set of faces by $F_{r,o}$. We denote all other faces of $\mathbb{G}_r$ by F_i. For any $f \in F$, whether its boundary has no edge in ∂E and it is identified to a face of F_i, or it is the union of faces of $\mathbb{G}_r$ with exactly one in F_i, that we abusively also denote by f. For any oriented edge e of $\mathbb{G}_r$ belonging to ∂E, its right retract is the oriented edge of ∂E_r belonging to the face of $F_{r,o}$ on the right of e. When γ is a path with edges in ∂E, its *right retraction* is the concatenation of the right retraction of its edges. The left retraction is defined likewise.

We can now prove the existence of shortening homotopy sequence starting from any regular loop, using a 5 type of operations.

***Step 1**–Deleting contraction points:* Consider a regular loop $\mathfrak{l}$ with $\#V_{c,\mathfrak{l}} > 0$ of a regular map with faces set F. Any lift $\tilde{\alpha}$ of an inner loop $\alpha \prec \mathfrak{l}$ is a loop and we can consider its bulk. Denote by K the union of bulks for all inner loops. Any face bordering ∂E does not belong to K so that $K \subsetneqq F$ while $\mathfrak{l}$ is $\sim_K$-equivalent to the regular loop $\mathfrak{l}'$ with all inner loops erased.

***Step 2**–Backtrack erasure:* Assume that $\mathfrak{l}$ is a regular loop of a regular map $(\mathbb{G}, \mathbb{G}_g)$ such that there is $1 < i < |\mathfrak{l}|_D$ with $h_{i-1} = h_{i+1}$, where $(h_1, \dots, h_{|\mathfrak{l}|_D})$ is the tiling path of $\mathfrak{l}$. Consider the decomposition of $\tilde{\mathfrak{l}}$ as in (17). Let $\mathbb{G}'$ be the map $(\mathbb{G}, \mathbb{G}_g)$ with a rim added. Denote by e_i and e_o the last and first edge of γ_{i-1} and γ_{i+1}. Then $\overline{e}_i$ and $\underline{e}_o$ belong the same edge e of $\mathbb{G}_g$. Let $\beta' \in \mathbb{P}(\mathbb{G}')$ be the reduced path using only edges of the rim with $\underline{\beta'} = \overline{e}_i$ and $\overline{\beta'} = \underline{e_o}$. Denote by γ'_{i-1} and γ'_{i+1} the reduction of $\gamma_{i-1}e_i^{-1}$ and $e_o^{-1}\gamma_{i+1}$. The *backtrack erasure* for the backtracking (h_{i-1}, h_i, h_{i+1}) of $\mathfrak{l}$ is the regular loop

$$\mathfrak{l}' = \gamma_1 \dots \gamma_{i-2}\gamma'_{i-1}\beta'\gamma'_{i+1}\gamma_{i+2} \dots \gamma_{|\mathfrak{l}|_D}.$$

It can be obtained from $\mathfrak{l}$ by the following discrete homotopy. Since a lift of the paths β' and $e_i\gamma_i e_o$ starting in $D_{h_{i-1}}$ both ends in D_{h_i}, the loop of $e_i\gamma_i e_o {\beta'}^{-1}$ is contractible. Denote by K_{bt} its bulk. Then

$$\mathfrak{l} \sim_{F_{bt}} \mathfrak{l}'.$$

Since γ_i only intersect the rim of $\mathbb{G}'$ through the edge e, any face belonging to the rim whose boundary intersects two different edges of $\mathbb{G}_b$ is not in K_{bt}. It follows that $K_{bt} \neq F'$.

Step 3- *Vertex switch:* Let $\mathfrak{l}$ be a regular loop of a regular map $(\mathbb{G}, \mathbb{G}_g)$ and consider its decomposition as in (17). A *half turn* of $\mathfrak{l}$ is a sequence $\gamma_l, \dots, \gamma_{l+k}$ such that $2g \leq k \leq 4g-1$, and $D_{h_l}, D_{h_{l+1}}, \dots, D_{h_{l+k}}$ runs around a common vertex $v \in \mathbb{G}_g$. Consider such a long turn and let $\mathbb{G}' = (V', E', F')$ be the map obtained from $\mathbb{G}$ by adding twice a rim as described in the last paragraph. See Figure 8 for an example. Let e_i and e_o be respectively the last and the first edge of γ_l and γ_{l+k} in $\mathbb{G}'$. Besides, let $\beta_p \in \mathrm{P}(\mathbb{G}'^*)$ be the shortest reduced path from a face adjacent of e_i to a face adjacent of e_o that crosses first e_i and uses only faces of $F_{r,o}$ so that its lift starting from $D^*_{h_l}$ goes through $D^*_{h_l} \cup D^*_{h_{l+k}}$ and ends in $D^*_{h_{l+k}}$. Let $\beta' \in \mathrm{P}(\mathbb{G}')$ be the reduced path from $\underline{e_i}$ to $\underline{e_o}$, such that each edge of β' is bordering a face of β_p. Denote by γ'_l and γ'_{k+l} the reduction of $\gamma_l e_i^{-1}$ and $e_o^{-1}\gamma_{k+l}$. The *vertex switch* of $\mathfrak{l}$ for the considered half turn is the regular loop

$$\mathfrak{l}' = \gamma_0\gamma_1 \dots \gamma_{l-1}\gamma'_l\beta'\gamma'_{k+l}\gamma_{k+l+1} \dots \gamma_{|\mathfrak{l}|_D}.$$

It can be obtained from $\mathfrak{l}$ by the following discrete homotopy. Consider the loop $e_i\gamma_{l+1} \dots \gamma_{l+k-1}e_o{\beta'}^{-1}$. Since a lift of β' starting in D_{h_l} ends in $D_{h_{l+k}}$ it follows that $e_i\gamma_{l+1} \dots \gamma_{l+k-1}e_o{\beta'}^{-1}$ is contractible. Denote by K_{sw} its bulk.

Then,

$$e_i\gamma_{l+1} \dots \gamma_{l+k-1}e_o \sim_{K_{sw}} \beta'$$

and

$$\mathfrak{l} \sim_{K_{sw}} \gamma_0 \dots \gamma_{l-1}\gamma'_l e_i\gamma_{l+1} \dots \gamma_{l+k-1}e_o\gamma_{k+l+1} \dots \gamma_{|\mathfrak{l}|_D} \sim_{K_{sw}} \mathfrak{l}'.$$

Besides, $F_{sw} \neq F'$. Indeed, consider the map $\mathbb{G}_1$ obtained by adding a single rim to $\mathbb{G}$, so that $\mathbb{G}'$ is finer than $\mathbb{G}_1$. Let $F_{cr}, \tilde{F}_{cr}$ be the set of of faces of $\mathbb{G}_1$ neighbouring respectively $p(v)$ and v. The restriction of p to $\tilde{F}_{cr}$ is a homeomorphism onto F_{cr}. Since $k < 4g$, there is at least one face $\tilde{f}_{cr}$ of $\tilde{F}_{cr}$ that does not belong to $p^{-1}(F_{sw})$. Since β uses only faces of $F'_{r,o}$, any face of $F' \setminus F_{r,o}$ included in $f_{cr} = p(\tilde{f}_{cr})$ does not belong to K_{sw}.

The following lemma reformulates a result due to [7] relating $|\mathfrak{l}|_D$ to long turns of $\mathfrak{l}$ when $g \geq 2$.

Lemma 2.5. *Let $\mathfrak{l}$ be a regular loop of a regular map $(\mathbb{G}, \mathbb{G}_b)$. There is a finite sequence $\mathfrak{l}_1, \dots, \mathfrak{l}_n$ or regular loops obtained by vertex switches or backtracking erasures such that $\mathfrak{l}_1 = \mathfrak{l}$, $|\mathfrak{l}_1|_D \geq |\mathfrak{l}_2|_D \dots \geq |\mathfrak{l}_n|_D$ and*

$$|\mathfrak{l}_n|_D = |\mathfrak{l}|_{\Gamma_g}.$$

Proof. The case $g = 1$ is elementary. An argument goes as follows. The path in $\Gamma_1 = \mathbb{Z}^2$ associated to $\mathfrak{l}$ can be assumed up to axial symmetries that $h_{\mathfrak{l}}$ has non-negative coordinates. A backtracking of $\mathfrak{l}_\Gamma$ can be erased by a backtracking erasure of $\mathfrak{l}$. A path is geodesic if and only if all of its increments coordinates are non-negative. There are two consecutive increments with a negative followed by a positive sign. This pair corresponds to a backtrack or a half turn of $\mathfrak{l}$ if one or two coordinates change. Applying to a backtrack erasure or a switch at the half turn, the new loop has one less pair of increments with coordinates changing sign.

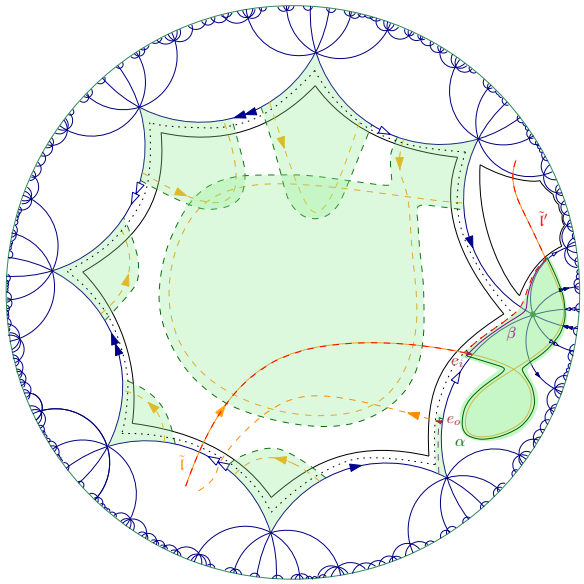

Figure 8: Discrete homotopy at a left turn of $\mathfrak{l}$ when $g = 2$ and $k = 7$. The latter vertex is shown as a green dot, contracted faces are shown in green. The second rim is displayed with dotted lines. A lift $\tilde{\mathfrak{l}}$ of the initial loop in displayed in plain orange line, while a lift $\tilde{\mathfrak{l}}'$ of the terminal loop is displayed in dashed red line.

When $g \geq 2$, the result follows from [7, Lemma 2.5]. In the setting of [7], a half turn of $\mathfrak{l}$ is a half cycle of the path in Γ_g associated to $\mathfrak{l}$. A switch at a half turn corresponds to a replacement of a half cycle with its complementary. Moreover in the setting of [7], replacing a long chain by its complementary chain can be obtained by successively replacing a long cycle by its complementary cycle. □

***Step 4**–From minimal tiling length to geodesic tiling paths:* We say that a regular path γ of a regular map has *minimal tiling length* when $|\gamma|_D = |\gamma|_\Gamma$. When $g \geq 2$, the following is a consequence of [7, Thm 2.8].

Lemma 2.6. *If $\mathfrak{l}$ is a regular loop of a regular map, there is a sequence of regular loops $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ with minimal tiling length equal to $|\mathfrak{l}_1|_\Gamma$ obtained by switches, such that $\mathfrak{l}_1 = \mathfrak{l}$ while the tiling path of $\mathfrak{l}_n$ is geodesic.*

Proof. When $g \geq 2$, in the setting of [7], our condition for a tiling path to be geodesic is equivalent for it to be a shortest path. Since switches at half turn imply switches for half cycles of the tiliing path in the setting of [7], the result follows from point (c) of [7, Thm 2.8].

When $g \geq 1$, for any regular loop with minimal tiling length, we can assume w.l.o.g. that both coordinates of the endpoint (a, b) of $\mathfrak{l}_\Gamma$ are non-negative. When γ is a path of $\mathbb{Z}^2$ with only positive coordinates, a corner swap of γ is the path obtained by replacing a sequence of the form $(x, y), (x + 1, y), (x + 1, y + 1)$ with $(x, y), (x, y+1), (x+1, y+1)$ or vice-versa. Any other path of $\mathbb{Z}^2$ with same endpoints can be obtained by corner swaps. Since a switch at a half turn of $\mathfrak{l}$ implies a corner

swap of its tiling path and that tiling paths with positive coordinates have minimal length in $\mathbb{Z}^2$, the claim follows. $\square$

***Step 5**–From geodesic tiling paths to geodesic paths:* Assume that $\mathfrak{l}$ is a regular loop such that $\mathfrak{l}_\Gamma$ is geodesic and set $n = |\mathfrak{l}|_D = |\mathfrak{l}|_\Gamma$. Let $\mathfrak{l}^{(*)}$ be a geodesic loop with $\mathfrak{l}_\Gamma^{(*)} = \mathfrak{l}_\Gamma$. Up to translation of the geodesic associated to $\mathfrak{l}^{(*)}$, we can assume that $\mathfrak{l}^{(0)}$ and $\mathfrak{l}^{(*)}$ are regular paths of a same regular map $(\mathbb{G}^{(0)}, \mathbb{G}_g^{(0)})$. Let $\eta \in P(\mathbb{G}^{(0)})$ that does not cross the boundary of the polygon, while $\underline{\eta} = \underline{\mathfrak{l}}$ and $\overline{\eta} = \underline{\mathfrak{l}}^{(*)}$, without using any edge of $\mathfrak{l}^{(*)}$. Denote by $(\mathbb{G}, \mathbb{G}_g)$ the regular map obtained by adding a rim to $(\mathbb{G}^{(0)}, \mathbb{G}_g^{(0)})$. Using the same notation as in (17), consider the tile paths decompositions of $\mathfrak{l}$ and $\mathfrak{l}^{(*)}$ adding an upper-script $(*)$ for the second decomposition. For any $0 \leq k \leq n-1$, let e_k and $e_k^{(*)}$ be the last edges of respectively γ_k and $\gamma_k^{(*)}$, denote by β_k the reduced path with edges in ∂E_r from $\underline{e_k^{(*)}}$ to $\underline{e_k}$ and define $\mathfrak{l}^{(k)}$ as the reduction of

$$\eta\gamma_0^{(*)} \dots \gamma_k^{(*)} {e_k^{(*)}}^{-1} \beta_k e_k \gamma_{k+1} \dots \gamma_n.$$

Let us set $\mathfrak{l}^{(n)} = \eta \mathfrak{l}^{(*)} \eta^{-1}$ and $\mathfrak{l}^{(-1)} = \mathfrak{l}$. Let α_k be the reduction of the loop $\eta^{-1}\gamma_0 e_0^{-1} \beta_0^{-1} e_0^{(*)} {\gamma_0^{(*)}}^{-1}$ when $k = 0$, ${e_{k-1}^{(*)}}^{-1} \beta_{k-1} e_{k-1} \gamma_k {e_k}^{-1} \beta_k^{-1} {\gamma_k^{(*)}}^{-1}$ when $0 < k < n$ and ${e_{n-1}^{(*)}}^{-1} \beta_{n-1} e_{k-1} \gamma_n \eta {\gamma_n^{(*)}}^{-1}$ when $k = n$. With this notation

$$\mathfrak{l} \sim_r \eta\alpha_0\gamma_0^*\alpha_1\gamma_1^* \dots \alpha_k\gamma_k^{(*)} {e_k^{(*)}}^{-1} \beta_k e_k \gamma_{k+1} \dots \gamma_n \text{ for } 0 \leq k < n$$

and

$$\mathfrak{l} \sim_r \eta\alpha_0\gamma_0^*\alpha_1\gamma_1^* \dots \alpha_n\gamma_n^{(*)}\eta^{-1}.$$

Therefore, for all $0 \leq k \leq n$

$$\mathfrak{l}^{(k-1)} = \alpha\beta \text{ and } \mathfrak{l}^{(k)} = \alpha\alpha_k\beta \tag{18}$$

for some paths $\alpha, \beta \in \mathrm{P}(\mathbb{G})$. For all $0 \leq k \leq n$, α_k is contractible. Denoting by K_k its associated bulk, (18) yields

$$\mathfrak{l}^{(k)} \sim_{K_k} \mathfrak{l}^{(k-1)} \text{ for all } 0 \leq k \leq n.$$

Besides, since α_k intersects at most two edges of $\mathbb{G}_g$, any face within the rim $f \in F_{r,o}$, which borders a different edge of $\mathbb{G}_g$, does not belong to K_k. Therefore $K_k \neq F$.

Proof of Proposition 2.4. For any regular loop $\mathfrak{l}$, the claimed shortening homotopy sequence can be obtained by applying first the deletion of contraction points, followed by Lemma 2.5, 2.6 and lastly a shortening homotopy sequence from a loop with geodesic tiling path to a loop conjugated to a geodesic loop. $\square$

The following lemma is not mandatory for our main argument and can be skipped at first read. Let us note that it is also possible to do the vertex switch operation (step 3) before deleting contraction points (Step 1) thanks to the following.

Lemma 2.7. *Consider $\mathfrak{l}$ is a regular loop within a regular map $(\mathbb{G}, \mathbb{G}_g)$ with faces set F. Denote respectively by K and E_{in} the union of bulks and the set of edges of its initial strand $\mathfrak{l}_D$. Then $F \setminus K$ is connected in $\mathbb{G}^* \setminus (\partial E \cup E_{in})$.*

Proof. Since $\mathfrak{l}$ is regular, any edge crossing ∂E does not belong to E_{in} and faces adjacent to ∂E belong to the same connected component X of $F \setminus K$ in $\mathbb{G}^* \setminus (\partial E \cup E_{in})$. Denote by $\tilde{X}$ the lift of X in D_1^*. Assume that $F \setminus K$ is not connected in $\mathbb{G}^* \setminus (\partial E \cup E_{in})$ and consider a connected component K' different from X. Then all edges of $\partial K'$ belong to E_{in}. Since the infinite connected component of $\tilde{\mathbb{G}}^* \setminus E_{in}$ is given by $\tilde{F} \setminus D^* \cup X$, the lift of K' in D_1^* is included in the bounded connected component of $\tilde{\mathbb{G}}^* \setminus E_{in}$, where we identified E_{in} with the set of edges of the lift of $\mathfrak{l}_D$ starting from D_1. It follows that K' is included in K, which is a contradiction. $\square$

2.5 Nested and marked loops

Nested loop: We say that a loop $\mathfrak{l}$ with n transverse intersection points is *nested* if it is regular and if there are sub-loops $\mathfrak{l}_1 \prec \mathfrak{l}_2 \prec \ldots \prec \mathfrak{l}_n$ such that $|\mathfrak{l}_1| < |\mathfrak{l}_2| < \ldots < |\mathfrak{l}_n|$. By convention, a constant loop is a nested loop. A regular loop is nested if and only if its transverse intersection points can be labeled $v_1, v_2, \ldots, v_n$ so that it visits them in the order $(v_1 v_2 \ldots v_{n-1} v_n v_n v_{n-1} \ldots v_2 v_1)$. See figure 9.

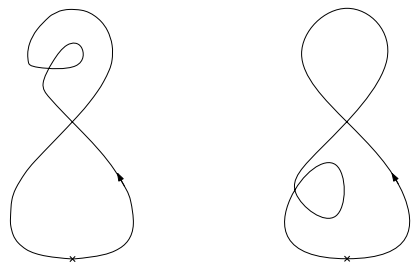

Figure 9: Left, a nested loop. Right, this is not a nested loop.

Remark. A nested loop is an example of a splittable loop as defined in [17, Section 6.5], originally introduced in [30] and called therein planar loops. Note that the right example in figure 9 is splittable but not nested.

Marked loops: A *marked loop* is a couple $(\mathfrak{l}, \gamma_{nest})$ of a regular loop and a regular path within a regular map such that

1. the first path γ_1 of the tiling decomposition of $\mathfrak{l}$ can be written $\gamma_1 = \gamma_{nest}\gamma'$, for some path γ'.
2. when γ_{nest} is non-constant, it is of the form $\alpha \mathfrak{l}_{nest} \beta$ where $\mathfrak{l}_{nest}$ is a nested loop and α, β are simple paths, such that the only intersection between α, β and $\mathfrak{l}_{nest}$ are at $\overline{\alpha}$ and $\underline{\beta}$.
3. The path γ_{nest} does not intersect transversally the two components of the initial strand $\mathfrak{l}_D$, that is $V_{\mathfrak{l}_D} \subset V_{\beta\alpha} \cup V_{\mathfrak{l}_{nest}}$.
4. The path γ_{nest} does not intersect any inner loop of $\alpha\beta$.

We call the loop $(\mathfrak{l}, \gamma_{nest})^\wedge = \gamma_0 \alpha \beta \gamma_1$ and the path $(\mathfrak{l}, \gamma_{nest})^{\wedge *} = \gamma_1 \gamma_0$ the *pruning* and the *cut* of $(\mathfrak{l}, \gamma_{nest})$. We shall often denote them abusively simply by $\mathfrak{l}^\wedge$ and $\mathfrak{l}^{\wedge *}$. We say that $(\mathfrak{l}, \gamma_{nest})$ is *non-overlapping* if γ_{nest} does not intersect the path $\mathfrak{l}^{\wedge *}$ but at its endpoints. For any marked loop $(\mathfrak{l}, \gamma_{nest})$, because of point 1., the loop $\mathfrak{l}_{nest}$ is contractible and we denote by $F_{nest} \subset F$ the associated bulk. A face of $F \setminus F_{nest}$ neighbouring F_{nest} is an *outer face.* When $\mathfrak{l}_{nest}$ is not constant, we call the simple

sub-loop of $\mathfrak{l}$ with length 1 the *central* loop of $(\mathfrak{l}, \gamma_{nest})$. Being a sub-loop of $\mathfrak{l}_{nest}$, it is contractible, faces belonging to its bulk are called *central*. A *moving edge* is an edge e of $\mathfrak{l}$ with the following property:

- When $\mathfrak{l}_{nest}$ is constant, e is any edge of γ_{nest}.
- Otherwise, e bounds a central face of $\mathfrak{l}_{nest}$.

When γ_{nest} is constant, we say that $\underline{\gamma}_{nest}$ is a moving vertex.

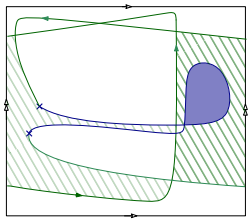

Figure 10: A marked loop. Its nested part is drawn in blue. There are exactly one central face coloured in blue and one outer face filled with dashed green lines.

The following is then a simple variation of Proposition 2.4.

Lemma 2.8. *For any marked loop $(\mathfrak{l}, \gamma_{nest})$ with $\mathfrak{l}^\wedge$ proper, there is a shortening homotopy sequence $\mathfrak{l}_1, \dots, \mathfrak{l}_m$ such that*

1. $\mathfrak{l}_1 \sim_c \mathfrak{l}$,
2. *For all $1 \leq k \leq m$, there is a nested sub-path $\gamma_{nest,k}$ of $\mathfrak{l}_k$, such that $(\mathfrak{l}_k, \gamma_{nest,k})$ is a marked loop and $\mathfrak{l}_k^\wedge$ is proper for $k \geq 2$.*
3. *There are proper faces subsets $K_1, \dots, K_m$, such that $\mathfrak{l}_k^\wedge \sim_{K_k} \mathfrak{l}_{k+1}^\wedge$ for all $1 \leq k < m$.*
4. *There is a marked loop $(\mathfrak{l}', \gamma'_{nest})$ such that $\mathfrak{l}_m \sim_\Sigma \mathfrak{l}'$ and $\mathfrak{l}'^\wedge$ is geodesic.*

2.6 Pull and twist moves

We introduce here two operations on loops in order to later modify shortening homotopy sequences to satisfy the constraint imposed by Makeenko–Migdal equations, namely to keep constant the algebraic area of loops introduced in section 2.2. This type of operation shall be required only when considering loops with vanishing homology.

Pull move: Consider a non-constant marked loop $(\mathfrak{l}, \gamma_{nest})$ in a regular map $(\mathbb{G}, \mathbb{G}_g)$ with $\mathfrak{l}^\wedge$ proper and a simple path $\gamma^* = (f_1, \dots, f_m)$ in the dual $\mathbb{G}^*$ that does not cross ∂E, such that the first edge of $\underline{\gamma}^*$ crosses a moving edge e. Let us define inductively a new map $\mathbb{G}'$ finer than $\mathbb{G}$, a new marked loop $(\mathfrak{l}', \gamma'_{nest})$, as well as a subset F_{stem} of faces of $\mathbb{G}'$. An example of the result in displayed in Figure 11. Let us first set $F_{stem} = \emptyset$. Let k be the largest $k < m$ such that (f_k, f_{k+1}) crosses an edge of $\mathfrak{l}_D$, setting $k = 1$ when γ^* does not cross any edge of γ_1.

1. Add first two new vertices to all edges crossed by α. Denote $e = e_0 e_1 e_2$ the edge decomposition of e in the new map.

2. Cut all faces of α but $\overline{\alpha}$ into three faces adding two non-crossing edges such that endpoints of a new edge do not belong to the same initial edge. Add to F_{stem} all new faces bounded by 2 new edges.
3. Cut the face $\overline{\alpha}$ into two faces, adding an edge connecting the two new vertices on $\partial\overline{\alpha}^*$. Add to F_{stem} the new face included in $\overline{\alpha}$ whose boundary has only two edges.
4. Denote by γ the simple path using only edges added in step 2 and 3 such that $\underline{\gamma} = \underline{e_1}$ and $\overline{\gamma} = \overline{e_1}$. Transform $\mathfrak{l}$ and γ_{nest} replacing the occurrence of the edge $\underline{e}$ by γ.
5. When $k = 1$ stop the procedure. Otherwise, repeat this operation for the path $\alpha' = (f_{k'}, \ldots, f_m)$, where k' is the largest $k' < k$ such that (f_k, f_{k+1}) crosses an edge of γ_1, setting $k' = 1$ when this set is empty.

The last marked loop produced in step 4. is called the *pull of* $(\mathfrak{l}, \gamma_{nest})$ *along* γ^*.

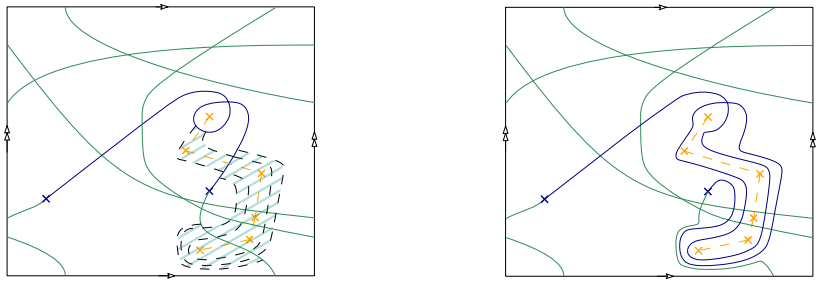

Figure 11: Left: A marked loop with the nested part drawn in blue. New edges of the modified regular map are drawn with dashed lines. The union of faces of F_{stem} is stroke with dashed lines. Right: Pull of the left marked loop along the path of the dual drawn in orange. The base point of the marked loops is displayed as a green cross.

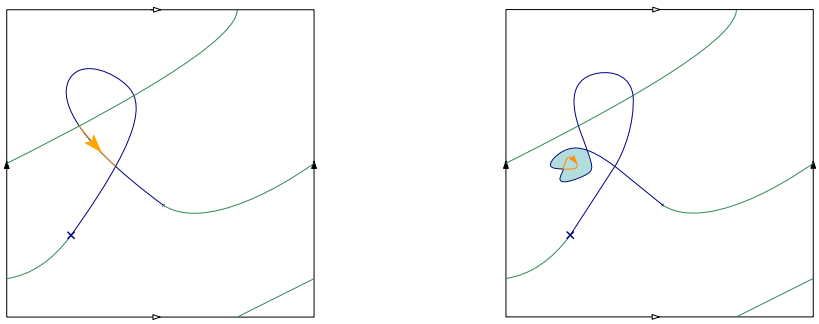

Figure 12: Left: A marked loop with the nested part drawn in blue. The chosen moving edge is drawn in orange. Right: n-twist of the left marked loop, with $n = -2$ and the chosen moving edge. The new moving edge is displayed in orange.

Twist move: Consider a marked loop $(\mathfrak{l}, \gamma_{nest})$. If γ_{nest} is constant, let us refine the map of $\mathbb{G}$ by adding a vertex to an edge e_r of $\mathfrak{l}_D$ that does belong to an inner loop. We reroot then $\mathfrak{l}$ at $\underline{e_r}$ and redefine γ_{nest} as the outgoing edge of $\mathfrak{l}_D$ at $\underline{e_r}$. Let

us now consider a marked loop with a moving edge e. Let us refine a regular and marked loop as follows. Add a vertex to e and cut the face left of e into two faces, adding an oriented edge e' with both endpoints equal to the new vertex, such that e' is the boundary a positively oriented face. The initial moving edge reads $e = e_1e_2$ in the new map. The *left twist* of $(\mathfrak{l}, \gamma_{nest})$ is the marked loop obtained by replacing the occurrence of e by $e_1e'e_e$ in both $\mathfrak{l}$ and γ_{nest}. The new marked loop has then e' has unique moving edge. We denote by F_{tw} the face bounded by e'. The right twist of $(\mathfrak{l}, \gamma_{nest})$ is defined similarly considering the right face and a negative orientation. When n is respectively positive or negative, the n-twist of a marked loop is obtained by applying respectively n left twists or $-n$ right-twists and denote then by F_{tw} the $|n|$ faces of the new map bounded solely by newly added edges. The set F_{tw} can be alternatively characterised as the smallest subset of faces of the new map such that the n-twist is $\sim_{F_{tw}}$-equivalent to $\mathfrak{l}$.

2.7 Vertex desingularisation

Consider a regular map $\mathbb{G}$. Assume that $\mathfrak{l}$ is a regular loop and $v \in V_{\mathfrak{l}}$ is an intersection point. We denote by $\mathfrak{l}_1$ and $\mathfrak{l}_2$ the two sub-loops of $\mathfrak{l}$ based at v such that $\mathfrak{l} \sim_c \mathfrak{l}_1\mathfrak{l}_2$. We then set

$$\delta_v \mathfrak{l} = \mathfrak{l}_1 \otimes \mathfrak{l}_2 \in \mathbb{C}[\mathrm{L}_c(\mathbb{G})]^{\otimes 2},$$

with the convention that $\mathfrak{l}_1$ is left of $\mathfrak{l}_2$ at v as displayed on Figure 2. By definition of Makeenko-Migdal vectors given in section 2.2, there are[25] linear forms $(\alpha_v)_{v\in V_{\mathfrak{l}}}$ and $(\beta_e)_{e\in E\setminus E_{\mathfrak{l}}}$ on $\mathfrak{m}_{\mathfrak{l}}$ such that

$$X = \sum_{v\in V_{\mathfrak{l}}} \alpha_v(X)\mu_v + \sum_{e\in E\setminus E_{\mathfrak{l}}} \beta_e(X) d\omega_e, \forall X \in \mathfrak{m}_{\mathfrak{l}}.$$

We then set

$$\delta_X \mathfrak{l} = \sum_{v\in V_{\mathfrak{l}}} \alpha_v(X)\delta_v \mathfrak{l}.$$

Let us compare how this operation combines with the different functions and moves on loops defined above.

Inner loops and tiling length: When $v \in V_{c,\mathfrak{l}}$ is a contractible loop, one them say $\mathfrak{l}_2$ is an inner loop of $\mathfrak{l}$, while $\mathfrak{l}_1$ is a regular loop with

$$\#V_{c,\mathfrak{l}_1} < \#V_{c,\mathfrak{l}}.$$

Otherwise, both $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are regular loops both crossing ∂E at least twice so that $|\mathfrak{l}_1|_D, |\mathfrak{l}_2|_D > 0$. Moreover

$$|\mathfrak{l}_1|_D + |\mathfrak{l}_2|_D = |\mathfrak{l}|_D$$

since both count the number of edges of ∂E crossed by $\mathfrak{l}$. Therefore,

$$|\mathfrak{l}_1|_D, |\mathfrak{l}_2|_D < |\mathfrak{l}|_D. \tag{19}$$

Homology: Since $\omega_{\mathfrak{l}} = \omega_{\mathfrak{l}_1} + \omega_{\mathfrak{l}_2}$, $[\mathfrak{l}] = [\mathfrak{l}_1] + [\mathfrak{l}_2]$. In particular, if $[\mathfrak{l}] \neq 0$, $[\mathfrak{l}_1] \neq 0$ or $[\mathfrak{l}_2] \neq 0$.

[25] We fix them arbitrarily, for instance using the pseudo-inverse of the Gram matrix of the spanning family $(\alpha_v)_{v\in V_{\mathfrak{l}}}$ and $(\beta_e)_{e\in E\setminus E_{\mathfrak{l}}}$.

Pull and twist moves: Consider a marked loop $(\mathfrak{l}^{(0)}, \gamma^{(0)}_{nest})$. Assume that $(\mathfrak{l}, \gamma_{nest})$ is a n-twist of a pull move of $(\mathfrak{l}^{(0)}, \gamma^{0}_{nest})$ and $v \in V_{\mathfrak{l}}$ be an intersection point of the new loop $\mathfrak{l}$. Note that the tiling decomposition of $\mathfrak{l}$ is identical to the one of $\mathfrak{l}^{(0)}$ but with the first part modified so that $|\mathfrak{l}|_D = |\mathfrak{l}^{(0)}|_D$. Also, by construction $\mathfrak{l}_{nest}$ does not intersect the root strand γ_D and $\mathfrak{l}^{\wedge}$ has as many contraction intersection points as ${\mathfrak{l}^{(0)}}^{\wedge}$. We deduce the following.

If $v \notin V_{c,\mathfrak{l}}$, then by (19),

$$|\mathfrak{l}_1|_D, |\mathfrak{l}_2|_D < |\mathfrak{l}|_D = |\mathfrak{l}^{(0)}|_D.$$

Since $\mathfrak{l}_{nest}$ is an inner loop of $\mathfrak{l}$, $V_{\mathfrak{l}_{nest}} \subset V_{c,\mathfrak{l}}$. If $v \in V_{c,\mathfrak{l}} \setminus V_{\mathfrak{l}_{nest}}$, $\mathfrak{l}_{nest}$ is a sub-loop of $\mathfrak{l}_1$ or $\mathfrak{l}_2$, say $\mathfrak{l}_1$. Then $(\mathfrak{l}_1, \mathfrak{l}_{nest})$ is a marked loop and

$$\#V_{c,\mathfrak{l}_1^{\wedge}}, \#V_{c,\mathfrak{l}_2} < \#V_{c,\mathfrak{l}^{\wedge}} = \#V_{c,{\mathfrak{l}^{(0)}}^{\wedge}}.$$

If $v \in V_{\mathfrak{l}_{nest}}$, $\mathfrak{l}_1$ or $\mathfrak{l}_2$, say $\mathfrak{l}_2$, is a sub-loop of $\mathfrak{l}_{nest}$. Then

$$\#V_{c,\mathfrak{l}_1^{\wedge}} = \#V_{c,\mathfrak{l}^{\wedge}} = \#V_{c,\mathfrak{l}^{\wedge(0)}} \text{ and } |\mathfrak{l}_1|_D = |\mathfrak{l}|_D = |\mathfrak{l}^{(0)}|.$$

These last identities prevents the use of a direct induction argument in section 4.2, but allow the one of a Grönwall estimate in Proposition 4.6.

3 Yang–Mills measure and Makeenko–Migdal equations

3.1 Metric and heat kernel on classical groups

We recall here briefly the definition and main properties of the heat kernel on classical groups that will be needed to define the discrete Yang–Mills measure. These results are quite standard, and can also be found for instance in [36, Section 1]. In this text, for any $N \geq 1$, we denote by $\mathbf{CG}_N$ the family of classical compact matrix groups $\mathrm{U}(N), \mathrm{SU}(N), \mathrm{SO}(N)$ and $\mathrm{Sp}(N)$, following the same conventions as in section 2.1.2 of [16].

When G denotes a compact Lie group, its Lie algebra $\mathfrak{g}$ is endowed with an invariant inner product $\langle \cdot, \cdot \rangle$. Setting

$$\mathcal{L}_X f(g) = \frac{d}{dt}\Big|_{t=0} f(ge^{tX}),\ \forall f \in C^{\infty}(G) \text{ and } g \in G,$$

the Laplacian associated to $\langle \cdot, \cdot \rangle$ is the operator defined by

$$\Delta_G f = \sum_{1 \leq i \leq d} \mathcal{L}_{X_i} \circ \mathcal{L}_{X_i}(f), \forall f \in C^{\infty}(G),$$

where $(X_i)_{1 \leq i \leq d}$ is an arbitrary orthonormal basis.

Definition 3.1. The *heat kernel* on G is the solution $p : (0, \infty) \times G \to \mathbb{R}_+, (t, g) \mapsto p_t(g)$ of the heat equation, with $p_t \in C^{\infty}(G)$ for all $t > 0$ and

$$\begin{cases} \partial_t p_t(g) & = \Delta_G p_t(g),\ \forall g \in G,\ \forall t > 0, \\ \lim_{t \downarrow 0} p_t(g) dg & = \delta_{I_N}, \end{cases} \tag{20}$$

where the convergence in the second line holds weakly.

It defines a semigroup for the convolution product, that is

$$p_t * p_s = p_{t+s}, \forall t, s > 0. \tag{21}$$

It inherits the following properties from the conjugation invariance of the scalar product: for all $g, h \in G$ and $t > 0$,

$$p_t(hgh^{-1}) = p_t(g) \tag{22}$$

and

$$p_t(g^{-1}) = p_t(g). \tag{23}$$

When $G_N \in \mathbf{CG}_N$, we fix an invariant inner product $\langle \cdot, \cdot \rangle$ as in (1) of [16, Section 2.1.2].

3.2 Area weighted maps, Yang–Mills measure and area continuity

We recall here a definition of the discrete and continuous Yang–Mills measure in two dimensions on arbitrary surfaces, with a focus on the former.

Area vectors and area weighted maps: When $\mathbb{G} = (V, E, F)$ is a topological map, an area vector is a function $a : F \to \mathbb{R}_+$. We say that $(\mathbb{G}, a)$ is an *area weighted map* with volume $\sum_{f \in F} a_f$. When K is a subset of faces of $\mathbb{G}$ we then write $a(K) = \sum_{f \in K} a(f)$ its volume. When $m = (\mathbb{G}, a)$ and $m' = (\mathbb{G}', a')$ are area weighted maps with faces set F and F', m' is finer than m if $\mathbb{G}'$ is finer than $\mathbb{G}$ and $a_f = \sum_{f' \in F' : f' \subset F} a'_{f'}$. When $T > 0$, we denote by

$$\Delta_{\mathbb{G}}(T) = \{a : F \to \mathbb{R}_+ : \sum_{f \in F} a_f = T\}$$

the closed simplex of area vectors of fixed volume T and its interior by

$$\Delta^o_{\mathbb{G}}(T) = \{a \in \Delta_{\mathbb{G}}(T) : a(f) > 0, \forall f \in F\}.$$

Its faces are given as follows. For any subset $K \subsetneqq F$, we set

$$\Delta_{K,\mathbb{G}}(T) = \{a \in \Delta_{\mathbb{G}}(T) : a(K) = 0\}$$

and

$$\Delta^o_{K,\mathbb{G}}(T) = \{a \in \Delta_{K,\mathbb{G}}(T) : a(f) > 0, \forall f \in F \setminus K\}.$$

When $(\mathbb{G}, B)$ is a map with boundary faces B, we set

$$\Delta_{\mathbb{G},B}(T) = \{a : F \setminus B \to \mathbb{R}_+ : \sum_{f \in F \setminus B} a_f = T\}$$

and

$$\Delta^o_{\mathbb{G},B}(T) = \{a \in \Delta_{\mathbb{G},B}(T) : a(f) > 0, \forall f \in F\}.$$

When $\mathbb{G}' = (V', E')$ is finer than $\mathbb{G}$, any face F of $\mathbb{G}$ can be identified with a subset of faces of $\mathbb{G}'$, and for any $a \in \Delta_{\mathbb{G}'}(T)$, we denote $r_{\mathbb{G}}(a) \in \Delta_{\mathbb{G}}(T)$ the associated area vector of $\mathbb{G}$. We then say that the area weighted map $(\mathbb{G}', a)$ is finer than $(\mathbb{G}, r_{\mathbb{G}}(a))$.

Multiplicative functions and Wilson loops: Given a map $\mathbb{G} = (V, E, F)$ and a compact group G, we say that a function $h : \mathrm{P}(\mathbb{G}) \to G$ is *multiplicative* if for any pair of paths γ_1, γ_2 with $\overline{\gamma}_1 = \underline{\gamma}_2$,

$$h_{\gamma_1\gamma_2} = h_{\gamma_2} h_{\gamma_1}. \tag{24}$$

We denote their set by $\mathcal{M}(\mathrm{P}(\mathbb{G}), G)$. Endowing it with pointwise multiplication, it is a compact group and fixing an orientation of the edges, the evaluation on these edges defines an isomorphism

$$\mathcal{M}(\mathrm{P}(\mathbb{G}), G) \simeq G^E.$$

The Haar measure on $\mathcal{M}(\mathrm{P}(\mathbb{G}), G)$ can be identified via this isomorphism to the tensor product of the Haar measure on G, we denote it simply by dh.

When $\mathbb{G}'$ is a map finer than $\mathbb{G}$, the restriction from $\mathrm{P}(\mathbb{G}')$ to $\mathrm{P}(\mathbb{G})$ defines a map

$$\mathcal{R}_{\mathbb{G}}^{\mathbb{G}'} : \mathcal{M}(\mathrm{P}(\mathbb{G}'), G) \to \mathcal{M}(\mathrm{P}(\mathbb{G}), G).$$

A *Wilson loop* is a function of the form

$$\begin{array}{ll} \mathcal{M}(\mathrm{P}(\mathbb{G}), G) & \longrightarrow \mathbb{C} \\ h & \longmapsto \chi(h_{\mathfrak{l}}) \end{array}$$

where $\chi : G \to \mathbb{C}$ is a function invariant by conjugation and $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$. By centrality, the value $\chi(h_{\mathfrak{l}})$ depends on $\mathfrak{l}$ only through its $\sim_c$-equivalence class l and we denote it by $\chi(h_l)$. When $G \in \mathbf{CG}_N$, for any loop $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$, we shall focus on the Wilson loop $W_{\mathfrak{l}}$ obtained considering as central function

$$\chi = \mathrm{tr}_N,$$

where $\mathrm{tr}_N = d_N^{-1}\mathrm{Tr}$ is the standard trace Tr in the natural matrix representation normalised by the size d_N of the matrix, that is $2N$ in the symplectic case and N otherwise.

Discrete Yang–Mills measure, non-singular case on closed surfaces: When $T > 0$, $\mathbb{G}$ is a map with boundary faces B and $a \in \Delta^o_{\mathbb{G},B}(T)$, the Yang–Mills measure is the probability measure $\mathrm{YM}_{\mathbb{G},B,a}$ on the compact group $\mathcal{M}(\mathrm{P}(\mathbb{G}), G)$ with density

$$Z^{-1}_{\mathbb{G},B,a} \prod_{f \in F \setminus B} p_{a_f}(h_{\partial f})$$

with respect to the Haar measure on $\mathcal{M}(\mathrm{P}(\mathbb{G}), G)$, where $Z_{\mathbb{G},B,a} = 1$ if $B \neq \emptyset$ and

$$Z_{\mathbb{G},a} = \int_{\mathcal{M}(\mathrm{P}(\mathbb{G}),G)} \prod_{f \in F} p_{a_f}(h_{\partial f}) dh$$

otherwise. In the above formula, ∂f is the boundary of the face for some arbitrary choice of root and orientation. This does not change the value of $p_{a_f}(h_{\partial f})$ thanks to (22) and (23). The fact that this density defines a probability measure when $B \neq \emptyset$ follows for instance from Lemma 3.2 below. We denote $\mathrm{YM}_{\mathbb{G},\emptyset,a}$ simply by $\mathrm{YM}_{\mathbb{G},a}$.

Lemma 3.1. *1. For any $a \in \Delta^o_{\mathbb{G}}(T)$, the constant $Z_{\mathbb{G},a}$ depends only on T and the genus g of $\mathbb{G}$, we denote it by $Z_{g,T}$.*

2. *When $m' = (\mathbb{G}', a'), m = (\mathbb{G}, a)$ are two area weighted maps with m' finer than m and $a' \in \Delta^o_{\mathbb{G}'}(T)$, then*

$$\mathcal{R}^{\mathbb{G}'}_{\mathbb{G}\ *}(\mathrm{YM}_{\mathbb{G}',a'}) = \mathrm{YM}_{\mathbb{G},a}.$$

Uniform continuity and compatibility: The Yang–Mills measure is also well defined on the faces on the simplex of area vectors. For any $r, g \geq 1$ let us consider the set $\mathrm{Hom}(\Gamma_{g,r}, G)$ of group morphisms. When endowed with point-wise multiplication it is a compact group and thanks to the presentation of Lemma 2.1,

$$\mathrm{Hom}(\Gamma_{g,r}, G) \simeq G^{r+2g-1}.$$

Moreover, this presentation allows to write the following integration formula.

Lemma 3.2 ([35]). *Assume that $(\mathbb{G}, a)$ is an area weighted map with r faces and that $(\mathfrak{l}_i, 1 \leq i \leq r)$ and $a_1, b_1, \ldots, a_g, b_g$ are as in Lemma 2.1. For any $1 \leq i \leq r$, denote by a_i the area of the face of $\mathfrak{l}_i$. Then for any continuous function $\chi : G^{2g+r} \to \mathbb{C}$ and any $a \in \Delta^o_{\mathbb{G}}(T)$ and $1 \leq k \leq r$,*

$$\begin{aligned}&\mathbb{E}_{\mathrm{YM}_{\mathbb{G},a}}(\chi(h_{\mathfrak{l}_1}, \ldots, h_{\mathfrak{l}_r}, h_{a_1}, \ldots, h_{b_g}))\\ &\quad = Z^{-1}_{g,T} \int_{G^{2g+r-1}} \chi(z_1, \ldots, z_r, x_1, \ldots, y_g) p_{a_k}(z_k) \prod_{i=1, i\neq k}^{r} p_{a_i}(z_i) dz_i \prod_{l=1}^{g} dx_l dy_l,\end{aligned}$$

where we set $z_k = (z_1 \ldots z_{k-1})^{-1}[a_1, b_1] \ldots [a_g, b_g](z_{k+1} \ldots z_r)^{-1}$. When B is a non-empty subset of faces of $\mathbb{G}$ and lassos with faces in its complement have labels $i_1, \ldots, i_p$,

$$\begin{aligned}&\mathbb{E}_{\mathrm{YM}_{\mathbb{G},B,a}}(\chi(h_{\mathfrak{l}_{i_1}}, \ldots, h_{\mathfrak{l}_{i_p}}, h_{a_1}, \ldots, h_{b_g}))\\ &\quad = \int_{G^{2g+p}} \chi(z_1, \ldots, z_p, x_1, \ldots, y_g) \prod_{i=1}^{p} p_{a_i}(z_i) dz_i \prod_{l=1}^{g} dx_l dy_l.\end{aligned}$$

The above expression yields the following continuity in the area parameter. For any vertex v of a map $\mathbb{G}$, the restriction of a multiplicative function to loops based at v depends only on the $\sim_r$-class of a loop and the restriction operation defines a map $\mathcal{R}_v : \mathcal{M}(\mathrm{P}(\mathbb{G}), G) \to \mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}, G))$. For any $a \in \Delta^o_{\mathbb{G}}(T)$, we set $\mathrm{YM}_{a,\mathbb{G},v} = \mathcal{R}_{v*}(\mathrm{YM}_{\mathbb{G},a})$.

Lemma 3.3. *The family of measures $(\mathrm{YM}_{a,\mathbb{G},v}, a \in \Delta^o_{\mathbb{G}}(T))$ on $\mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}), G)$ has a weakly continuous extension to $\Delta_{\mathbb{G}}(T)$. It has the following properties.*

1. *Consider $K \subset F$ with $K \neq F$, let $S \subset \{1, \ldots, r\}$ be the labels of the lassos with meander in $F \setminus K$ and set $s = \#S$. Then for any $a \in \Delta^o_K(T)$, any continuous function $\chi : G^{2g+r} \to \mathbb{C}$ and $k \in S$,*

$$\begin{aligned}&\mathbb{E}_{\mathrm{YM}_{\mathbb{G},a,v}}(\chi(h_{\mathfrak{l}_1}, \ldots, h_{\mathfrak{l}_r}, h_{a_1}, \ldots, h_{b_g}))\\ &\quad = \frac{1}{Z_{g,T}} \int_{G^{2g+r-1}} \chi(z_1, \ldots, z_r, x_1, \ldots, y_g) p_{a_k}(z_k) \prod_{i\in S, i\neq k} p_{a_i}(z_i) dz_i \prod_{l=1}^{g} dx_l dy_l,\end{aligned}$$

where we set $z_k = (z_1 \ldots z_{k-1})^{-1}[a_1, b_1] \ldots [a_g, b_g](z_{k+1} \ldots z_r)^{-1}$ and $z_i = 1$ for all $i \notin S$.

2. *Consider a weighted map* $(\mathbb{G}', a')$ *finer than* $(\mathbb{G}, a)$ *and denote the restriction map* $\mathcal{R}_{\mathbb{G}}^{\mathbb{G}'} : \mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}', G)) \to \mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}, G))$. *Then,*

$$\mathcal{R}_{\mathbb{G}}^{\mathbb{G}'}{}_*(\mathrm{YM}_{\mathbb{G}',a',v}) = \mathrm{YM}_{\mathbb{G},a,v}.$$

3. *Consider* $K \subset F$ *with* $K \neq F$ *and* $a \in \Delta_K(T)$. *Then for any loops* $\mathfrak{l}, \mathfrak{l}' \in \mathrm{RL}_v(\mathbb{G})$ *with* $\mathfrak{l} \sim_K \mathfrak{l}'$, $h_{\mathfrak{l}}$ *and* $h_{\mathfrak{l}'}$ *have same law under* $\mathrm{YM}_{\mathbb{G},a,v}$.

Continuous Yang–Mills measure: Thanks to the invariance by subdivision of the discrete Yang–Mills measure, given a Riemannian metric it is possible to take the projective limit of measures defined on graphs embedded in Σ whose edges are piecewise geodesic. It allows to define a multiplicative random process $(H_\gamma)_\gamma$ indexed by all piecewise geodesic paths, whose marginals are given by the discrete Yang-Mills measure.

This was done in [35], where the author is furthermore able to show a weak convergence result allowing to define uniquely the distribution of a multiplicative function $(H_\gamma)_{\mathrm{P}(\Sigma)}$ indexed by all path of finite length. Let us recall this result.

Denote by $\mathrm{P}(\Sigma)$ the set of Lipschitz functions $\gamma : [0,1] \to \Sigma$ with speed bounded from above and from below, considered up to bi-Lipshitz re-parametrisations of $[0,1]$. The set $\mathrm{P}(\Sigma)$ is endowed with the starting and endpoint maps, $\gamma \mapsto \underline{\gamma}, \overline{\gamma}$ and of the operations of concatenation and reversion as above. A path of Σ is an element of $\gamma \in \mathrm{P}(\Sigma)$. It is *simple* if for any parametrisation $p : [0,1] \to \Sigma$, $p : [0,1) \to \Sigma$ is injective. We consider then the set

$$\mathcal{M}(\mathrm{P}(\Sigma), G)$$

of multiplicative functions as in (24). It is a compact subset of $G^{\mathrm{P}(\Sigma)}$ when the latter is endowed with the product topology. A loop is a path $\ell \in \mathrm{P}(\Sigma)$ such that $\underline{\ell} = \overline{\ell}$. We denote their set by $\mathrm{L}(\Sigma)$. For any $x, y \in \Sigma$, we endow $\mathrm{P}_{x,y}(\Sigma) = \{\gamma \in \mathrm{P}(\Sigma) : \underline{\gamma} = x, \overline{\gamma} = y\}$ with a metric setting for any $\gamma_1, \gamma_2 \in \mathrm{P}_{x,y}(\Sigma)$,

$$d(\gamma_1, \gamma_2) = \inf_{p_1, p_2} \|p_1 - p_2\|_\infty + |\mathscr{L}(\gamma_1) - \mathscr{L}(\gamma_2)|$$

where the infimum is taken over all parametrisations p_1, p_2 of γ_1, γ_2 and for any $\gamma \in \mathrm{P}(\Sigma)$, $\mathscr{L}(\gamma)$ denotes the Riemannian length of γ. Endowing $\mathcal{M}(\mathrm{P}(\Sigma), G)$ with the cylindrical sigma field $\mathcal{B}_{\Sigma,G}$, we denote by $(H_\gamma)_{\gamma \in \mathrm{P}(\Sigma)}$ the canonical process. When $G = G_N$ is a classical compact matrix Lie group of size N, we write for any path $\gamma \in \mathrm{P}(\Sigma)$,

$$W_\gamma = \mathrm{tr}_N(H_\gamma).$$

When $(\mathbb{G}, a)$ is an area weighted map of genus $g \geq 0$, an *embedding* of $(\mathbb{G}, a)$ in a Riemann surface with volume vol, is a collection of simple paths $(\gamma_e)_{e \in E}$ of Σ indexed by edges of $\mathbb{G}$, that do not cross but at their endpoints with the following properties:

1. The ranges of all paths $(\gamma_e)_{e \in E}$ form the 1-cells of a CW complex isomorphic to the CW complex of $\mathbb{G}$.
2. Fixing such an isomorphism, each 2-cell of the complex associated to $(\gamma_e)_{e \in E}$ is a subset of Σ of Riemannian volume $a(f)$, whenever it is identified with a face f of $\mathbb{G}$.

When Σ is the Euclidean plane or the hyperbolic disc, while $\mathbb{G}$ is a map of genus 0, f_∞ is a face of $\mathbb{G}$ and $a \in \Delta_{\mathbb{G},\{f_\infty\}}(T)$, an embedding in Σ of the area weighted map $(\mathbb{G}, \{f_\infty\}, a)$ with one boundary component is a collection of simple paths $(\gamma_e)_{e\in E}$ of $\mathbb{R}^2$ indexed by edges of $\mathbb{G}$, that do not cross but at their endpoints with the following properties:

1. The ranges of all paths $(\gamma_e)_{e\in E}$ form the 1-cells of a CW complex isomorphic to the CW complex of $\mathbb{G}$, such that the unique unbounded 2-cell is mapped to f_∞.
2. Fixing such an isomorphism, each bounded 2-cell of the complex associated to $(\gamma_e)_{e\in E}$ is a subset of Σ of Riemannian volume $a(f)$, whenever it is identified with a face f of $\mathbb{G}$.

In each case, we say that $\mathbb{G}$ is embedded in Σ if there is an area vector a satisfying the property 2.

When $\mathbb{G} = (V, E, F)$ is a map, $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$, Σ is a two-dimensional Riemannian manifold and $\ell \in \mathrm{L}(\Sigma)$, we say that ℓ is a *drawing* of $\mathfrak{l} = e_1 \dots e_n$ if there is an embedding $(\gamma_e)_{E\in E}$ of $\mathbb{G}$ into Σ, such that ℓ is the concatenation $\gamma_{e_1} \dots \gamma_{e_n}$. The next two theorems are due to Lévy [35].

Theorem 3.4. *Let Σ be a compact Riemannian surface with area measure* vol*, G a fixed compact Lie group such that $\mathfrak{g}$ is endowed with a G-invariant inner product. There exists a unique measure YM_Σ on $(\mathcal{M}(\mathrm{P}(\Sigma), G), \mathcal{B}_\Sigma, G)$, with following properties.*

1. *If $(\gamma_e)_{e\in E}$ is an embedding in Σ of an area-weighted map $(\mathbb{G}, a)$ with edges E, the distribution of $(H_{\gamma_e})_{e\in E}$ is the discrete Yang–Mills measure $\mathrm{YM}_{\mathbb{G},a}$.*
2. *For any $x, y \in \Sigma$, if $(\gamma_n)_{n\geq 1}$ is a sequence of paths of $\mathrm{P}_{x,y}(\Sigma)$ with $\lim_{n\to\infty} d(\gamma_n, \gamma) = 0$ for some $\gamma \in \mathrm{P}(\Sigma)$, then under YM_Σ, the sequence of random variables $(H_{\gamma_n})_{n\geq 1}$ converges in probability to H_γ.*

The process $(H_\gamma)_{\gamma\in\mathrm{P}(\Sigma)}$ is called the Yang–Mills holonomy process*.*

Theorem 3.5. *Let Σ be a Euclidean plane $\mathbb{R}^2$ or the hyperbolic disc $D_\mathfrak{h}$, endowed with their area measure* vol*, G a fixed compact Lie group such that $\mathfrak{g}$ is endowed with a G-invariant inner product. There a exists a measure YM_Σ on $(\mathcal{M}(\mathrm{P}(\Sigma), G), \mathcal{B}_\Sigma, G)$, with following properties.*

1. *If $(\gamma_e)_{e\in E}$ is an embedding in Σ of an area-weighted map of genus 0 with one boundary $(\mathbb{G}, \{f_\infty\}, a)$ and edge set E, the distribution of $(H_{\gamma_e})_{e\in E}$ is the discrete Yang–Mills measure $\mathrm{YM}_{\mathbb{G},a}$.*
2. *For any $x, y \in \Sigma$, if (γ_n) is a sequence of paths of $\mathrm{P}_{x,y}(\Sigma)$ with $d(\gamma_n, \gamma) \underset{n\to\infty}{\to} 0$ for some $\gamma \in \mathrm{P}(\Sigma)$, then under YM_Σ, the sequence of random variables $(H_{\gamma_n})_{n\in\mathbb{N}}$ converges in probability to H_γ.*

The process $(H_\gamma)_{\gamma\in\mathrm{P}(\Sigma)}$ is called the Yang–Mills holonomy process*.*

In [9, 17], the authors showed that the proof of the above theorem can be adapted to yield the following extension result, when G is allowed to vary. Let us denote by $A(\Sigma)$ the subset of paths of $\mathrm{P}(\Sigma)$ with a geodesic bi-Lipschitz parametrisation.

Proposition 3.6. *Let G_N be a sequence of classical compact matrix Lie group. Assume the following two properties.*

1. *For any* $\gamma \in A(\Sigma)$, $\Phi(\gamma) = \lim_{N\to\infty} W_\gamma$ *where the convergence holds in probability under* YM_Σ *and* $\Phi(\gamma)$ *is constant.*
2. *There is a constant* $K > 0$ *independent of* N, *such that for any simple contractible loop* $\ell \in \mathrm{L}(\Sigma)$ *bounding an area* $t > 0$,

$$\mathbb{E}_{\mathrm{YM}_\Sigma}[1 - \Re(W_\ell)] \leq Kt.$$

Then $\Phi : A(\Sigma) \to \mathbb{C}$ *has a unique extension to* $\mathrm{P}(\Sigma)$ *such that for all* $x, y \in \Sigma$, $\Phi : \mathrm{P}_{x,y}(\Sigma) \to \mathbb{C}$ *is continuous and for any* $\gamma \in \mathrm{L}(\Sigma)$, W_γ *converges in probability towards* $\Phi(\gamma)$ *as* $N \to \infty$.

The argument given in section 5 of [17] for the sphere applies verbatim on any compact surface Σ to yields the above statement, we will not repeat it in the current version. The details of following extension result is also left in the current version.

Lemma 3.7. 1. *For any map* $\mathbb{G}$ *there is a regular graph* $\mathbb{G}'$ *finer than* $\mathbb{G}$.

2. *For any* $\gamma \in A(\Sigma)$ *there is an embedded graph* $\mathbb{G}$ *such that* γ *is the drawing of a path of* $\mathbb{G}$.
3. *For any area weighted map* $(\mathbb{G}, a)$ *and* $\gamma \in \mathrm{P}(\mathbb{G})$, *there is a regular area weighted map* $(\mathbb{G}', a')$ *finer than* $(\mathbb{G}, a)$, γ' *a regular path of* $\mathbb{G}'$ *and* K *a subset of faces of* $\mathbb{G}'$, *such that*

$$\gamma \sim_K \gamma'.$$

4. *For any compact Lie group* G *and any* $\gamma \in A(\Sigma)$, *there is a regular path* $\mathfrak{p}$ *in a regular graph* $\mathbb{G}$ *and* $a \in \Delta_{\mathbb{G}}(T)$ *such that under* YM_Σ, W_γ *has same law as* $W_{\mathfrak{p}}$ *under* $\mathrm{YM}_{\mathbb{G},a}$.

Together with the last proposition, this lemma reduces the study of Wilson loops for all loops of finite length to the case of regular loops.

3.3 Planar master field, main results and conjecture

In the above setting, the following was proved in [36] and[26] [29], see [60, 3] for a weaker statement with a smaller class of loops and of groups G_N. Recall the definition of the de-singularisation operation in section 2.7.

Theorem 3.8. *Assume that* G_N *is a classical compact matrix Lie group of size* N. *Assume that* $(\mathbb{G}, \{f_\infty\}, a)$ *is any area weighted map of genus* 0, *with one boundary component and* $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$, *or that* $\ell \in \mathrm{L}(\mathbb{R}^2)$, *then the following convergences hold in probability*[27] *and the limits are constant and independent of the type of series of* G_N:

$$\Phi_{\mathfrak{l}}^{f_\infty}(a) = \lim_{N\to\infty} W_{\mathfrak{l}} \text{ under } \mathrm{YM}_{\mathbb{G},\{f_\infty\},a}$$

and

$$\Phi_{\mathbb{R}^2}(\ell) = \lim_{N\to\infty} W_\ell \text{ under } \mathrm{YM}_{\mathbb{R}^2}.$$

The function $\Phi_{\mathbb{R}^2}$ *is characterised by the following properties:*

1. *For any* $x \in \mathbb{R}^2$, $\Phi_{\mathbb{R}^2} : \mathrm{P}_{x,x}(\mathbb{R}^2) \to \mathbb{C}$ *is continuous.*

[26] In [36], to get uniqueness (b) is replaced by an additional set of differential equations

[27] It is also shown in [36] that the following convergences are almost sure.

2. *Whenever* $\ell \in \mathrm{L}(\mathbb{R}^2)$ *is a drawing of a loop* $\mathfrak{l}$ *of an area weighted map of genus* 0 *with one boundary component* $(\mathbb{G}, \{f_\infty\}, a)$,

$$\Phi_{\mathbb{R}^2}(\ell) = \Phi_{\mathfrak{l}, f_\infty}(a).$$

3. *For any map of genus* 0 *with one boundary component* $(\mathbb{G}, \{f_\infty\})$, $T > 0$, *and any loop* $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$, Φ_ℓ *is uniformly continuous on* $\Delta_{\mathbb{G},\{f_\infty\}}(T)$ *and differentiable on* $\Delta^o_{\mathbb{G},\{f_\infty\}}(T)$ *such that*
   (a) *if* $\mathbb{G}$ *is regular,* $\mathfrak{l}$ *is a tame loop and* $v \in V_{\mathfrak{l}}$ *is a transverse intersection with* $\delta_v \mathfrak{l} = \mathfrak{l}_1 \otimes \mathfrak{l}_2$,

$$\mu_v.\Phi_{\mathfrak{l}, f_\infty} = \Phi_{\mathfrak{l}_1, f_\infty} \Phi_{\mathfrak{l}_2, f_\infty} \text{ in } \Delta^o_{\mathbb{G},\{f_\infty\}}(T).$$

   (b) *Whenever*

$$\Phi_{\mathbb{R}^2}(\ell) = e^{-\frac{t}{2}}.$$

See the appendix of [36] for a table of values of $\Phi_{\mathbb{R}^2}$. Alternatively, the master field can be characterised using free probability as follows.

Lemma 3.9. *Consider an area weighted map* $(\mathbb{G}, \{f_\infty\}, a)$ *of genus* 0 *with one boundary component. Assume* $\mathbb{G} = (V, E, F)$, $\#F = r + 1$, $F = \{f_1, \dots, f_r, f_\infty\}$ *and* $v \in V$. *For any* $\mathfrak{l} \in \mathrm{L}_v(\mathbb{G})$ *depends on* $\mathfrak{l}$ *only through its* $\sim_r$ *class. Setting*

$$\tau_v(\mathfrak{l}) = \Phi_{\mathfrak{l}, f_\infty}(a) \text{ and } \mathfrak{l}^* = \mathfrak{l}^{-1}, \forall \mathfrak{l} \in \mathrm{RL}_v(\mathbb{G})$$

and extending these maps linearly and sesqui-linearly, defines a non-commutative probability space $(\mathbb{C}[\mathrm{RL}_v(\mathbb{G})], \tau_v, *)$. *Assume that* $\mathfrak{l}_1, \dots, \mathfrak{l}_r, \mathfrak{l}_\infty$ *is a family of lassos as in Lemma 2.1 with* $\mathfrak{l}_i$ *bounding* f_i *for* $1 \le i \le r$ *and* $\mathfrak{l}_\infty$ *for* f_∞. *Then* τ_v *is the unique state on* $(\mathbb{C}[\mathrm{RL}_v(\mathbb{G})], *)$ *such that*

1. *for all* $n \in \mathbb{Z}^*$, $\tau_v(\mathfrak{l}_i^n) = \nu_{a(f_i)}(n)$,
2. $\mathfrak{l}_1, \dots, \mathfrak{l}_r$ *are freely independent under* τ_v.

Similarly the following lemma follows from the classical result of [6] and Lemma 3.2. It shows that the conclusion of the former one is valid when the genus condition is dropped.

Lemma 3.10. *Consider an area weighted regular map with boundary* $(\mathbb{G}, \{f_\infty\}, a)$ *of genus* $g \ge 1$. *Assume* $\mathbb{G} = (V, E, F)$, $\#F = r + 1$ *with* $F = \{f_1, \dots, f_r, f_\infty\}$ *and* $v \in V$. *Assume that* $a_1, \dots, b_g$ *and* $\mathfrak{l}_1, \dots, \mathfrak{l}_{r+1}$ *are* $2g$ *simple loops and* $r+1$ *lassos as in Lemma 2.1, with* $\mathfrak{l}_i$ *bounding* f_i *for* $1 \le i \le r$ *and* f_∞ *for* $i = r + 1$. *Assume that* G_N *is a sequence of classical compact matrix group of size* N. *Then for any* $T > 0$, $a \in \Delta_{\mathbb{G},\{f_\infty\}}(T)$ *and* $\mathfrak{l} \in \mathrm{RL}_v(\mathbb{G})$,

$$W_{\mathfrak{l}} \to \Phi^{1,g}_{\mathfrak{l}}(a) \text{ under } \mathrm{YM}_{\mathbb{G},\{f_\infty\},a},$$

where $\Phi^{1,g}_{\mathfrak{l}}(a)$ *is constant. Moreover there is a constant* $K > 0$ *independent of* $\mathbb{G}$ *and* $N \ge 1$, *such that for any face* $f \in F \setminus \{f_\infty\}$,

$$\mathbb{E}[1 - \Re(W_{\partial f})] \le K a(f). \tag{*}$$

The $*$*-algebra* $(\mathbb{C}[\mathrm{RL}_v(\mathbb{G})], *)$ *is endowed with a unique state* τ_v *satisfying*

$$\tau_v(\mathfrak{l}) = \Phi^{1,g}_{\mathfrak{l}}(a), \forall \mathfrak{l} \in \mathrm{RL}_v(\mathbb{G}).$$

Moreover, τ_v *is characterised by the following three properties:*

1. $\mathfrak{l}_1, \dots, \mathfrak{l}_r, a_1, \dots, b_g$ *are free under* τ_v.
2. *under* τ_v, $a_1, \dots, b_g$ *are* $2g$ *Haar unitaries.*
3. *For any* $1 \leq i \leq r$ *and* $n \in \mathbb{Z}^*$,

$$\tau_v(\mathfrak{l}_i^n) = \nu_{a(f_i)}(n).$$

A sketch of the proof is given in section 5. From Lemma 3.10 and the absolute continuity result of [16] follows a result on closed surfaces, for loops avoiding at least one handle.

Let us recall the definition of the universal cover $\tilde{\mathbb{G}} = (\tilde{V}, \tilde{E}, \tilde{F})$ of a regular graph $(\mathbb{G}, \mathbb{G}_b)$ given in section 2.3, with a canonical covering map $p : \tilde{F} \to F$. When $a \in \Delta_{\mathbb{G}}(T)$, let us set $\tilde{a} = a \circ p : \tilde{F} \to [0, T]$.

Theorem 3.11. *Assume that* $(\mathbb{G}, a)$ *is an area weighted map with cutting along a simple loop* $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$ *given by* $(\mathbb{G}_1, \{f_{1,\infty}\})$ *and* $(\mathbb{G}_2, \{f_{2,\infty}\})$*, with the same convention as in section 2.1. Then if* $\mathbb{G}_2$ *has genus* $g_2 \geq 1$*, for any loop* $\mathfrak{l} \in \mathrm{L}(\mathbb{G}_1)$ *and* $a \in \Delta_{\mathbb{G}}(T)$ *with* $a(F_2) \in (0, T)$,

$$W_{\mathfrak{l}} \underset{N\to\infty}{\to} \Phi_{\mathfrak{l}}(a) = \begin{cases} \Phi_{\tilde{\mathfrak{l}}}(\tilde{a}) & \text{if } \mathfrak{l} \sim_h c_{\underline{\mathfrak{l}}}, \\ 0 & \text{if } \mathfrak{l} \not\sim_h c_{\underline{\mathfrak{l}}}, \end{cases} \quad \text{in probability under } \mathrm{YM}_{\mathbb{G},a}, \tag{25}$$

where $\tilde{\mathfrak{l}}$ *is a lift of* $\mathfrak{l}$ *in* $\tilde{\mathbb{G}}$*. Moreover, when* $g_1 = 0$*, the convergence holds true uniformly in* $a \in \Delta_{\mathbb{G}}(T)$*. Besides, there is a constant* $K > 0$ *independent of* $\mathbb{G}$ *and* $N \geq 1$*, such that for any face* $f \in F_1$,

$$\mathbb{E}[1 - \Re(W_{\partial f})] \leq K a(f). \tag{26}$$

When $\mathbb{G}$ has genus 1 the above result gives information about loops included in a topological disc but does not say anything about other loops, for instance contractible loops obtained by concatenation of simple loops of non trivial homology. A more satisfying answer is then given by the following theorem.

Theorem 3.12. *Consider a classical compact matrix Lie group* G_N *of size* N*,* $\mathbb{T}_T$ *is a torus of volume* $T > 0$ *obtained as a quotient of the Euclidean plane* $\mathbb{R}^2$ *by the lattice* $\sqrt{T}\mathbb{Z}^2$*. Then, the following convergence holds in probability under* $\mathrm{YM}_{\mathbb{T}_T}$,

$$W_{\ell} \underset{N\to\infty}{\to} \Phi_{\mathbb{T}_T}(\ell) = \begin{cases} \Phi_{\mathbb{R}^2}(\tilde{\ell}) & \text{if } \ell \text{ is contractible}, \\ 0 & \text{otherwise}, \end{cases}$$

where for any loop $\ell \in \mathrm{L}(\mathbb{T}_T)$*,* $\tilde{\ell} \in \mathrm{P}(\mathbb{R}^2)$ *is a finite length path with projection to* $\mathbb{T}_T$ *given by* ℓ*. Besides,* $\Phi_{\mathbb{T}_T} : \mathrm{L}(\mathbb{R}^2) \to [-1, 1]$ *is the unique function satisfying*

1. *For any* $x \in \mathbb{T}_T$*,* $\Phi_{\mathbb{T}_T} : \mathrm{L}_x(\mathbb{T}_T) \to \mathbb{C}$ *is continuous for the length metric* d.
2. *For any regular loop* $\mathfrak{l}$ *in a regular map* $\mathbb{G}$ *of genus* 1*, there is a differentiable function*

$$\Phi_{\mathfrak{l}} : \Delta_{\mathbb{G}}(T) \to \mathbb{C}$$

such that for any transverse intersection $v \in V_{\mathfrak{l}}$*, with* $\delta_v \mathfrak{l} = \mathfrak{l}_1 \otimes \mathfrak{l}_2$,

$$\mu_v \Phi_{\mathfrak{l}} = \Phi_{\mathfrak{l}_1} \Phi_{\mathfrak{l}_2}. \tag{27}$$

3. *For any loop* $\ell \in \mathrm{L}(\mathbb{T}_T)$ *obtained by projection of a loop* $\tilde{\ell} \in \mathrm{L}(\mathbb{R}^2)$ *included in a fundamental domain of* $\sqrt{T}\mathbb{Z}^2$,

$$\Phi_{\mathbb{T}_T}(\ell) = \Phi_{\mathbb{R}^2}(\tilde{\ell}).$$

4. *For any non-contractible simple loop* $\ell \in \mathrm{L}(\mathbb{T}_T^2)$ *and* $n \in \mathbb{Z}^*$,

$$\Phi_{\mathbb{T}_T}(\ell^n) = 0.$$

When $g \geq 2$, we were unable to show a satisfying version of conjecture 1.3, but are able to prove the following conditional results.

Theorem 3.13. *Consider classical compact matrix Lie group* G_N *of size* N, $g \geq 2$ *and* $T > 0$. *Assume that for any regular area weighted map* $(\mathbb{G}, a)$ *of genus* g,

$$W_{\mathfrak{l}} \underset{N\to\infty}{\to} \Phi_{\tilde{\mathfrak{l}}}(\tilde{a}) \text{ in probability under } \mathrm{YM}_{\mathbb{G},a}, \tag{28}$$

whenever $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$ *such that*

1. *any lift* $\tilde{\mathfrak{l}} \in \mathrm{L}(\tilde{\mathbb{G}})$ *of* $\mathfrak{l}$ *is included in a fundamental domain, or*
2. $\mathfrak{l} = \gamma_{nest}\gamma$, *where* γ_{nest} *is a nested loop and* γ *is a geodesic path.*[28]

Then for any regular graph $\mathbb{G}$ *of genus* g, (28) *holds true for all* $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$.

Besides, the following weaker statement can be proved independently.

Proposition 3.14. *Consider* G_N *is a classical compact matrix Lie group of size* N *and* $g \geq 2$. *Assume that for any regular area weighted map* $(\mathbb{G}, a)$ *of genus* g,

$$W_{\mathfrak{l}} \underset{N\to\infty}{\to} 0 \text{ in probability under } \mathrm{YM}_{\mathbb{G},a}, \tag{29}$$

whenever $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$ *is a geodesic loop with non zero-homology. Then for any regular graph* $\mathbb{G}$ *of genus* g, (29) *holds true for all* $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$ *with non-zero homology.*

Remark. The above statements may give the impression that any possible master field is expressed in terms of the planar case. This is nonetheless not the case as the Wilson loops limit on the of sphere give raise to different limits [17]. See also the discussion in [16, Section 2.5].

3.4 Invariance in law and Wilson loop expectation

Before proceeding to the main part of this paper, let us give here a partial result that only holds in expectation, but relies on a simpler argument: the invariance in law by an action of the center of the structure group G_N. Consider a regular graph $\mathbb{G} = (V, E, F)$ with r faces, $v \in V$ and a basis $\mathfrak{l}_1, \ldots, \mathfrak{l}_r, a_1, \ldots, b_g$ of the free group $\mathrm{RL}_v(\mathbb{G})$ as in Lemma 2.1. For any $h \in G^{2g}$ and $\phi \in \mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}), G)$, let us denote by $h.\phi \in \mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}), G)$ the unique group morphism with

$$h.\phi(\mathfrak{l}_i) \mapsto \phi(\mathfrak{l}_i), \text{ for } 1 \leq i \leq r$$

and

$$h.\phi(a_i) = h_{2i-1}\phi(a_i) \text{ and } h.\phi(b_i) = h_{2i}\phi(b_i) \text{ for } 1 \leq i \leq g.$$

[28] See Fig. 9 and Section 2.3.

Let us denote by Z the center of G. When $h \in Z^{2g}$, it follows easily from point 2. of Lemma 2.2 that

$$h.\phi(\mathfrak{l}) = \phi_h([\mathfrak{l}]_{\mathbb{Z}})\phi(\mathfrak{l}), \forall \mathfrak{l} \in \mathrm{RL}_v(\mathbb{G}), \tag{30}$$

where $\phi_h \in \mathrm{Hom}(H_1(d^*, \mathbb{Z}), Z)$ is the unique group morphism such that

$$\phi_h([a_i]_{\mathbb{Z}}) = h_{2i-1} \text{ and } \phi_h([b_i]_{\mathbb{Z}}) = h_{2i} \text{ for } 1 \leq i \leq g.$$

Lemma 3.15. *Let $\mathbb{G}$ be regular map, $T > 0$, $a \in \Delta_{\mathbb{G}}(T)$. Denoting by $(H_{\mathfrak{l}})_{\mathfrak{l} \in \mathrm{RL}_v(\mathbb{G})}$ the canonical G-valued random variable on $\mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}), G)$, the following assertions hold true.*

1. *The measure $\mathrm{YM}_{a,\mathbb{G},v}$ on $\mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}), G)$ is invariant under the action of Z^{2g}.*
2. *Assume that $\chi : G \to \mathbb{C}$ is continuous and $\alpha : Z \to \mathbb{C}$ is such that $\chi(z.h) = \alpha_\chi(z)\chi(h), \forall (z, h) \in Z \times G$. Then*
   (a) for any $h \in Z^{2g}$ and $\mathfrak{l} \in \mathrm{RL}_v(\mathbb{G})$,
   $$\mathbb{E}_{\mathrm{YM}_{a,\mathbb{G},v}}[\chi(H_{\mathfrak{l}})] = \alpha_\chi \circ \phi_h([\mathfrak{l}]_{\mathbb{Z}})\mathbb{E}_{\mathrm{YM}_{a,\mathbb{G},v}}[\chi(H_{\mathfrak{l}})].$$
   (b) If there is $\phi \in \mathrm{Hom}(H_1(d^*, \mathbb{Z}), Z)$ with $\phi([\mathfrak{l}]_{\mathbb{Z}}) \neq 0$, then
   $$\mathbb{E}_{\mathrm{YM}_{a,\mathbb{G},v}}[\chi(H_{\mathfrak{l}})] = 0.$$
3. *When G is a classical compact matrix Lie group, for any $\mathfrak{l} \in \mathrm{RL}_v(\mathbb{G})$, $\mathbb{E}[W_\ell] = 0$ if one of the following conditions is satisfied:*
   (a) $G = \mathrm{U}(N)$ and $[\mathfrak{l}]_{\mathbb{Z}} \neq 0$.
   (b) $G = \mathrm{SU}(N)$ and $[\mathfrak{l}]_{\mathbb{Z}_n} \neq 0$
   (c) $G = \mathrm{SO}(2N)$ and $[\mathfrak{l}]_{\mathbb{Z}_2} \neq 0$.

Proof. The implication $2.a) \Rightarrow 2.b) \Rightarrow 3$ are elementary. Thanks to (30), $1 \Rightarrow 2.a)$. Lastly, consider 1. Denote by $d\phi$ the Haar measure on $\mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}), G)$ endowed with pointwise multiplication. By Lemma 3.3, it is enough to consider $a \in \Delta^o_{\mathbb{G}}(T)$ and denote by $a_1, \dots a_r$ the area enclosed by the meanders of $\mathfrak{l}_1, \dots, \mathfrak{l}_r$ and set $a_{r+1} = T - \sum_{i=1}^r a_i$. For any continuous function $\chi : \mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}), G) \to \mathbb{C}$ and $h \in Z^{2g}$, $d\phi$ is invariant by the action of Z^{2g} and

$$\begin{aligned}
&\int_{\mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}),G)} \chi(h^{-1}.\phi) d\mathrm{YM}_{a,\mathbb{G},v}(\phi) \\
&= \int_{\mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}),G)} \chi(h^{-1}.\phi) p_{a_{r+1}}(\phi((\mathfrak{l}_1 \dots \mathfrak{l}_r)^{-1}[a_1, b_1] \dots [a_g, b_g])) \prod_{i=1}^r p_{a_i}(\phi(\mathfrak{l}_i)) d\phi \\
&= \int_{\mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}),G)} \chi(\phi) p_{a_{r+1}}(h.\phi((\mathfrak{l}_1 \dots \mathfrak{l}_r)^{-1}[a_1, b_1] \dots [a_g, b_g])) \prod_{i=1}^r p_{a_i}(h.\phi(\mathfrak{l}_i)) d\phi \\
&= \int_{\mathrm{Hom}(\mathrm{RL}_v(\mathbb{G}),G)} \chi(\phi) p_{a_{r+1}}(h.\phi((\mathfrak{l}_1 \dots \mathfrak{l}_r)^{-1}[a_1, b_1] \dots [a_g, b_g])) \prod_{i=1}^r p_{a_i}(h.\phi(\mathfrak{l}_i)) d\phi,
\end{aligned}$$

where in the last line we used that $h.\phi([a_i, b_i]) = [\phi(a_i)h_{2i-1}, \phi(b_i)h_{2i-1}]) = \phi([a_i, b_i])$ for $1 \leq i \leq g$ and $h.\phi(\mathfrak{l}_j) = \phi(\mathfrak{l}_j)$, for $1 \leq j \leq r$. □

3.5 Makeenko–Migdal equations, existence and uniqueness problem

The main tool of the current article are approximate versions of equations (27), satisfied on any surface when $G_N \in \mathbf{CG}_N$ and $N \to \infty$. Let us introduce a setting to prove existence and uniqueness of these equations.

For any regular graph $\mathbb{G}$ and any vertex v of $\mathbb{G}$, let $\mathcal{A}_v(\mathbb{G})$ be the algebra with elements in $\mathbb{C}[\mathrm{L}_v(\mathbb{G})]$ endowed with the multiplication given by concatenation and setting $\mathfrak{l}^* = \mathfrak{l}^{-1}$ for all $\mathfrak{l} \in \mathrm{L}_v(\mathbb{G})$ and extending it skew-linearly. When w is another vertex, we consider the $*$-algebra $\mathcal{A}_{v,w}(\mathbb{G})$ with elements in $\mathbb{C}[\mathrm{L}_v(\mathbb{G})] \otimes \mathbb{C}[\mathrm{L}_w(\mathbb{G})]$ and multiplication and $*$-operation defined for all $(x_i, y_i) \in \mathcal{A}_v(\mathbb{G}) \times \mathcal{A}_w(\mathbb{G})$ by

$$(x_1 \otimes y_1).(x_2 \otimes y_2) = (x_1 x_2) \otimes (y_1 y_2) \text{ and } (x_1 \otimes y_1)^* = x_1^* \otimes y_1^*.$$

Let us fix $g \geq 1$ and $T > 0$. A *Wilson loop system* is a family of continuous functions $\phi_{\mathfrak{l}_1}, \phi_{\mathfrak{l}_1 \otimes \mathfrak{l}_2} : \Delta_{\mathbb{G}}(T) \to \mathbb{C}$ given for each regular graph $\mathbb{G}$ of genus g and each pair of loops $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathrm{L}(\mathbb{G})$, with the following properties:

1. For any constant loop c,
$$\phi_{\mathfrak{l}_1 \otimes c} = \phi_{\mathfrak{l}_1} \text{ and } \phi_c = 1.$$
2. For any pair of loops $\mathfrak{l}_1, \mathfrak{l}_2$ within a same regular graph of genus g,
$$\phi_{\mathfrak{l}_1 \otimes \mathfrak{l}_2} = \phi_{\mathfrak{l}_2 \otimes \mathfrak{l}_1}$$
depend on $\mathfrak{l}_1, \mathfrak{l}_2$ only through their $\sim_{r,c}$ equivalence class.
3. If $\mathbb{G}'$ is finer than $\mathbb{G}$ of genus g, then for all loops $\mathfrak{l}, \mathfrak{l}_1, \mathfrak{l}_2 \in \mathrm{L}(\mathbb{G})$
$$\phi_{\mathfrak{l}} \circ r_{\mathbb{G}}^{\mathbb{G}'} = \phi_{\mathfrak{l}} \text{ and } \phi_{\mathfrak{l}_1 \otimes \mathfrak{l}_2} \circ r_{\mathbb{G}}^{\mathbb{G}'} = \phi_{\mathfrak{l}_1 \otimes \mathfrak{l}_2}$$
where loops are identified in the right-hand-sides with elements of $\mathrm{L}(\mathbb{G}')$.
4. If $\mathbb{G}'$ is isomorphic to $\mathbb{G}$ of genus g, $a \in \Delta_{\mathbb{G}}(T)$ is mapped to $a' \in \Delta_{\mathbb{G}'}(T)$, while $\mathfrak{l}'_1, \mathfrak{l}'_2 \in \mathrm{L}(\mathbb{G})$ with $\mathfrak{l}_1 \sim_\Sigma \mathfrak{l}'_1, \mathfrak{l}_2 \sim_\Sigma \mathfrak{l}'_2$, through the same isomorphism map, then
$$\phi_{\mathfrak{l}_1}(a) = \phi_{\mathfrak{l}'_1}(a') \text{ and } \phi_{\mathfrak{l}_1 \otimes \mathfrak{l}_2}(a) = \phi_{\mathfrak{l}'_1 \otimes \mathfrak{l}'_2}(a').$$
5. If $\mathbb{G} = (V, E, F)$ is a regular graph of genus g, $\mathfrak{l}_1, \mathfrak{l}'_1, \mathfrak{l}_2 \in \mathrm{L}(\mathbb{G})$, $K \subset F$ with $\mathfrak{l}_1 \sim_K \mathfrak{l}'_1$, then
$$\phi_{\mathfrak{l}_1 \otimes \mathfrak{l}_2}(a) = \phi_{\mathfrak{l}'_1 \otimes \mathfrak{l}_2}(a), \forall a \in \Delta_{K,\mathbb{G}}(T).$$
6. For any regular graph $\mathbb{G}$ of genus g with vertices v and w, for any $a \in \Delta_{\mathbb{G}}(T)$, extending $\mathfrak{l}_1 \otimes \mathfrak{l}_2 \in \mathrm{L}_v(\mathbb{G}) \otimes \mathrm{L}_w(\mathbb{G}) \mapsto \phi_{\mathfrak{l}_1 \otimes \mathfrak{l}_2}(a)$ linearly defines a non-negative states $\phi_{a,v,w}$ on $\mathcal{A}_{v,w}(\mathbb{G})$ while for any $x \in \mathcal{A}_v(\mathbb{G})$,
$$\phi_{x \otimes x^*} \geq 0.$$

Whenever $G_N \in \mathbf{CG}_N$, from the above definition of the Yang-Mills measure, the collection
$$a \in \Delta_{\mathbb{G}}(T) \mapsto \mathbb{E}_{\mathrm{YM}_{\mathbb{G},a}}[W_{\mathfrak{l}}], \mathbb{E}_{\mathrm{YM}_{\mathbb{G},a}}[W_{\mathfrak{l}_1} W_{\mathfrak{l}_2}]$$
for all regular maps $\mathbb{G}$ of genus g and loops $\mathfrak{l}, \mathfrak{l}_1, \mathfrak{l}_2 \in \mathrm{L}(\mathbb{G})$, is a Wilson loop system. Let us note that the first part of point 6. together with point 1. implies that for any

regular graph of genus g with vertex v, for all $a \in \Delta_{\mathbb{G}}(T)$ $\mathfrak{l} \in \mathrm{L}_v \mapsto \phi_{\mathfrak{l}}(a)$ extends linearly to a state $\phi_{v,a}$ of $(\mathcal{A}_v, *)$. Moreover, it then follows from 1. and 2. that for any vertices v, w, and $(\mathfrak{l}_1, \mathfrak{l}_2) \in \mathrm{L}_v(\mathbb{G}) \times \mathrm{L}_w(\mathbb{G})$, $\mathfrak{l}_1$ and $\mathfrak{l}_1 \otimes \mathfrak{l}_2$ have a unitary distribution in $(\mathcal{A}_v, \phi_{v,a})$ and $(\mathcal{A}_{v,w}, \phi_{a,v,w})$. When ϕ is a Wilson loop system, for any regular graph $\mathbb{G}$ and any loop $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$, the second part of point 6. and point 1. yield

$$\mathscr{V}_{\phi,\mathfrak{l}} = \phi_{\mathfrak{l}\otimes\mathfrak{l}^{-1}} - |\phi_{\mathfrak{l}}|^2 = \phi_{\mathfrak{l}\otimes\mathfrak{l}^{-1}} - \phi_{\mathfrak{l}}\phi_{\mathfrak{l}^{-1}} \geq 0.$$

We say that a Wilson loop system ϕ is an *exact solution* of Makeenko–Migdal equations if

1. For any loop $\mathfrak{l}$ within a regular graph $\mathbb{G}$ of genus g, $\phi \in C^1(\Delta^o_{\mathbb{G}}(T))$ and for any $v \in V_{\mathfrak{l}}$,
$$\mu_v.\phi_{\mathfrak{l}} = \phi_{\mathfrak{l}_1}\phi_{\mathfrak{l}_2}.$$
2. For any pair of regular loops within the same graph, $\phi_{\alpha\otimes\beta} = \phi_\alpha\phi_\beta$.
3. For any regular loop $\mathfrak{l}$ with $\mathfrak{l} \not\sim_h c_{\underline{\mathfrak{l}}}$, $\phi_{\mathfrak{l}} = 0$.

We say that a sequence $(\phi^N)_{N\geq 1}$ of Wilson loop systems is an *approximate solution of Makeenko–Migdal equations* if for any regular graph of genus g, any loop $\mathfrak{l}$ in $\mathrm{L}(\mathbb{G})$, $\phi^N_{\mathfrak{l}}$ and $\mathscr{V}^N_{\phi,\mathfrak{l}}$ are in $C^1(\Delta^o_{\mathbb{G}}(T))$, and for any intersection point $v \in V_{\mathfrak{l}}$,

$$\mu_v.\phi^N_{\mathfrak{l}} = \phi^N_{\delta_v(\mathfrak{l})} + \frac{1}{N}R^N_{\mathfrak{l}} \tag{31}$$

while there is a constant $C > 0$ independent of $\mathfrak{l}$ and $N \geq 1$, such that

$$|\mu_v.\mathscr{V}^N_{\phi,\mathfrak{l}}| \leq \mathscr{V}_{\phi^N,\mathfrak{l}} + \mathscr{V}_{\phi^N,\mathfrak{l}_1} + \mathscr{V}_{\phi^N,\mathfrak{l}_2} + \frac{C}{N} \tag{32}$$

and

$$|\mu_v.\mathscr{V}^N_{\phi,\mathfrak{l}}| \leq \sqrt{\mathscr{V}_{\phi^N,\mathfrak{l}_1}\mathscr{V}_{\phi^N,\mathfrak{l}_2}} + |\phi^N_{\mathfrak{l}_1}|\sqrt{\mathscr{V}_{\phi^N,\mathfrak{l}_2}} + |\phi^N_{\mathfrak{l}_2}|\sqrt{\mathscr{V}_{\phi^N,\mathfrak{l}_1}} + \frac{C}{N} \tag{33}$$

where $\mathfrak{l}_1 \otimes \mathfrak{l}_2 = \delta_v\mathfrak{l}$ and $R^N_{\mathfrak{l}} : \Delta_{\mathbb{G}}(T) \to \mathbb{C}$ satisfy

$$\sup_{N\geq 1} \|R^N_{\mathfrak{l}}\|_\infty < +\infty.$$

Remark. Note that it follows from point 3. that if ϕ is a Wilson loop system and $\mathfrak{l}, \mathfrak{l}_1, \mathfrak{l}_2$ are a regular loops of a regular graph $\mathbb{G} = (V, E, F)$ with $e \in E^o \setminus (E^o_{\mathfrak{l}} \cup E^o_{\mathfrak{l}_1} \cup E^o_{\mathfrak{l}_2})$, then

$$d\omega_e.\phi_{\mathfrak{l}} = d\omega_e.\phi_{\mathfrak{l}_1\otimes\mathfrak{l}_2} = 0. \tag{34}$$

Consequently, for any regular loop $\mathfrak{l}$, using the same linear forms as in section 2.7, if ϕ^∞ and (ϕ^N) are respectively exact and approximate solutions of Makeenko-Migdal equations, for any regular loop $\mathfrak{l}$ and $X \in \mathfrak{m}_{\mathfrak{l}}$,

$$X.\phi^\infty_{\mathfrak{l}} = \phi_{\delta_X\mathfrak{l}} \text{ and } X.\phi^N_{\mathfrak{l}} = \phi^N_{\delta_X\mathfrak{l}} + \frac{\|X\|}{N}R^N_{\mathfrak{l}} \tag{35}$$

while

$$|X.\mathscr{V}_{\phi^N,\mathfrak{l}}| \leq C_{\mathfrak{l}}\|X\| \left(\sum_{v\in V_{\mathfrak{l}}} \left(\sqrt{\mathscr{V}_{\phi^N,\mathfrak{l}_1}\mathscr{V}_{\phi^N,\mathfrak{l}_2}} + |\phi^N_{\mathfrak{l}_1}|\sqrt{\mathscr{V}_{\phi^N,\mathfrak{l}_2}} + |\phi^N_{\mathfrak{l}_2}|\sqrt{\mathscr{V}_{\phi^N,\mathfrak{l}_1}} \right) + \frac{1}{N} \right) \tag{36}$$

and

$$|X.\mathscr{V}_{\phi^N,\mathfrak{l}}| \leq \|X\| C_{\mathfrak{l}} \left(\mathscr{V}_{\phi^N,\mathfrak{l}} + \sum_{v \in V_{\mathfrak{l}}} (\mathscr{V}_{\phi^N,\mathfrak{l}_{v,1}} + \mathscr{V}_{\phi^N,\mathfrak{l}_{1,2}}) + \frac{1}{N} \right) \tag{37}$$

where for any $v \in V_{\mathfrak{l}}$ we wrote $\partial_v \mathfrak{l} = \mathfrak{l}_{1,v} \otimes \mathfrak{l}_{2,v}$, $C_{\mathfrak{l}} > 0$ is independent of $N \geq 1$ and

$$\sup_{N \geq 1} \|R_{\mathfrak{l}}^N\|_\infty < +\infty.$$

The existence problem of these equations is a consequence of [21] and [36] for the approximate solutions, and given Theorem 3.8, of a simple computation for the exact ones.

Lemma 3.16. *Consider $g \geq 1, T > 0$.*

1. *Assume that $G_N \in \mathbf{CG}_N$, then setting for all regular graph $\mathbb{G}$, $a \in \Delta_{\mathbb{G}}(T)$ and all loops $\mathfrak{l}, \mathfrak{l}_1, \mathfrak{l}_2 \in \mathrm{L}(\mathbb{G})$*

$$\phi_{\mathfrak{l}}^N(a) = \mathbb{E}_{\mathrm{YM}_{\mathbb{G},a}}[W_{\mathfrak{l}}], \phi_{\mathfrak{l}_1 \otimes \mathfrak{l}_2}^N(a) = \mathbb{E}_{\mathrm{YM}_{\mathbb{G},a}}[W_{\mathfrak{l}_1} W_{\mathfrak{l}_2}]$$

   defines an approximate solution of the Makeenko–Migdal equations.

2. *Denoting by c_v the constant loop at a vertex v, setting for any regular graph $\mathbb{G}, a \in \Delta_{\mathbb{G}}(T)$ and $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$,*

$$\phi_{\mathfrak{l}}(a) = \begin{cases} \Phi_{\tilde{\mathfrak{l}}}(\tilde{a}) & \text{if } \mathfrak{l} \sim_h c_{\underline{\mathfrak{l}}}, \\ 0 & \text{if } \mathfrak{l} \not\sim_h c_{\underline{\mathfrak{l}}}, \end{cases}$$

   defines an exact solution of the Makeenko–Migdal equations.

Proof. Point 1. is a direct consequence of Proposition 7.3 below, together with Cauchy-Schwarz or arithmetic-geometric mean inequality to get (36) and (37). For point 2., we shall only check that the Makeenko–Migdal equations are satisfied and leave the other points to the Reader. Consider a regular graph $\mathbb{G}$ of genus g with $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$ and $v \in V_{\mathfrak{l}}$. Consider $\delta_v \mathfrak{l} = \mathfrak{l}_1 \otimes \mathfrak{l}_2$ and let us show that $\mu_v \phi_{\mathfrak{l}} = \phi_{\mathfrak{l}_1} \phi_{\mathfrak{l}_2}$. If $\mathfrak{l} \not\sim_h c_{\underline{\mathfrak{l}}}$, then the rerooting $\mathfrak{l}'$ at v of $\mathfrak{l}$ satisfies $\mathfrak{l}' \not\sim_h c_v$. Therefore $\mathfrak{l}_1 \not\sim_h c_v$ or $\mathfrak{l}_2 \not\sim_h c_v$ and we conclude that $\phi_{\mathfrak{l}} = \phi_{\mathfrak{l}'} = 0 = \phi_{\mathfrak{l}_1} \phi_{\mathfrak{l}_2}$. Assume now $\mathfrak{l} \sim_h c_{\underline{\mathfrak{l}}}$. Consider a fundamental cover $\tilde{\mathbb{G}} = (\tilde{V}, \tilde{E}, \tilde{F})$ of $\mathbb{G}$ with projection map p. For all $a \in \Delta_{\mathbb{G}}^o(T)$,

$$\mu_v.\phi_{\mathfrak{l}}(a) = \mu_v.(\Phi_{\tilde{\mathfrak{l}}}(\tilde{a})) = \sum_{\tilde{v} \in p^{-1}(v) \cap T_{\tilde{\mathfrak{l}}}} (\mu_{\tilde{v}}.\Phi_{\tilde{\mathfrak{l}}})(\tilde{a}),$$

where $T_{\tilde{\mathfrak{l}}}$ is the set of vertices of $\tilde{\mathbb{G}}$ visited by $\tilde{\mathfrak{l}}$. Since $\mathfrak{l}$ is regular, whether $\#p^{-1}(v) \cap T_{\tilde{\mathfrak{l}}} = 2$ and $\#(V_{\tilde{\mathfrak{l}}} \cap p^{-1}(v)) = 0$, or $\#(p^{-1}(v) \cap T_{\tilde{\mathfrak{l}}}) = \#(p^{-1}(v) \cap V_{\tilde{\mathfrak{l}}}) = 1$.

In the first case, $\mathfrak{l}_1 \not\sim_h c_v$ and $\mathfrak{l}_2 \not\sim_h c_v$, so that $\phi_{\mathfrak{l}_1} = \phi_{\mathfrak{l}_2} = 0$. Moreover for any $\tilde{v} \in p^{-1}(v) \cap T_{\tilde{\mathfrak{l}}}$ and $e_1, \ldots, e_4 \in \tilde{E}^o$ four cyclically ordered, outgoing edges at $\tilde{v}$, we may assume that $\tilde{\mathfrak{l}}$ uses e_1^{-1} and e_3 while $e_2, e_4 \notin E_{\tilde{\mathfrak{l}}}$. Therefore $d\omega_{e_2}.\Phi_{\tilde{\mathfrak{l}}} = d\omega_{e_4}.\Phi_{\tilde{\mathfrak{l}}} = 0$ and as $\mu_{\tilde{v}} = \pm(d\omega_{e_2} + d\omega_{e_4})$, $(\mu_{\tilde{v}}.\Phi_{\tilde{\mathfrak{l}}})(\tilde{a}) = 0 = \phi_{\mathfrak{l}_1}(a)\phi_{\mathfrak{l}_2}(a)$.

In the second case, for $\tilde{v} \in V_{\tilde{\mathfrak{l}}} \cap p^{-1}(v) = T_{\tilde{\mathfrak{l}}} \cap p^{-1}(v)$, by definition of the fundamental cover, $\mathfrak{l}_1 \sim_h c_v \sim_h \mathfrak{l}_2$. Then $\delta_{\tilde{v}} \tilde{\mathfrak{l}}' = \tilde{\mathfrak{l}}_1 \otimes \tilde{\mathfrak{l}}_2$, where $\tilde{\mathfrak{l}}_1, \tilde{\mathfrak{l}}_2$ are lift with initial condition v, so that using 3. a) of Theorem 3.8, we get

$$(\mu_{\tilde{v}}.\Phi_{\tilde{\mathfrak{l}}})(\tilde{a}) = \Phi_{\tilde{\mathfrak{l}}_1}(\tilde{a})\Phi_{\tilde{\mathfrak{l}}_2}(\tilde{a}) = \phi_{\mathfrak{l}_1}(a)\phi_{\mathfrak{l}_2}(a).$$

□

The main technical result of this article is the proof of the following uniqueness statements. Denote by $\mathfrak{L}_g$ the space of regular loops of regular maps of genus $g \geq 1$. Let us say that a subset $\mathcal{F}$ of $\mathfrak{L}_g$ is a *good boundary condition of the Makeenko–Migdal equations* if for any pair ϕ^∞ and $(\phi^N)_{N\geq 1}$ made of an exact and an approximate solutions of Makeenko–Migdal equations,

$$\lim_{N\to\infty} \|\phi^N_{\mathfrak{l}} - \phi^\infty_{\mathfrak{l}}\|_\infty + \|\mathscr{V}_{\phi^N,\mathfrak{l}}\|_\infty = 0, \forall \mathfrak{l} \in \mathcal{F} \tag{38}$$

implies

$$\lim_{N\to\infty} \|\phi^N_{\mathfrak{l}} - \phi^\infty_{\mathfrak{l}}\|_\infty + \|\mathscr{V}_{\phi^N,\mathfrak{l}}\|_\infty = 0, \forall \mathfrak{l} \in \mathfrak{L}_g. \tag{39}$$

Setting

$$\Psi^N_{\mathfrak{l}} = \phi^N_{(\mathfrak{l}-\phi^\infty_{\mathfrak{l}} c)\otimes(\mathfrak{l}-\phi^\infty_{\mathfrak{l}} c)^*} = \mathscr{V}_{\phi^N,\mathfrak{l}} + |\phi^N_{\mathfrak{l}} - \phi^\infty_{\mathfrak{l}}|^2, \tag{40}$$

where c is the constant loop at $\underline{\mathfrak{l}}$, this is equivalent to

$$\lim_{N\to\infty} \|\Psi^N_{\mathfrak{l}}\|_\infty = 0, \forall \mathfrak{l} \in \mathcal{F} \Rightarrow \lim_{N\to\infty} \|\Psi^N_{\mathfrak{l}}\|_\infty = 0, \forall \mathfrak{l} \in \mathfrak{L}_g.$$

Proposition 3.17. *For any genus $g \geq 1$ and total volume $T > 0$, the family of loops $\mathfrak{l} \in \mathfrak{L}_g$ with a sub-path γ such that $(\mathfrak{l}, \gamma)$ is a marked loop and $(\mathfrak{l}, \gamma)^\wedge$ is geodesic, is a good boundary condition.*

Denote by $\mathfrak{L}^*_g$ the subset of $\mathfrak{L}_g$ of loops $\mathfrak{l}$ with $[\mathfrak{l}]_{\mathbb{Z}} \neq 0$. Let us say that a subset $\mathcal{F}^*$ of $\mathfrak{L}^*_g$ is a *good boundary condition in homology* if for any pair ϕ^∞ and $(\phi^N)_{N\geq 1}$ made of an exact and an approximate solution of Makeenko–Migdal equations, using the same notation as in (40),

$$\lim_{N\to\infty} \|\Psi^N_{\mathfrak{l}}\|_\infty = 0, \forall \mathfrak{l} \in \mathcal{F}^* \Rightarrow \lim_{N\to\infty} \|\Psi^N_{\mathfrak{l}}\|_\infty = 0, \forall \mathfrak{l} \in \mathfrak{L}^*_g.$$

The following can be proven independently from Proposition 3.17.

Proposition 3.18. *For any genus $g \geq 1$ and total volume $T > 0$, the family of geodesic loops in $\mathfrak{L}^*_g$ is a good boundary condition in homology.*

When $g = 1$, for any loop $\mathfrak{l} \in \mathfrak{L}_g$, $\mathfrak{l} \sim_h c_{\underline{\mathfrak{l}}}$ if and only if $[\mathfrak{l}]_{\mathbb{Z}} = 0$ and any geodesic loop is of the form s^d where s is a simple loop and $d \geq 1$. Therefore the Proposition 3.18 and 3.17 have the following consequence.

Corollary 3.19. *Consider $g = 1$, $T > 0$, the set of regular loops $\mathfrak{l} \in \mathfrak{L}_g$ such that $|\mathfrak{l}|_D = 0$ or $\mathfrak{l} = s^d$ for some simple loop s and some integer $d \geq 1$ is a good boundary condition.*

Proof of Theorem 3.13 and Proposition 3.14. Since L^2 convergence implies convergence in probability, both statements follow from Lemma 3.16 and of respectively proposition 3.17 and 3.18. □

Proof of Theorem 3.12. Using the solutions given by 1. and 2. of Lemma 3.16, Theorem 3.11 implies that the boundary condition of corollary 3.19 are satisfied. Therefore the convergence in probability holds true for any regular loops. Using Lemma 3.7, it follows that the convergence holds for all $\gamma \in A(\Sigma) \cap \mathrm{L}(\Sigma)$. When $\gamma \in A(\Sigma) \setminus \mathrm{L}(\Sigma)$, under YM_Σ, W_γ is Haar distributed, so that $\mathbb{E}_{\mathrm{YM}_\Sigma}[|W_\gamma|^2] \to 0$ as $N \to \infty$ by [18]. To prove the convergence in probability for any path of finite length, it is now enough to combine the area bound (26) with Proposition 3.6. The uniqueness claim is proved identically considering in place of the above approximate solution, a constant sequence given by an exact solution. □

4 Proof of the main result, stability of convergence under deformation

In this section we give a proof first of Proposition 3.18, then of Proposition 3.17. We consider exact and approximate solutions ϕ^∞ and $(\phi^N)_{N\geq 1}$ of Makeenko–Migdal equations in genus $g \geq 1$ and volume $T > 0$, define Ψ^N as in (40) and consider the subset $\mathfrak{B}_g \subset \mathfrak{L}_g$ of loops $\mathfrak{l}$ with map $\mathbb{G}$, satisfying

$$\Psi^N_{\mathfrak{l}} \underset{N\to\infty}{\to} 0 \text{ uniformly on } \Delta_{\mathbb{G}}(T). \tag{41}$$

Our aim is to find a small subset $\mathfrak{C}_g$ of loops in $\mathfrak{L}_g$, such that $\mathfrak{C}_g \subset \mathfrak{B}_g$ implies $\mathfrak{B}_g = \mathfrak{L}_g$. In the first and second second sections, we shall use respectively the following bounds. Thanks to (35), (36) and (37), using the same notation, for any $\mathfrak{l} \in \mathfrak{L}_g$ and $X \in \mathfrak{m}_{\mathfrak{l}}$,

$$|X.\Psi^N_{\mathfrak{l}}| \leq \|X\| C'_{\mathfrak{l}} \left(\sum_{v\in V_{\mathfrak{l}}} \left(\sqrt{\Psi^N_{\mathfrak{l}_{v,1}}} + |\phi^\infty_{\mathfrak{l}_{v,1}}| \right) \left(\sqrt{\Psi^N_{\mathfrak{l}_{v,2}}} + |\phi^\infty_{\mathfrak{l}_{v,2}}| \right) + \frac{1}{N} \right) \tag{42}$$

and

$$|X.\Psi^N_{\mathfrak{l}}| \leq \|X\| C'_{\mathfrak{l}} \left(\Psi^N_{\mathfrak{l}} + \sum_{v\in V_{\mathfrak{l}}} (\Psi^N_{\mathfrak{l}_{v,1}} + \Psi^N_{\mathfrak{l}_{v,2}}) + \frac{1}{N} \right) \tag{43}$$

where $C'_{\mathfrak{l}} > 0$ is a constant independent of $N \geq 1$.

4.1 Non-null homology loops

Let us denote by $\mathfrak{B}^*_g$ the subset $\mathfrak{B}_g \cap \mathfrak{L}^*_g$. This sub-section purpose is to prove proposition 3.18. It is equivalent to the following statement. Denote by $\mathfrak{C}^*_g$ the subset of $\mathfrak{L}^*_g$ of regular loops with non-zero homology which are geodesic.

Theorem 4.1. *Assume that $\mathfrak{C}^*_g \subset \mathfrak{B}^*_g$. Then $\mathfrak{B}^*_g = \mathfrak{L}^*_g$.*

To prove this Theorem we shall use the following application of Makeenko–Migdal equations.

Lemma 4.2. *Let $\mathfrak{l}, \mathfrak{l}' \in \mathfrak{L}^*_g$ be two loops of a regular map $\mathbb{G}$ with faces set F, such that there is $K \subset F$ with $K \neq F$ and $\mathfrak{l} \sim_K \eta\mathfrak{l}'\eta^{-1}$ where η is a path with $\underline{\eta} = \underline{\mathfrak{l}'}$ and $\overline{\eta} = \underline{\mathfrak{l}}$. Assume that $\mathfrak{l}' \in \mathfrak{B}^*_g$ and that for any $v \in V_{\mathfrak{l}}$, if $\delta_v(\mathfrak{l}) = \mathfrak{l}_1 \otimes \mathfrak{l}_2$, then $\mathfrak{l}_1$ or $\mathfrak{l}_2$ belongs to $\mathfrak{B}^*_g$. Then $\mathfrak{l} \in \mathfrak{B}^*_g$.*

We split the proof Theorem 4.1 into two steps enacted by the two following propositions. The first one allows to contract inner loops. The second allows to follow a shortening sequence from proper loops to loops conjugated to a geodesic. Denote by $\mathfrak{P}^*_g$ of loops of $\mathfrak{L}^*_g$ which are proper or included in a fundamental domain. Theorem 4.1 follows directly from the following Proposition.

Proposition 4.3. *a) If $\mathfrak{P}^*_g \subset \mathfrak{B}^*_g$, then $\mathfrak{B}^*_g = \mathfrak{L}^*_g$.*

b) If $\mathfrak{C}^*_g \subset \mathfrak{B}^*_g$, then $\mathfrak{P}^*_g \subset \mathfrak{B}^*_g$.

Proof. Let us prove first point a). Assume $\mathfrak{P}^*_g \subset \mathfrak{B}^*_g$ and introduce for any $n \geq 0$ the subset $\mathfrak{L}^*_{n,g}$ of loops $\mathfrak{l} \in \mathfrak{L}^*_g$ with $\#V_{\mathfrak{l}} \leq n$. We shall prove by induction on n

that $\mathfrak{L}^*_{n,g} \subset \mathfrak{B}^*_g$. As a simple regular loop with non-zero homology belongs to $\mathfrak{P}^*_g$, $\mathfrak{L}^*_{0,g} \subset \mathfrak{B}^*_g$. Consider $n > 0$ and assume that the claim holds true for $n-1$. Consider $\mathfrak{l} \in \mathfrak{L}_{n,g}$ with $\#V_{c,\mathfrak{l}} > 0$. Let us write $\mathfrak{l} \sim_c \gamma_0.\mathfrak{l}_*.\gamma_2$ where $\mathfrak{l}_*$ is an inner loop of $\mathfrak{l}$ and γ_0, γ_2 are reduced loops. The loop $\mathfrak{l}' = \gamma_1\gamma_2$ is regular and homotopic to $\mathfrak{l}$, while $\mathfrak{l}_* \in V_{\mathfrak{l}} \setminus V_{\mathfrak{l}'}$. Therefore $\mathfrak{l}' \in \mathfrak{L}^*_{n-1,g}$ and by induction $\mathfrak{l}' \in \mathfrak{B}^*_g$. Being included in D^o, $\mathfrak{l}_*$ is contractible. Denoting by K its bulk, $K \neq F$ and $\mathfrak{l} \sim_K \mathfrak{l}'$. Moreover for any transversed intersection $v \in V_{\mathfrak{l}}$, if $\delta_v.\mathfrak{l} = \mathfrak{l}_1 \otimes \mathfrak{l}_2$ where $\mathfrak{l}_1$ and $\mathfrak{l}_2$, then

$$[\mathfrak{l}_1] + [\mathfrak{l}_2] = [\mathfrak{l}] \text{ and } \#V_{\mathfrak{l}_1} + \#V_{\mathfrak{l}_2} + 1 = \#V_{\mathfrak{l}}.$$

Since $[\mathfrak{l}] \neq 0$, we can assume that $[\mathfrak{l}_1] \neq 0$. Then, thanks to the last equation $\mathfrak{l}_1 \in \mathfrak{L}^*_{g,n-1}$ and by induction $\mathfrak{l}_1 \in \mathfrak{B}^*_g$. By Lemma 4.2, it follows that $\mathfrak{l} \in \mathfrak{B}^*_g$, which concludes the proof of the first point.

Consider now b), assume that $\mathfrak{C}^*_g \subset \mathfrak{B}^*_g$ and introduce for any $n \geq 0$ the subset $\mathfrak{P}^*_{n,g}$ of proper loops $\mathfrak{l} \in \mathfrak{P}^*_g$ with $|\mathfrak{l}|_D \leq n$. We shall prove by induction on n that $\mathfrak{P}^*_{n,g} \subset \mathfrak{B}^*_g$. When $\mathfrak{l} \in \mathfrak{P}^*_{g,0}$, then $\mathfrak{l}$ is included in a fundamental domain and $\mathfrak{l} \in \mathfrak{C}^*_g$. By assumption $\mathfrak{l} \in \mathfrak{B}^*_g$.

Assume that $n > 0$ and $\mathfrak{P}^*_{n-1,g} \subset \mathfrak{B}^*_g$, and consider $\mathfrak{l} \in \mathfrak{P}^*_{n,g}$. According to Proposition 2.4, there is a geodesic loop $\mathfrak{l}' \in \mathfrak{C}^*_g$ and shortening homotopy sequence $\mathfrak{l}_1, \ldots, \mathfrak{l}_m$ of proper loops with $\mathfrak{l}_1 = \mathfrak{l}$ and $\mathfrak{l}_m \sim_K \eta\mathfrak{l}'\eta^{-1}$ for some path η and proper subset of faces K. By assumption $\mathfrak{l}' \in \mathfrak{B}^*_g$. Therefore, by induction and Lemma 4.2, it is enough to prove that for any $1 \leq k \leq m$ and $v \in V_{\mathfrak{l}_m}$, then $\delta_v.\mathfrak{l}_m = \alpha \otimes \beta$ for some regular loops with α or β belonging to $\mathfrak{B}^*_g$. Now since $\mathfrak{l}_m$ is proper, (19) yields

$$|\alpha|_D, |\beta|_D < |\mathfrak{l}_m|_D \leq |\mathfrak{l}_1|_D \leq n.$$

Therefore by induction α and β belong to $\mathfrak{B}^*_g$. This concludes the proof of b). □

To complete the proof of Theorem 4.1, it remains to do the following.

Proof of Lemma 4.2. Setting

$$a'(f) = \begin{cases} \frac{T}{\#F - \#F_1} & \text{if } f \notin K, \\ 0 & \text{if } f \in K. \end{cases} \tag{44}$$

defines an element of $\Delta_K(T)$. According to the compatibility Lemma **??** and using that $\mathfrak{l}' \in \mathfrak{B}^*_g$,

$$\Psi^N_{\mathfrak{l}}(a') = \Psi^N_{\eta\mathfrak{l}'\eta^{-1}}(a') = \Psi^N_{\mathfrak{l}'}(a') \longrightarrow 0 \text{ as } N \to +\infty. \tag{*}$$

Now since $[\mathfrak{l}] \neq 0$ and $a, a' \in \Delta_{\mathbb{G}}(T)$, according to Lemma 2.3, $X = a - a' \in \mathfrak{m}_{\mathfrak{l}}$. For any $v \in V_{\mathfrak{l}}$, by assumption, we have $\delta_v.\mathfrak{l} = \mathfrak{l}_1 \otimes \mathfrak{l}_2$ with $\mathfrak{l}_1 \in \mathfrak{B}^*_g$. Therefore using the inequality (42), each term of the summand vanishes uniformly on $\Delta_{\mathbb{G}}(T)$ as $N \to \infty$ and for any $t \in (0,1)$,

$$|\partial_t \Psi^N_{\mathfrak{l}}(a + tX)| = |X.\Psi^N_{\mathfrak{l}}(a + tX)| \leq C_{\mathfrak{l}}\|X\|\varepsilon_N \leq C_{\mathfrak{l}}(\|a\| + \|a'\|)\varepsilon_N \tag{**}$$

where $\varepsilon_N \to 0$. Thanks to the boundary condition (*), we conclude that

$$\Psi^N_{\mathfrak{l}}(a) = \Psi^N_{\mathfrak{l}'}(a') + \int_0^1 \partial_t \Psi_{\mathfrak{l}}(a + tX)dt$$

converges to 0 uniformly in $a \in \Delta_{\mathbb{G}}(T)$, as $N \to \infty$, that is $\mathfrak{l} \in \mathfrak{B}^*_g$. □

4.2 Null homology loops

The purpose of this sub-section is to prove Proposition 3.17. It is equivalent to the following statement. Denote by $\mathfrak{C}_g^\vee$ the subset of $\mathfrak{L}_g$ of regular loops $\mathfrak{l}$, such that there is a nested sub-path γ_{nest} of $\mathfrak{l}$ making $(\mathfrak{l}, \gamma_{nest})$ a marked loop on a map of genus g and with $\mathfrak{l}^\wedge$ geodesic.

Theorem 4.4. *If $\mathfrak{C}_g^\vee \subset \mathfrak{B}_g$, then $\mathfrak{B}_g = \mathfrak{L}_g$.*

To prove this Theorem, we shall use the following Lemma, formally analog to Lemma 4.2. Though, unlike Lemma 4.2, due to the new constraint on the Makeenko–Migdal vectors, we shall work here with marked loops which allow to change the nested part in order to keep the contraint satisfied while performing the required homotopy. This will break the induction on the number of intersection points. This issue leads us to use of a different "complexity" function for loops, and is implemented by a multiple induction on its parameters. The modification of the nested part yields a constant complexity when applying Makeenko–Migdal equations. An idea is then to apply Grönwall's lemma, taking advantage of the nested structure of the modification. This is presented in step 4 of the proof of Lemma 4.5 .

Denote by $\mathfrak{L}_g^{\mathfrak{m}}$ the set of marked loops on a regular map of genus g.

Lemma 4.5. *Assume that $\mathfrak{l} \in \mathcal{B}_g$ for any nested loop $\mathfrak{l} \in \mathfrak{L}_g$. Let $(\alpha, \alpha_{nest}), (\beta, \beta_{nest}) \in \mathfrak{L}_g^m$ be two marked loops on a regular map and K proper subset of faces, such that $\alpha_{nest} = \beta_{nest}$ and $\alpha^{\wedge *} \sim_K \beta^{\wedge *}$, while*

(i) $\beta' \in \mathfrak{B}_g$, whenever $(\beta', \beta'_{nest}) \in \mathfrak{L}_g^{\mathfrak{m}}$ with $\beta'^{\wedge *} = \beta^{\wedge *}$,

(ii) for all $v \in V_\alpha$ with $\delta_v(\alpha) = \alpha_1 \otimes \alpha_2$, $\alpha_1, \alpha_2 \in \mathfrak{B}_g$,

(iii) if $\alpha^\wedge$ is not proper, whenever $(\alpha', \alpha'_{nest})$ is a twist of (α, α_{nest}), denoting by V_{tw} the intersection points of the twisted part of α'_{nest},

$$\forall v \in V_{\alpha'} \setminus V_{tw}, \delta_v(\alpha') = \alpha_1 \otimes \alpha_2, \text{ with } \alpha_1, \alpha_2 \in \mathfrak{B}_g,$$

(iv) if $\alpha^\wedge$ is a proper loop, whenever $(\alpha', \alpha'_{nest})$ is a twist of a pull move of (α, α_{nest}), denoting by V_{tw} the intersection points of the twisted part of α'_{nest},

$$\forall v \in V_{\alpha'} \setminus V_{tw}, \delta_v(\alpha') = \alpha_1 \otimes \alpha_2, \text{ with } \alpha_1, \alpha_2 \in \mathfrak{B}_g.$$

When $\alpha^\wedge$ is not proper, assume furthermore that (α, α_{nest}) has a moving edge or vertex that is not adjacent to K.

Then $\alpha \in \mathfrak{B}_g$.

Using this lemma, the rest of the proof is a refinement of the null-homology case using a suitable complexity on marked loops. We shall study the subset $\mathfrak{B}_g^{\mathfrak{m}} \subset \mathfrak{L}_g^{\mathfrak{m}}$ of marked loops $(\mathfrak{l}, \mathfrak{l}_{nest}) \in \mathfrak{L}_g^{\mathfrak{m}}$ with $\mathfrak{l} \in \mathfrak{B}_g$. Denote by $\mathfrak{C}_g^{\mathfrak{m}}$ the set of marked loops $(\mathfrak{l}, \mathfrak{l}_{nest}) \in \mathfrak{L}_g^{\mathfrak{m}}$ with $\mathfrak{l}^\wedge \in \mathfrak{C}_g$. Denote by $\mathfrak{P}_g^{\mathfrak{m}}$ the set of marked loops $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{L}_g^{\mathfrak{m}}$ with $\mathfrak{l}^\wedge$ proper or included in a fundamental domain. With this definition, if $\mathfrak{C}_g^\vee \subset \mathfrak{B}_g^0$, then $\mathfrak{C}_g^{\mathfrak{m}} \subset \mathfrak{B}_g^{\mathfrak{m}}$, whereas if $\mathfrak{B}_g^{\mathfrak{m}} = \mathfrak{L}_g^{\mathfrak{m}}$, then $\mathfrak{L}_g = \mathfrak{B}_g$. Theorem 4.4 is then a direct consequence of the following Proposition.

Proposition 4.6. *a) If $\mathfrak{P}_g^{\mathfrak{m}} \subset \mathfrak{B}_g^{\mathfrak{m}}$, then $\mathfrak{B}_g^{\mathfrak{m}} = \mathfrak{L}_g^{\mathfrak{m}}$.*

b) If $\mathfrak{C}_g^{\mathfrak{m}} \subset \mathfrak{B}_g^{\mathfrak{m}}$, then $\mathfrak{P}_g^{\mathfrak{m}} \subset \mathfrak{B}_g^{\mathfrak{m}}$.

Proof. We shall prove both statements by multiple inductions. Denote by $\mathbb{N}$ the set of non-negative integers and for any positive integer $K > 0$, let us endow $\mathbb{N}^K$ with the lexicographic order $\prec$. Consider a family of statements $(P_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^K}$ and $1 \leq k \leq K$. Starting from any $\mathbf{n} \in \mathbb{N}^K$, any strictly decreasing sequence in $(\mathbb{N}^K, \prec)$ with more than $\mathbf{n}_k + \ldots + \mathbf{n}_K$ steps will have an element with at least one of the first k-coordinates vanishing. Assume that $P_{\mathbf{n}}$ holds true for all $\mathbf{n} \in \mathbb{N}^K$ with $\mathbf{n}_l = 0$ for some $1 \leq l \leq k$. To prove $P_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{N}^K$, it is then enough to prove that for all $\mathbf{n} \in \mathbb{N}^K$, if $P_{\mathbf{n}'}$ holds true for any $\mathbf{n}' \prec \mathbf{n}$ with $\mathbf{n}' \neq \mathbf{n}$, then $P_{\mathbf{n}}$ holds true. When $\mathcal{C} : \mathfrak{L}_g^{\mathfrak{m}} \to \mathbb{N}^K$ has been fixed, for all $\mathbf{n} \in \mathbb{N}^K$, we denote by $\mathfrak{L}_{\mathbf{n}}, \mathfrak{L}_{\mathbf{n}}^-$ and $\mathfrak{P}_{\mathbf{n}}, \mathfrak{P}_{\mathbf{n}}^-$ the marked loops $x = (\mathfrak{l}, \gamma_{nest})$ of $\mathfrak{L}_g^{\mathfrak{m}}$ and $\mathfrak{P}_g^{\mathfrak{m}}$, with respectively $\mathcal{C}(x) \prec \mathbf{n}$ and $\mathcal{C}(x) \underset{\neq}{\prec} \mathbf{n}$.

Let us prove first point a). Assume $\mathfrak{P}_g^{\mathfrak{m}} \subset \mathfrak{B}_g^{\mathfrak{m}}$. For any marked loop $x = (\mathfrak{l}, \gamma_{nest})$, let us set $\mathcal{C}(x) = (|\mathfrak{l}|_D, \#V_{\mathfrak{l}^\wedge}, \#V_{\mathfrak{l}_{nest}})$. Consider $\mathbf{n} \in \mathbb{N}^3$ and $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{L}_{\mathbf{n}}$. If $\mathbf{n}_2 = 0$, $\mathfrak{l}^\wedge$ has no contraction point, $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{P}_g^{\mathfrak{m}}$ and by assumption $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{B}_g^{\mathfrak{m}}$. If $\mathbf{n}_1 = 0$, $\mathfrak{l}$ is included in a fundamental domain so that $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{B}_g^{\mathfrak{m}}$.

Let us now fix $\mathbf{n} \in \mathbb{N}^3$ and assume that $\mathfrak{L}_{\mathbf{n}}^- \subset \mathfrak{B}_g^{\mathfrak{m}}$. Let us prove that $\mathfrak{L}_{\mathbf{n}} \subset \mathfrak{B}_g^{\mathfrak{m}}$. Consider $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{L}_{\mathbf{n}}$. We may assume that $\mathbf{n}_2 > 0$ and $(\mathfrak{l}, \gamma_{nest}) \notin \mathfrak{P}_g$. Consequently, we can write $\mathfrak{l} \sim_c \gamma_0.\mathfrak{l}_*.\gamma_2$ where $\mathfrak{l}_*$ is an inner loop of $\mathfrak{l}$ that does not intersect γ_{nest}, while γ_0, γ_2 are reduced loops. The loop $\mathfrak{l}' = \gamma_0\gamma_2$ is regular and homotopic to $\mathfrak{l}$, while $(\mathfrak{l}', \gamma_{nest})$ is a marked loop with $\underline{\mathfrak{l}}_* \in V_{\mathfrak{l}^\wedge} \setminus V_{\mathfrak{l}'^\wedge}$. Therefore $\#V_{\mathfrak{l}^\wedge} < \#V_{\mathfrak{l}^\wedge}$. Besides $|\mathfrak{l}'|_D = |\mathfrak{l}|_D$, so that $x \in \mathfrak{L}_{\mathbf{n}}^- \subset \mathfrak{B}_g^{\mathfrak{m}}$ for any marked loop $x \in \mathfrak{L}_g$ with $x^\wedge = (\mathfrak{l}', \gamma_{nest})^\wedge$.

Being included in D^o, $\mathfrak{l}_*$ is contractible. Denoting by K its bulk, $K \neq F$ and $\mathfrak{l} \sim_K \mathfrak{l}'$. Possibly adding a vertex to an edge of the first path of the tile decomposition of $\mathfrak{l}$ that does not belong to an inner loop of $\mathfrak{l}$ and rerooting $\mathfrak{l}$, we can assume w.l.o.g. that γ_{nest} is not constant. By definition of marked loops, a moving edge e does not belong to an inner loop and is therefore not adjacent to K. Since $(\mathfrak{l}, \gamma_{nest})$ is not proper, it remains to check the conditions (ii) and (iii) of Lemma 4.5 for the pair $(\mathfrak{l}, \gamma_{nest}), (\mathfrak{l}', \gamma_{nest})$. It is enough to check (iii). Consider $v \in V_{\mathfrak{l}_{tw}}$ for a twist move $(\mathfrak{l}_{tw}, \gamma_{tw})$ of $(\mathfrak{l}, \gamma_{nest})$ and write $\delta_v(\mathfrak{l}_{tw}) = \mathfrak{l}_1 \otimes \mathfrak{l}_2$. If v is an intersection of $\mathfrak{l}_{tw}^{\wedge_*}$ with γ_{tw}, since γ_{tw} does not intersect transversally $\mathfrak{l}_{tw,D}$, $|\mathfrak{l}_1|_D, |\mathfrak{l}_2|_D < |\mathfrak{l}_t| = |\mathfrak{l}|_D$. As for any $\alpha \in \mathfrak{L}_g$ with $|\alpha|_D < |\mathfrak{l}|_D$ $(\alpha, \underline{\alpha}) \in \mathfrak{L}_{\mathbf{n}}^- \subset \mathfrak{B}_g^{\mathfrak{m}}$, $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathfrak{B}_g$. If $v \in V_{\mathfrak{l}_{tw}^\wedge}$, $\mathfrak{l}_{tw,nest}$ is a sub-loop of $\mathfrak{l}_1$ or $\mathfrak{l}_2$, say $\mathfrak{l}_1$. Then $(\mathfrak{l}_1, \mathfrak{l}_{tw,nest})$ is a marked loop, while $\mathfrak{l}_2$ is a sub-loop of $\mathfrak{l}_{tw}^\wedge$ and

$$\#V_{\mathfrak{l}_1^\wedge}, \#V_{\mathfrak{l}_2} < \#V_{\mathfrak{l}^\wedge} \text{ and } |\mathfrak{l}_1|_D, |\mathfrak{l}_2|_D \leq |\mathfrak{l}|_D.$$

Therefore $(\mathfrak{l}_1, \mathfrak{l}_{tw,nest}), (\mathfrak{l}_2, \underline{\mathfrak{l}}_2) \in \mathfrak{L}_{\mathbf{n}}^-$. Lastly if $v \in V_{\mathfrak{l}_{nest}} \setminus V_{\mathfrak{l}_{tw}}$, $\mathfrak{l}_1$ or $\mathfrak{l}_2$, say $\mathfrak{l}_2$, is a sub-loop of $\mathfrak{l}_{t,nest}$. Since $\mathfrak{l}_{t,nest}$ is nested and $v \notin V_{tw}$, there is a sub-path $\gamma_{1,nest}$ of γ_{nest} such that $(\mathfrak{l}_1, \gamma_{1,nest})$ is a marked loop with $(\mathfrak{l}_1, \gamma_{1,nest})^\wedge = (\mathfrak{l}, \gamma_{nest})^\wedge$. Denoting by $\mathfrak{l}_{1,nest}$ its nested sub-loop,

$$\#V_{c,\mathfrak{l}_1^\wedge} = \#V_{c,\mathfrak{l}^\wedge}, \#V_{\mathfrak{l}_{1,nest}} < \#V_{\mathfrak{l}} \text{ while } |\mathfrak{l}_1|_D = |\mathfrak{l}|_D \text{ and } |\mathfrak{l}_2|_D = 0. \tag{45}$$

Therefore $(\mathfrak{l}_1, \gamma_{1,nest}), (\mathfrak{l}_2, \underline{\mathfrak{l}}_2) \in \mathfrak{L}_{\mathbf{n}}^-$. We conclude in the three cases that $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathfrak{B}_g$. This concludes the proof of point a).

Consider now b) and assume that $\mathfrak{C}_g^{\mathfrak{m}} \subset \mathfrak{B}_g^{\mathfrak{m}}$. For any marked loop $x = (\mathfrak{l}, \gamma_{nest}) \in \mathfrak{L}_g$, let us set $|x|_s = 0$ if there is a marked loop $y = (\mathfrak{l}', \gamma'_{nest})$ with $y^\wedge$ geodesic and $\mathfrak{l} \sim_\Sigma \mathfrak{l}'$. Otherwise let $|\mathfrak{l}|_s$ be the smallest $l \geq 1$ such that there is a shortening homotopy sequence $\mathfrak{l}_1, \ldots, \mathfrak{l}_{l+1}$, satisfying the property of Lemma 2.8. We then set for any marked loop $x = (\mathfrak{l}, \gamma_{nest})$, with nested loop $\mathfrak{l}_{nest}$,

$$\mathcal{C}(x) = (|\mathfrak{l}|_D, |x|_s, \#V_{\mathfrak{l}^{\wedge_*}}, \#V_{\mathfrak{l}_{nest}}).$$

Consider $\mathbf{n} \in \mathbb{N}^5$ and $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{P}_{\mathbf{n}}$. If $\mathbf{n}_1 = 0$, as argued above, $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{B}_g^{\mathfrak{m}}$. If $\mathbf{n}_2 = 0$, then $\mathfrak{l} \sim_\Sigma \mathfrak{l}'$ for some marked loop $y = (\mathfrak{l}', \gamma'_{nest}) \in \mathfrak{C}_g^m$. Then $\Psi_{\mathfrak{l}}^N = \Psi_{\mathfrak{l}'}^N$ and by assumption, $(\mathfrak{l}, \gamma_{nest}) \in \mathfrak{B}_g^{\mathfrak{m}}$.

Let us now fix $\mathbf{n} \in \mathbb{N}^5$ and assume that $\mathfrak{P}_{\mathbf{n}}^- \subset \mathfrak{B}_g$. Let us prove that $\mathfrak{P}_{\mathbf{n}} \subset \mathfrak{B}_g^{\mathfrak{m}}$. Consider $x = (\mathfrak{l}, \gamma_{nest}) \in \mathfrak{P}_{\mathbf{n}}$. Since $\mathcal{C}(x)$ does not depend on the number of transverse intersections between γ_{nest} and $\mathfrak{l}$, we can assume w.l.o.g. that $\underline{\gamma}_{nest} = \overline{\gamma}_{nest}$. Besides, we can also assume $|\mathfrak{l}|_s > 0$. Consider a shortening homotopy sequence $\sigma_1, \dots, \sigma_{|x|_s+1}$ as in Lemma 2.8, such that $\sigma_1 = \mathfrak{l}$ and $(\sigma_k, \mathfrak{l}_{nest})_{1 \le k \le |x|_s+1}$ are marked loops.

It is enough to prove that the assumptions of Lemma 4.5 are satisfied for the pair $x_1 = (\mathfrak{l}, \mathfrak{l}_{nest}), x_2 = (\sigma_2, \mathfrak{l}_{nest})$. Since $\mathfrak{L}_{\mathbf{n}}^- \subset \mathfrak{B}_g^{\mathfrak{m}}$, while $|\sigma_2^\wedge|_D = |\sigma_2|_D \le |\sigma_1|_D = |\mathfrak{l}|_D$ and $|x_2|_s < |x|_s \le \mathbf{n}_2$, point (i) is satisfied. Let us check (iv). Consider a twist $(\mathfrak{l}_{tp}, \gamma_{tp})$ of a pull move $(\mathfrak{l}_p, \gamma_p)$ of $(\sigma_1, \gamma_{1,nest})$, denote by V_{tw} the intersection point of the twisted part of $\gamma_{1,nest}$ and let us show that for all $v \in V_{\mathfrak{l}_{tp}} \setminus V_{tw}$, $\delta_v(\mathfrak{l}_{tp}) = \mathfrak{l}_a \otimes \mathfrak{l}_b$ for loops $\mathfrak{l}_a$ and $\mathfrak{l}_b$ of marked loops belonging to $\mathfrak{L}_{\mathbf{n}}^-$. Now since $\mathfrak{l}_{tp}^\wedge$ is proper, for any $v \in V_{\mathfrak{l}_{tp}^\wedge}$,

$$|\mathfrak{l}_a|_D, |\mathfrak{l}_b|_D < |\mathfrak{l}|_D.$$

When $v \in V_{\mathfrak{l}_{tp,nest}} \setminus V_{tw}$, the argument is almost identical to the non-proper case: $\mathfrak{l}_a$ or $\mathfrak{l}_b$, say $\mathfrak{l}_b$ is a sub-loop of $\mathfrak{l}_{tp,nest}$ and since $\mathfrak{l}_{tp,nest}$ is nested and $v \notin V_{tw}$, there is a sub-path $\gamma_{a,nest}$ of $\mathfrak{l}_a$ and $\gamma_{p,nest}$ such that $(\mathfrak{l}_a, \gamma_{a,nest})$ is a marked loop, while

$$\#V_{\mathfrak{l}_a^{\wedge *}} = \#V_{\sigma_1^{\wedge *}}, \#V_{\mathfrak{l}_{a,nest}} < \#V_{\mathfrak{l}_{1,nest}} \text{ and } |\mathfrak{l}_a|_D = |\mathfrak{l}|_D, |\mathfrak{l}_a^\wedge|_s = |\mathfrak{l}^\wedge|_s, |\mathfrak{l}_2|_D = 0.$$

In both cases, $\mathfrak{l}_a, \mathfrak{l}_b \in \mathfrak{L}_{\mathbf{n}}^-$ so that (iv) is satisfied. For (ii), the first argument in the last case analysis yields the claim. Since $\mathfrak{l}^\wedge$ is proper, this concludes the proof of b). □

To complete the proof of Theorem 4.4, it remains to do the following.

Proof of Lemma 4.5. When (α, α_{nest}) has a moving edge e or vertex $\underline{\alpha}_{nest}$ not adjacent to K, let us set $(\mathfrak{l}^{(0)}, \gamma_{nest}^{(0)}) = (\alpha, \alpha_{nest})$ and $e^{(0)} = e$, when α_{nest} is not constant.

When (α, α_{nest}) is proper, w.l.o.g., possibly adding a vertex to the first edge of α, we can assume that $\alpha_{nest} = \beta_{nest}$ is not constant. Let us denote by $(\mathbb{G}_\circ, \mathbb{G}_g)$ a regular map finer than the one of α and β possibly modified accordingly. Let us choose an arbitrary path γ^* of $\mathbb{G}_\circ^* \setminus \partial E$ whose first edge crosses e and with $\overline{\gamma^*} \notin K$. We then define $(\mathfrak{l}^{(0)}, \gamma_{nest}^{(0)})$ as the pull move of (α, α_{nest}) along γ^*. Denote by $\mathbb{G}^{(0)} = (V^{(0)}, E^{(0)}, F^{(0)})$ the associated map finer than $\mathbb{G}_\circ$. The marked loop $(\mathfrak{l}^{(0)}, \gamma_{nest}^{(0)})$ has then a moving edge $e^{(0)}$ with both adjacent faces disjoint from K. Recall that F_{stem} denotes the subset of faces of $\mathbb{G}^{(0)}$ associated to the pull move. Then $\mathfrak{l}^{(0)} \sim_{F_{stem}} \alpha$ and the restriction map $\Delta_{F_{stem}}(T) \to \Delta_{\mathbb{G}}(T)$ is surjective. In both cases, it is enough to prove that $\mathfrak{l}^{(0)} \in \mathfrak{B}_g$.

Step 1: Let us first argue that the convergence at stake holds true on a subset of $\overline{\Delta_{\mathbb{G}^{(0)}}(T)}$. Since the pull operation does not change the cut of the marked loop, $\mathfrak{l}^{(0)\wedge *} = \alpha^{\wedge *}$ and $\mathfrak{l}^{(0)} = \gamma_{nest}^{(0)} \alpha^{\wedge *} \sim_K \mathfrak{l}^{(0)\prime} = \gamma_{nest}^{(0)} \beta^{\wedge *}$. Also since $\beta^{\wedge *}$ is a regular path of $\mathbb{G}_\circ$ that does not use edges of α_{nest}, while the pull move changes solely α_{nest} using only edges disjoint from edges of $\mathbb{G}_\circ$, $(\mathfrak{l}^{(0)\prime}, \gamma_{nest}^{(0)})$ is a marked loop.

Since ${\mathfrak{l}^{(0)}}'^{\wedge_*} = \beta^{\wedge_*}$, by assumption, ${\mathfrak{l}^{(0)}}' \in \mathfrak{B}_g$. Using that $\mathfrak{l}^{(0)} \sim_K {\mathfrak{l}^{(0)}}'$, $\Psi^N_{\mathfrak{l}^{(0)}} \to 0$ uniformly on $\Delta_{\mathbb{G}^{(0)},K}(T)$.

Let us now use Makeenko–Migdal equations to enlarge this set of convergence. Thanks to Theorem 4.1, we can assume that $[\mathfrak{l}^{(0)}] = 0$. Let f_l, f_r be the faces left and right of the moving edge $e^{(0)}$. Recall from Lemma 2.2, that since $[\mathfrak{l}^{(0)}] = 0$, $\mathfrak{l}^{(0)}$ has a unique winding function $n_{\mathfrak{l}^{(0)}} \in \Omega^2(\mathbb{G}^{(0)})$ with $n_{\mathfrak{l}^{(0)}}(f_l^{(0)}) = \frac{1}{2}$. For any $a \in \Delta_{\mathbb{G}^{(0)}}(T)$ with $|\langle n_{\mathfrak{l}}, a\rangle| \leq \frac{T}{2}$, setting

$$a'(f) = \begin{cases} \frac{T}{2} + \langle n_{\mathfrak{l}^{(0)}}, a\rangle & \text{if } f = f_l, \\ \frac{T}{2} - \langle n_{\mathfrak{l}^{(0)}}, a\rangle & \text{if } f = f_r, \\ 0 & \text{if } f \in F \setminus \{f_l, f_r\}, \end{cases} \tag{46}$$

defines an element $a' \in \Delta_{\mathbb{G}^{(0)}}(T)$ with $\langle n_{\mathfrak{l}^{(0)}}, a\rangle = \langle n_{\mathfrak{l}^{(0)}}, a'\rangle$ and hence $X = a' - a \in \mathfrak{m}_{\mathfrak{l}^{(0)}}$. On the one hand, since $e^{(0)}$ is not adjacent to K, $a' \in \Delta_{\mathbb{G}^{(0)},K}(T)$. On the other hand, for all $v \in V_{\mathfrak{l}^{(0)}}$, thanks to point (ii), $\delta_v(\mathfrak{l}^{(0)}) = \mathfrak{l}_1 \otimes \mathfrak{l}_2$ with $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathfrak{B}_g$. Using exactly the same argument as in Lemma 4.2, it follows that $\Psi^N_{\mathfrak{l}^{(0)}} \to 0$ uniformly on $\{a \in \Delta_{\mathbb{G}^{(0)}} : |\langle n_{\mathfrak{l}^{(0)}}, a\rangle| \leq \frac{T}{2}\}$.

Step 2: From here onwards, the argument differs substantially from the one of Lemma 4.2. Choosing the same convention for $n_{\mathfrak{l}^{(0)-1}}$, $n_{\mathfrak{l}^{(0)-1}} = -n_{\mathfrak{l}^{(0)}}$ and since $\Psi^N_{\mathfrak{l}^{(0)-1}} = \Psi^N_{\mathfrak{l}^{(0)}}$, it is enough to show that as $N \to \infty$, $\Psi^N_{\mathfrak{l}^{(0)}} \to 0$ uniformly on

$$\Delta^{(0)}_+(T) = \left\{ a \in \Delta_{\mathbb{G}^{(0)}}(T) : \langle n_{\mathfrak{l}^{(0)}}, a\rangle \geq \frac{T}{2} \right\}.$$

Let us set $n = \max(1, \max_{f \in F^{(0)}} n_{\mathfrak{l}^{(0)}}(f))$ and define $(\mathfrak{l}, \gamma_{nest})$ as the n-twist of $(\mathfrak{l}^{(0)}, \gamma^{(0)}_{nest})$. Denote by $\mathbb{G} = (V, E, F)$ the associated map finer than $\mathbb{G}^{(0)}$. Recall that F_{tw} denotes the subset of n faces of F associated to the twist move such that $\mathfrak{l} \sim_{F_{tw}} \mathfrak{l}^{(0)}$. Denote by $f_o \in F \setminus F_{tw}$ the face included in f_l neighbouring F_{tw} and by $f_c \in F_{tw}$ the central face of $(\mathfrak{l}, \gamma_{nest})$. Faces of $F^{(0)} \setminus \{f_l\}$ are not changed by the twist and can be identified with $F \setminus (F_{tw} \cup \{f_o\})$. Recall that $[\mathfrak{l}] = [\mathfrak{l}^{(0)}] = 0$ and denote by $n_{\mathfrak{l}}$ the winding number function of $\mathfrak{l}$ with $n_{\mathfrak{l}}(f_r) = 0$. It satisfies

$$n_{\mathfrak{l}}(f_c) = n + \frac{1}{2}, \frac{1}{2} \leq n_{\mathfrak{l}}(f) \leq n - \frac{1}{2}, \forall f \in F_{tw} \setminus \{f_c\}$$

while

$$n_{\mathfrak{l}}(f) = n_{\mathfrak{l}^{(0)}}(f), \forall f \in F \setminus (F_{tw} \cup \{f_o\}) \text{ and } n_{\mathfrak{l}}(f_o) = \frac{1}{2} = n_{\mathfrak{l}^{(0)}}(f_l).$$

Consider now

$$\Delta_+(T) = \left\{ a \in \Delta_{\mathbb{G}}(T) : \langle n_{\mathfrak{l}}, a\rangle \geq \frac{T}{2} \right\}.$$

For any $a \in \Delta_{F_{tw}}(T)$, $n_{\mathfrak{l}}(a) = n_{\mathfrak{l}^{(0)}}(a)$ and the restriction map from $\Delta_{F_{tw},+}(T) = \Delta_{F_{tw}}(T) \cap \Delta_+(T)$ to $\Delta^{(0)}_+(T)$ is surjective. Therefore, since $\mathfrak{l} \sim_{F_{tw}} \mathfrak{l}^{(0)}$, it is enough to show that $\Psi^N_{\mathfrak{l}} \to 0$ uniformly on $\Delta_{F_{tw},+}(T)$.

Step 3: Let us argue as above that this convergence holds true on another set of areas. Since the pull and twist operations do not change the cut of the marked loop, $\mathfrak{l}^{\wedge *} = \alpha^{\wedge *}$ and $\mathfrak{l} = \gamma_{nest}\alpha^{\wedge *} \sim_K \mathfrak{l}' = \gamma_{nest}\beta^{\wedge *}$. Also since $\beta^{\wedge *}$ is a regular path of $\mathbb{G}_\circ$ that does not use edges of α_{nest}, while the pull and twist moves change solely α_{nest} using only edges disjoint from edges of $\mathbb{G}_\circ$, $(\mathfrak{l}', \gamma_{nest})$ is a marked loop. Since $\mathfrak{l}'^{\wedge *} = \beta^{\wedge *}$, by assumption, $\mathfrak{l}' \in \mathfrak{B}_g$. Using that $\mathfrak{l} \sim_K \mathfrak{l}'$,

$$\Psi_{\mathfrak{l}}^N \to 0 \text{ uniformly on } \Delta_{\mathbb{G},K}(T). \tag{47}$$

Step 4: Let us now enlarge the convergence set using Makeenko–Migdal equations. Setting $K_* = F \setminus \{f_c, f_o\}$, as $f_l \notin K$, $K_* \supset K$. Moreover since the winding function is larger than $\frac{1}{2}$ on $\{f_c, f_o\}$,

$$\Delta_{K_*}(T) \subset \Delta_+(T) \cap \Delta_K(T).$$

For any $a \in \Delta_{F_{tw},+}(T)$, setting

$$a'(f) = \begin{cases} \frac{\langle n_{\mathfrak{l}}, a\rangle - \frac{T}{2}}{n} & \text{if } f = f_c, \\ \frac{T(n+\frac{1}{2}) - \langle n_{\mathfrak{l}}, a\rangle}{n} & \text{if } f = f_o, \\ 0 & \text{if } f \in F \setminus \{f_o, f_{in}\}. \end{cases} \tag{48}$$

defines an element $a' \in \Delta_{K_*}(T)$ with $\langle n_{\mathfrak{l}}, a\rangle = \langle n_{\mathfrak{l}}, a'\rangle$ and hence

$$X = a - a' \in \mathfrak{m}_{\mathfrak{l}}.$$

Let us now bound $\delta_v \Psi_{\mathfrak{l}}$ for all $v \in V_{\mathfrak{l}}$. Denote by $V_{tw} = \{v_1, \dots, v_n\}$ the n intersection points of the twisted part of $\mathfrak{l}_{nest}$, ordering them so that $\mathfrak{l}_{nest} = (v_1 \dots v_n v_n \dots v_1)$. The dual graph induces an order $f_1, \dots f_n$ of F_{tw}, with $f_n = f_c$. Let us set for all $1 \le k \le n$, $F_k = \{f_k, \dots, f_n\}$. On the one hand, using (ii), for all $v \in V_{\mathfrak{l}} \setminus V_{tw}$, $\delta_v(\mathfrak{l}) = \mathfrak{l}_{v,1} \otimes \mathfrak{l}_{v,2}$, with $\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2} \in \mathfrak{B}_g$.

On the other hand, for all $1 \le k \le n$, $\delta_v(\mathfrak{l}) = \alpha_k \otimes \mathfrak{l}_k$, where α_k is a nested loop, hence $\alpha_k \in \mathcal{B}_g$ and $\mathfrak{l}_k$ is a sub-loop of $\mathfrak{l}$, with $\mathfrak{l}_1 = \mathfrak{l}^{(0)}$ and $\mathfrak{l}_k \sim_{F_k} \mathfrak{l}$ for all $1 \le k \le n$. Since $X \in \mathfrak{m}_{\mathfrak{l}}$, using the inequality (43), we find

$$|X.\Psi_{\mathfrak{l}}^N| \le C\left(\varepsilon_N + \Psi_{\mathfrak{l}}^N + \sum_{k=1}^n \Psi_{\mathfrak{l}_k}^N\right). \tag{49}$$

where $C > 0$ is a constant independent of N and $\varepsilon_N = \frac{1}{N} + \sup_{1\le k\le n_+} \|\Psi_{\alpha_k}^N\|_\infty + \sup_{v\in V_{\mathfrak{l}}\setminus V_{tw}} \left(\|\Psi_{\mathfrak{l}_{v,1}}^N\|_\infty + \|\Psi_{\mathfrak{l}_{v,2}}\|_\infty\right)$. Consider for all $t \in [0,1]$,

$$\Delta_{in}(t) = \{a \in \Delta_{\mathbb{G}}(tT) : a(f) = 0, \forall f \notin F_{tw} \cup \{f_o\}\}$$

and for all $a \in \Delta_{F_{tw},+}(T)$ fixed, set

$$H_a^N(t) = \sup_{b\in\Delta_{in}((1-t)T)} \Psi_{\mathfrak{l}}(ta + b), \forall 0 \le t \le 1.$$

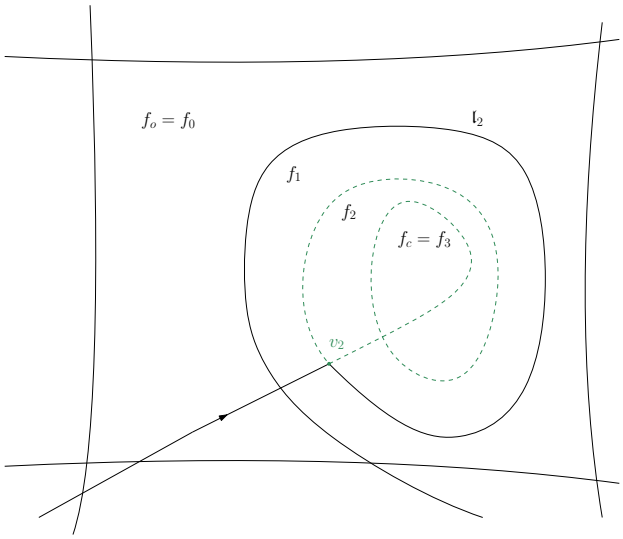

Figure 13: Example of a n-left twist with $n = 3$. We consider here $k = 2$, the area of F_2 needs to be "moved" into f_1. We have $a(f_1) = a(f_2) = a(f_3) = 0 = a'(f_1) = a'(f_2)$. For all $0 < s < t < 1$, define b_2 setting $b_2(f_1) = b(F_1) + (t-s)a'(F_1)$ and 0 for other faces. Denote $a_{s,t} = sa + (t-s)a' + b$ and $\tilde{a}_{s,t} = sa + b_2$. On the one hand, for any face $f \notin F_1$, $a_{s,t}(f) = a'_{s,t}(f)$ while $a_{s,t}(F_1) = a'_{s,t}(F_1)$, therefore $\Psi^N_{\mathfrak{l}_2}(a_{s,t}) = \Psi^N_{\mathfrak{l}_2}(\tilde{a}_{s,t})$. On the other hand, $\tilde{a}_{s,t}(F_2) = 0$ so that $\Psi^N_{\mathfrak{l}_2}(\tilde{a}_{s,t}) = \Psi^N_{\mathfrak{l}}(\tilde{a}_{s,t})$.

On the one hand, for any $t \in (0,1)$ and $b \in \Delta_{in}(T - tT)$,

$$\partial_s \Psi^N_{\mathfrak{l}}(sa + (t-s)a' + b) = X.\Psi^N_{\mathfrak{l}}(sa + (t-s)a' + b), \forall s \in (0,t).$$

On the other hand, for all $s \in (0,t)$, since $a(F_k) = 0$ and $\mathfrak{l}_k \sim_{F_k} \mathfrak{l}$ for all k, there are $b_1, \dots, b_n \in \Delta_{in}(T - sT) \cap \Delta_{F_k}(T)$ (see Figure 13) such that

$$\Psi_{\mathfrak{l}_k}(sa + (t-s)a' + b) = \Psi_{\mathfrak{l}}(sa + b_k), \forall 1 \leq k \leq n. \tag{50}$$

Combining the last two equalities with the bound (49), we find

$$H^N_a(t) \leq H^N_a(0) + \varepsilon_N C + (n+1)C \int_0^t H^N_a(s)ds, \forall t \in [0,1], a \in \Delta_{F_{tw},+}(T).$$

By Grönwall's inequality,

$$H^N_a(t) \leq (H^N_a(0) + \varepsilon_N C)\exp((n+1)Ct), \forall t \in [0,1]. \tag{51}$$

Since $\Delta_{in}(T) \subset \Delta_K(T)$, by (47)

$$\sup_{a \in \Delta_{F_{tw},+}} H^N_a(0) \leq \sup_{x \in \Delta_{\mathbb{G},K}(T)} \Psi_{\mathfrak{l}}(x)$$

vanishes as $N \to \infty$. Since $\varepsilon_N \to 0$ as $N \to \infty$, from (51),

$$\Psi^N_{\mathfrak{l}}(a) = H^N(1) \to 0$$

uniformly in $a \in \Delta_{F_{tw},+}(T)$. According to step 2, this concludes the proof. □

5 Proof of convergence after surgery

We give here the main arguments to prove Theorem 3.11.

Proof of Lemma 3.10. Thanks to the second part of Lemma 3.2, under $\mathrm{YM}_{\mathbb{G},\{f_\infty\},a}$, $(h_{\mathfrak{l}_1},\dots,h_{\mathfrak{l}_r},h_{a_1},\dots,h_{b_g})$ are independent random variables on G_N, such that for all $1\le i\le g$, h_{a_i},h_{b_i} are Haar distributed, while for any $1\le k\le r$, $h_{\mathfrak{l}_k}$ has same law as a Brownian motion at time $a(f_k)$. It is now standard, see [36, Section 3], that as $N\to\infty$, these tuple of matrices is asymptotically freely independent and its joint non-commutative distribution converges towards τ_v satisfying the properties (*), 1,2 and 3. □

Let us use the same notation as in Theorem 3.11. In what follows, we will denote by $\mathbb{E}$ (resp. $\mathbb{E}_i$, $\mathbb{E}'_i$) the expectation with respect to $\mathrm{YM}_{\mathbb{G},a}$ (resp. $\mathrm{YM}_{\mathbb{G}_i,a_i}$, $\mathrm{YM}_{\mathbb{G}'_i,a}$). In a previous paper, we proved that the restriction to $\mathbb{G}'_1$ of $\mathrm{YM}_{\mathbb{G},a}$ is absolutely continuous with respect to $\mathrm{YM}_{\mathbb{G}'_1,a}$.

Proposition 5.1 ([16], Corollary 4.3)**.** *Let $\mathfrak{l}\in\mathrm{RL}_v(\mathbb{G}'_1)$. For any $f:G_N\to\mathbb{C}$ bounded, measurable and central,*

$$\mathbb{E}[f(H_{\mathfrak{l}})]=\mathbb{E}'_1[f(H_{\mathfrak{l}})I(H_{\mathfrak{l}_0}^{-1})],\tag{52}$$

where $I:G_N\to\mathbb{C}$ is a bounded measurable function such that

$$\|I\|_\infty\le\frac{Z_{g_1,a(F_1)}Z_{g_2,a(F_2)}}{Z_{g,T}}.$$

Note that the bound in the previous proposition ensures that I is uniformly bounded, because for any considered sequence (G_N), the corresponding sequences of partition functions converge, therefore they are bounded.[29]

Proof of Theorem 3.11. Let $\mathfrak{l}$ be a loop in $\mathrm{L}_v(\mathbb{G}_1)$. According to Proposition 5.1,

$$\mathbb{E}[W_{\mathfrak{l}}]=\mathbb{E}'_1[W_{\mathfrak{l}}I(H_{\mathfrak{l}_0^{-1}})]$$

where I is uniformly bounded by a finite constant. From Lemma 3.10 we have the convergence of $W_{\mathfrak{l}}$ under $\mathrm{YM}_{\mathbb{G}_1,a}$, which implies the convergence in probability towards $\Phi_{\mathfrak{l}}^{1,g_1}(a_1)$. The proof of uniform convergence uses a simpler version of Proposition 3.6. Being very similar to the proof of uniformity in Theorem 2.12 in [16], its proof is not detailed here.

It remains to identify $\Phi_{\mathfrak{l}}^{1,g_1}(a_1)$ with $\Phi_{\mathfrak{l}}(a)$. Consider the linear functional $\tilde\tau$ on $(\mathbb{C}[\mathrm{RL}_v(\mathbb{G}_1)],*)$ extending linearly $\Phi(a)$. Consider a basis $\mathfrak{l}_1,\dots,\mathfrak{l}_r,\mathfrak{l}_{r+1},a_1,b_1,\dots,a_{g_1},b_{g_1}$ of $\mathrm{RL}_v(\mathbb{G}_1)$ as in Lemma 2.1.

It is enough to show that properties 1,2 and 3 are satisfied. Denote by $\tilde v_0$ a vertex of $\tilde{\mathbb{G}}$ with $v_0\in D=D_1$ and $p(v_0)=v$. The identity $\Phi_{\tilde\gamma^n}(\tilde a)=0,\forall n\neq 0$, for any lift $\tilde\gamma$ of $\gamma\in\{a_1,\dots b_g\}$ implies point 2. Point 3 follows point 1 of Lemma 3.9. Consider the free independence property. Note that $\mathfrak{l}_1,\dots,\mathfrak{l}_r$ have same joint distribution under τ_v and $\tilde\tau$. Hence, thanks to point 2 of Lemma 3.9, $\mathfrak{l}_1,\dots,\mathfrak{l}_r$ are freely independent under $\tilde\tau$.

[29] Besides, when $g_1,g_2\ge 2$ and $G_N\neq\mathrm{U}(N)$, it remains bounded uniformly in $a\in\Delta_{\mathbb{G}}(T)$, which allows then to drop the condition $a(F_1)\in(0,T)$ in Theorem 3.11.

Let us now recall that the images of $a_1, b_1, \dots, a_{g_1}, b_{g_1}$ in Γ_g span a free subgroup Γ' of Γ_g of rank $2g_1$, isomorphic to the fundamental group of the surface with one boundary given by the total space of $(\mathbb{G}_1, \{f_{1,\infty}\})$. To conclude it is enough to show that the sub-algebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{C}[\mathrm{RL}_v(\mathbb{G}_1)]$ spanned respectively by $S_c = \{\mathfrak{l}_1, \dots \mathfrak{l}_r\}$ and $S_{top} = \{a_1, b_1 \dots, a_{g_1}, b_{g_1}\}$ are free under $\tilde{\tau}$. Let us first note that $a_1, \dots, b_{g_1}$ are freely independent under $\tilde{\tau}$. Indeed, for any word w in S_{top} and their inverses, since the image of S_{top} is a free basis of Γ', the loop $\mathfrak{l}_w \in \mathrm{RL}_v(\mathbb{G}_1)$ associated to w satisfies $\mathfrak{l}_w \sim_h c_v$ if and only if w can be reduced to the empty word, so that $(a_1 \dots, b_{g_1})$ has same distribution as a tuple of $2g_1$ freely independent Haar unitaries. Moreover for any word w in S_{top} and their inverses, the collection of vertices and non-oriented edges visited by the lift $\tilde{\mathfrak{l}}_w$ with $\underline{\tilde{\mathfrak{l}}_w} = \tilde{v}_0$, form a tree of $\tilde{\mathbb{G}}$.

Consider now reduced words $w_1, \dots, w_k$ in S_{top} and their inverses. Denote by Γ_c the subgroup of $\mathrm{RL}_v(\mathbb{G}_1)$ spanned by S_c. For any $\alpha_1, \dots, \alpha_k \in \Gamma_c$, consider the loop obtained by the concatenation

$$\mathfrak{l} = \mathfrak{l}_{w_1}\alpha_1\mathfrak{l}_{w_2} \dots \mathfrak{l}_{w_k}\alpha_k. \tag{*}$$

Since $\mathfrak{l} \sim_h \mathfrak{l}_w$, where $w = w_1 \dots w_k$, and images of S_{top} form a free basis of Γ', we conclude as above that $\tilde{\tau}(\mathfrak{l}) = 0$ whenever $w_1 \dots w_k$ cannot be reduced to the empty word.

Otherwise, consider the tree T_w and the following property of the lift. Each loop in the decomposition (*) of $\mathfrak{l}$ has a unique lift so that $\tilde{\mathfrak{l}} = \tilde{\mathfrak{l}}_{w_1}\tilde{\alpha}_1\tilde{\mathfrak{l}}_{w_2} \dots \tilde{\mathfrak{l}}_{w_k}\tilde{\alpha}_k$, with $\underline{\tilde{\mathfrak{l}}} = \tilde{v}_0$. Their respective base points satisfy $\underline{\tilde{\mathfrak{l}}_{w_i}} = \underline{\tilde{\alpha}_i}$ for all i, whereas for any $i < j$, $\underline{\tilde{\mathfrak{l}}_{w_i}} = \underline{\tilde{\mathfrak{l}}_{w_j}}$ if and only if $w_{i+1} \dots w_j$ can be reduced to the empty word, that is $w_1 \dots w_i \sim_r w_1 \dots w_j$. Moreover for any i, $\tilde{\alpha}_i$ is included in $D_{w_1 \dots w_i}$ and can be decomposed as a product of lassos based at $\underline{\mathfrak{l}_{w_i}}$ with meanders in $D^*_{w_1 \dots w_i}$. Now for any vertex $\tilde{v}$ of T_w, let $\gamma_{\tilde{v}}$ be the unique path of T_w with $\underline{\gamma_i} = \tilde{v}_0$ and $\overline{\gamma}_i = \tilde{v}$. Then $\{(\gamma_{\tilde{v}}\tilde{\mathfrak{l}}_m\gamma_{\tilde{v}}^{-1})_{1 \le m \le r}, \tilde{v} \in T_w\}$ are collection of lassos of $\tilde{\mathbb{G}}$ based at $\tilde{v}_0$ which can be completed to form a free basis of an area weighted planar map coarser than $(\tilde{\mathbb{G}}, \tilde{a})$, with the same property as in Lemma 3.9. In particular, denoting by $\hat{\tau}$ the state with $\hat{\tau}(\tilde{\mathfrak{l}}) = \Phi_{\tilde{\mathfrak{l}}}(\tilde{a})$ for all $\tilde{\mathfrak{l}} \in \mathrm{RL}_{\tilde{v}}(\tilde{\mathbb{G}})$, thanks to property 2., the families $\{(\gamma_{\tilde{v}}\tilde{\mathfrak{l}}_m\gamma_{\tilde{v}}^{-1})_{1 \le m \le r}, \tilde{v} \in T_w\}$ are freely independent under $\hat{\tau}$. Denoting for any $\tilde{v} \in T_w$, $\mathcal{A}_{\tilde{v}}$ the algebra spanned by $(\gamma_{\tilde{v}}\tilde{\mathfrak{l}}_m\gamma_{\tilde{v}}^{-1})_{1 \le i \le r}$ and by $\Theta_{\tilde{v}} : \mathcal{A} \to \mathcal{A}_v$ the algebra morphism with $\Theta_{\tilde{v}}(\mathfrak{l}_i^{\varepsilon}) = \gamma_{\tilde{v}}\mathfrak{l}_i^{\varepsilon}\gamma_{\tilde{v}}^{-1}, \forall \varepsilon \in \{1, -1\}, 1 \le i \le r$, the algebras $(\mathcal{A}_{\tilde{v}})_{\tilde{v} \in T_w}$ are free under $\hat{\tau}$. Moreover, setting $\tilde{v}_l = \underline{\tilde{\mathfrak{l}}_{w_l}}, \forall 1 \le l \le k$, since w_l is reduced, $\tilde{v}_l \neq \tilde{v}_{l-1}$. Thanks to the latter free independence under $\hat{\tau}$, for all $x_1, \dots, x_k \in \mathcal{A}$ with $\tilde{\tau}(x_l) = 0, \forall 1 \le l \le k$,

$$\tilde{\tau}(w_1x_1 \dots w_kx_k) = \hat{\tau}(\Theta_{\tilde{v}_1}(x_1)\Theta_{\tilde{v}_2}(x_2) \dots \Theta_{\tilde{v}_k}(x_k)) = 0.$$

Since S_{top} are freely independent Haar unitaries under $\tilde{\tau}$, it follows by linearity that for any $y_1, \dots, y_k \in \mathcal{B}$ with $\tilde{\tau}(y_l) = 0$ for all $1 \le l \le k$,

$$\tilde{\tau}(y_1x_1 \dots y_kx_k) = 0.$$

It is enough to conclude that $\mathcal{A}$ and $\mathcal{B}$ are free under $\tilde{\tau}$. □

6 Liberation and master field on the torus

Let us give here a proof of corollary 1.11. For $T > 0$, let us consider the two dimensional torus $\mathbb{T}_T^2$ obtained as the quotient $\mathbb{R}^2/\sqrt{T}\mathbb{Z}^2$ endowed with the push-forward of the Euclidean metric, so that it has total volume T. Denote by α and β the loop of $\mathbb{T}_T^2$ obtained by projecting the segments from $(0,0)$ to respectively $(\sqrt{T},0)$ and $(0,\sqrt{T})$. Then, under YM_Σ, the law (a,b) on G^2 is given by (12). Therefore, for any word w in $\alpha,\beta,\alpha^{-1},\beta^{-1}$ denoting by $[w]\in\mathbb{Z}^2$ the signed number of occurences of α and β and by $\tilde{\gamma}_w$ the path of $\mathbb{R}^2$ starting from $(0,0)$ obtained by lifting the loop Σ formed by w, under YM_Σ, the following converge holds in probability as $N\to\infty$,

$$\tau_{\rho_N}(w)\to\begin{cases}\Phi_{\mathbb{R}^2}(\tilde{\gamma}_w) & \text{if } [\gamma_w]=0\\ \\ 0 & \text{if } [\gamma_w]\neq 0.\end{cases}$$

The first statement of Corollary 1.11 follows considering the non-commutative distribution Φ_T of αt and β under the limit of τ_{ρ_N} as $N\to\infty$.

On the one hand, for any word w with $[w]=0$, γ_w is a loop and by continuity of the master (Point 1 of Theorem 3.8), $\Phi_T(w)=\Phi_{\mathbb{R}^2}(\gamma_w)\to 1$ as $T\to 0$. On the other hand, for any word w with $[w]\neq 0$, $\tilde{\gamma}_w$ is not a loop, $[\gamma_w]\neq 0$, and for all $T>0$, $\Phi_T(w)=0$. Therefore, for all word in $\alpha,\beta,\alpha^{-1},\beta^{-1}$, $\lim_{T\to 0}\Phi_T(w)=\tau_u\star_c\tau_u(w)$, since

$$\tau_u\star_c\tau_u(w)=\begin{cases}1 & \text{if } [w]=0,\\ \\ 0 & \text{otherwise.}\end{cases}$$

Consider now the second limit of corollary 1.11. When $(\mathbb{G},a)$ an area weighted map embedded in $\mathbb{R}^2$ with v a vertex of $\mathbb{G}$ sent to 0 by the embedding, consider the state $\hat{\tau}_T$ on $(\mathrm{RL}_v(\mathbb{G}),*)$ such that $\hat{\tau}_T(\mathfrak{l})=\Phi_{\mathbb{R}^2}(\ell_T)$, where ℓ is the drawing of $\mathfrak{l}$ while $\ell_T=\sqrt{T}\ell$. Consider a free-basis of lassos $\mathfrak{l}_1,\dots\mathfrak{l}_r$ of $\mathrm{RL}_v(\mathbb{G})$, with meanders given by distinct faces of area $a_1,\dots,a_r$. Under $\hat{\tau}_T$, $\mathfrak{l}_1,\dots,\mathfrak{l}_r$ are r independent unitary Brownian motion marginals at time $\sqrt{T}a_1,\dots,\sqrt{T}a_r$. It follows easily from its definition in moments, that the free unitary Brownian motion at time s converges weakly towards a Haar unitary as $s\to\infty$. Since $a\in\Delta^o(T)$, $(\mathfrak{l}_1,\dots,\mathfrak{l}_r)$ converges weakly toward r freely independent unitary variables as $T\to\infty$. Therefore, for any reduced loop $\mathfrak{l}$, $\lim_{T\to\infty}\hat{\tau}_T(\mathfrak{l})=1$ if $\mathfrak{l}$ is the constant loop and 0 otherwise. Now for any word w in $\alpha,\beta,\alpha^{-1},\beta^{-1}$, with $[w]=0$, it follows that

$$\lim_{T\to\infty}\Phi_T(w)=\begin{cases}1 & \text{if } \gamma_w\sim_r c \text{ with } c \text{ constant},\\ \\ 0 & \text{otherwise.}\end{cases}$$

Since $\gamma_w\sim_r c$ where c is a constant loop if and only if w can be reduced to the empty word, it follows that $\lim_{T\to\infty}\Phi_T(w)=\tau_u\star\tau_u(w)$.

Let us now recall a way introduced in [5] to compute the evaluation of $\tau_{\mathcal{A}}\star_t\tau_{\mathcal{B}}$ given $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, solving systems of ODEs in the parameter t and present an argument for (13). Let us say that a non-commutative monomial P in $(X_{1,i})_{i\in I},(X_{2,j})_{j\in I}$ is alternated if it is of the form $X_{\varepsilon_1,i_1}X_{\varepsilon_2,i_2}\dots X_{\varepsilon_n,i_n}$ with $\varepsilon_k\neq\varepsilon_{k+1}$ for all $1\leq k<n$. Denote by d_{X_2} is degree in the variables $(X_{2,j})_{i\in I}$. For such a monomial, let us

set

$$\Delta_{ad}.P = -\frac{d_{X_2}(P)}{2}(P\otimes 1 + 1\otimes P) + \sum_{Q_1,Q_2,i} X_{2,i}\otimes Q_1Q_2,$$
$$- \sum_{P_{1,1},P_{1,2},P_2,i,j} \big[X_{2,i}P_2\otimes(P_{1,1}X_{2,j}P_{1,2}) + (P_{1,1}X_{2,i}P_{1,2})\otimes P_2X_{2,j}$$
$$- (P_{1,1}P_{1,2})\otimes(X_{2,i}P_2X_{2,j}) - (P_{1,1}X_{2,i}X_{2,j}P_{1,2})\otimes P_2\big]$$

where the first sum is over all monomials Q_1, Q_2 and $i\in I$ such that $P = Q_1X_{2,i}Q_2$, while the second is over all monomials $P_{1,1}, P_{1,2}, P_2$ and $i,j\in I$ such that $P = P_{1,1}X_{2,i}P_2X_{2,j}P_{1,2}$. With these notations, Theorem 3.4 of [5] states that for all alternated non-commutative monomial P in $(X_{1,i})_{i\in I}, (X_{2,j})_{j\in I}$, $\tau_{\mathcal{A}}\star_t\tau_{\mathcal{B}}(P)$ is differentiable with

$$\partial_t\tau_{\mathcal{A}}\star_t\tau_{\mathcal{B}}(P) = (\tau_{\mathcal{A}}\star_t\tau_{\mathcal{B}})^{\otimes 2}(\Delta_{ad}.P), \forall t\geq 0.$$

For instance assume that for all $t\geq 0$, (a,b) is a t-free couple within a non-commutative probability space $(\mathcal{C},\tau_t)$, such that a and b are Haar unitaries for all $t>0$. Then for any $n\geq 1$,

$$\partial_t\tau_t(ab^n) = -\tau_t(ab^n) + \tau(a)\tau_t(b^n) = -\tau_t(ab^n), \forall t\geq 0$$

and since $\tau_0(ab^n) = \tau_0(a)\tau_0(b^n) = 0$,

$$\tau_t(ab^n) = 0.$$

Likewise

$$\partial_t\tau_t(ab^na^*(b^*)^n) = -2\tau_t(ab^na^*(b^*)^n) + \tau_t(b^n)\tau_t(aa^*(b^*)^n) + \tau_t(ab^na^*)\tau_t((b^*)^n)$$
$$- \tau_t(a(b^*)^n)\tau_t(b^na^*) - \tau_t(a(b^*)^n)\tau_t(b^na^*) + \tau_t(a)\tau_t(b^na^*(b^*)^n) + \tau_t(ab^n(b^*)^n)\tau_t(a^*)$$
$$= -2\tau_t(ab^na^*(b^*)^n).$$

Since $\tau_0(ab^na^*(b^*)^n) = \tau_0(aa^*)\tau_0(b^n(b^*)^n) = 1$, this implies

$$\tau_t(ab^na^*(b^*)^n) = e^{-2t}. \tag{53}$$

A similar argument together with (11) imply the following lemma.

Lemma 6.1. *1. For any word w in a, b, a^{-1}, b^{-1}, if $[w]\neq 0$,*

$$\tau_t(w) = 0.$$

2. For any $n\geq 1$,

$$\partial_t\tau_t([a,b]^n) = -2n\tau_t([a,b]^n) - 2n\sum_{k=1}^{n-1}\tau_t([a,b]^k)\tau([a,b]^{n-k}).$$

3. For any $n\in\mathbb{Z}$ and $t\geq 0$,

$$\tau_t([a,b]^n) = \nu_{4t}(|n|).$$

The last equality of corollary 1.11 follows from last point of the above Lemma. Besides for any $t > 0, T > 0$

$$\tau_u \star_t \tau_u(XYX^*Y^*) = e^{-2t} \text{ and } \Phi_T(XYX^*Y^*) = e^{-\frac{T}{2}},$$

so that if $\tau_u \star_t \tau_u = \Phi_T$ then $T = 4t$. But (53) implies

$$\tau_u \star_t \tau_u(XY^2X^*Y^{-2}) = e^{-2t} > e^{-4t} = \Phi_{4t}(XY^2X^*Y^{-2}).$$

Therefore for all $t, T > 0$, $\Phi_T \neq \tau_u \star_t \tau_u$.

7 Appendix

7.1 Makeenko–Migdal equations

Let us recall some tensor identities, instrumental to prove Makeenko–Migdal relations.

Definition 7.1. Consider a Lie algebra $\mathfrak{g}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. The *Casimir element* of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is the tensor $C_{\mathfrak{g}} \in \mathcal{M}_d(\mathbb{C}) \otimes_{\mathbb{R}} \mathcal{M}_d(\mathbb{C})$ defined by

$$C_{\mathfrak{g}} = \sum_{X \in \mathcal{B}} X \otimes X, \tag{54}$$

where $\mathcal{B}$ is an orthonormal basis of $\mathfrak{g}$ for the inner product $\langle \cdot, \cdot \rangle$.

It is simple to check that the definition of the Casimir element does not depend on the choice of the basis but only on the inner product $\langle \cdot, \cdot \rangle$. We focus on the setting recalled in section 3.1; we consider the Lie algebra $\mathfrak{g}_N$ of a group $G_N \in \mathbf{CG}_N$ with the inner product (1) considered in [16, Section 2.1.]. We set the value β to be respectively 1 and 4 when G_N is $O(N)$ and $Sp(N)$ and 2 otherwise, that is when G_N is $SU(N)$ or $U(N)$. We set $\gamma = 1$ when $G_N = SU(N)$ and 0 otherwise.

Most of the following results can be proved by a direct computation using an arbitrary chosen basis. For any $(a, b) \in \{1, \dots, N\}^2$, the elementary matrix $E_{ab} \in \mathcal{M}_N(\mathbb{R})$ is defined by $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

We shall need the following standard result result on the Casimir element in this setting, which gives computation rules for traces of products and product of traces involving elements of $\mathcal{B}$.

Lemma 7.1. *For any $A, B \in G_N$ we have :*

$$\sum_{X \in \mathcal{B}} \mathrm{tr}(AXBX) = -\mathrm{tr}(A)\mathrm{tr}(B) - \frac{\beta - 2}{\beta N}\mathrm{tr}(AB^{-1}) + \frac{\gamma}{N^2}\mathrm{tr}(AB) \tag{55}$$

and

$$\sum_{X \in \mathcal{B}} \mathrm{Tr}(AX)\mathrm{Tr}(BX) = -\mathrm{tr}(AB) - \frac{\beta - 2}{\beta N}\mathrm{tr}(AB^{-1}) + \gamma\mathrm{tr}(A)\mathrm{tr}(B). \tag{56}$$

Proof. We only sketch the proof in order to show where the expressions come from. First of all, remark that by linearity they only need to be proved for $A = E_{ij}$ and $B = E_{k\ell}$. We have for instance

$$\sum_{X \in \mathcal{B}} \mathrm{tr}(AXBX) = \frac{1}{N} \sum_{X \in \mathcal{B}} \sum_{a,b,c,d} A_{ab}X_{bc}B_{cd}X_{da} = \frac{1}{N}(C_{\mathfrak{g}})_{jk\ell i},$$

where we have set

$$\big(\sum_i X^i \otimes Y^i\big)_{abcd} = \sum_i X^i_{ab} Y^i_{cd}.$$

Using the expression of $C_{\mathfrak{g}}$ for each value of $\mathfrak{g}$ leads to Eq. (55). By similar computations we also obtain Eq. (56). □

In the unitary case, the formulas in Lemma 7.1 are known as the "magic formulas", as stated in [21] for instance, and appeared already in [50]; they are crucial to the derivation of Makeenko–Migdal equations for Wilson loops, that we briefly recall in the next section. Although we do not detail it, there exist a beautiful interpretation Lemma 7.1 in terms of Schur–Weyl duality; the interested reader can refer to [34] or [15] for an explanation and discussion of this fact and to [36, Chap. I, Section 1.2] about the above Lemma.

7.2 Makeenko–Migdal equations

Given a topological map $\mathbb{G}$ of genus g with m edges, a vertex of $\mathcal{G}$ will be said to be an *admissible crossing* if it possesses four outgoing edges labelled e_1, e_2, e_3, e_4 counterclockwise.

Definition 7.2. Let $\mathbb{G}$ be map of genus g with m edges, and v be an admissible crossing. A function $f : G^m \to \mathbb{C}$ has an *extended gauge invariance at* v if for any $x \in G$,

$$f(a_1, a_2, a_3, a_4, \mathbf{b}) = f(a_1 x, a_2, a_3 x, a_4, \mathbf{b}) = f(a_1, a_2 x, a_3, a_4 x, \mathbf{b}), \tag{57}$$

where a_i denotes the variable associated to the edge e_i and $\mathbf{b}$ denotes the tuple of other edge variables than e_1, e_2, e_3, e_4.

The extended gauge-invariance was first introduced by Lévy in [35] to prove Makeenko–Migdal equations in the plane, then used in [22] to give alternative, local proofs of these equations, which allowed in [21] to prove their validity on any surface; these last equations were then applied in [17, 29].

Theorem 7.2 (Abstract Makeenko–Migdal equations)**.** *Let $(\mathbb{G}, a)$ be an area-weighted map of area T and genus g with m edges, and $f : G^m \to \mathbb{C}$ be a function with extended gauge invariance at an admissible crossing v. Denote by f_1 (resp. f_2, f_3, f_4) the face of $\mathbb{G}$ whose boundary contains (e_1, e_2) (resp. (e_2, e_3), (e_3, e_4), (e_4, e_1)). Denote by t_i the area of the face f_i, choose an orthonormal basis $\mathcal{B}$ of $\mathfrak{g}$ with respect to the chosen inner product, and set*

$$(\nabla^{a_1} \cdot \nabla^{a_2} f)(a_1, a_2, a_3, a_4, \mathbf{b}) = \sum_{X \in \mathcal{B}} \frac{\partial^2}{\partial s \partial t} f(a_1 e^{sX}, a_2 e^{tX}, a_3, a_4, \mathbf{b})\big|_{s=t=0}.$$

We have

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \int_{G^m} f d\mu = -\int_{G^m} \nabla^{a_1} \cdot \nabla^{a_2} f d\mu. \tag{58}$$

Equation (58) might be confusing, as it involves partial derivatives with respect to variables that do not appear explicitly in the function $\int_{G^m} f d\mu$; it becomes in fact

clearer after being translated in terms of the area simplex. We define the differential operator μ_v on functions $\Delta_{\mathbb{G}}(T) \to \mathbb{C}$ by

$$\mu_v = \frac{\partial}{\partial a_1} - \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3} - \frac{\partial}{\partial a_4},$$

using the labelling of $a = (a_1, \dots, a_p) \in \Delta_{\mathbb{G}}(T)$ such that a_i corresponds to the face f_i. Equation (58) becomes then

$$\mu_v E(f) = -E(\nabla^{a_1} \cdot \nabla^{a_2} f),$$

and now everything only depends on the areas of the faces. We want to apply these abstract Makeenko–Migdal equations to functionals of Wilson loops, in order to obtain the convergence to the master field. We define, for k unrooted loops $\mathfrak{l}_1, \dots, \mathfrak{l}_k \in \mathrm{L}_c(\mathbb{G})$, the k-point function $\phi^G_{\mathfrak{l}_1,\dots,\mathfrak{l}_k} : \Delta_{\mathbb{G}}(T) \to \mathbb{C}$ by

$$\phi^G_{\mathfrak{l}_1 \otimes \dots \otimes \mathfrak{l}_k} = \mathbb{E}(W_{\mathfrak{l}_1} \cdots W_{\mathfrak{l}_k}).$$

and extend it linearly to $\mathbb{C}[\mathrm{L}_c(\mathbb{G})]^{\otimes k}$. The following proposition offers an estimate of the face-area variation of the functions $\phi^G_{\mathfrak{l}_1 \otimes \dots \otimes \mathfrak{l}_k}$.

Proposition 7.3 (Makeenko–Migdal equations for Wilson loops)**.** *Assume that $G_N \in \mathbf{CG}_N$ and $\langle \cdot, \cdot \rangle$ are fixed as in section 3.1. Let $(\mathbb{G}, a)$ be a weighted map of area T and genus g with m edges, and v be an admissible crossing in $\mathbb{G}$.*

1. *If v is a self-intersection of a single loop $\mathfrak{l}_1$ such that the edges $(e_j^{\pm 1}, 1 \le j \le 4)$ are visited in the following order: $e_1, e_4^{-1}, e_2, e_3^{-1}$, then define $\mathfrak{l}_{11}$ the subloop of $\mathfrak{l}_1$ starting at e_1 and finishing at e_4^{-1}, $\mathfrak{l}_{12}$ the subloop starting at e_2 and finishing at e_3^{-1}. We have, for any loops $\mathfrak{l}_2, \dots, \mathfrak{l}_k$ that do not cross v,*

$$\mu_v \phi^G_{\mathfrak{l}_1 \otimes \dots \otimes \mathfrak{l}_k} = \phi^G_{\mathfrak{l}_{11} \otimes \mathfrak{l}_{12} \otimes \mathfrak{l}_2 \otimes \dots \otimes \mathfrak{l}_k} + \frac{2-\beta}{\beta N} \phi^G_{\mathfrak{l}_{11}\mathfrak{l}_{12}^{-1} \otimes \mathfrak{l}_2 \otimes \dots \otimes \mathfrak{l}_k} + \frac{\gamma}{N^2} \phi_{\mathfrak{l}_1 \otimes \dots \otimes \mathfrak{l}_k}, \quad (59)$$

$$\mu_v \phi^G_{\mathfrak{l}_1 \otimes \mathfrak{l}_1^{-1}} = \phi^G_{\mathfrak{l}_{11} \otimes \mathfrak{l}_{12} \otimes \mathfrak{l}_1^{-1}} + \phi^G_{\mathfrak{l}_1 \otimes \mathfrak{l}_{11}^{-1} \otimes \mathfrak{l}_{12}^{-1}} + \frac{R_{\mathfrak{l}_1}}{N}, \quad (60)$$

   where the $|R_{\mathfrak{l}_1}| \le 10$ uniformly on $\Delta_{\mathbb{G}}(T)$.

2. *If v is the intersection between two loops $\mathfrak{l}_1$ and $\mathfrak{l}_2$ such that $\mathfrak{l}_1$ starts at e_1 and finishes at e_3^{-1}, and $\mathfrak{l}_2$ starts at e_2 and finishes at e_4^{-1}, then define $\mathfrak{l}$ the loop obtained by concatenation of $\mathfrak{l}_1$ and $\mathfrak{l}_2$. We have, for any loops $\mathfrak{l}_3, \dots, \mathfrak{l}_k$ that do not cross v,*

$$\mu_v \phi^G_{\mathfrak{l}_1 \otimes \mathfrak{l}_2 \otimes \dots \otimes \mathfrak{l}_k} = \frac{R_{\mathfrak{l}_1 \otimes \mathfrak{l}_2 \otimes \dots \otimes \mathfrak{l}_k}}{N^2} \quad (61)$$

   with $|R_{\mathfrak{l}_1 \otimes \mathfrak{l}_2 \otimes \dots \otimes \mathfrak{l}_k}| \le 3$ uniformly on $\Delta_{\mathbb{G}}(T)$.

It was proved for all classical Lie algebras if $\mathbb{G}$ is a planar combinatorial graph by Lévy in [36, Prop. 6.16] when the loops form what he called a skein. If $\mathbb{G}$ is a map of genus 0 and $\mathfrak{g}$ is the Lie algebra of $\mathrm{U}(N)$, this result was proved by one of the authors with Norris in [17, Prop. 4.3]. See also [22, Prop.]

Proof of Prop. 7.3. Let us start with the first case, which is when v is a self-intersection of a loop $\mathfrak{l}_1$. We take $E_o = \{e_1, e_2, e_3, e_4, e'_1, \dots, e'_{m-4}\}$ as an orientation

of E, with e_1, e_2, e_3, e_4 the four outgoing edges from v. We identify any multiplicative function $h \in \mathcal{M}(P(\mathbb{G}), G)$ to a tuple $(a_1, a_2, a_3, a_4, \mathbf{b})$ by setting $a_i = h_{e_i}$ and $\mathbf{b} = (h_{e'_i})_{1 \leq i \leq m-4}$ the tuple of all other images of edges by h. There are words $\alpha, \beta, w_2, \ldots, w_k$ in the elements of $\mathbf{b}$ such that

$$h_{\mathfrak{l}_1} = a_3^{-1}\alpha a_2 a_4^{-1}\beta a_1, h_{\mathfrak{l}_i} = w_i \ \forall 2 \leq i \leq k.$$

It appears that $\phi^G_{\mathfrak{l}_1,\ldots,\mathfrak{l}_k} = E(f)$, where f is the extended gauge-invariant function

$$f : \left\{ \begin{array}{ccc} G^m & \to & \mathbb{C} \\ (a_1, a_2, a_3, a_4, \mathbf{b}) & \mapsto & \mathrm{tr}(a_3^{-1}\alpha a_2 a_4^{-1}\beta a_1)\mathrm{tr}(w_2)\cdots\mathrm{tr}(w_k). \end{array} \right.$$

Then, by the abstract Makeenko–Migdal equation (58), we get

$$\mu_v E(f) = -E(\nabla^{a_1} \cdot \nabla^{a_2} f),$$

and by definition

$$\nabla^{a_1} \cdot \nabla^{a_2} f = \left(\sum_X \mathrm{tr}(a_3^{-1}\alpha a_2 X a_4^{-1}\beta a_1 X) \right) \mathrm{tr}(w_2)\cdots\mathrm{tr}(w_k)$$

where X runs through an orthonormal basis of $\mathfrak{g}$. A straightforward application of (55) from Lemma 7.1 yields (59), by noticing that $h_{\mathfrak{l}_{11}} = a_4^{-1}\beta a_1$ and $h_{\mathfrak{l}_{12}} = a_3^{-1}\alpha a_2$.

Similarly, we have $\phi^G_{\mathfrak{l},\mathfrak{l}^{-1}} = E(f')$, where

$$f' : \left\{ \begin{array}{ccc} G^m & \to & \mathbb{C} \\ (a_1, a_2, a_3, a_4, \mathbf{b}) & \mapsto & \mathrm{tr}(a_3^{-1}\alpha a_2 a_4^{-1}\beta a_1)\mathrm{tr}(a_1^{-1}\beta^{-1} a_4 a_2^{-1}\alpha^{-1} a_3). \end{array} \right.$$

We have

$$\begin{aligned} \nabla^{a_1} \cdot \nabla^{a_2} f' = \sum_X \big\{ & \mathrm{tr}(a_3^{-1}\alpha a_2 X a_4^{-1}\beta a_1 X)\mathrm{tr}(a_1^{-1}\beta^{-1} a_4 a_2^{-1}\alpha^{-1} a_3) \\ & - \mathrm{tr}(a_3^{-1}\alpha a_2 a_4^{-1}\beta a_1 X)\mathrm{tr}(a_1^{-1}\beta^{-1} a_4 X a_2^{-1}\alpha^{-1} a_3) \\ & - \mathrm{tr}(a_3^{-1}\alpha a_2 X a_4^{-1}\beta a_1)\mathrm{tr}(X a_1^{-1}\beta^{-1} a_4 a_2^{-1}\alpha^{-1} a_3) \\ & + \mathrm{tr}(a_3^{-1}\alpha a_2 a_4^{-1}\beta a_1)\mathrm{tr}(X a_1^{-1}\beta^{-1} a_4 X a_2^{-1}\alpha^{-1} a_3) \big\}, \end{aligned}$$

and a simultaneous application of (55) and (56) leads to the result. We detail the case of $\mathrm{SU}(N)$ and leave the others as an exercise: if we set $A = h_{\mathfrak{l}_{11}}$ and $B = h_{\mathfrak{l}_{12}}$, then

$$\begin{aligned} \sum_X \mathrm{tr}(AXBX)\mathrm{tr}(B^{-1}A^{-1}) &= -\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}((AB)^{-1}) + \frac{1}{N}\mathrm{tr}(AB)\mathrm{tr}((AB)^{-1}) \\ \sum_X \mathrm{tr}(ABX)\mathrm{tr}(B^{-1}XA^{-1}) &= -\frac{1}{N^2}\mathrm{tr}([A,B]) + \frac{1}{N}\mathrm{tr}(AB)\mathrm{tr}(A^{-1}B^{-1}) \\ \sum_X \mathrm{tr}(AXB)\mathrm{tr}(XB^{-1}A^{-1}) &= -\frac{1}{N^2}\mathrm{tr}([A,B]^{-1}) + \frac{1}{N}\mathrm{tr}(BA)\mathrm{tr}(B^{-1}A^{-1}) \\ \sum_X \mathrm{tr}(AB)\mathrm{tr}(XB^{-1}XA^{-1}) &= -\mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}((AB)) + \frac{1}{N}\mathrm{tr}(AB)\mathrm{tr}(A^{-1}B^{-1}). \end{aligned}$$

We can then take the expectation of the alternated sum of these expressions, and as all traces are bounded by 1 because they apply to special unitary matrices, we find that all terms with a coefficient $\frac{1}{N}$ or $\frac{1}{N^2}$ fall into $O(\frac{1}{N})$ which does not depend on any loop[30], so that

$$\phi^{\mathrm{SU}(N)}_{\mathfrak{l}_1\otimes\mathfrak{l}_1^{-1}} = \phi^{\mathrm{SU}(N)}_{\mathfrak{l}_{11}\otimes\mathfrak{l}_{12}\otimes(\mathfrak{l}_{11}\mathfrak{l}_{12})^{-1}} + \phi^{\mathrm{SU}(N)}_{\mathfrak{l}_{11}^{-1}\otimes\mathfrak{l}_{12}^{-1}\otimes(\mathfrak{l}_{11}\mathfrak{l}_{12})} + O(\frac{1}{N}).$$

Let us now turn to the second case, when v is the intersection of $\mathfrak{l}_1$ and $\mathfrak{l}_2$. We take $E_o = \{e_1, e_2, e_3, e_4, e'_1, \dots, e'_{m-4}\}$ as an orientation of E, with e_1, e_2, e_3, e_4 the four outgoing edges from v. There are words $\alpha, \beta, w_2, \dots, w_k$ in the elements of $\mathbf{b}$ such that

$$h_{\mathfrak{l}_1} = a_3^{-1}\alpha a_1, h_{\mathfrak{l}_2} = a_4^{-1}\alpha a_2, h_{\mathfrak{l}_i} = w_i \ \forall 3 \le i \le k.$$

We have $\phi^G_{\mathfrak{l}_1,\dots,\mathfrak{l}_k} = E(f)$, where f is the extended gauge-invariant function

$$f : \begin{cases} G^m & \to & \mathbb{C} \\ (a_1, a_2, a_3, a_4, \mathbf{b}) & \mapsto & \mathrm{tr}(a_3^{-1}\alpha a_1)\mathrm{tr}(a_4^{-1}\beta a_2)\mathrm{tr}(w_2)\cdots\mathrm{tr}(w_k), \end{cases}$$

then

$$\mu_v E(f) = -E(\nabla^{a_1}\cdot\nabla^{a_2} f),$$

where

$$\nabla^{a_1}\cdot\nabla^{a_2} f = \left(\sum_X \mathrm{tr}(a_3^{-1}\alpha a_1)\mathrm{tr}(X a_4^{-1}\beta a_2 X)\right)\mathrm{tr}(w_2)\cdots\mathrm{tr}(w_k).$$

The result follows then from (56). □

By letting $N \to \infty$ in Prop. 7.3, one immediately gets the following.

Corollary 7.4 (Makeenko–Migdal equations for a master field)**.** *Assume for some some sequence $G_N, N \ge 1$ with $G_N \in \mathbf{CG}_N$ for all $N \ge 1$, we have for all maps $\mathbb{G}$ of genus $g \ge 1$ and $\mathfrak{l} \in \mathrm{L}(\mathbb{G})$, $\lim_{N\to\infty}\Phi^{G^N}_{\mathfrak{l}}$ and $\lim_{N\to\infty}\Phi^{G_N}_{\mathfrak{l}\otimes\mathfrak{l}^{-1}} = |\Phi_{\mathfrak{l}}|^2$ uniformly on $\Delta_{\mathbb{G}}(T)$, then Φ defines an exact solution of the Makeenko-Migdal solution as defined in section 3.5.*

To address uniqueness questions, it is convenient to work with *centered Wilson loops*. Define, for any $\mathfrak{l}_1, \dots, \mathfrak{l}_k$ in an area-weighted graph $(\mathbb{G}, a)$,

$$\psi^G_{\mathfrak{l}_1\otimes\dots\otimes\mathfrak{l}_k} = \mathbb{E}\left[\prod_{i=1}^k (W_{\mathfrak{l}_i} - \Phi_{\mathfrak{l}_i})\right].$$

Proposition 7.5 (Makeenko–Migdal equations for centered Wilson loops)**.** *Assume $g \ge 0$, $T > 0$, $\mathfrak{l} \in \mathfrak{L}_g$, $v \in V_{\mathfrak{l}}$ with $\delta_v\mathfrak{l} = \mathfrak{l}_1 \otimes \mathfrak{l}_2$. Then for any $G \in \mathbf{CG}_N$,*

$$\begin{aligned}\mu_v\psi^G_{\mathfrak{l}\otimes\mathfrak{l}^{-1}} =& \psi^G_{\mathfrak{l}_1\otimes\mathfrak{l}_2\otimes\mathfrak{l}^{-1}} + \psi^G_{\mathfrak{l}_1^{-1}\otimes\mathfrak{l}_2^{-1},\mathfrak{l}} + \psi^G_{\mathfrak{l}_1\otimes\mathfrak{l}^{-1}}\Phi_{\mathfrak{l}_2} + \psi^G_{\mathfrak{l}_1^{-1}\otimes\mathfrak{l}}\Phi_{\mathfrak{l}_2^{-1}} \\ &+ \psi^G_{\mathfrak{l}_2\otimes\mathfrak{l}^{-1}}\Phi_{\mathfrak{l}_1} + \psi_{\mathfrak{l}_2^{-1}\otimes\mathfrak{l}}\Phi_{\mathfrak{l}_1^{-1}} + \frac{R_{\mathfrak{l}}}{N},\end{aligned} \tag{62}$$

[30] we add up a finite number of terms, 6 to be precise, which are bounded by $\frac{1}{N}$, so their sum is bounded by $\frac{6}{N}$ which is indeed independent from the loops or the face-area vector.

where the $|R_{\mathfrak{l}}| \leq 10$ *uniformly on* $\Delta_{\mathbb{G}}(T)$*. There is a constant* $C_{\mathfrak{l}}$ *independent of* G*, such that for all* $X \in \mathfrak{m}_{\mathfrak{l}}$*,*

$$\begin{aligned}\mu_v \psi^G_{\mathfrak{l}\otimes\mathfrak{l}^{-1}} =& \psi_{\delta_X(\mathfrak{l})\otimes\mathfrak{l}^{-1}} + \psi^G_{\mathfrak{l},\delta_X(\mathfrak{l}^{-1})} + \psi_{\mathfrak{l}_1\otimes\mathfrak{l}^{-1}}\Phi_{\mathfrak{l}_2} + \psi_{\mathfrak{l}_1^{-1}\otimes\mathfrak{l}}\Phi_{\mathfrak{l}_2^{-1}} \\ &+ \psi^G_{\mathfrak{l}_2\otimes\mathfrak{l}^{-1}}\Phi_{\mathfrak{l}_1} + \psi_{\mathfrak{l}_2^{-1}\otimes\mathfrak{l}}\Phi_{\mathfrak{l}_1^{-1}} + \frac{R_{\mathfrak{l}}}{N},\end{aligned}$$

with $|R_{\mathfrak{l}}| \leq 10$ *uniformly on* $\Delta_{\mathbb{G}}(T)$*.*

Acknowledgments

The first author acknowledges partial support from "The Dr Perry James (Jim) Browne Research Centre on Mathematics and its Applications."

References

[1] Sergio Albeverio, Raphael H∅egh-Krohn, and Helge Holden. Stochastic Lie group-valued measures and their relations to stochastic curve integrals, gauge fields and Markov cosurfaces. In *Stochastic processes—mathematics and physics (Bielefeld, 1984)*, volume 1158 of *Lecture Notes in Math.*, pages 1–24. Springer, Berlin, 1986.

[2] Sergio Albeverio, Raphael H∅egh-Krohn, and Helge Holden. Stochastic multiplicative measures, generalized Markov semigroups, and group-valued stochastic processes and fields. *J. Funct. Anal.*, 78(1):154–184, 1988.

[3] Michael Anshelevich and Ambar N. Sengupta. Quantum free Yang–Mills on the plane. *J. Geom. Phys.*, 62(2):330–343, 2012.

[4] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[5] Florent Benaych-Georges and Thierry Lévy. A continuous semigroup of notions of independence between the classical and the free one. *Ann. Probab.*, 39(3):904–938, 2011.

[6] Philippe Biane. Free Brownian motion, free stochastic calculus and random matrices. In *Free probability theory (Waterloo, ON, 1995)*, volume 12 of *Fields Inst. Commun.*, pages 1–19. Amer. Math. Soc., Providence, RI, 1997.

[7] Joan S. Birman and Caroline Series. Dehn's algorithm revisited, with applications to simple curves on surfaces. In *Combinatorial group theory and topology (Alta, Utah, 1984)*, volume 111 of *Ann. of Math. Stud.*, pages 451–478. Princeton Univ. Press, Princeton, NJ, 1987.

[8] Jean-Michel Bismut and François Labourie. Symplectic geometry and the Verlinde formulas. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 97–311. Int. Press, Boston, MA, 1999.

[9] Guillaume Cébron, Antoine Dahlqvist, and Franck Gabriel. The generalized master fields. *J. Geom. Phys.*, 119:34–53, 2017.

[10] Sourav Chatterjee. Rigorous solution of strongly coupled so(n) lattice gauge theory in the large n limit. *Communications in Mathematical Physics*, 366(1):203–268, 2019.

[11] Ilya Chevyrev. Yang-Mills measure on the two-dimensional torus as a random distribution. *Comm. Math. Phys.*, 372(3):1027–1058, 2019.

[12] Benoît Collins, Alice Guionnet, and Edouard Maurel-Segala. Asymptotics of unitary and orthogonal matrix integrals. *Adv. Math.*, 222(1):172–215, 2009.

[13] Benoît Collins and Piotr Śniady. Integration with respect to the Haar measure on unitary, orthogonal and symplectic group. *Comm. Math. Phys.*, 264(3):773–795, 2006.

[14] Antoine Dahlqvist. Free energies and fluctuations for the unitary Brownian motion. *Comm. Math. Phys.*, 348(2):395–444, 2016.

[15] Antoine Dahlqvist. Integration formulas for Brownian motion on classical compact Lie groups. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(4):1971–1990, 2017.

[16] Antoine Dahlqvist and Thibaut Lemoine. 2021.

[17] Antoine Dahlqvist and James R. Norris. Yang–mills measure and the master field on the sphere. *Communications in Mathematical Physics*, 377(2):1163–1226, 2020.

[18] Persi Diaconis and Steven N. Evans. Linear functionals of eigenvalues of random matrices. *Trans. Amer. Math. Soc.*, 353(7):2615–2633, 2001.

[19] Bruce K. Driver. YM_2: continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.*, 123(4):575–616, 1989.

[20] Bruce K. Driver. A functional integral approaches to the Makeenko-Migdal equations. *Comm. Math. Phys.*, 370(1):49–116, 2019.

[21] Bruce K. Driver, Franck Gabriel, Brian C. Hall, and Todd Kemp. The Makeenko-Migdal equation for Yang-Mills theory on compact surfaces. *Comm. Math. Phys.*, 352(3):967–978, 2017.

[22] Bruce K. Driver, Brian C. Hall, and Todd Kemp. Three proofs of the Makeenko-Migdal equation for Yang-Mills theory on the plane. *Comm. Math. Phys.*, 351(2):741–774, 2017.

[23] William M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.

[24] Rajesh Gopakumar and David J. Gross. Mastering the master field. *Nuclear Phys. B*, 451(1-2):379–415, 1995.

[25] David J. Gross and Andrei Matytsin. Some properties of large-n two-dimensional yang-mills theory. *Nuclear Physics B*, 437(3):541–584, 1995.

[26] David J. Gross and Washington Taylor, IV. Two-dimensional QCD is a string theory. *Nuclear Phys. B*, 400(1-3):181–208, 1993.

[27] Leonard Gross. The Maxwell equations for Yang-Mills theory. In *Mathematical quantum field theory and related topics (Montreal, PQ, 1987)*, volume 9 of *CMS Conf. Proc.*, pages 193–203. Amer. Math. Soc., Providence, RI, 1988.

[28] Leonard Gross, Christopher King, and Ambar Sengupta. Two-dimensional Yang-Mills theory via stochastic differential equations. *Ann. Physics*, 194(1):65–112, 1989.

[29] Brian C. Hall. The large-N limit for two-dimensional Yang–Mills theory. *Comm. Math. Phys.*, 363(3):789–828, 2018.

[30] V. A. Kazakov. Wilson loop average for an arbitrary contour in two-dimensional $\mathrm{U}(N)$ gauge theory. *Nuclear Phys. B*, 179(2):283–292, 1981.

[31] V. A. Kazakov and I. K. Kostov. Nonlinear strings in two-dimensional $\mathrm{U}(\infty)$ gauge theory. *Nuclear Phys. B*, 176(1):199–215, 1980.

[32] Thierry Lévy. Yang-Mills measure on compact surfaces. *Mem. Amer. Math. Soc.*, 166(790):xiv+122, 2003.

[33] Thierry Lévy. Discrete and continuous Yang-Mills measure for non-trivial bundles over compact surfaces. *Probab. Theory Related Fields*, 136(2):171–202, 2006.

[34] Thierry Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Adv. Math.*, 218(2):537–575, 2008.

[35] Thierry Lévy. Two-dimensional Markovian holonomy fields. *Astérisque*, (329):172, 2010.

[36] Thierry Lévy. The master field on the plane. *Astérisque*, (388):ix+201, 2017.

[37] Thierry Lévy. *Two-Dimensional Quantum Yang–Mills Theory and the Makeenko–Migdal Equations*, pages 275–325. Springer International Publishing, Cham, 2020.

[38] Karl Liechty and Dong Wang. Nonintersecting brownian motions on the unit circle. *The Annals of Probability*, 44(2):1134–1211, 2016.

[39] Kefeng Liu. Heat kernels, symplectic geometry, moduli spaces and finite groups. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 527–542. Int. Press, Boston, MA, 1999.

[40] Michael Magee. Random Unitary Representations of Surface Groups I: Asymptotic expansions. *arXiv e-prints*, page arXiv:2101.00252, January 2021.

[41] Michael Magee. Random Unitary Representations of Surface Groups II: The large n limit. *arXiv e-prints*, page arXiv:2101.03224, January 2021.

[42] Yuri Makeenko and Alexander A. Migdal. Exact equation for the loop average in multicolor QCD. *Physics Letters B*, 88:135–137, 1979.

[43] Pierre Mergny and Marc Potters. Rank one HCIZ at high temperature: interpolating between classical and free convolutions. 2021. arXiv preprint: https://arxiv.org/abs/2101.01810.

[44] Alexander A. Migdal. Recursion equations in gauge field theories. *Sov. Phys. JETP*, pages 413–418, 1975.

[45] James A. Mingo and Roland Speicher. *Free probability and random matrices*, volume 35 of *Fields Institute Monographs*. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2017.

[46] Jürgen Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.*, 120:286–294, 1965.

[47] Ambar Sengupta. Gauge theory on compact surfaces. *Mem. Amer. Math. Soc.*, 126(600):viii+85, 1997.

[48] Ambar Sengupta. Yang-Mills on surfaces with boundary: quantum theory and symplectic limit. *Comm. Math. Phys.*, 183(3):661–705, 1997.

[49] Ambar N. Sengupta. The volume measure for flat connections as limit of the yang–mills measure. *Journal of Geometry and Physics*, 47(4):398–426, 2003.

[50] Ambar N. Sengupta. Traces in two-dimensional QCD: the large-N limit. In *Traces in number theory, geometry and quantum fields*, Aspects Math., E38, pages 193–212. Friedr. Vieweg, Wiesbaden, 2008.

[51] Isadore M. Singer. On the master field in two dimensions. In *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, volume 131 of *Progr. Math.*, pages 263–281. Birkhäuser Boston, Boston, MA, 1995.

[52] John Stillwell. *Classical topology and combinatorial group theory*, volume 72 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[53] Gerard 't Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 72(3):461 – 473, 1974.

[54] Dan Voiculescu. Limit laws for random matrices and free products. *Invent. Math.*, 104(1):201–220, 1991.

[55] Dan Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory. VI. Liberation and mutual free information. *Adv. Math.*, 146(2):101–166, 1999.

[56] Dan Voiculescu. Lectures on free probability theory. In *Lectures on probability theory and statistics (Saint-Flour, 1998)*, volume 1738 of *Lecture Notes in Math.*, pages 279–349. Springer, Berlin, 2000.

[57] Kenneth G. Wilson. Confinement of quarks. *Phys. Rev.*, 10, 1974.

[58] Edward Witten. The $1/N$ expansion in atomic and particle physics. *NATO Sci. Ser. B*, 59:403–419, 1980.

[59] Edward Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 141(1):153–209, 1991.

[60] Feng Xu. A random matrix model from two-dimensional Yang–Mills theory. *Comm. Math. Phys.*, 190(2):287–307, 1997.