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Bilaplacian on a Riemannian manifold and Levi-Civita connection

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1 Introduction

It is classical in stochastic analysis that the horizontal lift of a diffusion is useful in order to construct canonically the Brownian motion on a Riemannian manifold ([1], [2]). We extend in this note this classical relation on the Brownian motion and the (degenerated) process associated to the horizontal Laplacian to the case of a Bilaplacian. See [7] in the subelliptic case.

We consider a compact Riemannian oriented manifold M endowed with its normalized Riemannian measure dx. x is the generic element of M which is of dimension m. We consider the special orthonormal frame bundle $SO(M)$ endowed with the Levi-Civita connection with canonical projection π on M. u is the generic element of $SO(M)$. We consider the canonical vector fields X_i on $SO(M)$ and the associated horizontal Laplacian

$$L = \sum_{i=1}^{m} X_i^2 \tag{1}$$

Δ is the Laplace-Beltrami operator on M and Δ^2 the associated Bilaplacian. The Bilaplacian is elliptic, symmetric on $L^2(dx)$ and by elliptic theory ([5], [6])generates a unique contraction semi group P_t^{Δ} on $L^2(dx)$ with generic element f.

We can glue on the fiber the normalized "Haar" measures (We omitt the details) such that we get a probability measure du on $SO(M)$. L^2 is symmetric, densely defined on $L^2(du)$ and therefore admits a self-adjoint extension which generates a contraction semi-group P_t^L on $L^2(du)$.

Theorem 1 *If f is a smooth function on M, we have if $\pi u = x$*

$$P_t^{\Delta}[f](x) = P_t^L[f \circ \pi](u) \tag{2}$$

This theorem enters in our general program to extend stochastic analysis tools to the general theory of linear semi-group (See [8] and [9] for reviews).

2 Proof of the theorem

If u_0 belongs to $SO(\mathbb{R}^m)$ we get clearly

$$P_t^L[f \circ \pi](uu_0) = P_t^L[f \circ \pi(.u_0)](u) = P_t^L[f \circ \pi](u) \tag{3}$$

Therefore, $u \to P_t^L[f \circ \pi](u)$ defines a function on M. This function when t is moving defines a semi-group. Namely

$$P_{t+s}^L[f \circ \pi](u) = P_t^L[P_s^L[f \circ \pi]](u) = P_t^L[P_s^L[f \circ \pi] \circ \pi](u) \tag{4}$$

where is the right-hand side $P_s^L[f \circ \pi]$ is seen as a function on M.

We are therefore in presence of two semi-groups on $L^2(dx)$. But

$$L(f \circ \pi) = (\Delta f) \circ \pi \tag{5}$$

such that

$$L^2(f \circ \pi) = (\Delta^2 f) \circ \pi \tag{6}$$

Since there is only one semi-group generated by Δ^2, the result holds.

$\diamondsuit$

References

[1] K.D. Elworthy, *Stochastic differential equations on manifolds*. LMS Lectures Notes 70. Cambridge, England: Cambridge-University Press (1982.

[2] N. Ikeda and S.Watanabe, *Stochastic differential equations and diffusion processes* Sec. Edi, Amsterdam, Netherland: North-Holland (1989).

[3] S. Kobayashi and N. Nomizu, *Foundations of differential geometry I* New-York, U.S.A.: Interscience 1963.

[4] S.Kobayashi and N. Nomizu, *Foundations of differential geometry II* New-York, U.S.A.: Interscience, 1969.

[5] L. Hoermander, *The analysis of linear partial operators III*. Berlin, Germany: Springer, 1984.

[6] L. Hoermander, *The analysis of linear partial operators IV*. Berlin, Germany: Springer, 1984.

[7] R. Léandre, "A geometrical hypoelliptic diffusion", In *Differential geometry and its applications*.J. Bures and al eds.Prague, Cech republic: matfyzpress, 2004, pp 517-523.

[8] R. Léandre, "Malliavin Calculus of Bismut type in semi-group theory", *Far-East. J. Math. Sci*, 30, pp 1-26 (2008).

[9] R. Léandre, "Stochastic analysis for a non-markovian generator: an introduction", *Russian Journal of Mathematical Physics*, 22, pp 39–52, 2015.