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# Bilaplacian on a Riemannian manifold and Levi-Civita connection

Rémi Léandre

Laboratoire de Mathématiques, Université de Franche-Comté,  
25030, Besançon, FRANCE.  
email: remi.leandre@univ-fcomte.fr

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## 1 Introduction

It is classical in stochastic analysis that the horizontal lift of a diffusion is useful in order to construct canonically the Brownian motion on a Riemannian manifold ([1], [2]). We extend in this note this classical relation on the Brownian motion and the (degenerated) process associated to the horizontal Laplacian to the case of a Bilaplacian. See [7] in the subelliptic case.

We consider a compact Riemannian oriented manifold  $M$  endowed with its normalized Riemannian measure  $dx$ .  $x$  is the generic element of  $M$  which is of dimension  $m$ . We consider the special orthonormal frame bundle  $SO(M)$  endowed with the Levi-Civita connection with canonical projection  $\pi$  on  $M$ .  $u$  is the generic element of  $SO(M)$ . We consider the canonical vector fields  $X_i$  on  $SO(M)$  and the associated horizontal Laplacian

$$L = \sum_{i=1}^m X_i^2 \tag{1}$$

$\Delta$  is the Laplace-Beltrami operator on  $M$  and  $\Delta^2$  the associated Bilaplacian. The Bilaplacian is elliptic, symmetric on  $L^2(dx)$  and by elliptic theory ([5], [6]) generates a unique contraction semi group  $P_t^\Delta$  on  $L^2(dx)$  with generic element  $f$ .

We can glue on the fiber the normalized "Haar" measures (We omitt the details) such that we get a probability measure  $du$  on  $SO(M)$ .  $L^2$  is symmetric, densely defined on  $L^2(du)$  and therefore admits a self-adjoint extension which generates a contraction semi-group  $P_t^L$  on  $L^2(du)$ .

**Theorem 1** *If  $f$  is a smooth function on  $M$ , we have if  $\pi u = x$*

$$P_t^\Delta[f](x) = P_t^L[f \circ \pi](u) \tag{2}$$

This theorem enters in our general program to extend stochastic analysis tools to the general theory of linear semi-group (See [8] and [9] for reviews).

## 2 Proof of the theorem

If  $u_0$  belongs to  $SO(\mathbb{R}^m)$  we get clearly

$$P_t^L[f \circ \pi](uu_0) = P_t^L[f \circ \pi(.u_0)](u) = P_t^L[f \circ \pi](u) \quad (3)$$

Therefore,  $u \rightarrow P_t^L[f \circ \pi](u)$  defines a function on  $M$ . This function when  $t$  is moving defines a semi-group. Namely

$$P_{t+s}^L[f \circ \pi](u) = P_t^L[P_s^L[f \circ \pi]](u) = P_t^L[P_s^L[f \circ \pi] \circ \pi](u) \quad (4)$$

where is the right-hand side  $P_s^L[f \circ \pi]$  is seen as a function on  $M$ .

We are therefore in presence of two semi-groups on  $L^2(dx)$ . But

$$L(f \circ \pi) = (\Delta f) \circ \pi \quad (5)$$

such that

$$L^2(f \circ \pi) = (\Delta^2 f) \circ \pi \quad (6)$$

Since there is only one semi-group generated by  $\Delta^2$ , the result holds.

◇

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