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Letter on Uncertainty Relations for Mesoscopic Coherent Light

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Thermodynamic uncertainty relations unveil useful connections between fluctuations in thermal systems and entropy production. This letter extends these ideas to the disparate field of *zero temperature* quantum mesoscopic physics where fluctuations are due to coherent effects and entropy production is replaced by a cost function defined using a novel disorder reversal operator. A simple expression is obtained for the average cost function, which depends on the dimensionless conductance $g_{\mathcal{L}}$ and on a geometrical factor \mathcal{B} controlled by boundary conditions. Contrary to thermodynamic machines aimed at minimising fluctuations to increase precision, it is desirable in mesoscopic devices to increase coherent effects. The cost function indicates that increasing coherent effects can be achieved by playing with the geometry and boundary conditions through \mathcal{B} and not only by decreasing the bulk conductance $g_{\mathcal{L}}$.

This letter is a condensed presentation of our published work [1], which contains an introduction to coherent effects in quantum mesoscopic physics and step by step proofs of the results.

Stationary thermal systems display fluctuating currents j_{α} when taken far from equilibrium. A phenomenological formulation of these currents initiated by Onsager, is based on an entropy production rate Σ_{th} related to these currents and their associated forces. Useful and simple inequalities, $\mathcal{F}_{\alpha}(\Sigma_{th})_T \geq 2$ between the entropy production and fluctuating currents have been recently obtained where $\mathcal{F}_{\alpha} \equiv (\langle j_{\alpha}^2 \rangle_T - \langle j_{\alpha} \rangle_T^2) / \langle j_{\alpha} \rangle_T^2$ and $\langle \dots \rangle_T$ is the thermal average. These inequalities termed thermodynamic uncertainty relations (TUR), have triggered significant effort [2–4] exploring their generality [5–12] and the universal, i.e. independent of specific details, lower bound. They provide quantitative criteria to evaluate the tradeoff between fluctuations and their cost, so as to produce currents with a certain precision. The TUR were successfully applied to assess energy input required to operate a clock or bounding the number of steps in an enzymatic cycle [2, 13], and deriving the efficiency of molecular motors [14]. Finally, TUR inspired further studies of entropy production bounds under certain constraints [15–19].

This work proposes a modified version of the TUR applicable to the disparate (zero temperature) quantum mesoscopic physics, a field devoted to study the behaviour of waves (quantum or classical) propagating in random media. There, incoherent i.e. phase averaged, wave propagation is described by phenomenological approaches where the disorder averaged wave intensity obeys a diffusion equation. Yet, phase coherent effects are not erased by the disorder average and they are at the origin of spectacular measurable effects, e.g. Anderson

localization (weak and strong) [20], coherent backscattering, universal conductance fluctuations [21] and Sharvin & Sharvin effect [21, 22]. These phase coherent effects are faithfully described by adding to the (incoherent) diffusion equation a noise term $\xi(\mathbf{r})$ whose amplitude and spatial correlation account for mesoscopic quantum coherent effects [23, 24]. This noise and the associated Langevin equation allow to establish a correspondence between quantum coherent mesoscopic physics and effective fluctuating non equilibrium systems, where phase coherent induced fluctuating currents here play the role of thermal fluctuations. This correspondence makes mesoscopic fluctuating currents eligible on their own to a non thermal kind of uncertainty relations, henceforth coined quantum mesoscopic uncertainty relations (QMUR).

One purpose of this letter is to define a cost function Σ , analogous to the entropy production rate Σ_{th} , so as to set a lower bound and a trade-off for phase coherent induced fluctuations $\langle f^2 \rangle$ for relevant mesoscopic quantities f , namely,

$$\langle f^2 \rangle_c \langle \Sigma \rangle \geq \langle f \rangle^2 \quad (1)$$

where $\langle f^2 \rangle_c = \langle f^2 \rangle - \langle f \rangle^2$ and $\langle \dots \rangle$ denotes an average over disorder realizations. The remainder of this letter is devoted to establishing the QMUR (20) and to generalise it under the form (11). Then, we establish an expression for the average cost function $\langle \Sigma \rangle$ and apply it to show the genuine interest of the QMUR to optimise quantum mesoscopic features in different setups. We will conclude by a discussion of some consequences of our findings, e.g. a cross fertilization between nonequilibrium physics and

quantum mesoscopics, a class of systems of relatively easy experimental access, thus enlarging the playground of uncertainty relations.

Quantum mesoscopic physics is essentially the study of coherent wave propagation in a random medium, namely a study of solutions of wave equations, e.g. Helmholtz or Schrödinger, with proper boundary conditions [25]. Despite its simple formulation, this problem is notoriously rich and difficult and a wealth of equivalent phenomenological descriptions has been proposed. They all start from microscopic quantities, e.g. the specific intensity $I_\mu(\hat{\mathbf{s}}, \mathbf{r})$ [21], defined as the radiation intensity at point \mathbf{r} , propagating along a direction $\hat{\mathbf{s}}$, the total i.e. angular averaged intensity $I_\mu(\mathbf{r}) = \overline{I_\mu(\hat{\mathbf{s}}, \mathbf{r})}$ and the current $\mathbf{j}_\mu(\mathbf{r}) = v \overline{I_\mu(\hat{\mathbf{s}}, \mathbf{r})} \hat{\mathbf{s}}$, where v is a properly defined group velocity [26].

Radiative transfer is a successful and easily implemented coarse grained phenomenological description of the macroscopic intensity $I_D(\mathbf{r})$ and the associated current \mathbf{j}_D , obtained by keeping the incoherent, i.e. phase independent, contribution of the disorder averaged microscopic quantities [21]. They are related by a Fick's law, $\mathbf{j}_D(\mathbf{r}) = -D \nabla I_D(\mathbf{r})$, which together with current conservation, leads to a steady state diffusion equation, $-D \Delta I_D(\mathbf{r}) = 0$ with boundary conditions ensuring the vanishing of the incoming diffusive flux, namely

$$I_D(\mathbf{r}) + \frac{2l}{3} \hat{\mathbf{n}} \cdot \nabla I_D(\mathbf{r}) - 5I_0(\mathbf{r}) = 0 \quad (2)$$

for any \mathbf{r} at the interface, and with $\hat{\mathbf{n}}$ a unit vector normal to it (see supplementary material). I_0 is the Drude intensity [21] – the fraction of incident light source not multiply scattered – decaying exponentially inside the medium. Radiative transfer is useful since, through the diffusion coefficient $D = vl/3$, it depends on a small number of parameters, the elastic mean free path l which accounts for scattering features of the random medium, the wave number k and an appropriate group velocity [27]. This extensively studied approach proved accurate to describe incoherent behaviour in the weak disorder limit $kl \gg 1$. Its main drawback is its neglecting of all interference effects, e.g. speckle patterns or weak and strong localisation, which are washed out by the disorder averaging. This state of affairs can be improved using a semi-classical description which enables to systematically include coherent effects in the radiative transfer model [21]. Essentially, all this stems from the remark that spatially long ranged coherent effects result mostly from short range phase dependent quantum crossings occurring at scales $\ll l$ (see Fig.1.b). Stated otherwise, the large scale coarse grained hydrodynamic description of incoherent light can be modified to include coherent effects by inserting a local, properly tailored, noise function so as to reproduce expected long range coherent effects. The details of this generally cumbersome but well understood procedure are sketched in the SM of [24]. Building

on this remark, an elegant and systematic description has been proposed [23], based on the Langevin equation,

$$\mathbf{j}(\mathbf{r}) = -D \nabla I(\mathbf{r}) + \boldsymbol{\xi}(\mathbf{r}) \quad (3)$$

for the mesoscopic and stochastic (i.e. non disorder averaged) quantities $I(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ defined at an intermediate scale $\geq l$. This stochastic approach while phenomenological in nature, is equivalent to a perturbation theory for microscopic quantities with respect to the small and dimensionless parameter $1/g\mathcal{L}$, where

$$g\mathcal{L} \equiv \frac{k^2 l}{3\pi} \mathcal{L} \quad (4)$$

and \mathcal{L} is a length characteristic of the geometry of the random medium [21], [28]. The zero-average and time-independent noise $\boldsymbol{\xi}(\mathbf{r})$ includes all the information relative to phase coherence induced by quantum crossings. Its spatial correlations are systematically calculable as powers of $1/g\mathcal{L}$. The $g\mathcal{L}$ -independent behaviour accounts for the incoherent diffusive limit. To lowest order $1/g\mathcal{L}$, the noise is gaussian [23, 24], $\langle \xi_\alpha(\mathbf{r}) \xi_\beta(\mathbf{r}') \rangle = \sigma(I_D) \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}')$ with a mobility,

$$\sigma(I) = \frac{2\pi l v^2}{3k^2} I^2(\mathbf{r}) \quad (5)$$

similar to thermal diffusive processes as noted in [29]. The noise, relative to the mean current, scales like $1/\sqrt{g\mathcal{L}} \ll 1$, and is therefore weak. See (see supplementary material). We emphasize that the noise $\boldsymbol{\xi}(\mathbf{r})$ accounts for phase dependent mesoscopic stochastic quantities and not for the random distribution of scatterers. The Langevin equation (3) based on the two parameters D and σ , provides a complete hydrodynamic description of the coherent light flow in the random medium and it extends Fick's law to the fluctuating mesoscopic quantities $I(\mathbf{r}) \equiv I_D(\mathbf{r}) + \delta I(\mathbf{r})$ and $\mathbf{j}(\mathbf{r}) \equiv \mathbf{j}_D(\mathbf{r}) + \delta \mathbf{j}(\mathbf{r})$.

Equipped with the Langevin equation and motivated by the corresponding Einstein relation between D and σ , we define a "free energy" function $F(I)$ such that $F''(I) \equiv d^2 F/dI^2 = D/\sigma$ [30]. We are now in a position to formulate the mean cost function $\langle \Sigma \rangle$ and to push further the analogy with TUR in the spirit of the Onsager description, i.e. along the lines of [2]. We recall that a system brought slightly out of equilibrium by the application of forces X_α , creates current densities $\langle j_\alpha \rangle_T$ linearly related to the forces, $\langle j_\alpha \rangle_T = \sum_\beta L_{\alpha\beta} X_\beta$, so that the products $j_\alpha X_\alpha$ are additive terms in the corresponding entropy creation. The Langevin description and the Einstein relation allow to formally identify a force $\mathbf{X} = \nabla(-F'(I_D)) = -(D/\sigma_D) \nabla I_D$, where $\sigma_D \equiv \sigma(I_D)$. This force induces the average current $\mathbf{j}_D = \sigma_D \mathbf{X} = -D \nabla I_D$. The average cost function, i.e. the entropy creation, is thus given by the volume integral of average quantities,

$$\langle \Sigma \rangle = \int_V d\mathbf{r} \mathbf{X} \cdot \mathbf{j}_D = - \int_V d\mathbf{r} \frac{D}{\sigma_D} \nabla I_D \cdot \mathbf{j}_D. \quad (6)$$

Note that in this effective Onsager description, quantum crossings are responsible for the stochastic behaviour and thus lead to a finite cost function.

The hydrodynamic description of mesoscopic coherent effects together with the Onsager formalism allow for a straightforward derivation of the QMUR (20). Consider the scalar stochastic mesoscopic quantity $f \equiv \int_V \mathbf{j} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is an arbitrary unit vector. Its disorder average, $\langle f \rangle = \int_V \mathbf{j}_D \cdot \hat{\mathbf{n}}$, can also be obtained from the inner product defined by $\langle \mathbf{a}, \mathbf{b} \rangle_{\sigma_D} \equiv \int_V d^3\mathbf{r} \mathbf{a}(\mathbf{r}) \cdot \mathbf{b}(\mathbf{r}) \sigma_D^{-1}$. Indeed, choosing $\mathbf{a} = \sigma_D \mathbf{X}$ and $\mathbf{b} = \sigma_D \hat{\mathbf{n}}$ leads to $\langle \sigma_D \mathbf{X}, \sigma_D \hat{\mathbf{n}} \rangle_{\sigma_D} = \int_V d^3\mathbf{r} \sigma_D (-D/\sigma_D) \nabla I_D \cdot \hat{\mathbf{n}} = \int_V \mathbf{j}_D \cdot \hat{\mathbf{n}} = \langle f \rangle$. The Cauchy-Schwartz relation,

$$\langle \sigma_D \hat{\mathbf{n}}, \sigma_D \hat{\mathbf{n}} \rangle_{\sigma_D} \langle \sigma_D \mathbf{X}, \sigma_D \mathbf{X} \rangle_{\sigma_D} \geq \langle \sigma_D \mathbf{X}, \sigma_D \hat{\mathbf{n}} \rangle_{\sigma_D}^2 \quad (7)$$

is exactly the announced QMUR (20) noting that (37) implies $\langle \Sigma \rangle = \int_V d\mathbf{r} \mathbf{j}_D^2 / \sigma_D = \langle \sigma_D \mathbf{X}, \sigma_D \mathbf{X} \rangle_{\sigma_D}$ and, by construction, $\langle f^2 \rangle_c = \int_V d\mathbf{r} \sigma_D = \langle \sigma_D \hat{\mathbf{n}}, \sigma_D \hat{\mathbf{n}} \rangle_{\sigma_D}$ (see supplementary material).

Choosing a physical quantity f linearly related to the mesoscopic stochastic current \mathbf{j} may seem restrictive. Note though, that standard physically relevant mesoscopic quantities correspond indeed to such linear combinations, e.g. the recently studied force induced by coherent fluctuations [24]. A generalised expression for QMUR will be given in (11).

While the straightforward Onsager-like derivation of (20) grasps the essential meaning, it nevertheless suffers from a few caveats. The QMUR are not yet in their most general form, the bound may still be tightened and more importantly, this formulation lacks a physical meaning of the cost function Σ in (37). To address these points and gain more physical insight, we present an alternative proof based on a path integral formalism. Starting from (3), the probability to observe a spatial trajectory $\{I(\mathbf{r}), \mathbf{j}(\mathbf{r})\}$ is [31]

$$\mathcal{P}\{\{I, \mathbf{j}\}\} \sim \exp \left[- \int_V d\mathbf{r} \frac{(\mathbf{j}(\mathbf{r}) + D \nabla I(\mathbf{r}))^2}{2\sigma} \right], \quad (8)$$

where $\nabla \cdot \mathbf{j} = 0$ is implicitly assumed. A simple scaling shows that for $kl \gg 1$, the noise term in (3) is small (see supplementary material). This implies that disorder averaged quantities are dominated by the mean field equations of (8) – the average i.e. incoherent trajectories $\{I_D(\mathbf{r}), \mathbf{j}_D(\mathbf{r})\}$. Note that the discretization of the noise (Ito/Stratonovich) is immaterial here at leading order due to the weak noise Langevin equation (3).

For thermal systems, given a space and time dependent trajectory Γ with a probability $P(\Gamma)$, the entropy production variable is $\Sigma_{\text{th}}(\Gamma) = \frac{1}{2} \log \frac{P(\Gamma)}{P(\theta\Gamma)}$ where θ is the time-reversal operator [32]. Since $\langle \Sigma_{\text{th}} \rangle_T = \int d\Gamma P(\Gamma) \Sigma_{\text{th}}(\Gamma) = \frac{1}{2} \int d\Gamma (P(\Gamma) - P(\theta\Gamma)) \Sigma_{\text{th}}(\Gamma)$ and since the integrand is always non-negative, the expectation $\langle \Sigma_{\text{th}} \rangle_T \geq 0$. The entropy production Σ_{th} vanishes only under time-reversal

symmetry $P(\Gamma) = P(\theta\Gamma) \forall \Gamma$. Generalising these ideas to time-independent mesoscopic systems requires an equivalent for time-reversed processes. For coherent waves in a random medium, we consider the "disorder reversed process"; the Langevin description suggests that, given a process $\Gamma = \{I, \mathbf{j}\}$ associated to one realization of disorder, there exists another realization of disorder corresponding to the trajectory $\Theta\Gamma = \{I, -\mathbf{j}\}$, where Θ is the "disorder reversal" operator. At a microscopic level, Θ acts on the specific intensity by reversing its propagation, $\Theta I_\mu(\hat{\mathbf{s}}, \mathbf{r}) = I_\mu(-\hat{\mathbf{s}}, \mathbf{r})$. Equipped with the notion of disorder reversal, we define the cost function,

$$\Sigma = \frac{1}{2} \log \frac{\mathcal{P}\{\{I, \mathbf{j}\}\}}{\mathcal{P}\{\{I, -\mathbf{j}\}\}} \quad (9)$$

analogous to the thermodynamic entropy production [33]. Relations (8) and (9) lead immediately to (37), $\langle \Sigma \rangle = \int d\mathbf{r} \frac{\mathbf{j}_D^2}{\sigma(I_D)}$, namely the cost function deduced from the Onsager approach, once noticing that for $g_{\mathcal{L}} \gg 1$, the probability is dominated by its mean value.

In thermal systems, a uniform boundary condition (e.g. equal temperature) leads to (equilibrium) time-reversal symmetry. Here, however, for fixed boundary conditions (2), quantum crossings do not allow for disorder reversal symmetry, even for the case of a uniformly illuminated sphere. Hence, the cost function $\langle \Sigma \rangle > 0$ for any finite value of $g_{\mathcal{L}}$ and the disorder symmetry is always broken by coherent effects (see Fig.1.b) [34]

The path integral formalism allows to establish the QMUR (20) under a generalised form. We define the stochastic mesoscopic quantity

$$f = \int d^3\mathbf{r} z(\mathbf{j}, I), \quad (10)$$

with z an arbitrary function thus generalising the linear dependence of f considered previously. Using an approach similar to [7], the cumulant generating function $\mu(\lambda) = \log \langle e^{\lambda f} \rangle$ attains a lower bound, which leads to the generalised QMUR (see supplementary material),

$$\langle f^2 \rangle_c \langle \Sigma \rangle \geq \langle \partial_{\mathbf{j}} f \rangle^2, \quad (11)$$

where $\langle \partial_{\mathbf{j}} f \rangle = \int d^3\mathbf{r} \mathbf{j}_D \frac{\delta}{\delta \mathbf{j}_D} z(\mathbf{j}_D, I_D)$. Note that (11) reduces to (20) for the linear choice $z = \frac{1}{v^2} \mathbf{j} \cdot \hat{\mathbf{n}}$ in (10), a result corresponding to the radiative force induced by coherent fluctuations [24].

An integration by parts in (37) together with the diffusion equation for I_D , lead to (see supplementary material),

$$\langle \Sigma \rangle = g_{\mathcal{L}} \mathcal{B}, \quad (12)$$

where $\mathcal{B} = -\frac{1}{2} \oint_{\mathcal{S}} \frac{\nabla I_D(\hat{\mathbf{r}}) \cdot d\hat{\mathbf{S}}}{I_D(\hat{\mathbf{r}})}$ is determined by the boundary conditions (2) and all lengths have been rescaled by the characteristic geometric length $\hat{\mathbf{r}} = \mathbf{r}/\mathcal{L}$. Relation

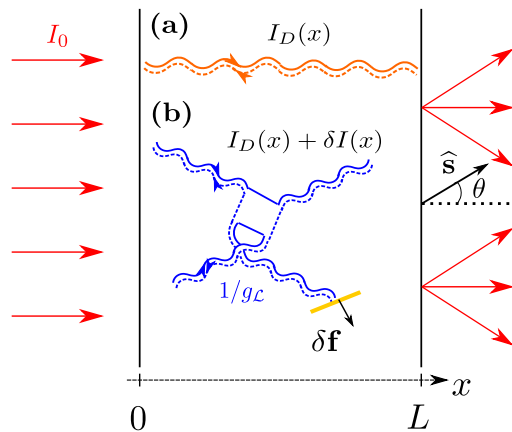


FIG. 1. **Broken disorder symmetry.** A slab of section S and width L is filled with a scattering medium so that $g_{\mathcal{L}} = k^2 l S / 3\pi L$. It is illuminated by a monochromatic plane wave of intensity I_0 . (a) The diffusive intensity I_D with boundary conditions (2), is built out of paired multiple scattering amplitudes, solutions of the Helmholtz equation, with opposite phases represented by full and dotted wave shaped lines. (b) Coherent intensity fluctuations $\delta I(x)$ are accounted by quantum crossings described by the noise (5). Exchange of multiple scattering amplitudes and new pairings occur within a small ($\propto 1/g_{\mathcal{L}}$) volume. Quantum crossings are not symmetric under disorder reversed trajectories with fixed boundary conditions (2) which thus emerge with a cost Σ . Intensity fluctuations induced by quantum crossings have been observed, e.g by measuring light transmitted along a direction $\hat{\mathbf{s}}$ [35, 36] and could be measured by means of the predicted fluctuations $\delta \mathbf{f}$ of the radiative force exerted on a suspended membrane (yellow in the figure) [24].

(12) constitutes an important result of this letter and it calls for a remark. The average cost function $\langle \Sigma \rangle$ depends non perturbatively on the parameter $g_{\mathcal{L}}$ which drives the strength of coherent effects and is large in the weak disorder limit $g_{\mathcal{L}} \gg 1$. The boundary term \mathcal{B} is independent of $g_{\mathcal{L}}$, and can be changed by modifying the geometry of the random medium. Finally, we highlight that, as opposed to thermal systems where the purpose is to minimize fluctuations, it is beneficial in mesoscopic setups to increase coherent effects. For example, for imaging in turbid media, coherent effects are not an obstruction but are instead used to focus light waves [37].

As stated in the introduction, an important feature of thermal uncertainty relations is the trade-off between fluctuations and their entropy cost. Turning to the quantum mesoscopic versions (20) and (11), the trade-off becomes how to enhance relative coherent fluctuations $\langle f^2 \rangle_c / \langle f \rangle^2$ by monitoring $\langle \Sigma \rangle$. It is thus useful to determine the $g_{\mathcal{L}}$ -dependence of each term in the inequalities. $\langle f \rangle$ depends on the macroscopic average current \mathbf{j}_D , and is therefore independent of $g_{\mathcal{L}}$. The Langevin equation (3) and the noise (5) imply that $\langle f^2 \rangle_c$ scales like $1/g_{\mathcal{L}}$ times a function which depends on I_D , \mathbf{j}_D , on the volume and on boundary conditions (see forthcoming examples).

The dependence of $\langle \Sigma \rangle$ is explicitly given in (12). Since $\langle f^2 \rangle_c$ scales like $1/g_{\mathcal{L}}$, it is clear that decreasing $g_{\mathcal{L}}$ will increase the fluctuations as is well known. The novelty of the QMUR is to show how to play with the boundary conditions or the geometry i.e. on \mathcal{B} in a controlled way, to monitor the fluctuations $\langle f^2 \rangle_c$ without changing the disorder strength.

We now illustrate these considerations in two situations. First, we present a direct check of the generalised QMUR (11) for the transmission coefficient $T(\theta)$ of light in a direction θ for the slab geometry of Fig.1. In this well studied and easily measurable one-dimensional setup of characteristic length $\mathcal{L} = S/L$, the diffusive intensity and current are $I_D(x) = \frac{I_0}{2\pi} \frac{1+u-3x/2l}{2+u}$ and $j_D = \frac{3DI_0}{4\pi(2+u)}$ with $u \equiv 3L/2l$, so that from (37), the average entropy cost is $\langle \Sigma \rangle = g_{\mathcal{L}} \frac{u^2}{2(1+u)}$ (see supplementary material). The mesoscopic stochastic transmission coefficient $T(\theta) = s_x \frac{I(\hat{\mathbf{s}}, L)}{I_0}$ with $s_x = \cos \theta$, depends on the light current and intensity using the radiative transfer equation [21], $I(\hat{\mathbf{s}}, x) = I(x) + \frac{1}{D} \hat{\mathbf{s}} \cdot \mathbf{j}(x)$. It thus abides (10) with $z(I, \mathbf{j}) = \delta(x-L) s_x (I(x) + \frac{1}{D} \hat{\mathbf{s}} \cdot \mathbf{j}(x)) / I_0$. In this geometry, the fluctuations $\langle T^2(\theta) \rangle_c$ induced by quantum crossings and the lower bound $\langle \partial_j T(\theta) \rangle$ in (11), take respectively the form $\langle T^2(\theta) \rangle_c = \frac{4}{3g_{\mathcal{L}}} \left(\frac{15s_x}{8\pi u} \right)^2 (s_x + 2/3)^2$ and $\langle \partial_j T(\theta) \rangle = \frac{15s_x^2}{8\pi(u+2)}$ hence following the scaling with $g_{\mathcal{L}}$ discussed earlier. The lower bound reaches its maximum for $\theta = 0$, however the inequality (11) is strict for any value of u in the multiple scattering regime (see supplementary material).

Secondly, we briefly discuss the recently studied radiative force induced by mesoscopic coherent fluctuations of light [24]. The radiative force exerted on a suspended membrane, of surface dS , immersed in the medium is given by $\delta \mathbf{f} = \hat{\mathbf{n}} v^{-2} \int_{dS} \mathbf{j} \cdot \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector normal to dS . As a result of coherent effects described by quantum crossings, this force displays significant and measurable fluctuations [24]. The spatially averaged force, $f = v^{-2} \int_V \mathbf{j} \cdot \hat{\mathbf{n}}$, satisfies the QMUR (20). Indeed using $\langle f^2 \rangle_c = v^{-4} \int_V \sigma(\mathbf{r})$ and the expressions for I_D and \mathbf{j}_D in a slab geometry, given earlier, we obtain

$$\frac{\langle f^2 \rangle_c \langle \Sigma \rangle}{\langle f \rangle^2} = \frac{(u+1)^3 - 1}{3u(u+1)}, \quad (13)$$

which is always greater than 1. The equality is not attained for $u \geq 1$; experimentally, it is reasonable to achieve $u \sim 10$, for which the ratio (13) is ~ 4 .

The purpose of this letter was to show that bounds set by thermal uncertainty relations (TUR) are not restricted to time-dependent thermal fluctuations. We proved the existence of another set of uncertainty relations (QMUR) associated to coherent mesoscopic fluctuations of waves, either classical or quantum, in the multiple scattering regime at weak disorder. We identified the cost function – the mesoscopic equivalent of entropy production – and defined it both at the coarse grained level of the

Langevin equation and at the microscopic level using the wave intensity and current density as observables. It remains an open question to find a mesoscopic setup that saturates the bound (11). Beyond these fundamental implications, our findings have a threefold interest. First, the uncertainty relations (20) and (11) provide a way to monitor coherent light fluctuations using the cost function $\langle \Sigma \rangle$ and its dependence upon boundary conditions or geometry through \mathcal{B} in (12) and not only the dimensionless conductance $g_{\mathcal{L}}$. This is of interest since coherent fluctuations are used as probes in biology and soft matter physics [37, 38]. Secondly, importing methods from statistical mechanics to mesoscopic physics, such as uncertainty relations [2–4] and lower bounds for the fluctuations, may prove helpful for imaging and wave control in complex media [39–41]. Finally, in thermal systems it is often hard to measure entropy production and to determine the conditions for a tight bound of TUR. Conversely, the significant progress made in recent years in the ability to control the light flow in random media [42–44], paves the way for experimental verifications of QMUR and measurements of the mesoscopic cost function.

We also wish to stress that the present Langevin description applies beyond the case of scalar coherent light propagation so as to include e.g polarization effects, anisotropic scattering and electronic quantum transport. But, extending the applicability of QMUR close to a Anderson localisation transition ($g_{\mathcal{L}} \sim 1$) where the Langevin approach is expected to break down appears more challenging. Yet, noting the unexpected connection between the cost function and Fisher information [45–47] (see supplementary material) is a possible path to explore to study QMUR for $g_{\mathcal{L}} \sim 1$.

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- [1] A. Soret, O. Shpielberg, and E. Akkermans, “Uncertainty relations for mesoscopic coherent light,” *Journal of Statistical Mechanics: Theory and Experiment* **2021**, 123302 (2021).
- [2] A. C. Barato, U. Seifert, *Phys. Rev. Lett.* **114**, 158101 (2015).
- [3] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, *Phys. Rev. Lett.* **116**, 120601 (2016).
- [4] J. M. Horowitz and T. R. Gingrich, *Nat. Phys.* **16**, 15–20 (2020).
- [5] T. Koyuk, U. Seifert and P. Pietzonka, *J. Phys. A: Math. Theor.* **52**, 02LT02 (2019).
- [6] A. Timpanaro, G. Guarnieri, J. Goold and G. T. Landi, *Phys. Rev. Lett.* **123**, 090604 (2019).
- [7] A. Dechant, S.-I. Sasa, *J. Stat. Mech.: Theory and Experiment* **2018**, 063209 (2018).
- [8] O. Niggemann and U. Seifert, *J. Stat. Phys.* , 1–33 (2020).
- [9] K. Macieszczak, K. Brandner and J. P. Garrahan, *Phys. Rev. Lett.* **121**, 130601 (2018).
- [10] J. M. Horowitz and T. R. Gingrich, *Phys. Rev. E* **96**, 020103 (2017).
- [11] G. Falasco, M. Esposito and J.-C. Delvenne, *New J. Phys.* **22**, 053046 (2020).
- [12] K. Proesmans and C. Van den Broeck, *EPL* **119**, 20001 (2017).
- [13] U. Seifert, *Physica A: Statistical Mechanics and its Applications* **504**, 176–191 (2018).
- [14] P. Pietzonka, A. C. Barato and U. Seifert, *Phys. Rev. E* **93**, 052145 (2016).
- [15] G. Bisker, M. Polettoni, T. R. Gingrich and J. M. Horowitz, *Journal of Statistical Mechanics: Theory and Experiment* **2017**, 093210 (2017).
- [16] O. Raz, Y. Subaşı and C. Jarzynski, *Phys. Rev. X* **6**, 021022 (2016).
- [17] D. M. Busiello, C. Jarzynski and O. Raz, *New Journal of Physics* **20**, 093015 (2018).
- [18] O. Shpielberg and T. Nemoto, *Phys. Rev. E* **100**, 032104 (2019).
- [19] G. Falasco and M. Esposito, *Phys. Rev. Lett.* **125**, 120604 (2020).
- [20] P. W. Anderson, *Phys. Rev.* **109**, 1492 (1957).
- [21] E. Akkermans and G. Montambaux, *Mesoscopic physics of electrons and photons* (Cambridge University Press, 2007).
- [22] D. Yu. Sharvin and Yu. V. Sharvin, *JETP Lett.* **34**, (1981).
- [23] B. Z. Spivak and A. Yu. Zyuzin, *Sov. Phys. JETP* **93**, 994–1006 (1987).
- [24] A. Soret, K. Le Hur, and E. Akkermans, *Phys. Rev. Lett.* **124**, 136803 (2020).
- [25] This description generally applies to d -dimensional media, yet we focus on the specific $d = 3$ for simplicity unless stated otherwise.
- [26] To maintain a homogeneous description throughout the paper, we opt for the language of propagation of scalar waves. A generalisation to other types of waves, classical or quantum is straightforward.
- [27] The microscopic and macroscopic group velocities may differ as discussed in [48]. We shall not differentiate between the two.
- [28] For a slab of section S and width L as in Fig.1, $\mathcal{L} = S/L$. See the SM in [24] for further examples.
- [29] O. Shpielberg and E. Akkermans, *Phys. Rev. Lett.* **116**, 240603 (2016).
- [30] B. Derrida, *J. Stat. Mech.* **2007**, P07023 (2007).
- [31] P. C. Martin, E. D. Siggia and H. A. Rose, *Phys. Rev. A* **8**, 423 (1973).
- [32] The normalization factor $1/2$ has no physical meaning; it is chosen to elegantly present the QMUR.
- [33] U. Seifert, *Rep. Prog. Phys.* **75**, 126001 (2012).
- [34] To lowest order in $1/g_{\mathcal{L}}$ considered for the noise (5), the average cost function $\langle \Sigma \rangle$ depends on the incoherent quantities I_D and \mathbf{j}_D so that it vanishes for a uniform illumination. Higher order terms are expected to break this universality.
- [35] F. Scheffold, W. Hartl, G. Maret, E. Matijevic, *Phys. Rev. B* **56**, 10942 (1997).
- [36] F. Scheffold and G. Maret, *Phys. Rev. Lett.* **81**, 5800 (1998).
- [37] A. P. Mosk, A. Lagendijk, G. Leroosey and M. Fink, *Nat.*

- Phot. **6**, 283–292 (2012).
- [38] G. Cox, *Optical Imaging Techniques in Cell Biology* (Taylor and Francis, 2012).
 - [39] S. M. Popoff, G. Lerosey, R. Carminati, M. Fink, A.C. Boccara, S. Gigan, Nat. Phot. **8**, 784–790 (2014).
 - [40] N. Fayard, A. Cazé, R. Pierrat, and R. Carminati, Phys. Rev. A **92**, (2015).
 - [41] I. Starshynov et. al., Phys. Rev. X **8**, (2018).
 - [42] I. M. Vellekoop and A. P. Mosk, Optics Lett. **32**, 2309–2311 (2007).
 - [43] S. Rotter and S. Gigan, Rev. Mod. Phys. **89**, (2017).
 - [44] I. M. Vellekoop, A. Lagendijk and A. P. Mosk, Nat. Phot. **4**, 320–322 (2010).
 - [45] Y. Hasegawa and T. Van Vu, Phys. Rev. E **99**, 062126 (2019).
 - [46] S. Ito and A. Dechant, Phys. Rev. X **10**, 021056 (2020).
 - [47] Arnab Pal, Shlomi Reuveni, and Saar Rahav, “Thermodynamic uncertainty relation for systems with unidirectional transitions,” Phys. Rev. Research **3**, 013273 (2021).
 - [48] M. P. van Albada, B. A. van Tiggelen, A. Lagendijk, and A. Tip, Phys. Rev. Lett. **66**, 3132–3135 (1991).

Supplementary material for Uncertainty Relations for Mesoscopic Coherent Light

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FIELD THEORY PROOF FOR THE QMUR

In what follows, we derive, in the spirit of [7], the QMUR and its generalization to the cumulant generating function. We complement the theoretical bound with a numerical study of the cumulant generating bound for the slab geometry, presented in the main text.

Let us start from the Langevin equation (3) and implicitly assume $\nabla \cdot \mathbf{j} = 0$. The Martin-Siggia-Rose technique allows to evaluate the probability of observing the realization $\Gamma = \{\mathbf{j}(\mathbf{x}), I(\mathbf{x})\}$ as in eq.(8),

$$\begin{aligned} \mathcal{P}[\Gamma] &\sim e^{-\int d^3\mathbf{r} \mathcal{L}[\Gamma]} \\ \mathcal{L}[\Gamma] &= \frac{(\mathbf{j} - \mathbf{J})^2}{2\sigma}, \end{aligned}$$

with $\mathbf{J} = -D\nabla I$. Define a tilted action due to the force \mathbf{Y} using $\mathcal{L}_{\mathbf{Y}} = (\mathbf{j} - \mathbf{J} - \mathbf{Y})^2/2\sigma$. It is easy to verify that $\mathcal{L}_{\mathbf{Y}}$ emanates from the tilted Langevin equation $\mathbf{j} = \mathbf{J} + \mathbf{Y} + \xi$, where ξ is the noise term defined in the main text. It is straightforward to verify that

$$\mathcal{L}[\mathbf{j}, I] = \mathcal{L}_{\mathbf{Y}}[\mathbf{j}, I] + \frac{|\mathbf{Y}|^2}{2\sigma} + \frac{\mathbf{Y}}{\sigma} \cdot (\mathbf{j} - \mathbf{J} - \mathbf{Y}). \quad (14)$$

Let us define the cumulant generating function for a general physical observable

$$\mu(\lambda) = \log\langle e^{\lambda f} \rangle, \quad (15)$$

where $f = \int d^3\mathbf{r} z(\mathbf{j}, I)$, and z an arbitrary function. Using (14), we find

$$\mu(\lambda) = \log\langle e^{\lambda f} e^{-\int d^3\mathbf{r} \frac{\mathbf{Y}^2}{2\sigma} + \frac{\mathbf{Y}}{\sigma} \cdot (\mathbf{j} - \mathbf{J} - \mathbf{Y})} \rangle_{\mathbf{Y}}, \quad (16)$$

where $\langle O \rangle_{\mathbf{Y}}$ is the averaging of O with respect to the weight $\mathcal{P}_{\mathbf{Y}}[\Gamma]$ defined with the tilted $\mathcal{L}_{\mathbf{Y}}$. Using the Jensen inequality, and identifying $\langle \frac{\mathbf{Y}}{\sigma} \cdot (\mathbf{j} - \mathbf{J} - \mathbf{Y}) \rangle_{\mathbf{Y}} = 0$, leads to the inequality

$$\mu(\lambda) \geq \lambda \langle f \rangle_{\mathbf{Y}} - \left\langle \int d^3\mathbf{r} \frac{\mathbf{Y}^2}{2\sigma} \right\rangle_{\mathbf{Y}}. \quad (17)$$

In fact, using the Ito convention allows to identify $\mathbf{j} - \mathbf{J} - \mathbf{Y}$ as the noise term in the tilted \mathbf{Y} dynamics even without relying on the saddle [7]. This in turn implies that $\langle \frac{\mathbf{Y}}{\sigma} \cdot (\mathbf{j} - \mathbf{J} - \mathbf{Y}) \rangle_{\mathbf{Y}} = 0$. Now, let us choose $\mathbf{Y} = \alpha \mathbf{j}_D$. Since $\mathbf{Y} = \alpha \mathbf{j}_D$ is divergence free, the optimal solution for I does not change. Therefore,

$$\begin{aligned} \left\langle \int d^3\mathbf{r} \frac{j_D^2}{\sigma} \right\rangle_{\alpha \mathbf{j}_D} &= \langle \Sigma \rangle \\ \langle f \rangle_{\alpha \mathbf{j}_D} &= \int d^3\mathbf{r} z(\mathbf{j}_D(1 + \alpha), I_D). \end{aligned} \quad (18)$$

From (17) and (18) we find the inequality

$$\mu(\lambda) \geq \lambda \langle f \rangle_{\alpha \mathbf{j}_D} - \frac{1}{2} \alpha^2 \langle \Sigma \rangle, \quad (19)$$

Let us assume $\alpha \ll 1$ to derive the generalized QMUR. In this limit the right hand side of the inequality (19) is maximised for $\alpha = \lambda \langle \partial_{\mathbf{j}} f \rangle / \langle \Sigma \rangle$ where we recall $\langle \partial_{\mathbf{j}} f \rangle \equiv \int d^3\mathbf{r} \mathbf{j}_D \frac{\delta}{\delta \mathbf{j}_D} z(\mathbf{j}_D, I_D)$, a scalar quantity.

The small α assumption implies $\lambda \ll 1$. Expanding $\mu(\lambda)$ to second order in λ we recover eq.(11), the generalized QMUR

$$\langle f^2 \rangle_c \langle \Sigma \rangle \geq \langle \partial_{\mathbf{j}} f \rangle^2. \quad (20)$$

For $z(\mathbf{j}, I) = \mathbf{j} \cdot \mathbf{u}(I)$ with an arbitrary function \mathbf{u} , $\alpha = \lambda \langle \partial_{\mathbf{j}} f \rangle / \langle \Sigma \rangle$ optimizes the lower bound

$$\mu(\lambda) \geq \lambda \langle f \rangle + \lambda^2 \frac{\langle f \rangle^2}{2 \langle \Sigma \rangle}. \quad (21)$$

Two notes are in order. First, the lower bound to the cumulant generating function (21) is the counterpart to the large deviation result reported in [3]. Second, note that a leading order expansion in λ of (21) leads to the QMUR of eq.(1).

Finally, we report a numerical confirmation of (21) for the fluctuation induced coherent force with the slab geometry in Fig.2, where $z = \frac{1}{v^2} \mathbf{j} \cdot \hat{\mathbf{n}}$ which corresponds to (21). .

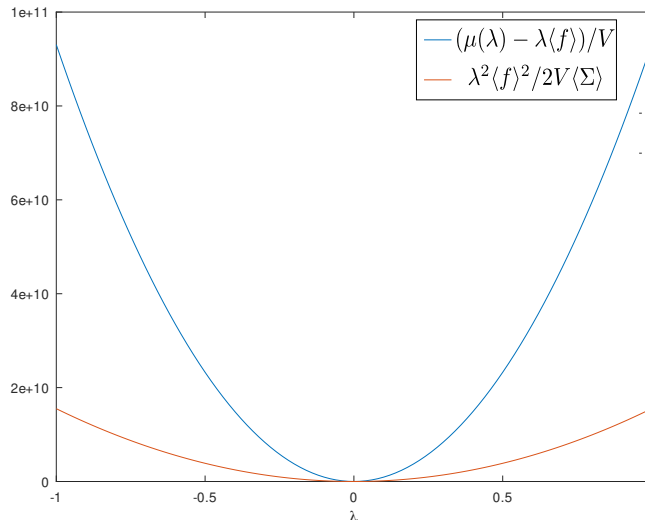


FIG. 2. Numerical verification of the lower bound to the cumulant generating function (21) of the fluctuation induced coherent force for the slab geometry. The figure presents $\mu(\lambda) - \lambda\langle f \rangle$ (blue) and its lower bound $\lambda^2\langle f \rangle^2/2V\langle \Sigma \rangle$ (red), both divided by the volume. Here we consider $L/l_0 = 5$. The bound is tightest in the slab geometry when L/l_0 is as small as physically possible.

CUMULANTS OF THE FLUCTUATION INDUCED COHERENT FORCE

Let us consider the cumulant generating function (CGF) of $f = \int d^d \mathbf{x} \mathbf{j}(\mathbf{x}) \cdot \hat{n}$, namely

$$\mu(\lambda) = \log\langle e^{\lambda f} \rangle. \quad (22)$$

The purpose of this section is to show that $\partial_\lambda \mu(0) = \langle f \rangle = \int d^d \mathbf{x} \mathbf{j}_D(\mathbf{x}) \cdot \hat{n}$ and $\partial_{\lambda\lambda} \mu(0) = \int d^d \mathbf{x} \sigma_D(\mathbf{x})$.

To do so, we first write explicitly the path integral formulation for the cumulant generating function

$$e^{\mu(\lambda)} = \int \mathcal{D}I \mathcal{D}j \mathcal{D}p \exp\left(-\int d^d \mathbf{x} \frac{1}{2\sigma} (\mathbf{j} + D\nabla I)^2 - \lambda \mathbf{j} \cdot \hat{n} + p \nabla \cdot \mathbf{j}\right). \quad (23)$$

The introduction of the p variable – a Lagrange multiplier – ensures a divergence free current in the bulk. Integrating the Gaussian integral in j , we find

$$e^{\mu(\lambda)} = \int \mathcal{D}I \mathcal{D}p e^{\int d^d \mathbf{x} \mathcal{H}(I,p)(\mathbf{x})}, \quad (24)$$

where $\mathcal{H} = -D\nabla I \nabla p + \frac{1}{2}\sigma(\nabla p)^2$ and we redefine $p \rightarrow p + \lambda n$ with $n \equiv \hat{n}_x x + \hat{n}_y y + \hat{n}_z z$ and $\hat{\mathbf{n}} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)$. Since we are dealing with a weak noise theory, the CGF is dominated by a saddle point solution, given by the saddle equations

$$\frac{\delta \mathcal{H}}{\delta p} = 0 \Rightarrow \nabla \cdot (D\nabla I - \sigma \nabla p) = 0 \quad (25)$$

$$\frac{\delta \mathcal{H}}{\delta I} = 0 \Rightarrow D\Delta p + \frac{1}{2}\sigma'(\nabla p)^2 = 0, \quad (26)$$

where $\sigma'(I) = \frac{\delta}{\delta I} \sigma(I)$. The boundary conditions for I is left unchanged as in the main text and such that $p = \lambda n$ on the boundary. Notice that what we have done is simply moving from a Lagrangian picture to a Hamiltonian one. The Hamiltonian picture is more straightforward in this case, where we aim to calculate the first two cumulants of the CGF at $\lambda = 0$. A general solution of (25) is hard to obtain. However, to evaluate $\mu(\lambda)$ to second order in λ , it is sufficient to consider the perturbative solution

$$\begin{aligned} I(\mathbf{x}) &= I_D(\mathbf{x}) + \lambda \delta I_1(\mathbf{x}) + O(\lambda^2) \\ p(\mathbf{x}) &= \lambda \delta p_1(\mathbf{x}) + O(\lambda^2). \end{aligned} \quad (27)$$

Solving the saddle equations to first order in λ we find

$$\begin{aligned} D\delta I_1(\mathbf{x}) &= \hat{\mathbf{n}} \cdot \int g(\mathbf{x}, \mathbf{y}) d\mathbf{y} \nabla_{\mathbf{y}} \sigma_D(\mathbf{y}) \\ \nabla \delta p_1 &= \hat{\mathbf{n}}, \end{aligned} \quad (28)$$

where $\Delta_{\mathbf{x}} g(\mathbf{x}, \mathbf{y}) = \delta^d(\mathbf{x} - \mathbf{y})$ defines the Green function of the Laplacian with vanishing boundary conditions. Plugging the solutions (28) into (24), we find to second order in λ that indeed $\langle f \rangle = \int d^d \mathbf{x} \mathbf{j}_D(\mathbf{x}) \cdot \hat{\mathbf{n}}$ and $\langle f^2 \rangle_c = \int d^d \mathbf{x} \sigma_D(\mathbf{x})$.

BOUNDARY CONDITIONS

In this section, we discuss the boundary conditions for the diffusive intensity I_D . The exact boundary conditions for multiply scattered light intensity are not trivial, and we refer to the section A5.2 in [21] for a detailed derivation. Moreover, in this exact description, the light intensity satisfies a diffusion equation with a source term, unlike the convention used in the main text, where we assumed $-D\Delta I_D = 0$. The purpose of this section is to obtain an alternative set of boundary conditions (31), which, associated with the source free diffusion equation, give a good approximation for the intensity I_D , simplifying the derivation of the QMUR.

The idea behind the boundary conditions for the diffusive light intensity is to formalize that, since diffusion processes happen inside the disordered medium, there can be no incoming diffusive intensity at the interface. For a random medium, illuminated by an external light source of intensity I_0 , propagating in the direction $\hat{\mathbf{k}}$, the diffusive intensity is the solution of the following problem,

$$\begin{aligned} \Delta I_D(\mathbf{r}) &= -\frac{v}{D} I_0(\mathbf{r}) \\ I_D(\mathbf{r}) + \frac{2l}{3} \hat{\mathbf{n}} \cdot \nabla I_D(\mathbf{r}) &= 0 \text{ for any } \mathbf{r} \text{ at the interface} \end{aligned} \quad (29)$$

where $\hat{\mathbf{n}}$ is the normal unit vector at the point \mathbf{r} on the surface. $I_0(\mathbf{r})$ is the ballistic component of the intensity, corresponding to the fraction of the incoming radiation which propagates without any collisions on the scatterers; it decays exponentially with the distance to the surface, $I_0(\mathbf{r}) \propto e^{-\mathbf{r} \cdot \hat{\mathbf{k}}/l}$.

We wish to reformulate the boundary conditions in order to have a source-less diffusion equation for I_D – or equivalently $\nabla \cdot \mathbf{j}_D = 0$ – which is more convenient for the derivation of the QMUR in the main text. We begin by noticing that $\Delta I_D(\mathbf{r}) \simeq 0$ for \mathbf{r} at a distance $> l$ from the surface. The idea is to neglect the layer of width l at the boundary, and to solve for I_D, \mathbf{j}_D in the bulk, where we can assume $\Delta I_D = 0$, and impose as boundary conditions the solutions of the exact problem (29) at the distance l from the boundary, see Fig.3.

To avoid confusion, we note I'_D, \mathbf{j}'_D the approximated solutions in the bulk, such that $\nabla \cdot \mathbf{j}'_D = 0$.

We obtain the boundary conditions for I'_D, \mathbf{j}'_D by calculating the incoming current $j_D^+ = \mathbf{j}_D^+ \cdot \hat{\mathbf{n}}_{\text{in}}$ of the real problem (29) at the distance l from the boundary. By definition, $j_D^+ = v \langle \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_{\text{in}} I_D(\hat{\mathbf{s}}, \mathbf{r}) \rangle_{\hat{\mathbf{s}}^+}$ where the average is taken over the half space $\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_{\text{in}} \geq 0$. On the other hand, j_D^+ is related to I_D by means of the radiative transfer equation [21],

$$\begin{aligned} I_D(\hat{\mathbf{s}}, \mathbf{r}) &= I_D(\mathbf{r}) - l \hat{\mathbf{s}} \cdot \nabla I_D(\mathbf{r}) \\ \Rightarrow j_D^+(\mathbf{r}) &= \frac{v}{2} I_D(\mathbf{r}) - \frac{vl}{3} \hat{\mathbf{n}}_{\text{in}} \cdot \nabla I_D(\mathbf{r}) \end{aligned} \quad (30)$$

We derive $j_D^+(\mathbf{r})$ by solving (29), and obtain the boundary conditions for I'_D, \mathbf{j}'_D ,

$$I'_D(\mathbf{r}) + \frac{2l}{3} \hat{\mathbf{n}} \cdot \nabla I'_D(\mathbf{r}) = \frac{2}{v} j_D^+(\mathbf{r}) \text{ for any } \mathbf{r} \text{ at a distance } l \text{ from the interface} \quad (31)$$

where $\hat{\mathbf{n}} = -\hat{\mathbf{n}}_{\text{in}}$ is the normal vector of the fictive interface, see Fig.3.

We now derive explicitly the new boundary conditions (31) for a slab geometry, considered in the main text.

Example: slab geometry

Consider the case of an infinite slab, of width L , illuminated by a homogeneous light beam of intensity I_0 , see Fig.3. In this geometry, the radiative transfer equation (30) becomes

$$\begin{aligned} I_D(\hat{\mathbf{s}}, x) &= I_D(x) - l \partial_x I_D(x) \\ \Rightarrow j_D^+(x) &= \frac{v}{2} I_D(x) - \frac{vl}{3} \partial_x I_D(x) \end{aligned} \quad (32)$$

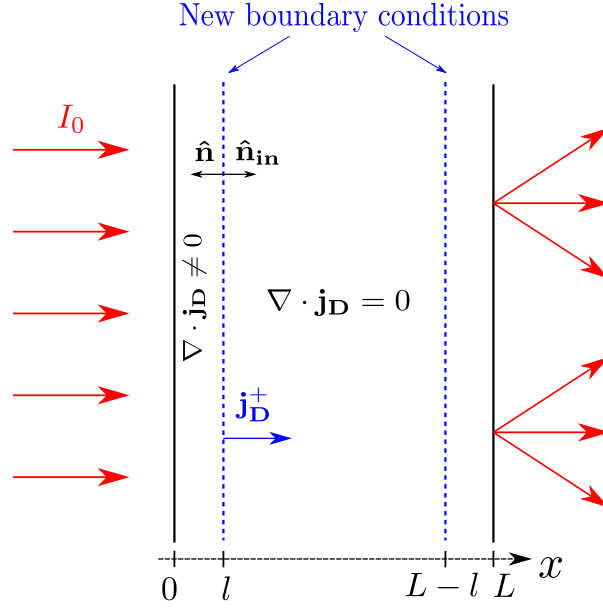


FIG. 3. A slab of scattering medium is illuminated by a plane wave. The diffusive current obeys a continuity equation $\nabla \cdot \mathbf{j}_D = vI_0(\mathbf{r})/l \simeq 0$ for \mathbf{r} at a distance $> l$ from the boundary. We solve for I_D, \mathbf{j}_D in the bulk, assuming the current to be divergence free, and shifting the boundary conditions to the fictive boundary defined as the surface at a distance l from the boundary (blue dotted lines).

In this geometry, the Drude intensity is given by $I_0(\mathbf{r}) = I_0 e^{-x/l}/4\pi$. Solving the exact problem (29), we find, in the limit $L \gg l$,

$$I_D(l) - \frac{2l}{3} \partial_x I_D(l) = \frac{5I_0}{4\pi} \quad (33)$$

We therefore define the boundary conditions to be $j_D^+ = \frac{5I_0}{4\pi}$ at the new boundary (the surface at a distance l from the boundary), which, using eq.(32), can be formulated as

$$I'_D(\mathbf{r}) + \frac{2l}{3} \hat{\mathbf{n}} \cdot \nabla I'_D(\mathbf{r}) = 5I_0(\mathbf{r}), \quad (34)$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the surface, and we recover eq.(2). Let's now compare the exact and approximated solutions. The approximated solution to (34) is

$$I'_D(x) = \frac{I_0}{4\pi} \frac{-5 + 5e^{-L/l}}{4l/3 + L} x + \frac{I_0}{4\pi} \frac{5L + \frac{10l}{3}(1 + e^{-L/l})}{4l/3 + L} \quad (35)$$

In comparison, the exact solution, obtained from (29), is

$$I_D(x) = \frac{I_0}{4\pi} \frac{-5 + e^{-L/l}}{4l/3 + L} x + \frac{I_0}{4\pi} \frac{5L + \frac{2l}{3}(5 + e^{-L/l})}{4l/3 + L} - \frac{3I_0}{4\pi} e^{-x/l}, \quad (36)$$

hence the two solutions (35) and (36) differ only by exponentially decreasing terms, see Fig.4.

DERIVATION OF $\langle \Sigma \rangle$

Consider a scattering medium contained in a volume V , of external surface S , illuminated by a light source (external or internal). We show here that

$$\langle \Sigma \rangle = g_{\mathcal{L}} \mathcal{B} \quad (37)$$

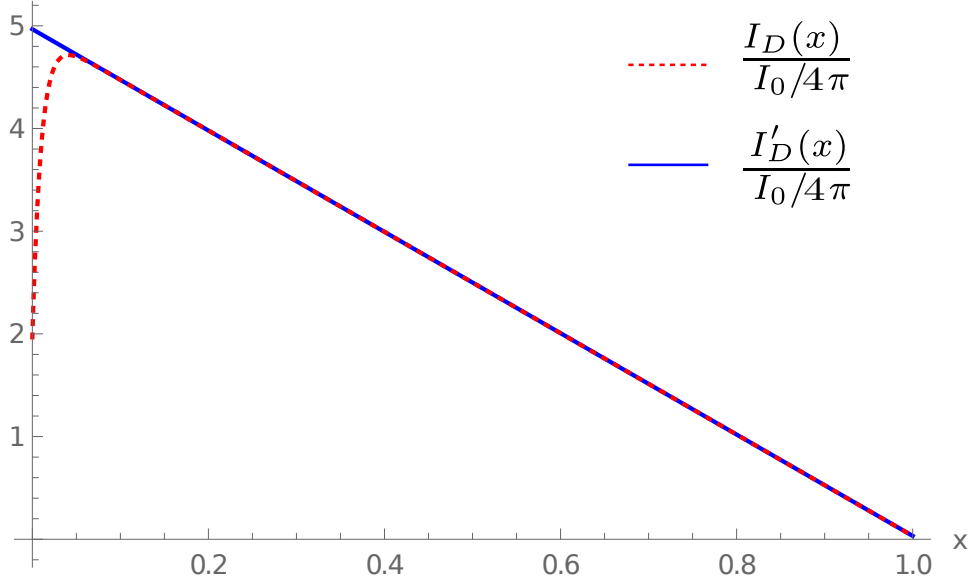


FIG. 4. Exact and approximated solutions I_D and I'_D respectively for a slab geometry, for $l/L = 0.01$, as functions of the rescaled variable $x \rightarrow x/L$, $x \in [0, 1]$. The solutions are normalized by $I_0/4\pi$.

as claimed in eq.(12) of the main text. We remind that $\mathcal{B} = -\frac{1}{2} \oint_S \frac{\nabla I_D(\tilde{\mathbf{r}}) \cdot d\tilde{\mathbf{S}}}{I_D(\tilde{\mathbf{r}})}$, where the tilde implies the scaling of all length scales with respect to the characteristic length of the system \mathcal{L} . Therefore \mathcal{B} is a dimensionless number, depending on the boundary conditions. We remind that the dimensionless conductance $g_{\mathcal{L}}$ is given by $g_{\mathcal{L}} = k^2 l \mathcal{L} / 3\pi$.

The proof consists of writing $\langle \Sigma \rangle$ in terms of I_D alone, then using an integration by parts, and finally using the boundary conditions eq.(31). The first step is achieved using Fick's law $\mathbf{j}_D = -D \nabla I_D$; starting from eq.(6), we obtain

$$\begin{aligned}
 \langle \Sigma \rangle &= \int_V d^3 \mathbf{r} \frac{|\mathbf{j}_D(\mathbf{r})|^2}{\sigma(I_D(\mathbf{r}))} \\
 &= \frac{D^2}{c_0} \int_V d^3 \mathbf{r} \frac{|\nabla I_D(\mathbf{r})|^2}{I_D(\mathbf{r})^2} \\
 &= \frac{D^2}{c_0} \int_V d^3 \mathbf{r} \nabla \left(\frac{-1}{I_D(\mathbf{r})} \right) \cdot \nabla I_D(\mathbf{r})
 \end{aligned} \tag{38}$$

with $c_0 = \frac{2\pi l v^2}{3k^2}$. An integration by parts yields

$$\langle \Sigma \rangle = \frac{D^2}{c_0} \oint_S \frac{-1}{I_D(\mathbf{r})} \nabla I_D(\mathbf{r}) \cdot d\mathbf{S} - \frac{D^2}{c_0} \int_V d^3 \mathbf{r} \frac{-1}{I_D(\mathbf{r})} \Delta I_D(\mathbf{r}) \tag{39}$$

where $d\mathbf{S} = dS \hat{\mathbf{n}}(\mathbf{r})$ with $\hat{\mathbf{n}}(\mathbf{r})$ the normal vector to the infinitesimal surface dS located at the point \mathbf{r} on the boundary. The second term of the right hand side of eq.(39) vanishes since $-\Delta I_D = 0$. To obtain the form eq.(37), we rescale the surface integral in the right hand term of eq.(39) by the characteristic length of the system \mathcal{L} ; we note $\tilde{\mathbf{r}} = \mathbf{r}/\mathcal{L}$ and $\tilde{\mathbf{S}} = d\mathbf{S}/\mathcal{L}^2$ the rescaled variables. We find

$$\begin{aligned}
 \langle \Sigma \rangle &= -\frac{k^2 \mathcal{L} l}{6\pi} \oint_{\partial \tilde{V}} \frac{1}{I_D(\tilde{\mathbf{r}})} \nabla I_D(\tilde{\mathbf{r}}) \cdot d\tilde{\mathbf{S}} \\
 &= -\frac{g_{\mathcal{L}}}{2} \oint_{\partial \tilde{V}} \frac{1}{I_D(\tilde{\mathbf{r}})} \nabla I_D(\tilde{\mathbf{r}}) \cdot d\tilde{\mathbf{S}}
 \end{aligned} \tag{40}$$

to recover eq.(37). Note that, by virtue of the boundary conditions eq.(31), the integral $\oint_{\partial \tilde{V}} \frac{1}{I_D(\tilde{\mathbf{r}})} \nabla I_D(\tilde{\mathbf{r}}) \cdot d\tilde{\mathbf{S}}$ is negative.

UNCERTAINTY RELATION FOR THE TRANSMISSION COEFFICIENT

In this section, we apply the QMUR given in its general form in eq.(20) to the transmission coefficient for the slab geometry Fig.1. The fraction of light transmitted in the direction $\hat{\mathbf{s}}$ is by definition given by [21]

$$T(\theta) = \hat{s}_x \frac{I(\hat{\mathbf{s}}, L)}{I_0} \quad (41)$$

where $\theta \in [-\pi/2, \pi/2]$ is the angle between $\hat{\mathbf{s}}$ and the x axis. The specific intensity satisfies the radiative transfer equation, which, in the absence of light source inside the medium, is given by

$$I(\hat{\mathbf{s}}, x) = I(x) + \frac{l}{D} \hat{\mathbf{s}} \cdot \mathbf{j}(x). \quad (42)$$

We can therefore write $T(\theta)$ in the form $T(\theta) = \frac{1}{S} \int_V d\mathbf{r} \delta(x-L) \hat{s}_x \frac{I(x) + \frac{l}{D} \hat{\mathbf{s}} \cdot \mathbf{j}(x)}{I_0}$ where $z(I, \mathbf{j}) = \delta(x-L) \hat{s}_x (I(x) + \frac{l}{D} \hat{\mathbf{s}} \cdot \mathbf{j}(x)) / I_0$, and use the eq.(20) to obtain

$$\langle T^2(\theta) \rangle_c \langle \Sigma \rangle \geq \langle \partial_{\mathbf{j}} T(\theta) \rangle^2, \quad (43)$$

where the lower bound of the QMUR is given by

$$\begin{aligned} \langle \partial_{\mathbf{j}} T(\theta) \rangle &= \left\langle \frac{1}{S} \int_V d\mathbf{r} \delta(x-L) \frac{\hat{s}_x l}{D I_0} \hat{\mathbf{s}} \cdot \mathbf{j} \right\rangle \\ &= \frac{\hat{s}_x l}{D I_0} \frac{1}{S} \int d\mathbf{S} \hat{\mathbf{s}} \cdot \mathbf{j}_D. \end{aligned} \quad (44)$$

Using the mesoscopic Fick's law $\mathbf{j}_D = -D \nabla I_D$ and the solution eq.(35) for I_D in a slab geometry, we obtain $\langle \partial_{\mathbf{j}} T(\theta) \rangle = \frac{\hat{s}_x^2}{S} \frac{15(1-e^{-L/l})}{8\pi(u+2)} \simeq \frac{\hat{s}_x^2}{S} \frac{15}{8\pi(u+2)}$ where $u = 3L/2l$ and $\hat{s}_x = \cos(\theta)$.

Saturation of the bound

In the slab geometry, the correlation function of the transmission coefficient is given by (see section 12.4 in [21])

$$\langle T^2(\theta) \rangle_c = \frac{4}{3g_{\mathcal{L}}} \left(\frac{15\hat{s}_x}{8\pi u} \right)^2 (\hat{s}_x + 2/3)^2. \quad (45)$$

Reinjecting this expression, together with the lower bound eq.(44) and $\Sigma = g_{\mathcal{L}} \frac{u^2}{2(1+u)}$ (derived in the main text), in the QMUR (43), and rearranging the terms to separate those depending on u and s_x , we find

$$\frac{2}{3} \left(\frac{\cos \theta + 2/3}{\cos \theta} \right)^2 \geq \frac{1+u}{(2+u)^2}. \quad (46)$$

Since $u > 0$ and $\theta \in [-\pi/2, \pi/2]$, we have $\sup \left(\frac{1+u}{(2+u)^2} \right) = 1/4$ and $\inf \left(\frac{2}{3} \left(\frac{\cos \theta + 2/3}{\cos \theta} \right)^2 \right) = 50/27$, hence the inequality (46) is strict for any value of u and θ in the multiple scattering regime.

THE CRAMÉR-RAO BOUND INTERPETATION

The purpose of this section is to rederive the QMUR using the Cramér-Rao bound, identifying the Cost function $\langle \Sigma \rangle$ as the Fisher information.

The Fisher information is a way of measuring the amount of information that an observable random variable Γ carries about an unknown parameter θ upon which the probability of Γ depends. The Cramér-Rao bound is given for any function $\Theta(\Gamma)$,

$$\frac{\text{Var}_{\theta} [\Theta(\Gamma)]}{(\partial_{\theta} \langle \Theta(\Gamma) \rangle_{\theta})^2} \geq 1/\mathcal{I}(\theta). \quad (47)$$

We can prove the Cramér-Rao bound using the Cauchy-Schwarz inequality, and will do so later on, in .

To apply the Cramér-Rao bound to the mesoscopic case, let us consider the tilted diffusion $D_\theta = De^\theta$. Furthermore, we define the probability $P_\theta(\Gamma)$ by replacing $D \rightarrow D_\theta$. This implies replacing $\mathcal{L} \rightarrow \mathcal{L}_\theta = (j - e^\theta J)^2/2\sigma$. We define the Fisher information $\mathcal{I}(\theta) = \langle (\partial_\theta \log P_\theta)^2 \rangle_\theta$.

One can then show that $\mathcal{I}(0) = \langle \Sigma \rangle$. Then, it is simple enough to show that setting $\Theta = \int d\mathbf{x} \mathbf{j} \cdot \hat{n}$ leads to the QMUR in eq.(1). What we have gained here is an interpretation of the Cost function $\langle \Sigma \rangle$ as the Fisher information of changing the diffusion coefficient.

Proving the general Cramér-Rao bound

Let us define for the function $\Theta(\Gamma)$, $\psi(\theta) \equiv \langle \Theta(\Gamma) \rangle_\theta$. Furthermore, we define the inner product $\langle a, b \rangle_\theta = \int d\Gamma a(\Gamma)b(\Gamma)P_\theta(\Gamma)$. We notice that

$$\langle (\Theta(\Gamma) - \psi(\theta)), \partial_\theta \log P_\theta \rangle_\theta = \partial_\theta \psi(\theta). \quad (48)$$

Then, applying the Cauchy-Schwarz inequality, we find

$$|\langle \Theta(\Gamma) - \psi(\theta), \Theta(\Gamma) - \psi(\theta) \rangle_\theta|^2 |\langle \partial_\theta \log P_\theta, \partial_\theta \log P_\theta \rangle_\theta|^2 \geq (\partial_\theta \psi(\theta))^2. \quad (49)$$

Identifying $\langle \Theta(\Gamma) - \psi(\theta), \Theta(\Gamma) - \psi(\theta) \rangle_\theta = \text{Var}_\theta [\Theta(\Gamma)]$ and $\langle \partial_\theta \log P_\theta, \partial_\theta \log P_\theta \rangle_\theta = \mathcal{I}(\theta)$ we recover (47).

DIMENSIONLESS SCALING OF THE LANGEVIN EQUATION

The purpose of this section is to show that the strength of the noise ξ in the Langevin equation (3),

$$\mathbf{j} = -D\nabla I(\mathbf{r}) + \xi \quad (50)$$

is, upon proper rescaling, proportional to the dimensionless parameter $1/g_{\mathcal{L}} \ll 1$. To that purpose, we rescale the spatial coordinates with respect to the length scale \mathcal{L} : $\tilde{\mathbf{r}} = \mathbf{r}/\mathcal{L}$, $\tilde{\nabla} = \nabla_{\tilde{\mathbf{r}}} = \mathcal{L}\nabla$. Furthermore, we rescale the Langevin equation by dividing by the diffusion constant D and by I_0 , a typical strength of the external illumination defining $\tilde{I} = I/I_0$. This implies

$$\tilde{\mathbf{j}} = -\tilde{\nabla} \tilde{I}(\tilde{\mathbf{r}}) + \tilde{\xi}, \quad (51)$$

where $(\tilde{\mathbf{j}}, \tilde{\xi}) \equiv \frac{\mathcal{L}}{DI_0}(\mathbf{j}, \xi)$. Using the fact that $\delta(\mathcal{L}(\tilde{\mathbf{r}}_1 - \tilde{\mathbf{r}}_2)) = \delta(\tilde{\mathbf{r}}_1 - \tilde{\mathbf{r}}_2)/\mathcal{L}^3$, we obtain

$$\langle \tilde{\xi}_\alpha(\tilde{\mathbf{r}}_1) \tilde{\xi}_\beta(\tilde{\mathbf{r}}_2) \rangle = \frac{2}{g_{\mathcal{L}}} \tilde{I}^2(\tilde{\mathbf{r}}_1) \delta(\tilde{\mathbf{r}}_1 - \tilde{\mathbf{r}}_2) \delta_{\alpha\beta}. \quad (52)$$

Recall that $g_{\mathcal{L}} = \frac{k^2 l^2}{3\pi} \frac{\mathcal{L}}{l} \gg 1$ due to the limits taken $kl \gg 1$ and $\mathcal{L} > l$.