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DEFORMATION THEORY OF HOLOMORPHIC CARTAN GEOMETRIES, II

INDRANIL BISWAS, SORIN DUMITRESCU, AND GEORG SCHUMACHER

ABSTRACT. In this continuation of [BDS], we investigate the deformations of holomorphic Cartan geometries where the underlying complex manifold is allowed to move. The space of infinitesimal deformations of a flat holomorphic Cartan geometry is computed. We show that the natural forgetful map, from the infinitesimal deformations of a flat holomorphic Cartan geometry to the infinitesimal deformations of the underlying flat principal bundle on the topological manifold, is an isomorphism.

1. INTRODUCTION

In [BDS] we studied the deformations of holomorphic Cartan geometries on a fixed compact complex manifold. Here we consider the more general deformations of holomorphic Cartan geometries where the underlying compact complex manifold is allowed to move. We also investigate the deformations of the flat holomorphic Cartan geometries.

Let G be a connected Lie group and $H \subset G$ a connected closed Lie subgroup. As a consequence of the foundational work of Cartan and Ehresmann, a flat Cartan geometry with model (G, H) on a compact manifold M is determined by the following geometrical objects: a smooth principal G -bundle E_G over M endowed with a flat connection and a principal H -subbundle $E_H \subset E_G$ transverse to the integrable horizontal distribution associated to the flat connection [Eh] (see also the survey [BD]).

Fix a base point $x_0 \in M$ and a point $z \in (E_H)_{x_0}$ in the fiber of E_H over x_0 . The above properties imply that the pull-back of this principal G -bundle E_G to the universal cover \widetilde{M} of M for x_0 is isomorphic to the trivial principal bundle $\widetilde{M} \times G \rightarrow \widetilde{M}$ and the pull-back of E_H to \widetilde{M} is defined by a smooth map $\widetilde{M} \rightarrow G/H$. The above mentioned transversality condition is equivalent to the statement that this map $\widetilde{M} \rightarrow G/H$ is a local diffeomorphism; it is customary to call this map $\widetilde{M} \rightarrow G/H$ the developing map of the (flat) Cartan geometry. The flat Cartan geometry on M with model (G, H) produces a monodromy homomorphism $\rho : \pi_1(M, x_0) \rightarrow G$. The developing map is $\pi_1(M, x_0)$ -equivariant with respect to the action of $\pi_1(M, x_0)$ on \widetilde{M} via the deck transformations and the action of $\pi_1(M, x_0)$ on G/H via the monodromy morphism ρ and the left-translation action of G on G/H [Eh] (see the expository works [Sh, BD]).

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The above geometrical description of Ehresmann leads to the so-called Ehresmann-Thurston principle which states that the Riemann-Hilbert map associating to each flat Cartan geometry its monodromy morphism $\rho : \pi_1(M) \longrightarrow G$ (uniquely determined up to inner automorphisms of G) is a local homeomorphism between the moduli space of flat Cartan geometries with model (G, H) on X and the space of group homomorphisms from $\pi_1(M)$ to G (modulo the action of G acting on the target G by inner conjugation) [BG, p. 115, Theorem 2.1] (see also [CEG]).

Now let G be a connected complex Lie group and $H \subset G$ a connected complex closed Lie subgroup. Then the model manifold G/H inherits a G -invariant complex structure. Any flat Cartan geometry with model (G, H) induces on M an underlying complex structure (for which the above G -bundle E_G , its flat connection and the transverse H -subbundle E_H are all holomorphic). Hence there is a natural forgetful map from the deformation space of flat Cartan geometries with model (G, H) to the Kuranishi space of M . In the particular case where M is a surface and $G = \mathrm{PSL}(2, \mathbb{C})$, with $H \subset G$ being the stabilizer of a point in the complex projective line, a flat Cartan geometry with model (G, H) determines a complex projective structure on S and hence an underlying structure of Riemann surface. Led by the work of Klein and Poincaré, the complex projective structures had a major role in the formulation and (some of) the proofs of the uniformization theorem for Riemann surfaces (see [Gu] or [StG]). Indeed, the uniformization theorem for Riemann surfaces ensures the existence on any Riemann surface of a compatible complex projective structure (meaning it defines the same complex structure) with injective developing map.

Much more recently, the deformation space of flat Cartan geometries with model $G = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ and $H = \mathrm{SL}(2, \mathbb{C})$, diagonally embedded in G , and the associated forgetful map to the Kuranishi space were studied by Ghys in [Gh]. Using this method, Ghys computed in [Gh] the Kuranishi space of the parallelizable manifolds $\mathrm{SL}(2, \mathbb{C})/\Gamma$, where Γ is a uniform lattice in $\mathrm{SL}(2, \mathbb{C})$. Ghys proved that the forgetful map realizes an isomorphism between the deformation space of $\mathrm{SL}(2, \mathbb{C})/\Gamma$ as flat Cartan geometry with the above model (G, H) and the Kuranishi space of the complex manifold $\mathrm{SL}(2, \mathbb{C})/\Gamma$. Moreover, the deformation space of $\mathrm{SL}(2, \mathbb{C})/\Gamma$ as flat Cartan geometry is modeled on the germ, at the trivial morphism, of the algebraic variety of group homomorphisms from Γ into $\mathrm{SL}(2, \mathbb{C})$. In particular, for any uniform lattice Γ with positive first Betti number this germ has positive dimension. Hence the corresponding parallelizable manifolds $\mathrm{SL}(2, \mathbb{C})/\Gamma$ admit nontrivial deformations of the underlying complex structure. Those examples of flexible parallelizable manifolds associated to semi-simple complex Lie groups are exotic (by Raghunathan's rigidity results [Ra] a compact quotient of a complex Lie group by a lattice has a rigid complex structure, if no local factor is isomorphic to $\mathrm{SL}(2, \mathbb{C})$).

Our result below is a reformulation of the Ehresmann-Thurston principle in the context of flat holomorphic Cartan geometries with model (G, H) , where G is a connected complex Lie group and $H \subset G$ a connected complex closed Lie subgroup.

Let E_H be a holomorphic principal H -bundle on a compact complex manifold M ; the holomorphic principal G -bundle on M obtained by extending the structure group of E_H

using the inclusion map $H \hookrightarrow G$ will be denoted by E_G . Let

$$\vartheta : \text{At}(E_H) \xrightarrow{\sim} \text{ad}(E_G)$$

be a flat holomorphic Cartan geometry on M modeled on the pair (G, H) , where $\text{At}(E_H)$ is the Atiyah bundle for E_H and $\text{ad}(E_G)$ is the adjoint bundle for E_G . The isomorphism ϑ produces a flat holomorphic connection θ' on the principal G -bundle E_G . Let

$$\mathcal{I}_{CG}$$

denote the space of all infinitesimal deformations of the flat holomorphic Cartan geometry (M, E_H, θ) in the category of the flat holomorphic Cartan geometries. Let

$$\mathcal{I}_{FC}$$

denote the space of all infinitesimal deformations of the flat principal G -bundle (E_G, θ') on the topological manifold M , where θ' is the above mentioned flat connection on E_G given by ϑ . Associating to any flat holomorphic Cartan geometry (E_H, θ) the corresponding flat principal G -bundle on the underlying topological manifold we obtain a homomorphism

$$\varphi : \mathcal{I}_{CG} \longrightarrow \mathcal{I}_{FC}.$$

(see (5.1) and (5.2)).

We prove the following (see Theorem 5.1):

Theorem 1.1. *The above homomorphism φ is an isomorphism.*

More generally, we study here the deformation space of (non necessarily flat) holomorphic Cartan geometries where the underlying complex structure of the manifold is allowed to move. We generalize in this broader context results previously obtained in [BDS] (see Proposition 4.1 and Corollary 4.2).

It should be mentioned that Proposition 4.1 and Corollary 4.2 correct and generalize Theorem 3.4 in [BDS]¹.

2. CARTAN GEOMETRY

The holomorphic tangent (respectively, cotangent) bundle of a complex manifold N will be denoted by TN (respectively, Ω_N^1).

Let G be a connected complex Lie group; its Lie algebra will be denoted by \mathfrak{g} . Let $H \subset G$ be a connected complex closed Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$.

Take a connected complex manifold M . Let

$$f : E_H \longrightarrow M \tag{2.1}$$

be a holomorphic principal H -bundle on M . The action of H on E_H produces a holomorphic action of H on the holomorphic tangent bundle TE_H . Let

$$df : TE_H \longrightarrow f^*TM$$

¹We thank Yasuhiro Wakabayashi who pointed out that the kernel of the exact sequence in the statement of the Theorem 3.4 is not the correct one (see [Wa, Remark 6.4.5]). It should be replaced with $H^0(X, \text{Hom}(TX, \text{ad}(E_G)))$.

be the differential of the projection f in (2.1). Using the action of H on E_H , the kernel of df is identified with the trivial holomorphic vector bundle on E_H with fiber \mathfrak{h} ; such an identification is known as the Maurer-Cartan form.

A holomorphic Cartan geometry on M of type G/H is a pair (E_H, θ) , where E_H is a holomorphic principal H -bundle on M and

$$\theta : TE_H \longrightarrow E_H \times \mathfrak{g} \quad (2.2)$$

is a holomorphic homomorphism, such that

- (1) θ is an isomorphism,
- (2) θ is H -equivariant for the above action of H on TE_H and the diagonal action of H on $E_H \times \mathfrak{g}$ constructed using the adjoint action of H on \mathfrak{g} and the above mentioned action of H on E_H , and
- (3) the restriction of θ to $\ker(df)$ coincides with the above identification of $\ker(df)$ with $E_H \times \mathfrak{h}$.

(See [Sh].)

Since θ in (2.2) is H -equivariant, it descends to a homomorphism between appropriate vector bundles over M . We now recall an equivalent reformulation of the above definition.

As in (2.1), let E_H be a holomorphic principal H -bundle on M . The action of H on TE_H produces an action of H on the direct image f_*TE_H over the trivial action of H on M . Let

$$\text{At}(E_H) = (f_*TE_H)^H = (TE_H)/H \longrightarrow M \quad (2.3)$$

be the Atiyah bundles for E_H [At, p. 187, Theorem 1], where $(f_*TE_H)^H \subset f_*TE_H$ is the H -invariant subbundle. Let

$$\text{ad}(E_H) := (f_*\ker(df))^H = \ker(df)/H \subset (TE_H)/H = \text{At}(E_H) \quad (2.4)$$

be the adjoint bundle of E_H . So we have a short exact sequence of holomorphic vector bundles on M

$$0 \longrightarrow \text{ad}(E_H) \xrightarrow{h_1} \text{At}(E_H) \xrightarrow{df} TM \longrightarrow 0 \quad (2.5)$$

which is known as the *Atiyah exact sequence* for E_H . We recall that a holomorphic connection on E_H is a holomorphic splitting of the exact sequence in (2.5), meaning a holomorphic homomorphism $\psi : TM \longrightarrow \text{At}(E_H)$ such that $(df) \circ \psi = \text{Id}_{TM}$, where df is the projection in (2.5) (see [At, p 188, Definition]).

Let $E_H(\mathfrak{h})$ (respectively, $E_H(\mathfrak{g})$) be the holomorphic vector bundle over M associated to the principal H -bundle E_H for the adjoint action of H on \mathfrak{h} (respectively, \mathfrak{g}). We note that $E_H(\mathfrak{h})$ is the adjoint vector bundle $\text{ad}(E_H)$. Let

$$E_G := E_H \times^H G \longrightarrow M \quad (2.6)$$

be the holomorphic principal G -bundle on M obtained by extending the structure group of E_H using the inclusion map of H in G . The above vector bundle $E_H(\mathfrak{g})$ evidently coincides with the adjoint bundle $\text{ad}(E_G)$ for E_G . The inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$ produces a short exact sequence of holomorphic vector bundles on M

$$0 \longrightarrow \text{ad}(E_H) \xrightarrow{h_2} \text{ad}(E_G) \longrightarrow \text{ad}(E_G)/\text{ad}(E_H) \longrightarrow 0. \quad (2.7)$$

A holomorphic Cartan geometry on M of type G/H is a pair (E_H, ϑ) , where E_H is a holomorphic principal H -bundle on M and

$$\vartheta : \text{At}(E_H) \longrightarrow \text{ad}(E_G) \quad (2.8)$$

is a holomorphic isomorphism of vector bundles, such that

$$\vartheta \circ h_1 = h_2, \quad (2.9)$$

where h_1 and h_2 are the homomorphisms in (2.5) and (2.7) respectively.

Using (2.3) it is straightforward to check that the above definition of a holomorphic Cartan geometry on M of type G/H is equivalent to the definition given earlier.

For any isomorphism ϑ as in (2.8) satisfying the equation in (2.9), we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{ad}(E_H) & \xrightarrow{h_1} & \text{At}(E_H) & \longrightarrow & TM & \longrightarrow & 0 \\ & & \parallel & & \downarrow \vartheta & & \downarrow & & \\ 0 & \longrightarrow & \text{ad}(E_H) & \xrightarrow{\iota_1} & \text{ad}(E_G) & \longrightarrow & \text{ad}(E_G)/\text{ad}(E_H) & \longrightarrow & 0 \end{array} \quad (2.10)$$

[Sh, Ch. 5]; the above homomorphism $TM \longrightarrow \text{ad}(E_G)/\text{ad}(E_H)$ induced by ϑ is an isomorphism because ϑ is so.

Let

$$0 \longrightarrow \text{ad}(E_G) \xrightarrow{h_3} \text{At}(E_G) := (TE_G)/G \longrightarrow TM \longrightarrow 0 \quad (2.11)$$

be the Atiyah exact sequence for the principal G -bundle E_G in (2.6). The Atiyah bundle $\text{At}(E_G)$ in (2.11) can also be constructed using $\text{At}(E_H)$ and $\text{ad}(E_G)$; we now recall this construction. Consider the embedding

$$\text{ad}(E_H) \hookrightarrow \text{At}(E_H) \oplus \text{ad}(E_G) \quad (2.12)$$

that sends any v to $(h_1(v), -h_2(v))$, where h_1 and h_2 are the homomorphisms in (2.5) and (2.7) respectively. The Atiyah bundle $\text{At}(E_G)$ is the quotient bundle

$$\text{At}(E_G) = (\text{At}(E_H) \oplus \text{ad}(E_G))/\text{ad}(E_H) \quad (2.13)$$

for this embedding. The map h_3 in (2.11) is given by the inclusion h_2 or h_1 of $\text{ad}(E_G)$ in $\text{At}(E_H) \oplus \text{ad}(E_G)$; note that they produce the same homomorphism to the above quotient bundle $(\text{At}(E_H) \oplus \text{ad}(E_G))/\text{ad}(E_H)$.

Let (E_H, ϑ) be a holomorphic Cartan geometry of type G/H on M . Then the homomorphism

$$\text{At}(E_H) \oplus \text{ad}(E_G) \longrightarrow \text{ad}(E_G), \quad (v, w) \longmapsto \vartheta(v) + w$$

produces a homomorphism

$$\theta' : \text{At}(E_G) = (\text{At}(E_H) \oplus \text{ad}(E_G))/\text{ad}(E_H) \longrightarrow \text{ad}(E_G), \quad (2.14)$$

because it vanishes on the subbundle $\text{ad}(E_H)$ in (2.12); see (2.13). It is straightforward to check that $\theta' \circ h_3 = \text{Id}_{\text{ad}(E_G)}$, where h_3 is the homomorphism in (2.11). Consequently, θ' gives a holomorphic splitting of the exact sequence in (2.11). Therefore, θ' is a holomorphic connection on the principal G -bundle E_G [At].

The curvature $\text{Curv}(\theta')$ of the connection θ' is a holomorphic section

$$\text{Curv}(\theta') \in H^0(X, \text{ad}(E_G) \otimes \Omega_M^2),$$

where $\Omega_M^i := \bigwedge^i(TM)^*$.

The Cartan geometry (E_H, ϑ) is called *flat* if

$$\text{Curv}(\theta') = 0$$

[Sh, Ch. 5, § 1, p. 177].

3. CONNECTION AND DIFFERENTIAL OPERATORS

Take a connected complex manifold M . Let

$$f : E_H \longrightarrow M$$

be a holomorphic principal H -bundle, and let

$$\theta : TE_H \longrightarrow E_H \times \mathfrak{g}$$

be a holomorphic isomorphism defining a holomorphic Cartan geometry of type G/H on M ; see (2.2). Take a nonempty open subset $U \subset M$. Let

$$\varpi \in H^0\left(f^{-1}(U), (TE_H)|_{f^{-1}(U)}\right)^H = H^0(f^{-1}(U), T(f^{-1}(U)))^H$$

be an H -invariant holomorphic vector field on $f^{-1}(U)$. So we have

$$\theta(\varpi) \in H^0(f^{-1}(U), f^{-1}(U) \times \mathfrak{g}) = H^0\left(f^{-1}(U), (E_H \times \mathfrak{g})|_{f^{-1}(U)}\right). \quad (3.1)$$

In other words, $\theta(\varpi)$ is a \mathfrak{g} -valued holomorphic function on $f^{-1}(U)$. Take any point $z \in f^{-1}(U)$, and also take a holomorphic tangent vector

$$v \in T_z E_H.$$

For the function $\theta(\varpi)$ in (3.1), consider

$$v(\theta(\varpi)) + [\theta(v), \theta(\varpi)] \in \mathfrak{g} \quad (3.2)$$

for the above tangent vector v .

First treat the case where $v \in \text{kernel}(df)(z)$; here f , as before, is the projection of E_H to M . Recall that the third condition in the definition of a Cartan geometry says that the restriction of θ to $\text{kernel}(df)$ coincides with the identification of $\text{kernel}(df)$ with $E_H \times \mathfrak{h}$ given by the action of H (the Maurer–Cartan form).

Since θ is H -equivariant, and ϖ is H -invariant, it follows that $\theta(\varpi)$ transforms as the inverse adjoint representation, of H on \mathfrak{g} , under the H -action. So under the infinitesimal \mathfrak{h} -action it equals $-[\theta(v), \theta(\varpi)]$ for the action of $v \in \mathfrak{h}$. This implies that

$$v(\theta(\varpi)) + [\theta(v), \theta(\varpi)] = 0, \quad (3.3)$$

because $v \in \text{kernel}(df)(z)$.

Next we investigate the action of H on the construction in (3.2). Take any $g \in H$. Denote

$$z' = zg,$$

and let

$$v' = g \cdot v \in T_{z'}E_H$$

be the image of v under the automorphism of TE_H given by the action of g . It is straightforward to check that

$$v'(\theta(\varpi)) + [\theta(v'), \theta(\varpi)] = \text{Ad}(g^{-1})(v(\theta(\varpi)) + [\theta(v), \theta(\varpi)]); \quad (3.4)$$

to clarify, $\text{Ad}(g)$ is the automorphism of \mathfrak{g} corresponding to the automorphism of G defined by $y \mapsto gyg^{-1}$.

From the combination of (3.3) and (3.4) it follows that the construction in (3.2) produces a homomorphism of sheaves

$$\Theta : \text{At}(E_H) \longrightarrow \text{ad}(E_G) \otimes \Omega_M^1. \quad (3.5)$$

To explain this homomorphism Θ , take ϖ and v as in (3.2). Then ϖ gives a section of $\text{At}(E_H)|_U$ (see (2.3)), which will be denoted by $\tilde{\varpi}$; also, we have $df(v) \in T_{f(z)}M$, where, as before, df is the differential of the projection f . The homomorphism Θ in (3.5) is uniquely determined by the condition that the element

$$\Theta(\tilde{\varpi})(df(v)) \in \text{ad}(E_G)_{f(z)}$$

coincides with the image of $(z, v(\theta(\varpi)) + [\theta(v), \theta(\varpi)])$ in $\text{ad}(E_G)_{f(z)}$, where $v(\theta(\varpi)) + [\theta(v), \theta(\varpi)] \in \mathfrak{g}$ is the element in (3.2). Recall that $\text{Ad}(E_G)$ is the quotient of $E_H \times \mathfrak{g}$ where two elements (z_1, v_1) and (z_2, v_2) of $E_H \times \mathfrak{g}$ are identified if there is an element $g \in H$ such that $z_2 = z_1g$ and $v_2 = \text{Ad}(g^{-1})(v_1)$, and hence $(z, v(\theta(\varpi)) + [\theta(v), \theta(\varpi)])$ gives an element of $\text{ad}(E_G)_{f(z)}$. From (3.3) and (3.4) it follows that the above characterization of $\Theta(\tilde{\varpi})(df(v)) \in \text{ad}(E_G)_{f(z)}$ uniquely defines the homomorphism in (3.5).

From the construction of Θ in (3.5) it follows immediately that Θ is a holomorphic differential operator of order one.

The Cartan geometry (E_H, θ) produces a holomorphic isomorphism

$$\vartheta : \text{At}(E_H) \longrightarrow \text{ad}(E_G)$$

(see (2.8)). Let

$$\Theta \circ \vartheta^{-1} : \text{ad}(E_G) \longrightarrow \text{ad}(E_G) \otimes \Omega_M^1 \quad (3.6)$$

be the composition of ϑ^{-1} with the homomorphism Θ in (3.5).

Recall the holomorphic connection θ' on E_G in (2.14). Any holomorphic connection on E_G induces a holomorphic connection on any holomorphic fiber bundle associated to E_G . In particular, the connection θ' produces a holomorphic connection

$$\Theta' : \text{ad}(E_G) \longrightarrow \text{ad}(E_G) \otimes \Omega_M^1 \quad (3.7)$$

on the adjoint bundle $\text{ad}(E_G)$.

Proposition 3.1. *The holomorphic connection Θ' in (3.7) coincides with the homomorphism $\Theta \circ \vartheta^{-1}$ constructed in (3.6).*

Proof. We first recall the construction of the homomorphism Θ' in (3.7). As in (2.6), let

$$q : E_G \longrightarrow M$$

be the holomorphic principal G -bundle on M obtained by extending the structure group of E_H using the inclusion of H in G . Using the action of G on E_G , the vector bundle $\text{kernel}(dq) \subset TE_G$, where $dq : TE_G \longrightarrow q^*TM$ is the differential of the projection q , is identified with the trivial vector bundle $E_G \times \mathfrak{g} \longrightarrow E_G$. Take any

$$s \in H^0(U, \text{ad}(E_G)|_U),$$

where $U \subset M$ is a nonempty open subset. We recall that

$$\text{ad}(E_G) = (q_*\text{kernel}(dq))^G = \text{kernel}(dq)/G$$

(see (2.4)). Using this isomorphism

$$\text{ad}(E_G) = \text{kernel}(dq)/G \subset (TE_G)/G = \text{At}(E_G),$$

the above section s of $\text{ad}(E_G)|_U$ produces a holomorphic vector field

$$\tilde{s} \in H^0\left(q^{-1}(U), (TE_G)|_{q^{-1}(U)}\right). \quad (3.8)$$

We note that \tilde{s} satisfies the following two conditions:

- \tilde{s} lies in the subspace

$$H^0\left(q^{-1}(U), \text{kernel}(dq)|_{q^{-1}(U)}\right) \subset H^0\left(q^{-1}(U), (TE_G)|_{q^{-1}(U)}\right),$$

where dq as before is the differential of the projection q , and

- the action of G on E_G preserves \tilde{s} .

Take a holomorphic vector field

$$v \in H^0(U, TU).$$

Let

$$\tilde{v} \in H^0\left(q^{-1}(U), (TE_G)|_{q^{-1}(U)}\right)$$

be the horizontal lift of v for the holomorphic connection θ' on E_G in (2.14). Now consider the Lie bracket

$$[\tilde{v}, \tilde{s}] \in H^0\left(q^{-1}(U), (TE_G)|_{q^{-1}(U)}\right).$$

It is straightforward to verify the following statements:

- The vector field $[\tilde{v}, \tilde{s}]$ is G -invariant. Indeed, this is a consequence of the fact that both \tilde{v} and \tilde{s} are G -invariant vector fields.
- We have

$$[\tilde{v}, \tilde{s}] \in H^0\left(q^{-1}(U), \text{kernel}(dq)|_{q^{-1}(U)}\right).$$

Indeed, this follows from the facts that $\tilde{s} \in H^0\left(q^{-1}(U), \text{kernel}(dq)|_{q^{-1}(U)}\right)$ and \tilde{v} is G -invariant.

- The equality

$$[\widetilde{\beta \cdot v}, \widetilde{s}] = (\beta \circ q) \cdot [\widetilde{v}, \widetilde{s}]$$

holds for any holomorphic function β on U . To see this, note that $\widetilde{s}(\beta \circ q) = 0$ because

$$\widetilde{s} \in H^0\left(q^{-1}(U), \text{kernel}(dq)|_{q^{-1}(U)}\right).$$

The above equality follows immediately from this.

- $[\widetilde{v}, \widetilde{\beta \cdot s}] = (\beta \circ q) \cdot [\widetilde{v}, \widetilde{s}] + (v(\beta) \circ q) \cdot \widetilde{s}$ for any holomorphic function β on U .

Note that the first two of the above four statements together imply that $[\widetilde{v}, \widetilde{s}]$ gives a holomorphic section of $\text{ad}(E_G)|_U$; this holomorphic section of $\text{ad}(E_G)|_U$ will be denoted by $\widehat{[\widetilde{v}, \widetilde{s}]}$.

The homomorphism Θ' in (3.7) is uniquely determined by the following equation:

$$\langle \Theta'(s), v \rangle = \widehat{[\widetilde{v}, \widetilde{s}]}, \quad (3.9)$$

where $\langle -, - \rangle$ denotes the contraction of forms by vector fields; note that both sides of (3.9) are sections of $\text{ad}(E_G)|_U$. The third and fourth statements above imply that Θ' defined by (3.9) is actually a connection on $\text{ad}(E_G)$.

Using the earlier mentioned identification between $\text{kernel}(dq)$ and the trivial vector bundle $E_G \times \mathfrak{g} \rightarrow E_G$, the vertical vector field \widetilde{s} in (3.8) (vertical for the projection q) defines a \mathfrak{g} -valued holomorphic function on $q^{-1}(U)$. This \mathfrak{g} -valued holomorphic function on $q^{-1}(U)$ will be denoted by \widetilde{s}_1 . Consider the Lie bracket of vector fields

$$[\widetilde{v}, \widetilde{s}] \in H^0\left(q^{-1}(U), \text{kernel}(dq)|_{q^{-1}(U)}\right);$$

it defines a \mathfrak{g} -valued holomorphic function on $q^{-1}(U)$. This \mathfrak{g} -valued holomorphic function on $q^{-1}(U)$ coincides with the derivative $\widetilde{v}(\widetilde{s}_1)$ of the \mathfrak{g} -valued function \widetilde{s}_1 in the direction of the vector field \widetilde{v} .

Since E_G is the principal G -bundle obtained by extending the structure group of the principal H -bundle E_H using the inclusion map $H \hookrightarrow G$, we have an inclusion map

$$\text{At}(E_H) \hookrightarrow \text{At}(E_G); \quad (3.10)$$

this inclusion map sends any $v \in \text{At}(E_H)$ to $(v, 0) \in (\text{At}(E_H) \oplus \text{ad}(E_G))/\text{ad}(E_H)$; see (2.13). The connection θ' on E_G in (2.14) produces a holomorphic splitting

$$\theta' : \text{At}(E_G) \rightarrow \text{ad}(E_G)$$

(see (2.14)). Combining this with the homomorphism in (3.10), we get a homomorphism

$$\text{At}(E_H) \rightarrow \text{ad}(E_G). \quad (3.11)$$

From the construction of the connection θ' in (2.14) it follows immediately that this homomorphism in (3.11) coincides with the homomorphism ϑ in (2.8).

Let $w \in H^0(q^{-1}(U), TE_G|_{q^{-1}(U)})^G$ be a holomorphic vector field on $q^{-1}(U) \subset E_G$. As before, take a holomorphic vector field $v \in H^0(U, TU)$, and let

$$\widetilde{v} \in H^0\left(q^{-1}(U), (TE_G)|_{q^{-1}(U)}\right)$$

be the horizontal lift of v for the holomorphic connection θ' on E_G in (2.14). Let

$$[\tilde{v}, w]' \in H^0\left(q^{-1}(U), \ker(dq)|_{q^{-1}(U)}\right) \subset H^0\left(q^{-1}(U), (TE_G)|_{q^{-1}(U)}\right)$$

be the projection of the Lie bracket $[\tilde{v}, w]$ to the vertical component for the connection θ' in (2.14). If w is horizontal for the connection θ' , then we have $[\tilde{v}, w]' = 0$, because the connection θ' is flat which means that the horizontal distribution for θ' is integrable. Therefore, for any

$$w_1, w_2 \in H^0(q^{-1}(U), TE_G|_{q^{-1}(U)})^G,$$

we have

$$[\tilde{v}, w_1]' = [\tilde{v}, w_2]'$$

if the vertical components of w_1 and w_2 coincide. We noted above that the homomorphism in (3.11) coincides with the homomorphism ϑ in (2.8). Now the proposition follows by comparing the constructions of Θ' and Θ . \square

4. DEFORMATIONS OF CARTAN GEOMETRY

Let S be a complex space. A *holomorphic family* of Cartan geometries of type G/H parametrized by S consists of the following:

- (1) $\phi : M_S \rightarrow S$ is a holomorphic family of compact complex manifolds parametrized by S .
- (2) $F : \mathcal{E}_H \rightarrow M_S$ is a holomorphic principal H -bundle. The relative holomorphic tangent bundle for the projection $\phi \circ F$ will be denoted by \mathcal{T} . So \mathcal{T} is the subbundle of $T\mathcal{E}_H$ given by the kernel of the differential $d(\phi \circ F)$ of the map $\phi \circ F$.
- (3) $\theta_S : \mathcal{T} \rightarrow \mathcal{E}_H \times \mathfrak{g}$ is a holomorphic isomorphism of \mathcal{T} with the trivial holomorphic bundle $\mathcal{E}_H \times \mathfrak{g} \rightarrow \mathcal{E}_H$ satisfying the following two conditions:
 - θ_S is H -equivariant for the action of H on $T\mathcal{E}_H$ given by the action of H on \mathcal{E}_H and the diagonal action of H on $\mathcal{E}_H \times \mathfrak{g}$ constructed using the adjoint action of H on \mathfrak{g} and the action of H on \mathcal{E}_H , and
 - the restriction of θ_S to $\ker(dF)$, where dF is the differential of the projection F , coincides with the identification of $\ker(dF)$ with $\mathcal{E}_H \times \mathfrak{h}$ given by the action of H on \mathcal{E}_H .

It should be clarified that the above definition is more general than [BDS, p. 516, Definition 2.2]. In [BDS, Definition 2.2] the family of complex manifolds $\phi : M_S \rightarrow S$ is taken to be a constant family of the form $M \times S \rightarrow S$.

Let (E_H, θ) be a holomorphic Cartan geometry of type G/H on a compact complex manifold M . Let S be a complex space with a distinguished point $s_0 \in S$. A *deformation* of (M, E_H, θ) parametrized by S is a holomorphic family of Cartan geometries $(M_S, \phi, \mathcal{E}_H, F, \theta_S)$ of type G/H parametrized by S (see above) together with

- a holomorphic isomorphism of M with $M_{s_0} := \phi^{-1}(s_0)$, and
- a holomorphic isomorphism of principal H -bundles

$$E_H \xrightarrow{\sim} \mathcal{E}_H|_{M_{s_0}}$$

that takes θ to the restriction of θ_S to $(\phi \circ F)^{-1}(s_0)$.

An *isomorphism* between two deformations $(M_S, \phi, \mathcal{E}_H, F, \theta_S)$ and $(M'_S, \phi', \mathcal{E}'_H, F', \theta'_S)$ of (M, E_H, θ) consists of a holomorphic isomorphism $M_S \xrightarrow{\sim} M'_S$ parametrized by S together with a holomorphic isomorphism of principal H -bundles

$$\delta : \mathcal{E}_H \xrightarrow{\sim} \mathcal{E}'_H$$

satisfying the following conditions:

- the isomorphism $\mathcal{T} \xrightarrow{\sim} \mathcal{T}'$ given by δ , where \mathcal{T}' relative holomorphic tangent bundle for the projection $\phi' \circ F'$, takes θ_S to θ'_S ,
- the diagram

$$\begin{array}{ccc} M & \xrightarrow{\text{Id}} & M \\ \downarrow & & \downarrow \\ M_S & \longrightarrow & M'_S \end{array}$$

commutes, and

- the diagram

$$\begin{array}{ccc} E_H & \xrightarrow{\text{Id}} & E_H \\ \downarrow & & \downarrow \\ \mathcal{E}_H & \xrightarrow{\delta} & \mathcal{E}'_H \end{array}$$

commutes.

When the parameter space S is the nonreduced space $\text{Spec } \mathbb{C}[t]/t^2$, then the deformation of (M, E_H, θ) is called an *infinitesimal deformation*.

Let (E_H, θ) be a holomorphic Cartan geometry of type G/H on a compact complex manifold M . Consider the homomorphism Θ in (3.5). Let \mathcal{C}_\bullet denote the two-term complex of sheaves on M

$$\mathcal{C}_\bullet : \mathcal{C}_0 := \text{At}(E_H) \xrightarrow{\Theta} \mathcal{C}_1 := \text{ad}(E_G) \otimes \Omega_M^1,$$

where \mathcal{C}_i is at the i -th position. Using the inclusion map h_1 in (2.5) we get the following short exact sequence of complexes of sheaves on M

$$\begin{array}{ccc} \begin{array}{c} 0 \\ \downarrow \\ \mathcal{C}'_\bullet : \mathcal{C}'_0 := \text{ad}(E_H) \\ \downarrow h_1 \\ \mathcal{C}_\bullet : \mathcal{C}_0 = \text{At}(E_H) \\ \downarrow df \\ TM \\ \downarrow \\ 0 \end{array} & \xrightarrow{\Theta} & \begin{array}{c} 0 \\ \downarrow \\ \mathcal{C}'_1 := \text{ad}(E_G) \otimes \Omega_M^1 \\ \downarrow \text{Id} \\ \mathcal{C}_1 = \text{ad}(E_G) \otimes \Omega_M^1 \\ \downarrow \\ 0 \\ \downarrow \\ 0 \end{array} \end{array} \quad (4.1)$$

(the restriction of Θ to $\text{ad}(E_H) \subset \text{At}(E_H)$ is also denoted by Θ). From the short exact sequence of complexes in (4.1) we get the following long exact sequence of hypercohomologies

$$\longrightarrow \mathbb{H}^1(\mathcal{C}'_\bullet) \xrightarrow{\gamma_1} \mathbb{H}^1(\mathcal{C}_\bullet) \xrightarrow{\gamma_2} H^1(M, TM) \longrightarrow \dots \quad (4.2)$$

Proposition 4.1.

- (1) *The space of all infinitesimal deformations of the holomorphic Cartan geometry (M, E_H, θ) are parametrized by the first hypercohomology $\mathbb{H}^1(\mathcal{C}_\bullet)$.*
- (2) *The space of infinitesimal deformations of the holomorphic Cartan geometry (E_H, θ) , keeping the complex manifold M fixed, are parametrized by the first hypercohomology $\mathbb{H}^1(\mathcal{C}'_\bullet)$.*
- (3) *The homomorphism γ_2 in (4.2) is the natural map that sends an infinitesimal deformation of (M, E_H, θ) to the infinitesimal deformation of M obtained from it by simply forgetting E_H and θ . In other words, γ_2 gives the infinitesimal deformation of the underlying compact complex manifold when the Cartan geometry (M, E_H, θ) deforms.*
- (4) *The homomorphism γ_1 in (4.2) is the natural map that sends an infinitesimal deformation of (E_H, θ) to the infinitesimal deformation of (M, E_H, θ) obtained from it by keeping the complex manifold M unchanged.*

Proof. This is a straightforward consequence of the computations in [BDS] and [Ch]. \square

Consider the isomorphism $\vartheta : \text{At}(E_H) \longrightarrow \text{ad}(E_G)$ given by (E_H, θ) (see (2.8)). Using Proposition 3.1 and (2.10), the diagram in (4.1) is transformed to the diagram

$$\begin{array}{ccc}
& 0 & 0 \\
& \downarrow & \downarrow \\
\mathcal{B}'_\bullet : & \mathcal{B}'_0 := \text{ad}(E_H) & \xrightarrow{\Theta'} \mathcal{B}'_1 := \text{ad}(E_G) \otimes \Omega_M^1 \\
& \downarrow & \downarrow \\
\mathcal{B}_\bullet : & \mathcal{B}_0 := \text{ad}(E_G) & \xrightarrow{\Theta'} \mathcal{B}_1 := \text{ad}(E_G) \otimes \Omega_M^1 \\
& \downarrow & \downarrow \\
& TM = \text{ad}(E_G)/\text{ad}(E_H) & \longrightarrow 0 \\
& \downarrow & \downarrow \\
& 0 & 0
\end{array} \quad (4.3)$$

where Θ' is the homomorphism in (3.7); the restriction of Θ' to $\text{ad}(E_H) \subset \text{ad}(E_G)$ is also denoted by Θ' . Let

$$\longrightarrow \mathbb{H}^1(\mathcal{B}'_\bullet) \xrightarrow{\alpha_1} \mathbb{H}^1(\mathcal{B}_\bullet) \xrightarrow{\alpha_2} H^1(M, TM) = H^1(M, \text{ad}(E_G)/\text{ad}(E_H)) \longrightarrow \dots \quad (4.4)$$

be the corresponding long exact sequence of hypercohomologies.

The following is a reformulation of Proposition 4.1.

Corollary 4.2.

- (1) *The space of all infinitesimal deformations of the holomorphic Cartan geometry (M, E_H, θ) is parametrized by the first hypercohomology $\mathbb{H}^1(\mathcal{B}_\bullet)$.*

- (2) *The space of infinitesimal deformations of the holomorphic Cartan geometry (E_H, θ) , keeping the complex manifold M fixed, is parametrized by the first hypercohomology $\mathbb{H}^1(\mathcal{B}'_\bullet)$.*
- (3) *The homomorphism α_2 in (4.4) is the natural map that sends an infinitesimal deformation of (M, E_H, θ) to the infinitesimal deformation of M obtained from it by simply forgetting E_H and θ .*
- (4) *The homomorphism α_1 in (4.4) is the natural map that sends an infinitesimal deformation of (E_H, θ) to the infinitesimal deformation of (M, E_H, θ) obtained from it by keeping the complex manifold M unchanged.*

5. FLAT CARTAN GEOMETRY

Let $(M_S, \phi, \mathcal{E}_H, F, \theta_S)$ be a holomorphic family of Cartan geometries of type G/H parametrized by S ; see Section 4. Let $\mathcal{E}_G \rightarrow M_S$ be the holomorphic principal G -bundle obtained by extending the structure group of \mathcal{E}_H using the inclusion map of H in G . Recall the construction of the holomorphic connection θ' in (2.14), given any holomorphic Cartan geometry of type G/H . This construction produces a relative holomorphic connection on the holomorphic principal G -bundle \mathcal{E}_G , relative for the projection $\mathcal{E}_G \rightarrow S$. This relative holomorphic connection on \mathcal{E}_G will be denoted by θ'_S .

Let (E_H, θ) be a flat holomorphic Cartan geometry of type G/H on a compact complex manifold M . Let S be a complex space. A *holomorphic family of flat Cartan geometries* of type G/H parametrized by S is a holomorphic family of Cartan geometries $(M_S, \phi, \mathcal{E}_H, F, \theta_S)$ of type G/H parametrized by S such that the relative holomorphic connection θ'_S on \mathcal{E}_G is flat.

Associating to any flat holomorphic Cartan geometry (E_H, θ) of type G/H on M , we have a flat principal G -bundle (E_G, θ') on the topological manifold M , where θ' is constructed in (2.14). Let

$$\mathcal{I}_{CG} \tag{5.1}$$

denote the space of all infinitesimal deformations of the flat holomorphic Cartan geometry (M, E_H, θ) in the category of the flat holomorphic Cartan geometries. Let

$$\mathcal{I}_{FC} \tag{5.2}$$

denote the space of all infinitesimal deformations of the flat principal G -bundle (E_G, θ') on the topological manifold M . The above association of a flat G -bundle (E_G, θ') to a flat holomorphic Cartan geometry (E_H, θ) produces a homomorphism

$$\varphi : \mathcal{I}_{CG} \longrightarrow \mathcal{I}_{FC} \tag{5.3}$$

(see (5.1) and (5.2)).

Theorem 5.1. *The homomorphism φ in (5.3) is an isomorphism.*

Proof. As before, (E_H, θ) is a flat holomorphic Cartan geometry of type G/H on a compact complex manifold M . Consider the holomorphic connection Θ' on $\text{ad}(E_G)$ in (3.7) given by

(E_H, θ) . We note that Θ' is flat because the holomorphic connection θ' on E_G in (2.14) is flat. Since Θ' is flat, it produces the following complex $\tilde{\mathcal{B}}_\bullet$ of sheaves on M :

$$\tilde{\mathcal{B}}_\bullet : \tilde{\mathcal{B}}_0 := \text{ad}(E_G) \xrightarrow{\Theta'} \tilde{\mathcal{B}}_1 := \text{ad}(E_G) \otimes \Omega_M^1 \xrightarrow{\Theta'} \tilde{\mathcal{B}}_2 := \text{ad}(E_G) \otimes \Omega_M^2. \quad (5.4)$$

We note that this $\tilde{\mathcal{B}}_\bullet$ and the complex \mathcal{B}_\bullet in (4.3) fit in the following short exact sequences of complexes of sheaves on M :

$$\begin{array}{ccccc} & 0 & & 0 & & 0 \\ & \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}_\bullet : & \mathcal{D}_0 := 0 & \longrightarrow & \mathcal{D}_1 := 0 & \longrightarrow & \mathcal{D}_2 := \text{ad}(E_G) \otimes \Omega_M^2 \\ & \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{B}}_\bullet : & \tilde{\mathcal{B}}_0 := \text{ad}(E_G) & \xrightarrow{\Theta'} & \tilde{\mathcal{B}}_1 := \text{ad}(E_G) \otimes \Omega_M^1 & \xrightarrow{\Theta'} & \tilde{\mathcal{B}}_2 := \text{ad}(E_G) \otimes \Omega_M^2 \\ & \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}_\bullet : & \mathcal{B}_0 := \text{ad}(E_G) & \xrightarrow{\Theta'} & \mathcal{B}_1 := \text{ad}(E_G) \otimes \Omega_M^1 & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow \\ & 0 & & 0 & & 0 \end{array}$$

Let

$$\mathbb{H}^1(\mathcal{D}_\bullet) \longrightarrow \mathbb{H}^1(\tilde{\mathcal{B}}_\bullet) \xrightarrow{\Phi} \mathbb{H}^1(\mathcal{B}_\bullet) \quad (5.5)$$

be the corresponding long exact sequence of hypercohomologies. Since

$$\mathbb{H}^i(\mathcal{D}_\bullet) = H^{i-2}(M, \text{ad}(E_G) \otimes \Omega_M^2),$$

we have $\mathbb{H}^1(\mathcal{D}_\bullet) = 0$, and hence (5.5) gives an injective homomorphism

$$\Phi : \mathbb{H}^1(\tilde{\mathcal{B}}_\bullet) \longrightarrow \mathbb{H}^1(\mathcal{B}_\bullet).$$

From Corollary 4.2(1) we know that $\mathbb{H}^1(\mathcal{B}_\bullet)$ coincides with the space of all infinitesimal deformations of the holomorphic Cartan geometry (M, E_H, θ) . Now, $\mathbb{H}^1(\tilde{\mathcal{B}}_\bullet)$ coincides with the space of all infinitesimal deformations of the flat holomorphic Cartan geometry (M, E_H, θ) in the category of the flat holomorphic Cartan geometries. The injective homomorphism Φ in (5.5) is the natural map that considers an infinitesimal deformation of the flat holomorphic Cartan geometry (M, E_H, θ) in the category of the flat holomorphic Cartan geometries as simply an infinitesimal deformation of the holomorphic Cartan geometry (M, E_H, θ) .

The kernel of the homomorphism

$$\Theta' : \text{ad}(E_G) \longrightarrow \text{ad}(E_G) \otimes \Omega_M^1$$

is the local system on M given by the sheaf of flat sections of $\text{ad}(E_G)$ for the flat connection Θ' . This locally constant sheaf of flat sections of $\text{ad}(E_G)$ will be denoted by $\underline{\text{ad}}(E_G)$. The space of infinitesimal deformations \mathcal{I}_{FC} of the flat connection (see (5.3)) has the following description

$$\mathcal{I}_{FC} = H^1(M, \underline{\text{ad}}(E_G)) \quad (5.6)$$

(see [Go1], [GM]).

From the homomorphism of complexes of sheaves

$$\begin{array}{ccccc}
 \underline{\mathrm{ad}}(E_G) & \longrightarrow & 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 \tilde{\mathcal{B}}_{\bullet} : \tilde{\mathcal{B}}_0 := \underline{\mathrm{ad}}(E_G) & \xrightarrow{\theta'} & \tilde{\mathcal{B}}_1 := \underline{\mathrm{ad}}(E_G) \otimes \Omega_M^1 & \xrightarrow{\theta'} & \tilde{\mathcal{B}}_2 := \underline{\mathrm{ad}}(E_G) \otimes \Omega_M^2
 \end{array}$$

we conclude that

$$H^1(M, \underline{\mathrm{ad}}(E_G)) = \mathbb{H}^1(\tilde{\mathcal{B}}_{\bullet}).$$

Now the theorem follow from 5.6 and Corollary 4.2(1). \square

Using the Riemann–Hilbert correspondence, the space \mathcal{I}_{FC} in (5.2) is identified with the space of all infinitesimal deformations of the pair (E_G, θ') in the category of flat holomorphic G –connections on the fixed complex manifold M .

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REFERENCES

- [At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.
- [BG] N. Bergeron and T. Gelander, A note on local rigidity, *Geom. Dedicata* **107** (2004), 111–131.
- [BD] I. Biswas and S. Dumitrescu, Generalized holomorphic Cartan geometries, *European J. Math.* **6** (special issue dedicated to the memory of Stefan Papadima) (2020), 661–680.
- [BDS] I. Biswas, S. Dumitrescu and G. Schumacher, Deformation theory of holomorphic Cartan geometries, *Indag. Math.* **31** (2020), 512–524.
- [CEG] R. D. Canary, D. B. A. Epstein and P. Green, Notes on notes of Thurston, *Analytical and geometric aspects of hyperbolic space* (Coventry/Durham, 1984), 3–92, London Math. Soc. Lecture Note Ser., 111, Cambridge Univ. Press, Cambridge, 1987.
- [Ch] T. Chen, The associated map of the nonabelian Gauss–Manin connection, *Cent. Eur. Jour. Math.* **10** (2012), 1407–1421.
- [Eh] C. Ehresmann, Sur les espaces localement homogènes, *L'Enseign. Math.* **35** (1936), 317–333.
- [Gh] E. Ghys, Déformations des structures complexes sur les espaces homogènes de $SL(2, \mathbb{C})$, *Jour. Reine Angew. Math.* **468** (1995), 113–138.
- [Go1] W. M. Goldman, The symplectic nature of fundamental groups of surfaces, *Adv. in Math.* **54** (1984), 200–225.
- [GM] W. M. Goldman and J. J. Millson, The deformation theory of representations of fundamental groups of compact Kähler manifolds, *Inst. Hautes Études Sci. Publ. Math.* **67** (1988), 43–96.
- [Gu] R. C. Gunning, *On uniformization of complex manifolds: the role of connections*, Princeton Univ. Press, 1978.
- [Ra] M. S. Raghunathan, Vanishing theorems for cohomology groups associate to discrete subgroups of semi-simple Lie groups, *Osaka Math. Jour.* **3** (1966), 243–256, corrections *ibid.* **16**, (1979), 295–299.

- [Sh] R. W. Sharpe, *Differential Geometry : Cartan's Generalization of Klein's Erlangen Program*, Graduate Text Math., 166, Springer-Verlag, New York, Berlin, Heidelberg, 1997.
- [StG] H.-P. de Saint Gervais, *Uniformization of Riemann Surfaces. Revisiting a hundred year old theorem*, E.M.S., 2016.
- [Wa] Y. Wakabayashi, Frobenius-Ehresmann structures and Cartan geometries in positive characteristic, arXiv:2109.02826.

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