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► **To cite this version:**

Semaan Amine, Stéphane Caro, Philippe Wenger. Constraint and Singularity Analysis of the Exechon. The Joint International Conference of the XI International Conference on Mechanisms and Mechanical Transmissions (MTM) and the International Conference on Robotics (Robotics'12), Jan 2012, Clermont-Ferrand, France. pp.141-150, 10.4028/www.scientific.net/AMM.162.141 . hal-03533368

**HAL Id: hal-03533368**

**<https://hal.science/hal-03533368v1>**

Submitted on 18 Jan 2022

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# Constraint and Singularity Analysis of the Exechon

Semaan Amine<sup>1, a</sup>, Stéphane Caro<sup>1, b</sup> and Philippe Wenger<sup>1, c</sup>

<sup>1</sup>Institut de Recherche en Communications et Cybernétique de Nantes,  
1, rue de la Noë, 44321 Nantes, Cedex3, France

<sup>a</sup>semaan.amine@ircsyn.ec-nantes.fr,

<sup>b</sup>stephane.caro@ircsyn.ec-nantes.fr (corresponding author),

<sup>c</sup>philippe.wenger@ircsyn.ec-nantes.fr,

**Keywords:** parallel manipulator, singularity, Grassmann-Cayley algebra, Exechon.

**Abstract.** This paper deals with the constraint and the singularity analysis of the Exechon. Using the screw theory, the constraint and actuation wrenches acting on the moving platform are analyzed. The motion pattern of the Exechon is characterized based on a representation of the constraint wrenches in the projective space. A wrench graph representing the constraint and actuation wrench systems in the projective space is obtained. Based on this wrench graph, a superbracket of the Exechon is formulated. Finally, this superbracket is explored to provide the geometric conditions for the parallel singularities of the Exechon.

## Introduction

When the manipulation tasks require less than six *dof*, the use of a lower-mobility parallel manipulator (PM) may bring some advantages (very fast machine, simple architecture) in addition to the known features of PMs. The parallel singularities of lower-mobility PMs are critical configurations characterized by either a loss of control or a change of the motion performed by the moving platform. Thus, the determination of these singularities is a central issue in the field of parallel robots. The kinematic characteristics of lower-mobility PMs make their constraint analysis, singularity analysis and design a more complex problem in comparison with their 6-*dof* counterparts.

For a  $(n < 6)$ -*dof* PM, in a general configuration, the actuators apply a  $n$ -system,  $\mathcal{W}^a$ , of *actuation wrenches*, whereas the limbs apply a  $(6-n)$ -system,  $\mathcal{W}^c$ , of *constraint wrenches* on the moving platform. Based on screw theory [1, 2, 3], Ling and Huang [4], and then Joshi and Tsai [5], developed a method to derive a  $6 \times 6$  Jacobian matrix  $\mathbf{J}$  for a lower-mobility PM. Accordingly, in a general configuration, the rows of  $\mathbf{J}$  for a  $(n < 6)$ -*dof* PM are composed of  $n$  linearly independent actuation wrenches plus  $(6-n)$  linearly independent constraint wrenches.

Parallel singularities are related to the rank deficiency of  $\mathbf{J}$ . Classical methods for the parallel singularity analysis consist in a direct analysis of  $\mathbf{J}$  by exploring the vanishing conditions of its determinant. These methods mostly fail to provide satisfactory results since the determinant of the Jacobian matrix is usually unwieldy to assess, even with a computer algebra system. Thus, alternative approaches, using Grassmann-Cayley Algebra (GCA) [6, 7, 8, 9] and/or Grassmann Geometry (GG) [9, 10], were proposed in the literature to deal with the vanishing conditions of this determinant. These approaches are based on the correspondence between the lines composing the Jacobian matrix and Plücker vectors, namely, six-component vectors representing lines in the 3-dimensional projective space.

The GCA provides an approach for the singularity analysis of a PM through the *superbracket*, which is bracket representation of the determinant of the Jacobian matrix. By exploring this superbracket, the parallel singularity conditions of a PM can be obtained.

The classification of singularities of PMs has stimulated the interest of many researchers [5, 12, 13, 14, 15]. In this paper, we adopt a classification similar to the one used in [5, 14]. Accordingly, a lower-mobility PM can exhibit two types of parallel singularities, namely, constraint singularities and actuation singularities. Constraint singularities of a  $n < 6$ -dof PM correspond to the degeneracy of its  $(6-n)$ -constraint wrench system  $\mathcal{W}^c$ . In such configurations, at least one of the initial limited motions will be no longer constrained and the moving platform of the PM gains one or several *dof*. In turn, actuation singularities correspond to configurations in which  $\mathbf{J}$  is rank deficient while the PM is not in a constraint singularity, namely, while its constraint wrench system  $\mathcal{W}^c$  does not degenerate. In such configurations, the moving platform of the PM has some instantaneous uncontrolled motions.

This paper deals with the constraint and singularity analysis of the Exechon [16]. To this vein and to better understand the geometric characteristics of the Exechon, it is necessary to use the concept of points at infinity [7, 9] in the projective space  $\mathbb{P}^3$ . This concept allows to visualize the relations between the constraint wrenches (the intersection of two parallel constraint forces and the representation of the constraint moment, which is a line at infinity) and to formulate a superbracket expression that provides geometric conditions for parallel singularities.

The paper is organized as follows. First, the main properties of points and lines at infinity in  $\mathbb{P}^3$  as well as the expression of the superbracket decomposition of GCA are recalled. Then, the constraint analysis of the Exechon is performed. The motion of the moving platform is not easy to grasp. Thus, a representation of the constraint wrench system in  $\mathbb{P}^3$  helps understand the PM's motion pattern. A wrench graph representing the actuation and the constraint wrenches in  $\mathbb{P}^3$  is also obtained. This wrench allows formulating a superbracket of the PM. Finally, this superbracket is explored to provide geometric conditions for the parallel singularities of the Exechon.

## Background

The 3-dimensional projective space  $\mathbb{P}^3$  is characterized by the affine space  $\mathbb{R}^3$  in addition to the plane at infinity  $\Omega_\infty$ . It is noteworthy that the coordinates of a projective element are determined up to scale. A projective point has four homogeneous coordinates whereas a projective line has six Plücker coordinates represented by its Plücker coordinate vector. The following properties highlight the relations between projective elements:

- A finite point,  $A$ , is represented by its homogeneous coordinates vector  $a = (a_1, a_2, a_3, 1)^T$ , the first three coordinates being its Cartesian coordinates in  $\mathbb{R}^3$ ;
- A finite line,  $\mathcal{L}$ , is represented by its Plücker coordinates vector  $\mathcal{F} = (\mathbf{s}; \mathbf{r} \times \mathbf{s})$ ; where  $\mathbf{s}$  is the unit vector of  $\mathcal{L}$ ,  $\mathbf{r}$  is the position vector of any point on  $\mathcal{L}$  and  $(\mathbf{r} \times \mathbf{s})$  represents the moment of  $\mathcal{L}$  with respect to the origin;
- Let underlined points denote points at infinity. Any finite line,  $\mathcal{F} = (\mathbf{s}; \mathbf{r} \times \mathbf{s})$ , has a unique point at infinity  $\underline{c} = (\mathbf{s}; 0)$ . This point only depends on the line direction and is determined up to scale. Accordingly, if  $a$  and  $b$  are two finite points on  $F$ , then  $\underline{c} = b - a$ . Furthermore, all finite lines directed along  $\mathbf{s}$  intersect at one common point at infinity, namely,  $\underline{c}$ ;
- All finite planes of normal vector  $\mathbf{m}$ , have a common line at infinity. This line is given by:  $\mathcal{M} = (0; \mathbf{m})$  and passes through the point at infinity on any finite line orthogonal to  $\mathbf{m}$ ;

- Two lines at infinity  $\mathcal{M}_1 = (0; \mathbf{m}_1)$  and  $\mathcal{M}_2 = (0; \mathbf{m}_2)$  intersect at a unique point at infinity  $\underline{g} = (\mathbf{m}_1 \times \mathbf{m}_2; 0)$ .

## GCA and superbracket

The GCA was developed by H. Grassmann (1809-1877) as a calculus for linear varieties operating on *extensors* with the *join*  $\vee$  and *meet*  $\wedge$  operators. The latter are associated with the *span* and *intersection* of vector spaces of extensors characterized by their *step*. GCA makes it possible to work at the symbolic level, and therefore, to produce coordinate-free algebraic expressions for the singularity conditions of spatial PMs. For further details on GCA, the reader is referred to [6, 17] and references therein.

### The superbracket and its decomposition

Many researchers in the field of parallel robots have explored the determinant of the  $6 \times 6$  Jacobian matrix  $\mathbf{J}$  to analyze the singularities. The columns of  $\mathbf{J}^T$  of a PM are usually six Plücker vectors. Each Plücker vector, being of six components, is the support of an extensor of step 1 in the 5-dimensional projective space  $\mathbb{P}^5$  (a Plücker vector corresponds to a point in  $\mathbb{P}^5$ ). The join (also called *superjoin*) of these six 1-extensors in  $\mathbb{P}^5$  is equal to the determinant of  $\mathbf{J}^T$ , up to scale. This determinant matrix, whose columns are the Plücker coordinates of six lines (2-extensors), is called the *superbracket* in GCA  $\Lambda(V^{(2)})$  [17]. Thus, a singularity occurs when this superbracket vanishes.

The superbracket is an expression involving 12 points selected on six projective lines (2-extensors) and can be developed into a linear combination of 24 bracket monomials [6, 18], each one being the product of three brackets of four projective points:

$$[ab, cd, ef, gh, ij, kl] = \sum_{i=1}^{24} y_i \quad (1)$$

where

$$\begin{array}{lll}
y_1 = -[abcd][efgi][hjkl] & y_2 = [abcd][efhi][gjkl] & y_3 = [abcd][efgj][hikl] \\
y_4 = -[abcd][efhj][gikl] & y_5 = [abce][dfgh][ijkl] & y_6 = -[abde][cfgh][ijkl] \\
y_7 = -[abcf][degh][ijkl] & y_8 = [abdf][cegh][ijkl] & y_9 = -[abce][dghi][fjkl] \\
y_{10} = [abde][cghi][fjkl] & y_{11} = [abcf][dghi][ejkl] & y_{12} = [abce][dghj][fikl] \\
y_{13} = -[abdf][cghi][ejkl] & y_{14} = -[abde][cghj][fikl] & y_{15} = -[abcf][dghj][eikl] \\
y_{16} = [abdf][cghj][eikl] & y_{17} = [abcg][defi][hjkl] & y_{18} = -[abdg][cefi][hjkl] \\
y_{19} = -[abch][defi][gjkl] & y_{20} = -[abcg][defj][hikl] & y_{21} = [abdj][cefi][gjkl] \\
y_{22} = [abdg][ceff][hikl] & y_{23} = [abch][defj][gikl] & y_{24} = -[abdj][ceff][gikl]
\end{array}$$

A bracket  $[abcd]$  is null if and only if (iff) the projective points  $a, b, c$  and  $d$  are coplanar. The bracket of four projective points is defined as the determinant of the matrix whose columns are the homogeneous coordinates of these points.

### Constraint analysis of the Exechon

The Exechon is a machine tool recently developed by Neumann [16] based on the Tricept PM. The kinematic analysis of the Exechon was addressed in [19, 20]. The basic architecture of the

Exechon consists of a 3-dof PM mounted in series with a spherical wrist. This paper deals with the constraint analysis of the first part only, called hereafter the Exechon PM.

The Exechon PM presented in Fig.1 has three limbs:  $L_1 = UPR \equiv R_1^1 R_2^1 P^1 R_3^1$ ,  $L_2 = UPR \equiv R_1^2 R_2^2 P^2 R_3^2$  and  $L_3 = SPR \equiv S^3 P^3 R^3$  such that:

- The axis of  $R_1^1$  is directed along  $\mathbf{s}_1$  and coincides with the axis of  $R_1^2$ ;
- The axes of  $R_2^1$ ,  $R_3^1$ ,  $R_2^2$  and  $R_3^2$  are directed along  $\mathbf{s}_2$ ;
- The  $P^i$ -joint,  $i=1,2,3$ , is directed along  $\mathbf{p}_i$ ;
- The axis of  $R^3$  is directed along  $\mathbf{s}_3$ , which is orthogonal to  $\mathbf{s}_2$ .

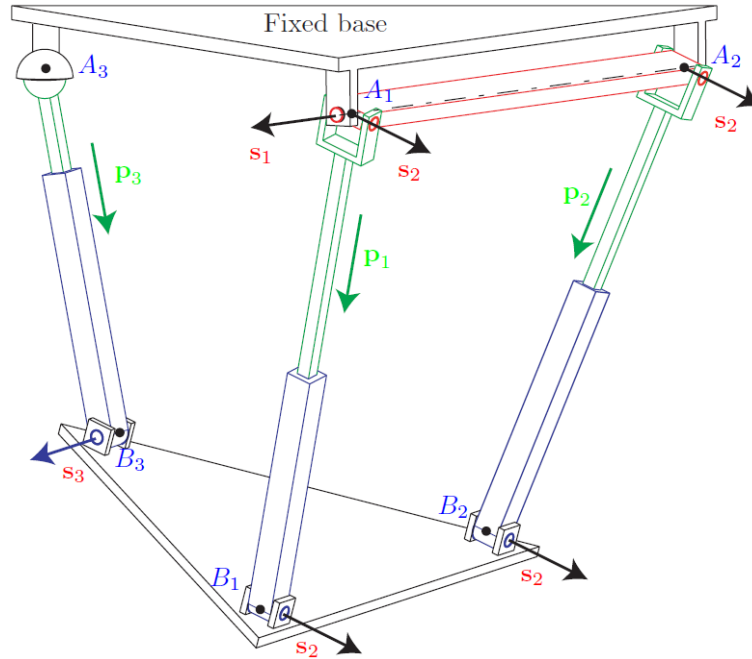


Figure 1: Schematic representation of the Exechon PM

## Twist systems and constraint wrench systems associated with each limb

### Limb $L_1$

The twist system  $\mathcal{T}_1$  of limb  $L_1$  is spanned by the twists associated with its kinematic joints  $R_1^1, R_2^1, P^1$  and  $R_3^1$ , namely,  $\hat{\mathcal{E}}_{01}^1 = (\mathbf{s}_1, \mathbf{r}_{A_1} \times \mathbf{s}_1)$ ,  $\hat{\mathcal{E}}_{02}^1 = (\mathbf{s}_2, \mathbf{r}_{A_1} \times \mathbf{s}_2)$ ,  $\hat{\mathcal{E}}_{\infty 1}^1 = (\mathbf{p}_1)$  and  $\hat{\mathcal{E}}_{03}^1 = (\mathbf{s}_2, \mathbf{r}_{B_1} \times \mathbf{s}_2)$ , respectively where the superscript  $\hat{(\cdot)}$  denotes a unit screw. Since  $\hat{\mathcal{E}}_{02}^1$  and  $\hat{\mathcal{E}}_{03}^1$  have parallel axes,  $\text{span}(\hat{\mathcal{E}}_{02}^1, \hat{\mathcal{E}}_{03}^1) = \text{span}(\hat{\mathcal{E}}_{02}^1, \hat{\mathcal{E}}_{\infty 12}^1)$  [8], with  $\hat{\mathcal{E}}_{\infty 12}^1 = (\mathbf{s}_2 \times \mathbf{p}_1)$ . Therefore,  $L_1$  is a 2T2R limb, namely, it generates two independent translations and two independent rotations. The constraint wrench system  $\mathcal{W}_1^c$  of limb  $L_1$  is spanned by screws that are reciprocal to all the twists of  $\mathcal{T}_1$ :

$$\mathcal{W}_1^c = \text{span}(\hat{\mathcal{F}}_{1c}, \hat{\mathcal{M}}_{1c}) \quad (2)$$

where  $\hat{\mathcal{F}}_{1c} = (\mathbf{s}_2, \mathbf{r}_{A_1} \times \mathbf{s}_2)$  and  $\hat{\mathcal{M}}_{1c} = (0, \mathbf{s}_1 \times \mathbf{s}_2)$ .

### Limb $L_2$

The twist system  $\mathcal{T}_2$  of limb  $L_2$  is spanned by:  $\hat{\mathcal{E}}_{01}^2 = (\mathbf{s}_1, \mathbf{r}_{A_2} \times \mathbf{s}_1)$ ,  $\hat{\mathcal{E}}_{02}^2 = (\mathbf{s}_2, \mathbf{r}_{A_2} \times \mathbf{s}_2)$ ,  $\hat{\mathcal{E}}_\infty^2 = (0, \mathbf{p}_2)$ , and  $\hat{\mathcal{E}}_{03}^2 = (\mathbf{s}_2, \mathbf{r}_{B_2} \times \mathbf{s}_2)$ . The wrench system  $\mathcal{W}_2^c$  of limb  $L_2$  is given by:

$$\mathcal{W}_2^c = \text{span}(\hat{\mathcal{F}}_{2c}, \hat{\mathcal{M}}_{2c}) \quad (3)$$

where  $\hat{\mathcal{F}}_{2c} = (\mathbf{s}_2, \mathbf{r}_{A_2} \times \mathbf{s}_2)$  and  $\hat{\mathcal{M}}_{2c} = (0, \mathbf{s}_1 \times \mathbf{s}_2)$ . It should be noticed that  $\hat{\mathcal{M}}_{1c}$  and  $\hat{\mathcal{M}}_{2c}$  are identical. Moreover,  $\hat{\mathcal{F}}_{1c}$  and  $\hat{\mathcal{F}}_{2c}$  have parallel axes. Thus,  $\text{span}(\hat{\mathcal{F}}_{1c}, \hat{\mathcal{F}}_{2c}) = \text{span}(\hat{\mathcal{F}}_c^1, \hat{\mathcal{M}}_{12c})$ , with  $\hat{\mathcal{M}}_{12c} = (0, \mathbf{s}_1 \times \mathbf{s}_2) = \hat{\mathcal{M}}_{1c} = \hat{\mathcal{M}}_{2c}$ .

### Limb $L_3$

The twist system  $\mathcal{T}_3$  of limb  $L_3$  is spanned by:  $\hat{\mathcal{E}}_\infty^3 = (0, \mathbf{p}_3)$  and  $\hat{\mathcal{E}}_{04}^3 = (\mathbf{s}_3, \mathbf{r}_{B_3} \times \mathbf{s}_3)$  in addition to any three independent  $\hat{\mathcal{E}}_0$  whose axes pass through point  $A_3$ . The constraint wrench system  $\mathcal{W}_3^c$  of limb  $L_3$  is given by:

$$\mathcal{W}_3^c = \text{span}(\hat{\mathcal{F}}_{3c}) \quad (4)$$

where  $\hat{\mathcal{F}}_{3c} = (\mathbf{s}_3, \mathbf{r}_{A_3} \times \mathbf{s}_3)$ .

### Constraint wrench system of the Exechon PM

Since  $\hat{\mathcal{M}}_{1c} \equiv \hat{\mathcal{M}}_{3c}$ , the limbs of the Exechon PM apply four different constraints to its moving platform. The constraint wrench system of this PM is given by:

$$\mathcal{W}_{PM}^c = \text{span}(\hat{\mathcal{F}}_{1c}, \hat{\mathcal{F}}_{2c}, \hat{\mathcal{F}}_{3c}, \hat{\mathcal{M}}_{1c}) \quad (5)$$

where  $\hat{\mathcal{F}}_{1c} = (\mathbf{s}_2, \mathbf{r}_{A_1} \times \mathbf{s}_2)$ ,  $\hat{\mathcal{F}}_{2c} = (\mathbf{s}_2, \mathbf{r}_{A_2} \times \mathbf{s}_2)$ ,  $\hat{\mathcal{F}}_{3c} = (\mathbf{s}_3, \mathbf{r}_{A_3} \times \mathbf{s}_3)$  and  $\hat{\mathcal{M}}_{1c} = (0, \mathbf{s}_1 \times \mathbf{s}_2)$ . Since  $\text{span}(\hat{\mathcal{F}}_{1c}, \hat{\mathcal{F}}_{2c}) = \text{span}(\hat{\mathcal{F}}_c^1, \hat{\mathcal{M}}_{1c})$ ,  $\mathcal{W}_{PM}^c$  is a 3-system. Therefore, the Exechon PM is an over-constrained 3-*dof* parallel manipulator. Indeed, the limbs apply redundant constraint on the moving platform.

### Motion of the moving platform in a general configuration

The constraint wrench system  $\mathcal{W}_{PM}^c$  of the Exechon PM provides information about the constraints applied to its moving platform in a general configuration. In order to characterize the motion of the moving platform in such a configuration, one should determine the twist system  $\mathcal{T}_{PM}$  of the moving platform, which is reciprocal to  $\mathcal{W}_{PM}^c$ .

Let  $a_1$ ,  $a_2$  and  $a_3$  be the extensors of step 1 associated with points  $A_1$ ,  $A_2$  and  $A_3$ , respectively and let  $\underline{s}_i = (\mathbf{s}_i, 0)$ ,  $i=1,2,3$ , be the 1-extensor representing the point at infinity in the direction of  $\mathbf{s}_i$ . Then,  $\hat{\mathcal{F}}_{1c} = (\mathbf{s}_1, \mathbf{r}_{A_1} \times \mathbf{s}_2)$  can be represented by  $a_1 \underline{s}_2$ ;  $\hat{\mathcal{F}}_{2c} = (\mathbf{s}_2, \mathbf{r}_{A_2} \times \mathbf{s}_2)$  can be represented by  $a_2 \underline{s}_2$ ;  $\hat{\mathcal{F}}_{3c} = (\mathbf{s}_3, \mathbf{r}_{A_3} \times \mathbf{s}_3)$  can be represented by  $a_3 \underline{s}_3$  and  $\hat{\mathcal{M}}_{1c} = (0, \mathbf{s}_1 \times \mathbf{s}_2)$  can be represented by  $\underline{s}_1 \underline{s}_2$ .

Let  $\mathcal{P}$  be the finite plane containing the axes of  $\hat{\mathcal{F}}_{1c}$  and  $\hat{\mathcal{F}}_{2c}$ . Since line  $A_1A_2$  (Fig. 1) is directed along  $\mathbf{s}_1$ , the point at infinity on this line is  $\underline{s}_1$ . Hence, the line at infinity of plane  $\mathcal{P}$  is  $\underline{s}_1\underline{s}_2 = \hat{\mathcal{M}}_{1c}$ . The constraint wrenches of the Exechon PM are illustrated in Fig. 2.

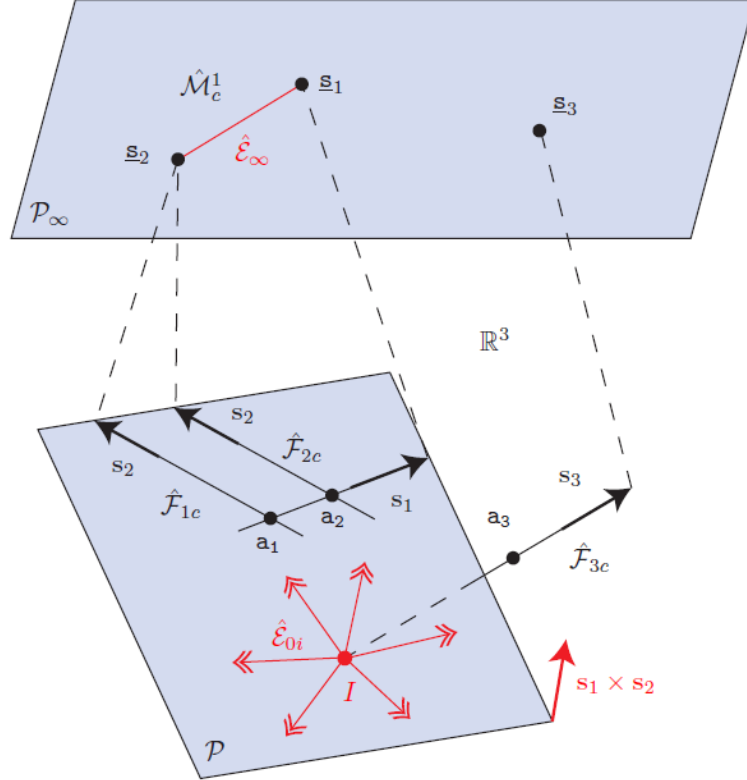


Figure 2: Twist system of the Exechon PM

Infinite-pitch twists in  $\mathcal{T}_{PM}$  should be orthogonal to the axes of  $\hat{\mathcal{F}}_{1c}$ ,  $\hat{\mathcal{F}}_{2c}$  and  $\hat{\mathcal{F}}_{3c}$ . Thus, they form a 1- $\infty$ -system spanned by:  $\hat{\mathcal{E}}_\infty = (0, \mathbf{s}_2 \times \mathbf{s}_3) = \underline{s}_1 \underline{s}_2$ . On the other hand, the axis of a zero-pitch twist  $\hat{\mathcal{E}}_0$  in  $\mathcal{T}_{PM}$  should be: coplanar with the axis of  $\hat{\mathcal{F}}_{3c}$ , coplanar with the axes of  $\hat{\mathcal{F}}_{1c}$  and  $\hat{\mathcal{F}}_{2c}$  and orthogonal to  $\mathbf{s}_1 \times \mathbf{s}_2$ .

In order to satisfy these two conditions, the axis of a  $\hat{\mathcal{E}}_{0i}$  in  $\mathcal{T}_{PM}$  should necessarily belong to plane  $\mathcal{P}$  and cross at the same time the axis of  $\hat{\mathcal{F}}_{3c}$ . Let  $I$  be the intersection point between plane  $\mathcal{P}$  and the axis of  $\hat{\mathcal{F}}_{3c}$ . Hence, zero-pitch twists in  $\mathcal{T}_{PM}$  form a planar pencil of finite lines in plane  $\mathcal{P}$  and passing through point  $I$  as shown in Fig. 2. Consequently,

$$\mathcal{T}_{PM} = \text{span}(\hat{\mathcal{E}}_\infty, \hat{\mathcal{E}}_{01}, \hat{\mathcal{E}}_{02}) \quad (6)$$

where  $\hat{\mathcal{E}}_{01}$  and  $\hat{\mathcal{E}}_{02}$  are any two distinct zero-pitch twists whose axes lie in plane  $\mathcal{P}$  and pass through point  $I$ . The moving platform of the Exechon PM provides a pure translation along  $\mathbf{s}_1 \times \mathbf{s}_2$  plus two independent rotations. These rotations are about any two distinct axes in plane  $\mathcal{P}$  and passing through point  $I$ .

## Actuation wrench system of the Exechon PM

Each limb of the Exechon applies one pure actuation force whose axis is along the direction of the corresponding actuated  $\text{P}$ -joint, as follows:  $\widehat{\mathcal{F}}_{1a} = (\mathbf{p}_1, \mathbf{r}_{A_1} \times \mathbf{p}_1)$ ;  $\widehat{\mathcal{F}}_{2a} = (\mathbf{p}_2, \mathbf{r}_{A_2} \times \mathbf{p}_2)$  and  $\widehat{\mathcal{F}}_{3a} = (\mathbf{p}_3, \mathbf{r}_{A_3} \times \mathbf{p}_3)$ . The actuation wrench system of the Exechon PM is given by:

$$\mathcal{W}_{PM}^a = \text{span}(\widehat{\mathcal{F}}_{1a}, \widehat{\mathcal{F}}_{2a}, \widehat{\mathcal{F}}_{3a}) \quad (7)$$

### Wrench graph of the Exechon PM

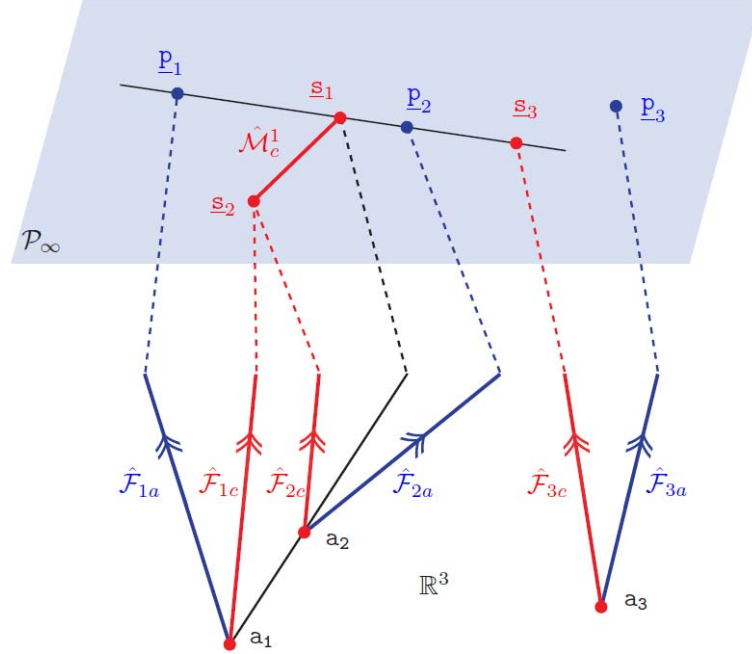


Figure 3: Wrench graph of the Exechon in  $\mathbb{P}_3$

Let  $\underline{p}_i = (\mathbf{p}_i, 0)$ ,  $i = 1, 2, 3$ , be the 1-extensor representing the point at infinity in the direction of  $\mathbf{p}_i$ . Therefore,  $\widehat{\mathcal{F}}_{1a} = (\mathbf{p}_1, \mathbf{r}_{A_1} \times \mathbf{p}_1)$  can be represented by the 2-extensor  $a_1 \underline{p}_1$ . Likewise,  $\widehat{\mathcal{F}}_{2a} = (\mathbf{p}_2, \mathbf{r}_{A_2} \times \mathbf{p}_2) = a_2 \underline{p}_2$  and  $\widehat{\mathcal{F}}_{3a} = (\mathbf{p}_3, \mathbf{r}_{A_3} \times \mathbf{p}_3) = a_3 \underline{p}_3$

The line at infinity  $\mathcal{L}_{s_2} = (0, \mathbf{s}_2)$  is the intersection of any plane having  $\mathbf{s}_2$  as normal vector with the plane at infinity  $\mathcal{P}_\infty$ . Thus, any finite line orthogonal to  $\mathbf{s}_2$  meets  $\mathcal{L}_{s_2}$ . Thus, line  $\mathcal{L}_{s_2}$  collects all the points at infinity defined by directions orthogonal to  $\mathbf{s}_2$ . Since vectors  $\mathbf{p}_1$ ,  $\mathbf{p}_2$ ,  $\mathbf{s}_3$  and  $\mathbf{s}_1$  are orthogonal to  $\mathbf{s}_2$ , points  $\underline{p}_1$ ,  $\underline{p}_2$ ,  $\underline{s}_3$  and  $\underline{s}_1$  belong to a same line, namely, to  $\mathcal{L}_{s_2} = (0, \mathbf{s}_2)$ . A wrench graph of the Exechon PM is obtained in Fig. 3.

### Singularity conditions of the Exechon PM

The Exechon is an over-constrained PM since the limbs can apply four distinct constraint wrenches but they constrain only three *dof*. In that case, the transpose of the Jacobian matrix can be written as a  $6 \times 7$  matrix, namely,

$$\mathbf{J}^T = \begin{bmatrix} \widehat{\mathcal{F}}_{1a} & \widehat{\mathcal{F}}_{2a} & \widehat{\mathcal{F}}_{3a} & \widehat{\mathcal{F}}_{1c} & \widehat{\mathcal{F}}_{2c} & \widehat{\mathcal{F}}_{3c} & \widehat{\mathcal{M}}_{1c} \end{bmatrix} \quad (8)$$



## Superbracket expression

A parallel singularity occurs when the rank of  $\mathbf{J}$  becomes smaller than six, namely, when the seven wrenches of Eq. 8 span a  $n < 6$ -system  $\widehat{\mathcal{F}}_{1c}$ ,  $\widehat{\mathcal{F}}_{2c}$  and  $\widehat{\mathcal{M}}_{1c}$  are always linearly dependent. Consequently, there are  $\binom{3}{2} = 3$  ways to select two among these three wrenches in the superbracket expression and thus, we can formulate 3 superbracket expressions for the Exechon PM. However, since  $\text{span}(\widehat{\mathcal{F}}_{1c}, \widehat{\mathcal{F}}_{2c}) = \text{span}(\widehat{\mathcal{F}}_{1c}, \widehat{\mathcal{M}}_{1c})$  in any robot configurations, these three superbracket expressions are always equivalent to each other. Thus, it is sufficient to analyze one of the three possible expressions.

## Superbracket decomposition and singularity conditions

Here we will analyze uniquely the superbracket of  $S_1$ , involving  $\widehat{\mathcal{F}}_{1c}$ ,  $\widehat{\mathcal{F}}_{2c}$  but not  $\widehat{\mathcal{M}}_{1c}$ , as follows:

$$S_1 = [a_1 \underline{p}_1, a_1 \underline{s}_2, a_2 \underline{p}_2, a_2 \underline{s}_2, a_3 \underline{p}_3, a_3 \underline{s}_3] \quad (9)$$

The decomposition of  $S_1$  leads to three non-zero monomials as follows:

$$S_1 = [a_1 \underline{p}_1 \underline{s}_2 a_2][a_1 a_2 \underline{s}_2 a_3][\underline{p}_2 \underline{p}_3 a_3 \underline{s}_3] - [a_1 \underline{p}_1 \underline{s}_2 \underline{p}_2][a_1 a_2 \underline{s}_2 a_3][a_2 \underline{p}_3 a_3 \underline{s}_3] - [a_1 \underline{p}_1 \underline{s}_2 a_2][a_1 a_2 \underline{p}_2 a_3][\underline{s}_2 \underline{p}_3 a_3 \underline{s}_3] \quad (10)$$

Now let consider the sum<sup>1</sup>  $[a_1 \underline{p}_1 \underline{s}_2 \dot{a}_2][a_1 a_2 \underline{s}_2 \dot{a}_3][\dot{\underline{p}}_2 \underline{p}_3 a_3 \underline{s}_3]$ . It is developed as follows:

$$\begin{aligned} [a_1 \underline{p}_1 \underline{s}_2 \dot{a}_2][a_1 a_2 \underline{s}_2 \dot{a}_3][\dot{\underline{p}}_2 \underline{p}_3 a_3 \underline{s}_3] = & [a_1 \underline{p}_1 \underline{s}_2 a_2][a_1 a_2 \underline{s}_2 a_3][\underline{p}_2 \underline{p}_3 a_3 \underline{s}_3] \\ & - [a_1 \underline{p}_1 \underline{s}_2 a_2][a_1 a_2 \underline{p}_2 a_3][\underline{s}_2 \underline{p}_3 a_3 \underline{s}_3] \\ & - \cancel{[a_1 \underline{p}_1 \underline{s}_2 \underline{s}_2][a_1 a_2 a_2 a_3][\underline{p}_2 \underline{p}_3 a_3 \underline{s}_3]} \\ & + \cancel{[a_1 \underline{p}_1 \underline{s}_2 \underline{s}_2][a_1 a_2 \underline{p}_2 a_3][a_2 \underline{p}_3 a_3 \underline{s}_3]} \\ & + [a_1 \underline{p}_1 \underline{s}_2 \underline{p}_2][\cancel{a_1 a_2 a_2 a_3}][\underline{s}_2 \underline{p}_3 a_3 \underline{s}_3] \\ & - [a_1 \underline{p}_1 \underline{s}_2 \underline{p}_2][a_1 a_2 \underline{s}_2 a_3][a_2 \underline{p}_3 a_3 \underline{s}_3] \end{aligned} \quad (11)$$

The crossed out brackets of Eq. 11 being equal to zero, we can notice that the superbracket of the Exechon PM can be written as:

$$S_1 = [a_1 \underline{p}_1 \underline{s}_2 \dot{a}_2][a_1 a_2 \underline{s}_2 \dot{a}_3][\dot{\underline{p}}_2 \underline{p}_3 a_3 \underline{s}_3] \quad (12)$$

From [6], it follows that the right hand side of Eq. 12 is the result of the meet of four planes:  $(a_1 \underline{p}_1 \underline{s}_2)$ ,  $(a_1 a_2 a_3)$ ,  $(\underline{p}_3 a_3 \underline{s}_3)$  and  $(a_2 \underline{s}_2 \underline{p}_2)$ . Hence,

$$S_1 = (a_1 \underline{p}_1 \underline{s}_2) \wedge (a_1 a_2 a_3) \wedge (\underline{p}_3 a_3 \underline{s}_3) \wedge (a_2 \underline{s}_2 \underline{p}_2) \quad (13)$$

The singularity conditions of the Exechon PM depend on some relations between four planes. Plane  $a_1 a_2 a_3$  is spanned by three fixed points  $A_1$ ,  $A_2$  and  $A_3$  lying in the plane of the fixed base. Planes  $\mathcal{P}_1 = a_1 \underline{p}_1 \underline{s}_2$ ,  $\mathcal{P}_2 = a_2 \underline{s}_2 \underline{p}_2$  and  $\mathcal{P}_3 = \underline{p}_3 a_3 \underline{s}_3$  contain the actuation and the constraint forces of limbs  $L_1$ ,  $L_2$  and  $L_3$ , respectively. Figure 4 shows planes  $\mathcal{P}_1$ ,  $\mathcal{P}_2$  and  $\mathcal{P}_3$  of the Exechon PM. Generally, the Exechon PM exhibits a parallel singularity whenever the four foregoing planes intersect at least at one point. This point can be a finite point or a point at infinity. As a matter of fact, the four planes

<sup>1</sup> The superscript  $\dot{}$  stands for permuted elements as pointed out in [17]

intersect at a point at infinity whenever one can find a finite line (or a direction) parallel to the four planes at once. Moreover, Eq. 13 also holds the following particular cases:

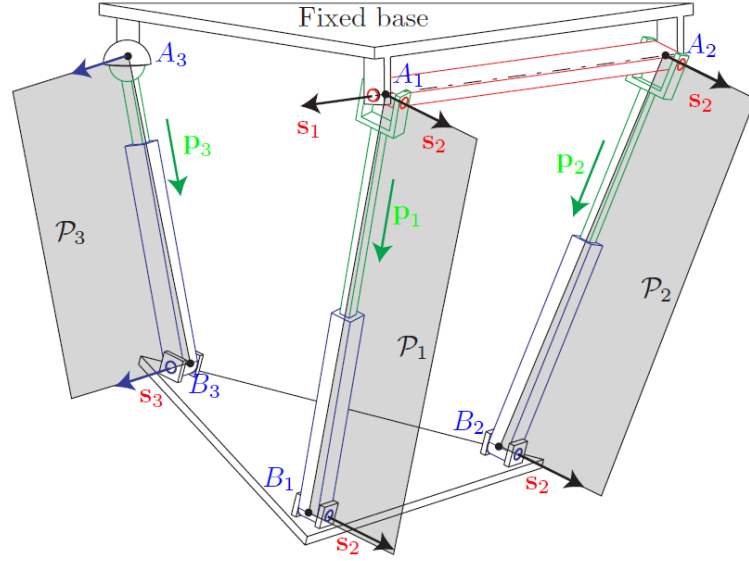


Figure 4: The Exechon PM, planes  $\mathcal{P}_1, \mathcal{P}_2$  and  $\mathcal{P}_3$

1. One of the four planes degenerates:

- Plane  $a_1 a_2 a_3$  degenerates. This cannot occur since  $a_1, a_2$  and  $a_3$  correspond to three fixed points.
- Plane  $\mathcal{P}_1 = a_1 \underline{p}_1 \underline{s}_2$ , whose Plücker coordinate vector is  $\begin{bmatrix} \mathbf{p}_1 \times \mathbf{s}_2 \\ \mathbf{r}_{A_1} \cdot (\mathbf{s}_1 \times \mathbf{s}_2) \end{bmatrix}$ , degenerates.

This plane contains the axes of the actuation force  $\widehat{\mathcal{F}}_{1a}$  and the constraint force  $\widehat{\mathcal{F}}_{1c}$  of limb  $L_1$  that intersect at a fixed point  $A_1$ . Thus, this plane degenerates if and only if  $\underline{p}_1 \equiv \underline{s}_2$ , i.e., if  $\mathbf{s}_1 \times \mathbf{s}_2 = 0$ , i.e., if the two forces  $\widehat{\mathcal{F}}_{1c}$  and  $\widehat{\mathcal{F}}_{1a}$  become coaxial. However, since vectors  $\mathbf{p}_1$  and  $\mathbf{s}_2$  are orthogonal in any robot configuration, the foregoing condition turns to be impossible.

- Plane  $\mathcal{P}_2 = a_2 \underline{s}_2 \underline{p}_2$ , whose Plücker coordinate vector is  $\begin{bmatrix} \mathbf{p}_2 \times \mathbf{s}_2 \\ \mathbf{r}_{A_2} \cdot (\mathbf{p}_2 \times \mathbf{s}_2) \end{bmatrix}$ , degenerates.
- This cannot occur since vectors  $\mathbf{p}_2$  and  $\mathbf{s}_2$  are orthogonal in any robot configuration.
- Plane  $\mathcal{P}_3 = \underline{p}_3 a_3 \underline{s}_3$ , whose Plücker coordinate vector is  $\begin{bmatrix} \mathbf{p}_3 \times \mathbf{s}_3 \\ \mathbf{r}_{A_3} \cdot (\mathbf{p}_3 \times \mathbf{s}_3) \end{bmatrix}$ , degenerates.

This cannot occur since vectors  $\mathbf{p}_3$  and  $\mathbf{s}_3$  are orthogonal in any robot configuration.

- Two of the four planes coincide.
- Three of the four planes intersect at a common line.
- The four planes intersect at a common point.

## Conclusions

This paper presented the constraint and singularity analysis of the Exechon. The constraint analysis was performed by using the concept of points and lines at infinity and the screw theory. The motion pattern of the moving platform was not easy to grasp. Thus, a representation of the constraint wrenches in the projective space proved very useful to describe the motion of the moving platform

in a general configuration. Then, a wrench graph representing all the wrenches of the manipulator in the projective space was obtained to help formulating the superbracket expression. Finally, this superbracket was explored to provide geometric conditions for the parallel singularities of the Exechon.

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