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ABSTRACT

The short-term ('operational') dynamics of pedestrian crowds are generally thought to involve no anticipation, except perhaps the avoidance of the most imminent collisions. We show that current models rooted in this belief fail to reproduce essential features experimentally observed when a static crowd is crossed by an intruder. We identify the missing ingredient as the pedestrians' ability to plan ahead far enough beyond the next interaction, which explains why instead of walking away from the intruder they accept to temporarily move transversely towards denser regions on the intruder's sides. On this basis, a minimal model based on mean-field game theory proves remarkably successful in capturing the experimental observa-

tions associated with this setting, but also other daily-life situations such as partial metro boarding. These findings are clear evidence that a long term game theoretical approach is key to capturing essential elements of the dynamics of crowds.

I. INTRODUCTION

Although crowd disasters (such as the huge stampedes that grieved the Hajj in 1990, 2006 and 2015 [1]) are more eye-catching to the public, the dynamics of pedestrian crowds are also of great relevance in less dire circumstances. They are central when it comes to designing and dimensioning busy public facilities, from large transport hubs to entertainment venues, and optimising the flows of people. Modelling pedestrian motion in these settings is a multi-scale endeavour, which requires determining where people are heading for (*strategic* level), what route they will take (*tactical* level), and finally how they will move along that route in response to interactions with other people (*operational* level) [2]. The *strategic* and *tactical* levels typically involve some planning in order to make a choice among a discrete or continuous set of options, such as targeted activities, destinations [2], paths (possibly knowing their expected level of congestion) [3], or, in the context of evacuations, egress alternatives [3, 4]. These choices are often handled as processes of maximisation (minimisation) of a utility (cost), which may depend on lower-level information such as pedestrian density or streaming velocity [5, 6].

The *operational* level deals with much shorter time scales and is generally believed to involve no planning ahead. Anticipatory effects are thus merely neglected in so-called *reactive* models, especially at high densities, possibly with the lingering idea that mechanical forces then prevail. For example, the popular social force model of Helbing and Molnar [7], still at the heart of several commercial software products, combines contact forces and pseudo-forces ("social" forces) which, in the original implementation, are only functions of the agents' current positions (and possibly orientations). Some degree of anticipation has since been introduced into these models to better describe collision avoidance, e.g., by making the pseudo-forces depend on future positions rather than current ones [8, 9]. In a dual approach, the most imminent collisions can be avoided by scanning the whole velocity space [10–12] or a subset of it [13] in search of the optimal velocity. In order to handle navigation through dense crowds, anticipated collisions beyond the most imminent one [14] or, at a more coarse-grained scale, local density inhomogeneities [5] can be taken into account in the optimisation. All these dynamic models, at best premised on a constant-velocity hypothesis, owe their high computational tractability to their relative shortsightedness : The simulated agents do not plan ahead in interaction with their counterparts.

In this Letter, we argue that, even at the *operational* level, crowds in some daily-life circumstances display signs of anticipation that may elude the foregoing short-sighted models ; this will be exemplified by the recently studied response of a static crowd when crossed by an 'intruder' [15, 16]. We purport to show that a minimal game theoretical approach, made tractable thanks to an elegant analogy between its mean-field formulation [17–19] and Schrödinger's equation [20, 21], can replicate the empirical observations for this example case, provided that it accounts for the anticipation of future costs. Beyond that particular example, the approach efficiently captures certain behaviours of crowds at the interface between the *operational* and *tactical* levels.

II. CROSSING A STATIC CROWD

Crossing a static crowd is a common experience in busy premises, from standing concerts and festivals to railway stations. Recently, small-scale controlled experiments [15] shed light on robust trends in the response of the crowd when crossed by a cylindrical intruder, as displayed in Fig. 1 (right column). The induced response consists of a fairly symmetric density field around the intruder, displaying depleted zones both upstream and downstream from the intruder, as well as higher-density regions on the sides. Indeed the crowd's displacements are mostly transverse : pedestrians tend to simply step aside. Incidentally, a qualitatively similar response was filmed at much larger scale in a dense crowd of protesters in Hong-Kong, which split open to let an ambulance through [22].

Such features strongly depart from the mechanical response observed *e.g.* in experiments [23, 24] or simulations [25] of penetration into a granular mono-layer below jamming, where grains are pushed forward by the intruder (see Figs. 1 (left column)) and accumulate downstream, instead of moving crosswise. More worryingly, these "mechanical" features [26] are also observed in simulations of pedestrian dynamics performed with the social-force model [7], which rests on tangential and normal forces at contact and radial repulsive forces for longer-ranged interactions.

Introducing collision anticipation in the pedestrian model helps reproduce the opening of an agent-free 'tunnel' ahead of the intruder, as illustrated with a 'time to (first) collision' model (second column of Fig. 1) directly inspired from [12], details of which can be found in SI. However, even though the displacements need not align with the contact forces in this agent-based model, the displacement pattern diverges from the experimental observations, with streamwise (walk-away) moves that prevail over transverse (step-aside) ones. Indeed, such models rely on 'short-sighted' agents, who do not see past the most imminent collision expected from constant-velocity extrapolation.

Results may vary with the specific collision-avoidance model and the selected parameters. Yet, our inability to reproduce prominent experimental features suggests that an ingredient is missing in these approaches based on short-time (first-collision) anticipation.

III. A GAME THEORETICAL APPROACH TO ACCOUNT FOR LOW-LEVEL PLANNING

To bring in the missing piece, we start by noticing that the observed behaviours are actually most intuitive : Pedestrians anticipate that it will cost them less effort to step aside and then resume their positions, even if it entails enduring high densities for some time, than to endlessly run away from an intruder that will not deviate from its course. But accounting for this requires a change of paradigm compared to the foregoing approaches. Game theory is an adequate framework to handle the conflicting impulses of interacting agents endowed with planning capacities : agents are now able to optimise their strategy taking into account the choices (or strategies) of others. So far, its use in pedestrian dynamics has mostly been restricted to evacuation tactics in discrete models [4, 27, 28]. Unfortunately, the problem becomes intractable when the number of interacting agents grows.

To overcome this quandary, we turn to Mean Field Games (MFG), introduced by Lasry and Lions [17, 18] as well as Huang et al. [19] in the wake of the mean-field approximations of statistical mechanics, and since used in a variety of fields,

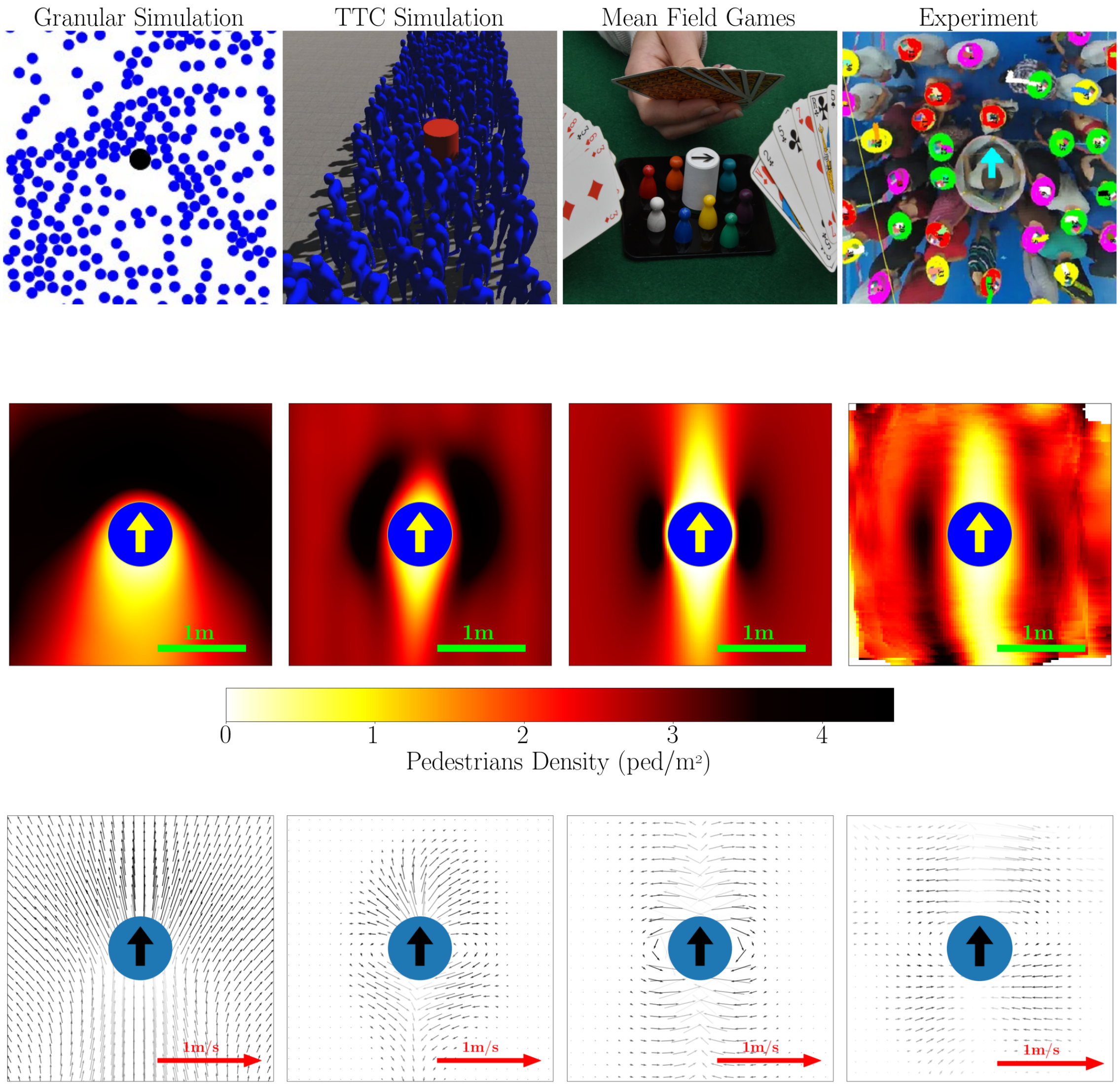

FIGURE 1: Density (middle row) and velocity (bottom row) fields induced in a static crowd by a cylindrical intruder that crosses it ; the transparency of the velocity arrows is linearly related to the local density. (Column 1) Simulations of a mono-layer of vibrated disks. (Column 2) Simulations of an agent-based model [a] wherein agents may anticipate the most imminent collision. (Column 3) Results of the mean-field game introduced in this paper. (Column 4) Controlled experiments of [15]. Note the relatively symmetrical density dip in front and behind the intruder, as well as the transverse moves. (Columns 1-3) The crowd's density and intruder's size have been adjusted to match the experimental data. Details of simulations and videos showcasing time evolution can be found in SI.

a. The snapshot illustrating the agent-based model was rendered using the Chaos visualisation software developed by INRIA (https ://project.inria.fr/crowdscience/project/ocsr/chaos/).

ranging from finance [29–31] to economics [32–34], epidemiology [35–37], sociology [35, 38, 39], or engineering [40–42]. While applications of MFG to crowd dynamics have already been proposed [3, 43–45], our goal here is to demonstrate the practical relevance of this approach at the *operational* level, using an elementary MFG belonging to one of the first class of models introduced by Lasry and Lions [17], and which can be thoroughly analysed thanks to its connection with the nonlinear Schrödinger equation.

In the mean field approximation, the "N-player" game is replaced by a generalized Nash equilibrium [46] where indiscriminate microscopic agents play against a macroscopic state of the system (a density field) formed by the infinitely many remaining agents. Consider a large set of pedestrians, the agents of our game, characterised by their spatial position (state variable) $\mathbf{X}^i \equiv (x^i, y^i) \in \mathbb{R}^2$, which we assume follows Langevin dynamics, viz.,

$$d\mathbf{X}^i_t = \mathbf{a}^i_t dt + \sigma d\mathbf{W}^i_t \,, \tag{1}$$

where the drift velocity (*control* variable) $\mathbf{a}^i_t$ reflects the agent's strategy. In Eq. (1), σ is a constant and components of $\mathbf{W}^i$ are independent white noises of variance one accounting for unpredictable events. All agents are supposed identical, apart from their initial positions $\mathbf{X}^i(t=0)$ and realisations of $\mathbf{W}^i$.

Each agent strives to adapt their velocity $\mathbf{a}^i_t$ in order to minimise a cost functional that we assume to take the simple form

$$c[\mathbf{a}^i](t, \mathbf{x}^i_t) = \left\langle \int_t^T \left[\frac{\mu \mathbf{a}^2}{2} - (g m_t(\mathbf{x}) + U_0(\mathbf{x} - \mathbf{v}t)) \right] d\tau \right\rangle_{\text{noise}} \,, \tag{2}$$

where the average denoted by $\langle \cdot \rangle_{\text{noise}}$ is performed over all realizations of the noise for trajectories starting at $\mathbf{x}^i_t$ at time t. In this expression, the term $\mu \mathbf{a}^2/2$, akin to a kinetic energy, represents the efforts required by the agent to enact their strategy, while the interactions with the other agents via the empirical density $m^{(e)}(t, \mathbf{x}) = \sum_i \delta(\mathbf{x} - \mathbf{X}^i(t))/N$ are controlled by a parameter $g < 0$. Finally, the space occupied by the intruding cylinder, which moves at a velocity $\mathbf{v} = (0, v)$, is characterised by a 'potential' $U_0(\mathbf{x}) = V_0 \Theta(\|\mathbf{x}\| - R)$ that tends to $V_0 \to -\infty$ inside the radius R of the cylinder and is zero elsewhere.

In the presence of many agents, the density self-averages to $m(t, \mathbf{x}) = \langle m^{(e)}(t, \mathbf{x}) \rangle_{\text{noise}}$ and the optimization problem (2) does not feature explicit coupling between agents anymore. It can then be solved by introducing the value function $u(t, \mathbf{x}) = \min_{a(\cdot)} c[\mathbf{a}](t, \mathbf{x})$, which obeys a Hamilton-Jacobi-Bellman [HJB] equation [18, 47], with an optimal control given by $\mathbf{a}^*(t, \mathbf{x}) = -\nabla u(t, \mathbf{x})/\mu$. Consistency imposes that $m(t, \mathbf{x})$ is solution of the Fokker-Planck [FP] equation associated with Eq. (1), given the drift velocity $\mathbf{a}(t, \mathbf{x}) = \mathbf{a}^*(t, \mathbf{x})$. As such, MFG can be reduced to a system of two coupled partial differential equations [17, 18, 20, 21]

$$\begin{cases} \partial_t u(t, \mathbf{x}) = \dfrac{1}{2\mu} \left[\nabla u(t, \mathbf{x})\right]^2 - \dfrac{\sigma^2}{2} \Delta u(t, \mathbf{x}) + g m(t, \mathbf{x}) + U_0(\mathbf{x} - \mathbf{v}t) & \text{[HJB]} \\ \partial_t m(t, \mathbf{x}) = \dfrac{1}{\mu} \nabla \left[m(t, \mathbf{x}) \nabla u(t, \mathbf{x})\right] + \dfrac{\sigma^2}{2} \Delta m(t, \mathbf{x}) & \text{[FP]} \end{cases} \tag{3}$$

The atypical “forward-backward” structure of Eqs. (3), highlighted by the opposite signs of Laplacian terms in the two equations, accounts for anticipation. The boundary conditions epitomise this structure : based on Eq.(2), the value function has terminal condition $u(t = T, \mathbf{x}) = 0$, while the density of agents evolves from a fixed initial distribution $m(t = 0, \mathbf{x}) = m_0(\mathbf{x})$. In previous work, we have evinced a formal, but insightful mapping of these MFG equations onto a nonlinear Schrödinger equation (NLS) [20, 21, 48], which has been studied for decades in fields ranging from non-linear optics [49] to Bose-Einstein condensation [50] and fluid dynamics [51].

We perform a change of variables $(u(t, \mathbf{x}), m(t, \mathbf{x})) \mapsto (\Phi(t, \mathbf{x}), \Gamma(t, \mathbf{x}))$ through $u(t, \mathbf{x}) = -\mu\sigma^2 \log \Phi(t, \mathbf{x})$, $m(t, \mathbf{x}) = \Gamma(t, \mathbf{x})\Phi(t, \mathbf{x})$ [21]. The first relation is the usual Cole-Hopf transform [52] ; the second corresponds to an "Hermitization" of Eqs. (3). In terms of the new variables (Φ, Γ), the MFG equations read

$$\begin{cases} -\mu\sigma^2 \partial_t \Phi = \dfrac{\mu\sigma^4}{2} \Delta\Phi + (U_0 + g\Gamma\Phi)\Phi \\ +\mu\sigma^2 \partial_t \Gamma = \dfrac{\mu\sigma^4}{2} \Delta\Gamma + (U_0 + g\Gamma\Phi)\Gamma \end{cases} . \tag{4}$$

Except for the missing imaginary factor associated with time derivation, these equations have exactly the structure of NLS describing the evolution of a quantum state $\Psi(t, \mathbf{x})$ of a Bose-Einstein condensate, with formal correspondence $\Psi \to \Gamma$, $\Psi^* \to \Phi$ and $\rho \equiv ||\Psi||^2 \to m \equiv \Phi\Gamma$. This system, however, retains the forward-backward structure of MFG evidenced by mixed initial and final boundary conditions $\Phi(T, \mathbf{x}) = 1$, $\Gamma(0, \mathbf{x})\, \Phi(0, \mathbf{x}) = m_0(\mathbf{x})$. Several methods have been developed to deal with NLS and most can be leveraged to tackle the MFG problem [21, 53].

Self-consistent solutions of Eqs. (4) are obtained by iteration over a backward-forward scheme. A video illustrating the evolution of the agents' density for a particular set of parameters, as well as details about the numerical scheme, can be found in SI.

Focusing on the *permanent* regime (a.k.a. the *ergodic state* [54]) , rather than on the transients associated with the intruder's entry or exit, further simplifies the resolution. In this regime, defined by time-independent density and velocity fields in the intruder's frame, the auxiliary functions Φ and Γ are not constant in time, but they assume the trivial dynamics $\Phi(t,\mathbf{x}) = \exp[\lambda t/\mu\sigma^2]\Phi_{\text{er}}$ and $\exp[-\lambda t/\mu\sigma^2]\Gamma_{\text{er}}$ where, in the frame of the intruder, Φ_{er} and Γ_{er} satisfy

$$\begin{cases} \dfrac{\mu\sigma^4}{2}\Delta\Phi_{\text{er}} - \mu\sigma^2\mathbf{v}\cdot\vec{\nabla}\Phi_{\text{er}} + [U_0(\mathbf{x}) + gm_{\text{er}}]\Phi_{\text{er}} = -\lambda\Phi_{\text{er}}, \\ \dfrac{\mu\sigma^4}{2}\Delta\Gamma_{\text{er}} + \mu\sigma^2\mathbf{v}\cdot\vec{\nabla}\Gamma_{\text{er}} + [U_0(\mathbf{x}) + gm_{\text{er}}]\Gamma_{\text{er}} = -\lambda\Gamma_{\text{er}} \end{cases}, \tag{5}$$

(with $m_{\text{er}} = \Phi_{\text{er}}\Gamma_{\text{er}}$ independent of time). Far from the intruder, pedestrians are at rest in the lab frame, and thus have constant velocity $-\mathbf{v}$ in the intruder frame. This imposes the asymptotic form $\Phi_{\text{er}}(\mathbf{x}) \to \sqrt{m_0}\exp[-\mathbf{v}\cdot\mathbf{x}/\sigma^2]$ and $\Gamma_{\text{er}}(\mathbf{x}) \to \sqrt{m_0}\exp[+\mathbf{v}\cdot\mathbf{x}/\sigma^2]$ far from the intruder, which then sets $\lambda = -gm_0 - \frac{3}{2}\mu v^2$.

IV. RESULTS

The ergodic Eqs. (5) have two remarkable features : (i) They give direct access to the permanent regime, and are straightforward to implement numerically since time dependence has disappeared. (ii) The solutions of Eqs. (5) are entirely specified by two dimensionless parameters.

Indeed, the intruder is characterised by its radius R and its velocity v. In the same way, pedestrians are characterized by a length scale $\xi = \sqrt{|\mu\sigma^4/2gm_0|}$, the distance over which the crowd density tends to recover its bulk value from a perturbation, a.k.a *healing length*, and a velocity scale $c_s = \sqrt{|g\bar{m}_0/2\mu|}$, the typical speed at which pedestrians tend to move[1]. Up to a scaling factor, solutions of Eqs. (5) can

1. Note that $\mu\xi c_s = \mu\sigma^2$ has the dimension of an action and plays the role of $\hbar$ in the original nonlinear Schrödinger equation.

be expressed as a function of the two ratios ξ/R and c_s/v instead of depending on the full set of parameters $(R, v, \mu, \sigma, m_0, g)$, which facilitates the exploration of the parameter space.

Figure 2 presents typical density and velocity fields simulated in the ergodic state, for parameters selected in each quadrant of the reduced space parametrised by $\log c_s/v$ and $\log \xi/R$ on the horizontal and vertical axis respectively. Intuitively, one understands that c_s governs the cost of motion for the agents while ξ gives the extent of the perturbation caused by the presence of the intruder. The main visual difference between the small and large c_s/v cases is the change in rotational symmetry, a fact that reflects a more fundamental change in strategy. For large values of c_s/v pedestrians do not mind moving, and they rather try to avoid congested areas for as long as possible, thus creating circulation around the intruder, as shown in the velocity plots. On the other hand, for small values of c_s/v, moving fast costs more ; therefore, in order to avoid the intruder, pedestrians have to move earlier, and will accept to temporarily side-step into a more crowded area, thereby causing a stretch of the density along the vertical direction. The experimental observations of [15] are best reproduced for small c_s/v and small ξ/R ($c_s = 0.22$ and $\xi = 0.15$), as shown in the third column of Fig. 1. The obtained agreement is remarkable, considering the minimalism of our MFG model.

V. DISCUSSION

The data plotted in Fig. 1 (third column) demonstrate that even basic MFG models can naturally capture and semi-quantitatively reproduce prominent features of the response of static crowds [15], which may be out of reach of more short-sighted pedestrian dynamics models.

Beyond this particular example, MFG are also applicable to a broader array of crowd-related problems. This will now be illustrated by exploring the daily-life situation of people waiting to board the coach in an underground station. This is readily achieved by suitably modifying the external potential $U_0(\mathbf{x})$ and the geometry of the system, as shown on Fig. 3, and introducing a terminal cost $c_T(\mathbf{x})$ [21, 53] that is lower aboard the metro than on the platform ; we manage to reproduce the

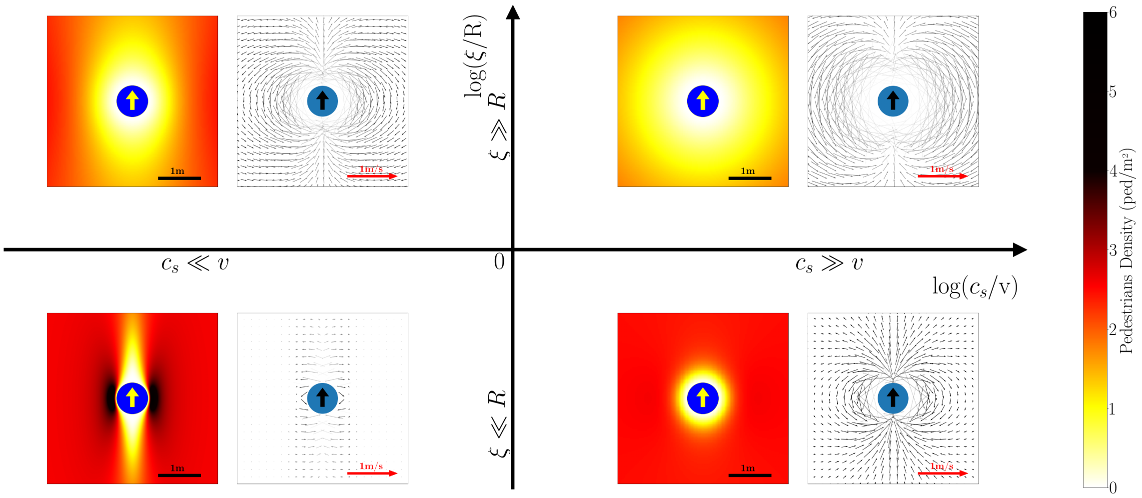

FIGURE 2: Typical density and velocity fields induced by the crossing intruder in the ergodic state, as predicted by the MFG model in different regions of the parameter space. Parameters taken in the small c_s/v and small ξ/R quadrant display good visual agreement with the experimental data.

boarding process in a qualitatively realistic way, up to the decision made by some agents to stay on the platform rather than board the overcrowded metro, which would be more difficult to implement in traditional approaches of crowd dynamics at the tactical level.

To conclude, let us recall that the foregoing results have been obtained with a simple, generic MFG model which depends linearly on density via $gm(t, \mathbf{x})$. This approximation can be refined and the MFG formalism is flexible enough to incorporate further elements to make it truer to life, including time-discounting effects [55, 56] and congestion [43, 57, 58]. Higher quantitative accuracy will be within reach of these more sophisticated approaches, possibly at the expense of less transparent outcomes compared to the elementary model used here. For sure, MFG will struggle to capture a variety of problems of crowd dynamics at the operational level, notably those for which the granularity of the crowd is central. However, the above demonstration of the realistic predictions of a generic MFG model, in cases where more conventional pedestrian dynamics are found wanting, highlights the role played by *optimization* and *anticipation* at the *operational* level of crowd dynamics, and justifies to claim entry for such an approach into the toolkit of practitioners of the field.

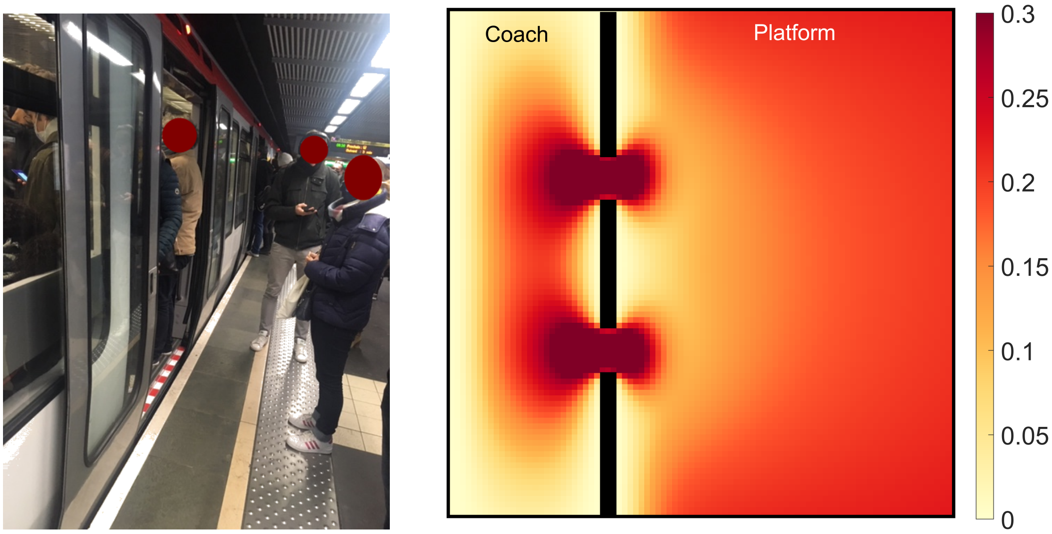

FIGURE 3: Boarding a crowded metro coach at rush hour. Left : Morning rush hour of November 18, 2021, on the platform of Metro A in Lyon, France. The doors are about to close and the gap between boarding passengers and those who preferred to wait for the next metro is clearly visible. Right : Snapshot from a MFG simulation at $t = 0.9T$. Players start uniformly distributed on the platform and would like to get on the coach before the doors close, at $t = T$. Just before that moment, the players closest to the doors choose to rush towards the coach and cram themselves in it despite the high density. Others prefer to stay on the platform (see SI for a movie of the whole process).

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