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1     **PERIODIC UNFOLDING FOR ANISOTROPICALLY BOUNDED**  
2     **SEQUENCES**

3     RICCARDO FALCONI\*, GEORGES GRISO†, AND JULIA ORLIK‡

4     **Abstract.** This paper is focused on the asymptotic behavior of sequences of functions, whose  
5     partial derivatives estimates in one or more directions are highly contrasted with respect to the  
6     periodic parameter  $\varepsilon$ . In particular, a direct application for the homogenization of a homogeneous  
7     Dirichlet problem defined on an anisotropic structure is presented. In general, the obtained results  
8     can be applied to thin structures where the behavior is different according to the observed direction.

9     **Key words.** Periodic unfolding method, homogenization, anisotropic Sobolev spaces, Dirichlet  
10    problem

11    **AMS subject classifications.** 31C25, 35B27, 46E35, 49J45

12    **1. Introduction.** The aim of this paper is to extend some of the results already  
13    developed in [1, 6, 13] about the periodic unfolding method to new classes of functions,  
14    the so-called "anisotropically bounded", including a direct application of the obtained  
15    results.

16    Given a small parameter  $\varepsilon$  and a bounded domain  $\Omega \subset \mathbb{R}^N$  with Lipschitz boundary,  
17    we consider the periodic paving of  $\Omega$  made with cells of size  $\varepsilon$ . In [13, Section 1.4], it is  
18    extensively investigated the asymptotic behavior of sequences  $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\Omega)$  whose  
19    gradient is isotropically bounded, that are, sequences whose gradient is bounded by  
20    the same order regardless of the direction observed. In this sense, the asymptotic  
21    behavior of sequences  $\{\phi_\varepsilon\}_\varepsilon$  such that

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla \phi_\varepsilon\|_{L^p(\Omega)} \leq C \quad \text{and} \quad \|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla \phi_\varepsilon\|_{L^p(\Omega)} \leq C$$

22    are shown in detail. In the present, we focus on the periodic unfolding for sequences  
23    whose assumptions on the gradient bound are anisotropic, e.g. where at least one or  
24    more directions are privileged and have more information than the others. In partic-  
25    ular, we define a domain partition for every  $x \in \Omega$  by setting  $x = (x', x'')$ , where the  
26    variable  $x'$  corresponds to the first  $N_1 < N$  directions, and we show the asymptotic  
27    behavior of sequences  $\{\phi_\varepsilon\}_\varepsilon$  such that

- 28    (i)  $\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x'} \phi_\varepsilon\|_{L^p(\Omega)} \leq C$ ;
- 29    (ii)  $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'} \phi_\varepsilon\|_{L^p(\Omega)} \leq C$ ;
- 30    (iii)  $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'} \phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x''} \phi_\varepsilon\|_{L^p(\Omega)} \leq C$ ;
- 31    (iv)  $\|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'} \phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x''} (\nabla_{x'} \phi_\varepsilon)\|_{L^p(\Omega)} \leq C$ .

32    These results allow to extend some more complex lemmas concerning the periodic  
33    unfolding (see for an instance [13, Lemma 11.11]) to this new classes of functions. In  
34    the end, a direct application of such lemmas is done by homogenizing via unfolding  
35    the homogeneous Dirichlet problem

$$\left\{ \begin{array}{l} \text{Find } u_\varepsilon \in H_0^1(\Omega) \text{ such that:} \\ \int_{\Omega} A_\varepsilon \begin{pmatrix} \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'} \phi \\ \varepsilon \nabla_{x''} \phi \end{pmatrix} dx = \int_{\Omega} f \phi dx, \quad \forall \phi \in H_0^1(\Omega), \end{array} \right.$$

36    whose nature is anisotropic.

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The homogenization via unfolding method, an equivalent to the two-scale convergence, has been presented the first time by D. Cioranescu and al. in [1], with further development in [2, 3, 4, 6] and extensively in [13]. Such method has largely found application (see e.g. [5, 7, 8, 9, 10, 11, 12]) and also for thin periodic structures like periodically perforated shells (see [18]), textiles made of long curved beams (see [15, 16]) and stable lattice structures made of beams (see [14, 17]).

The present paper provides additional tools for up-coming papers dealing with a wider range of periodic thin structures. Among them, we cite [20], where the assumption of loose contact between fibers on a textile made of beams arises different gradient estimates with respect to the observed direction, and [19], where the same peculiarity applies but on periodic lattice structures. More generally, such tools can be applied to many other problems related to partial differential equations on domains involving periodic grids, lattices, thin frames and fiber structures.

The paper is organized as follows. In section 2, the standard notation for the classical homogenization via unfolding method on bounded domains  $\Omega \in \mathbb{R}^N$  with Lipschitz boundary are listed. In section 3, we set a domain partition  $x = (x', x'')$  for every  $x \in \Omega$  and from [13, Chap. 7], we recall the unfolding with parameters and prove the main properties. In section 4, we give the asymptotic behavior for the classes of sequences bounded anisotropically (i)-(iv) described above. The aim of section 5 is to extend [13, Lemma 11.11] to the anisotropic case. Different anisotropic assumptions arise different regularity of the limit fields. Such results are fundamental if one considers the key role that such a Lemma has in the study of elasticity problem for thin structures (see [16, 18]). At last, in section 6, we consider the Dirichlet problem shown above. Using the results in the previous sections, existence and uniqueness of the limit problem are shown and through the homogenization via unfolding, the cell problems and the macroscopic limit problem are found.

**2. Preliminaries and notation.** Let  $\mathbb{R}^N$  be the euclidean space with usual basis  $(\mathbf{e}_1, \dots, \mathbf{e}_N)$  and  $Y = (0, 1)^N$  the open unit parallelotope associated with this basis. For a.e.  $z \in \mathbb{R}^N$ , we set the unique decomposition  $z = [z]_Y + \{z\}_Y$  such that

$$[z]_Y \doteq \sum_{i=1}^N k_i \mathbf{e}_i, \quad k_i \in \mathbb{Z}^N \quad \text{and} \quad \{z\}_Y \doteq z - [z]_Y \in Y.$$

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Let  $\{\varepsilon\}$  be a sequence of strictly positive parameters going to 0. We scale our paving by  $\varepsilon$  writing

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$$(2.1) \quad x = \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon \left\{ \frac{x}{\varepsilon} \right\}_Y \quad \text{for a.e. } x \in \mathbb{R}^N.$$

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Let now  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary. We consider the covering

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$$\Xi_\varepsilon \doteq \left\{ \xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega \right\}$$

64

and set

65

$$\widehat{\Omega}_\varepsilon \doteq \text{int} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \bar{Y}) \right\}, \quad \Lambda_\varepsilon \doteq \Omega \setminus \widehat{\Omega}_\varepsilon.$$

66

We recall the definitions of classical unfolding operator and mean value operator.

DEFINITION 2.1. (see [13, Definition 1.2]) For every measurable function  $\phi$  on  $\widehat{\Omega}_\varepsilon$ , the unfolding operator  $\mathcal{T}_\varepsilon$  is defined as follows:

$$\mathcal{T}_\varepsilon(\phi) \doteq \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

67 Note that such an operator acts on functions defined in  $\Omega$  by operating on their  
68 restriction to  $\widehat{\Omega}_\varepsilon$ .

DEFINITION 2.2. (see [13, Definition 1.10]) For every measurable function  $\widehat{\phi}$  on  $L^1(\Omega \times Y)$ , the mean value operator  $\mathcal{M}_Y$  is defined as follows:

$$\mathcal{M}_Y(\widehat{\phi})(x) \doteq \frac{1}{|Y|} \int_Y \widehat{\phi}(x, y) dy, \quad \text{for a.e. } x \in \Omega.$$

69 Let  $p \in [1, +\infty]$ . From [13, Propositions 1.8 and 1.11], we recall the properties of  
70 these operators:

$$\begin{aligned} 71 \quad \|\mathcal{T}_\varepsilon(\phi)\|_{L^p(\Omega \times Y)} &\leq |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Omega)} \quad \text{for every } \phi \in L^p(\Omega), \\ \|\mathcal{M}_Y(\widehat{\phi})\|_{L^p(\Omega)} &\leq |Y|^{-\frac{1}{p}} \|\widehat{\phi}\|_{L^p(\Omega \times Y)} \quad \text{for every } \widehat{\phi} \in L^p(\Omega \times Y). \end{aligned}$$

Since we will deal with Sobolev spaces, we give hereafter some definitions:

$$\begin{aligned} W_{per}^{1,p}(Y) &\doteq \{\phi \in W^{1,p}(Y) \mid \phi \text{ is periodic with respect to } y_i, i \in \{1, \dots, N\}\}, \\ W_{per,0}^{1,p}(Y) &\doteq \{\phi \in W_{per}^{1,p}(Y) \mid \mathcal{M}_Y(\phi) = 0\}, \\ L^p(\Omega; W^{1,p}(Y)) &\doteq \{\phi \in L^p(\Omega \times Y) \mid \nabla_y \phi \in L^p(\Omega \times Y)^N\}. \end{aligned}$$

72 **3. The unfolding with parameters.** We start by giving a rigorous definition  
73 of the domain split. Let  $(N_1, N_2)$  be in  $\mathbb{N} \times \mathbb{N}^*$  and such that  $N = N_1 + N_2$ . Denote

$$\begin{aligned} 74 \quad \mathbb{R}^{N_1} &= \left\{x' \in \mathbb{R}^N \mid x' = \sum_{i=1}^{N_1} x_i \mathbf{e}_i, \ x_i \in \mathbb{R}\right\}, \\ 75 \quad \mathbb{R}^{N_2} &= \left\{x'' \in \mathbb{R}^N \mid x'' = \sum_{i=N_1+1}^N x_i \mathbf{e}_i, \ x_i \in \mathbb{R}\right\}, \\ 76 \quad Y' &= \left\{y' \in \mathbb{R}^N \mid y' = \sum_{i=1}^{N_1} y_i \mathbf{e}_i, \ y_i \in (0, 1)\right\}, \\ 77 \quad Y'' &= \left\{y'' \in \mathbb{R}^N \mid y'' = \sum_{i=N_1+1}^N y_i \mathbf{e}_i, \ y_i \in (0, 1)\right\} \\ 78 \end{aligned}$$

and

$$\mathbb{Z}^{N_1} = \mathbb{Z}\mathbf{e}_1 \oplus \dots \oplus \mathbb{Z}\mathbf{e}_{N_1}, \quad \mathbb{Z}^{N_2} = \mathbb{Z}\mathbf{e}_{N_1+1} \oplus \dots \oplus \mathbb{Z}\mathbf{e}_N.$$

One has

$$\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \quad Y = Y' \oplus Y'', \quad \mathbb{Z}^N = \mathbb{Z}^{N_1} \oplus \mathbb{Z}^{N_2}.$$

For every  $x \in \mathbb{R}^N$  and  $y \in Y$ , we write

$$x = x' + x'' \in \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \quad y = y' + y'' \in Y' \oplus Y''.$$

From now on, however, we find easier to refer to such partition with the vectorial notation

$$x = (x', x'') \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \quad y = (y', y'') \in Y' \times Y''.$$

Similarly to (2.1), we apply the paving to a.e.  $x' \in \mathbb{R}^{N_1}$  and  $x'' \in \mathbb{R}^{N_2}$  setting

$$\begin{aligned} x' &= \varepsilon \left[ \frac{x'}{\varepsilon} \right]_{Y'} + \varepsilon \left\{ \frac{x'}{\varepsilon} \right\}_{Y'}, & \text{with } \left[ \frac{x'}{\varepsilon} \right]_{Y'} &\in \mathbb{Z}^{N_1}, \quad \left\{ \frac{x'}{\varepsilon} \right\}_{Y'} \in Y', \\ x'' &= \varepsilon \left[ \frac{x''}{\varepsilon} \right]_{Y''} + \varepsilon \left\{ \frac{x''}{\varepsilon} \right\}_{Y''}, & \text{with } \left[ \frac{x''}{\varepsilon} \right]_{Y''} &\in \mathbb{Z}^{N_2}, \quad \left\{ \frac{x''}{\varepsilon} \right\}_{Y''} \in Y''. \end{aligned}$$

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80 **DEFINITION 3.1.** For every  $\widehat{\phi} \in L^1(\Omega \times Y)$ , the partial mean value operators are  
81 defined as follows:

$$82 \quad \mathcal{M}_{Y'}(\widehat{\phi})(x, y'') \doteq \frac{1}{|Y'|} \int_{Y'} \widehat{\phi}(x, y', y'') dy', \quad \text{for a.e. } (x, y'') \in \Omega \times Y'',$$

$$83 \quad \mathcal{M}_{Y''}(\widehat{\phi})(x, y') \doteq \frac{1}{|Y''|} \int_{Y''} \widehat{\phi}(x, y', y'') dy'', \quad \text{for a.e. } (x, y') \in \Omega \times Y'.$$

84

85 Denote

$$\begin{aligned} L^p(\Omega, \nabla_{x'}) &\doteq \{ \phi \in L^p(\Omega) \mid \nabla_{x'} \phi \in L^p(\Omega)^{N_1} \}, \\ L^p(\Omega, \nabla_{x''}) &\doteq \{ \phi \in L^p(\Omega) \mid \nabla_{x''} \phi \in L^p(\Omega)^{N_2} \}, \\ L^p(\Omega, \nabla_{x'}; W^{1,p}(Y'')) &\doteq \{ \widetilde{\phi} \in L^p(\Omega \times Y'') \mid \nabla_{x'} \widetilde{\phi} \in L^p(\Omega \times Y'')^{N_1}, \\ &\quad \nabla_{y''} \widetilde{\phi} \in L^p(\Omega \times Y'')^{N_2} \}, \\ 86 \quad (3.1) \quad L^p(\Omega, \nabla_{x''}; W^{1,p}(Y')) &\doteq \{ \widetilde{\phi} \in L^p(\Omega \times Y') \mid \nabla_{x''} \widetilde{\phi} \in L^p(\Omega \times Y')^{N_2}, \\ &\quad \nabla_{y'} \widetilde{\phi} \in L^p(\Omega \times Y')^{N_1} \}, \\ L^p(\Omega \times Y''; W^{1,p}(Y')) &\doteq \{ \widehat{\phi} \in L^p(\Omega \times Y) \mid \nabla_{y'} \widehat{\phi} \in L^p(\Omega \times Y)^{N_1} \}, \\ L^p(\Omega \times Y'; W^{1,p}(Y'')) &\doteq \{ \widehat{\phi} \in L^p(\Omega \times Y) \mid \nabla_{y''} \widehat{\phi} \in L^p(\Omega \times Y)^{N_2} \}. \end{aligned}$$

87 We endow these spaces with the respective norms:

(3.2)

$$\| \cdot \|_{L^p(\Omega, \nabla_{x'})} \doteq \| \cdot \|_{L^p(\Omega)} + \| \nabla_{x'}(\cdot) \|_{L^p(\Omega)^{N_1}},$$

$$\| \cdot \|_{L^p(\Omega, \nabla_{x''})} \doteq \| \cdot \|_{L^p(\Omega)} + \| \nabla_{x''}(\cdot) \|_{L^p(\Omega)^{N_2}},$$

$$88 \quad \| \cdot \|_{L^p(\Omega, \nabla_{x'}; W^{1,p}(Y''))} \doteq \| \cdot \|_{L^p(\Omega \times Y'')} + \| \nabla_{x'}(\cdot) \|_{L^p(\Omega \times Y'')^{N_1}} + \| \nabla_{y''}(\cdot) \|_{L^p(\Omega \times Y'')^{N_2}},$$

$$\| \cdot \|_{L^p(\Omega, \nabla_{x''}; W^{1,p}(Y'))} \doteq \| \cdot \|_{L^p(\Omega \times Y')} + \| \nabla_{x''}(\cdot) \|_{L^p(\Omega \times Y')^{N_2}} + \| \nabla_{y'}(\cdot) \|_{L^p(\Omega \times Y')^{N_1}},$$

$$\| \cdot \|_{L^p(\Omega \times Y''; W^{1,p}(Y'))} \doteq \| \cdot \|_{L^p(\Omega \times Y)} + \| \nabla_{y'}(\cdot) \|_{L^p(\Omega \times Y)^{N_1}},$$

$$\| \cdot \|_{L^p(\Omega \times Y'; W^{1,p}(Y''))} \doteq \| \cdot \|_{L^p(\Omega \times Y)} + \| \nabla_{y''}(\cdot) \|_{L^p(\Omega \times Y)^{N_2}}.$$

89 We recall the unfolding with parameters tools already developed in [13, Chap. 7] and  
90 we define two partial unfolding operators.

**DEFINITION 3.2.** For every measurable function  $\phi$  on  $\Omega$ , the unfolding operator  $\mathcal{T}_\varepsilon''$  is defined as follows:

$$\mathcal{T}_\varepsilon''(\phi)(x', x'', y'') = \begin{cases} \phi\left(x', \varepsilon \left[ \frac{x''}{\varepsilon} \right]_{Y''} + \varepsilon y''\right) & \text{for a.e. } (x', x'', y'') \in \widehat{\Omega}_\varepsilon \times Y'', \\ 0 & \text{for a.e. } (x', x'', y'') \in \Lambda_\varepsilon \times Y''. \end{cases}$$

For every measurable function  $\psi$  on  $\Omega \times Y''$ , the unfolding operator  $\mathcal{T}'_\varepsilon$  is defined as follows:

$$\mathcal{T}'_\varepsilon(\psi)(x', x'', y', y'') = \begin{cases} \psi\left(\varepsilon\left[\frac{x'}{\varepsilon}\right]_{Y'} + \varepsilon y', x'', y''\right) & \text{for a.e. } (x', x'', y', y'') \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x', x'', y', y'') \in \Lambda_\varepsilon \times Y. \end{cases}$$

Note that in the partial unfolding operator  $\mathcal{T}''_\varepsilon(\phi)$  the variable  $x'$  plays the role of a parameter, while in  $\mathcal{T}'_\varepsilon(\psi)$  the role of parameters is played by the variables  $(x'', y'')$ .

LEMMA 3.3. *One has*

$$(3.3) \quad \mathcal{T}_\varepsilon = \mathcal{T}'_\varepsilon \circ \mathcal{T}''_\varepsilon, \quad \mathcal{M}_Y = \mathcal{M}_{Y'} \circ \mathcal{M}_{Y''}.$$

Moreover, for every  $\phi \in L^1(\Omega, \nabla_{x'})$  one has

$$(3.4) \quad \nabla_{x'} \mathcal{T}'_\varepsilon(\phi) = \mathcal{T}''_\varepsilon(\nabla_{x'} \phi) \quad \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y''.$$

*Proof.* Let  $\phi$  be measurable on  $\Omega$ . We have that

$$\begin{aligned} \mathcal{T}'_\varepsilon \circ \mathcal{T}''_\varepsilon(\phi)(x, y) &= \mathcal{T}'_\varepsilon\left(\phi\left(x', \varepsilon\left[\frac{x''}{\varepsilon}\right]_{Y''} + \varepsilon y''\right)\right) = \phi\left(\varepsilon\left[\frac{x'}{\varepsilon}\right]_{Y'} + \varepsilon y', \varepsilon\left[\frac{x''}{\varepsilon}\right]_{Y''} + \varepsilon y''\right) \\ &= \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) = \mathcal{T}_\varepsilon(\phi)(x, y) \quad \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y. \end{aligned}$$

For  $(x, y) \in \Lambda_\varepsilon \times Y$  the result is obvious.

Let  $\widehat{\phi}$  be in  $L^1(\Omega \times Y)$ . We have

$$\begin{aligned} \mathcal{M}_{Y'} \circ \mathcal{M}_{Y''}(\widehat{\phi})(x) &= \mathcal{M}_{Y'}\left(\frac{1}{|Y''|} \int_{Y''} \widehat{\phi}(x, y', y'') dy''\right) \\ &= \frac{1}{|Y'| |Y''|} \int_{Y'} \int_{Y''} \widehat{\phi}(x, y', y'') dy'' dy' = \frac{1}{|Y|} \int_Y \widehat{\phi}(x, y) dy \\ &= \mathcal{M}_Y(\widehat{\phi})(x) \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Let now  $\phi$  be in  $L^1(\Omega, \nabla_{x'})$ . We have

$$\begin{aligned} \nabla_{x'} \mathcal{T}'_\varepsilon(\phi)(x, y'') &= \nabla_{x'}\left(\phi\left(x', \varepsilon\left[\frac{x''}{\varepsilon}\right]_{Y''} + \varepsilon y''\right)\right) = \nabla_{x'} \phi\left(x', \varepsilon\left[\frac{x''}{\varepsilon}\right]_{Y''} + \varepsilon y''\right) \\ &= \mathcal{T}''_\varepsilon(\nabla_{x'} \phi)(x, y'') \quad \text{for a.e. } (x, y'') \in \widehat{\Omega}_\varepsilon \times Y''. \quad \square \end{aligned}$$

**4. Asymptotic behavior of anisotropically bounded sequences.** We start with sequences bounded in  $L^p(\Omega, \nabla_{x'})$ ,  $p \in [1, +\infty]$ , whose gradient is bounded with order  $\varepsilon^{-1}$  in  $L^p(\Omega)$  in the first  $N_1$  directions.

LEMMA 4.1. *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $L^p(\Omega, \nabla_{x'})$ ,  $p \in (1, +\infty)$ , satisfying*

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x'} \phi_\varepsilon\|_{L^p(\Omega)} \leq C,$$

where the constant does not depend on  $\varepsilon$ .

Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and a function  $\widehat{\phi}$  in the space  $L^p(\Omega \times Y''; W_{per}^{1,p}(Y'))$  such that

$$\begin{aligned} \phi_\varepsilon &\rightharpoonup \phi \quad \text{weakly in } L^p(\Omega), \\ \mathcal{T}_\varepsilon(\phi_\varepsilon) &\rightharpoonup \widehat{\phi} \quad \text{weakly in } L^p(\Omega \times Y''; W^{1,p}(Y')) \end{aligned}$$

120 where  $\phi = \mathcal{M}_Y(\widehat{\phi})$ .

121 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
122 the corresponding spaces.

123 *Proof.* The proof is similar to [13, Theorem 1.36].  $\square$

124 An analogous result holds for sequences uniformly bounded in  $L^p(\Omega, \nabla_{x'})$ ,  $p \in [1, +\infty]$ .

125 LEMMA 4.2. Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $L^p(\Omega, \nabla_{x'})$ ,  $p \in (1, +\infty)$ , such that

$$126 \quad \|\phi_\varepsilon\|_{L^p(\Omega, \nabla_{x'})} \leq C.$$

127 Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and  $\widetilde{\phi} \in L^p(\Omega \times Y'', \nabla_{x'})$ ,

128  $\widehat{\phi} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$  such that

$$\begin{aligned} & \phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega, \nabla_{x'}), \\ & \mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in } L^p(\Omega \times Y''; W^{1,p}(Y')), \\ 129 \quad & \mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{\phi} + \nabla_{y'}\widehat{\phi} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}, \\ & \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(\phi_\varepsilon) - \mathcal{M}_{Y'} \circ \mathcal{T}_\varepsilon(\phi_\varepsilon)) \rightharpoonup \nabla_{x'}\widetilde{\phi} \cdot y'^c + \widehat{\phi} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1} \end{aligned}$$

130 where  $\phi = \mathcal{M}_{Y''}(\widetilde{\phi})$  and  $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$ .

131 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
132 the corresponding spaces.

133 *Proof.* The proof is similar to [13, Corollary 1.37] and [13, Theorem 1.41].  $\square$

134 Now, we consider the sequences in  $W^{1,p}(\Omega)$ ,  $p \in [1, +\infty]$ , whose gradient is estimated  
135 with different order according to the considered direction.

136 LEMMA 4.3. Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $W^{1,p}(\Omega)$ ,  $p \in (1, +\infty)$ , satisfying

$$137 \quad (4.1) \quad \|\phi_\varepsilon\|_{L^p(\Omega, \nabla_{x'})} + \varepsilon \|\nabla_{x''}\phi_\varepsilon\|_{L^p(\Omega)} \leq C,$$

where the constant does not depend on  $\varepsilon$ .

Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and functions

$$\widetilde{\phi} \in L^p(\Omega, \nabla_{x'}; W_{per}^{1,p}(Y'')) \quad \text{and} \quad \widehat{\phi} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$$

138 such that

$$\begin{aligned} & \phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega, \nabla_{x'}), \\ & \mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y)), \\ 139 \quad (4.2) \quad & \mathcal{T}_\varepsilon(\nabla_{x'}\phi_\varepsilon) \rightharpoonup \nabla_{x'}\widetilde{\phi} + \nabla_{y'}\widehat{\phi} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}, \\ & \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(\phi_\varepsilon) - \mathcal{M}_{Y'} \circ \mathcal{T}_\varepsilon(\phi_\varepsilon)) \rightharpoonup \nabla_{x'}\widetilde{\phi} \cdot y'^c + \widehat{\phi} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1} \end{aligned}$$

140 where  $\phi = \mathcal{M}_{Y''}(\widetilde{\phi})$  and  $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$ .

141 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
142 the corresponding spaces.

143 *Proof.* From (4.1), up to a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , one has the  
144 existence of  $\phi \in L^p(\Omega, \nabla_{x'})$  such that (4.2)<sub>1</sub> holds.

145 Set  $\{\Phi_\varepsilon\}_\varepsilon = \{\mathcal{T}'_\varepsilon(\phi_\varepsilon)\}_\varepsilon$ . This sequence belongs to  $L^p(\widehat{\Omega}_\varepsilon, \nabla_{x'}; W^{1,p}(Y''))$  and from  
 146 estimate (4.1) and equality (3.4), it satisfies

$$147 \quad (4.3) \quad \|\Phi_\varepsilon\|_{L^p(\widehat{\Omega}_\varepsilon, \nabla_{x'}; W^{1,p}(Y''))} \leq C,$$

where the constant does not depend on  $\varepsilon$ .

Up to a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , there exists  $\tilde{\phi} \in L^p(\Omega; W^{1,p}_{per}(Y''))$  and  
 $\tilde{\Phi} \in L^p(\Omega \times Y'')^{N_1}$  (the periodicity of  $\tilde{\phi}$  is proved as in [13, Theorem 1.36]) such that

$$\begin{aligned} \Phi_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''} &\rightharpoonup \tilde{\phi} && \text{weakly in } L^p(\Omega; W^{1,p}(Y'')), \\ \nabla_{x'} \Phi_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''} &\rightharpoonup \tilde{\Phi} && \text{weakly in } L^p(\Omega \times Y'')^{N_1}. \end{aligned}$$

148 Let  $g$  be in  $C_c^\infty(\Omega \times Y'')^{N_1}$ . For  $\varepsilon$  sufficiently small such that  $\text{supp}(g) \subset \widehat{\Omega}_\varepsilon \times Y''$ , we  
 149 have

$$\begin{aligned} 150 \quad &\int_{\Omega \times Y''} \nabla_{x'} \Phi_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''} \cdot g \, dx dy'' = \int_{\widehat{\Omega}_\varepsilon \times Y''} \nabla_{x'} \Phi_\varepsilon \cdot g \, dx dy'' \\ 151 \quad &= - \int_{\widehat{\Omega}_\varepsilon \times Y''} \Phi_\varepsilon \nabla_{x'} g \, dx dy'' = - \int_{\Omega \times Y''} \Phi_\varepsilon \mathbf{1}_{\widehat{\Omega}_\varepsilon \times Y''} \nabla_{x'} g \, dx dy''. \end{aligned}$$

Then, passing to the limit yields

$$\int_{\Omega \times Y''} \tilde{\Phi} \cdot g \, dx dy'' = - \int_{\Omega \times Y''} \tilde{\phi} \cdot \nabla_{x'} g \, dx dy'', \quad \forall g \in C_c^\infty(\Omega \times Y'')^{N_1}.$$

This means that  $\tilde{\Phi} = \nabla_{x'} \tilde{\phi}$  a.e. in  $\Omega \times Y''$ , thus  $\nabla_{x'} \tilde{\phi} \in L^p(\Omega \times Y'')^{N_1}$  and therefore  
 $\tilde{\phi} \in L^p(\Omega, \nabla_{x'}; W^{1,p}_{per}(Y''))$ .

Now, we transform the sequence  $\{\Phi_\varepsilon\}_\varepsilon$  using the unfolding operator  $\mathcal{T}'_\varepsilon$ ,  $Y''$  being a  
 set of parameters.

From the above convergence and estimate (4.3), up to a subsequence of  $\{\varepsilon\}$ , still de-  
 noted  $\{\varepsilon\}$ , [13, Corollary 1.37] and [13, Theorem 1.41] give  $\widehat{\phi} \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'))$   
 such that (using the rule (3.3)<sub>1</sub>)

$$\begin{aligned} \mathcal{T}'_\varepsilon(\phi_\varepsilon) &= \mathcal{T}'_\varepsilon(\Phi_\varepsilon) \rightharpoonup \tilde{\phi} && \text{weakly in } L^p(\Omega; W^{1,p}(Y' \times Y'')), \\ \mathcal{T}'_\varepsilon(\nabla_{x'} \phi_\varepsilon) &= \mathcal{T}'_\varepsilon(\nabla_{x'} \Phi_\varepsilon) \rightharpoonup \nabla_{x'} \tilde{\phi} + \nabla_{y'} \widehat{\phi} && \text{weakly in } L^p(\Omega \times Y' \times Y'')^{N_1}, \\ \frac{1}{\varepsilon} (\mathcal{T}'_\varepsilon(\phi_\varepsilon) - \mathcal{M}_{Y'}(\mathcal{T}'_\varepsilon(\phi_\varepsilon))) &= \frac{1}{\varepsilon} (\mathcal{T}'_\varepsilon(\Phi_\varepsilon) - \mathcal{M}_{Y'}(\mathcal{T}'_\varepsilon(\Phi_\varepsilon))) \rightharpoonup \nabla_{x'} \tilde{\phi} \cdot y'^c + \widehat{\phi} \\ &&& \text{weakly in } L^p(\Omega \times Y' \times Y''). \end{aligned}$$

153 This ends the proof of (4.2).  $\square$

154 As the lemma below shows, an analogous asymptotic behavior is achieved starting  
 155 from sequences uniformly bounded in  $L^p(\Omega, \nabla_{x'})$ ,  $p \in [1, +\infty]$ , with some assumptions  
 156 on the gradient derivatives.

157 **LEMMA 4.4.** *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $L^p(\Omega, \nabla_{x'})$ ,  $p \in (1, +\infty)$ , satisfying*

$$158 \quad (4.4) \quad \|\phi_\varepsilon\|_{L^p(\Omega, \nabla_{x'})} + \varepsilon \|\nabla_{x''}(\nabla_{x'} \phi_\varepsilon)\|_{L^p(\Omega)} \leq C,$$

where the constant does not depend on  $\varepsilon$ .

Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , functions

$$\tilde{\phi} \in L^p(\Omega, \nabla_{x'}; W^{1,p}_{per}(Y'')) \quad \text{and} \quad \widehat{\psi} \in L^p(\Omega; W^{1,p}_{per}(Y))$$



such that  $\mathcal{M}_{Y'}(\widehat{\psi}) = 0$  a.e. in  $\Omega \times Y''$ ,

$$\nabla_{x'} \widetilde{\phi} \in L^p(\Omega; W_{per}^{1,p}(Y''))^{N_1}, \quad \nabla_{y'} \widehat{\psi} \in L^p(\Omega \times Y'; W_{per}^{1,p}(Y''))^{N_1}$$

159 and we have

$$\begin{aligned} & \phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega, \nabla_{x'}), \\ 160 \quad (4.5) \quad & \mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y)), \\ & \mathcal{T}_\varepsilon(\nabla_{x'} \phi_\varepsilon) \rightharpoonup \nabla_{x'} \widetilde{\phi} + \nabla_{y'} \widehat{\psi} \quad \text{weakly in } L^p(\Omega \times Y'; W^{1,p}(Y''))^{N_1} \end{aligned}$$

161 where  $\phi = \mathcal{M}_{Y''}(\widetilde{\phi})$ .

162 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
163 the corresponding spaces.

164 *Proof.* By estimate (4.4)<sub>1</sub> and Lemma 4.2, there exists a subsequence of  $\{\varepsilon\}$ , still  
165 denoted  $\{\varepsilon\}$ , and functions  $\widetilde{\phi} \in L^p(\Omega \times Y'', \nabla_{x'})$ ,  $\widehat{\psi} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$  such  
166 that

$$\begin{aligned} & \phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega, \nabla_{x'}), \\ 167 \quad (4.6) \quad & \mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \quad \text{weakly in } L^p(\Omega \times Y''; W^{1,p}(Y')), \\ & \mathcal{T}_\varepsilon(\nabla_{x'} \phi_\varepsilon) \rightharpoonup \nabla_{x'} \widetilde{\phi} + \nabla_{y'} \widehat{\psi} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}. \end{aligned}$$

Set  $\{\psi_\varepsilon\}_\varepsilon \doteq \{\nabla_{x'} \phi_\varepsilon\}_\varepsilon$ . By estimate (4.4), this sequence satisfies

$$\|\psi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x''} \psi_\varepsilon\|_{L^p(\Omega)} \leq C,$$

where the constant does not depend on  $\varepsilon$ .

Hence, applying Lemma 4.1 to the above sequence (but swapping  $Y'$  and  $Y''$ ), there  
exists a function  $\widehat{\psi} \in L^p(\Omega \times Y'; W_{per}^{1,p}(Y''))^{N_1}$  such that

$$\mathcal{T}_\varepsilon(\nabla_{x'} \phi_\varepsilon) = \mathcal{T}_\varepsilon(\psi_\varepsilon) \rightharpoonup \widehat{\psi} \quad \text{weakly in } L^p(\Omega \times Y'; W^{1,p}(Y''))^{N_1}.$$

This, together with convergence (4.6)<sub>3</sub> implies that the quantity  $\nabla_{x'} \widetilde{\phi} + \nabla_{y'} \widehat{\psi}$  belongs  
to  $L^p(\Omega \times Y'; W_{per}^{1,p}(Y''))^{N_1}$ . Since  $\widetilde{\phi}$  does not depend on  $y'$  and  $\widehat{\psi}$  is periodic with  
respect to  $y'$ , we have that

$$\nabla_{x'} \widetilde{\phi} = \mathcal{M}_{Y'}(\nabla_{x'} \widetilde{\phi}) + \mathcal{M}_{Y'}(\nabla_{y'} \widehat{\psi}) = \mathcal{M}_{Y'}(\widehat{\psi}),$$

168 thus  $\nabla_{x'} \widetilde{\phi} \in L^p(\Omega; W_{per}^{1,p}(Y''))^{N_1}$  and therefore  $\widetilde{\phi} \in L^p(\Omega, \nabla_{x'}; W_{per}^{1,p}(Y''))$ .

169 Moreover, the quantity  $\nabla_{y'} \widehat{\psi}$  belongs to  $L^p(\Omega \times Y'; W_{per}^{1,p}(Y''))^{N_1}$  and thus Lemma  
170 7.2 implies that there exists  $\widehat{\phi} \in L^p(\Omega; W_{per}^{1,p}(Y))$  with  $\nabla_{y'} \widehat{\phi} = \nabla_{y'} \widehat{\psi}$  such that (4.5)<sub>3</sub>  
171 hold. The proof follows by replacing  $\widehat{\phi}$  by the function  $\widehat{\psi} = \widehat{\phi} - \mathcal{M}_{Y'}(\widehat{\phi})$ , which  
172 belongs to the space  $L^p(\Omega; W_{per}^{1,p}(Y))$ .  $\square$

173 **5. Other unfolding results for anisotropically bounded sequences.** In  
174 this section we want to extend the lemma below to the anisotropic case.

175 **LEMMA 5.1.** (see [13, Lemma 11.11]) Let  $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$  be a sequence converging  
176 weakly to  $(u, v)$  in  $W^{1,p}(\Omega) \times W^{1,p}(\Omega)^N$ ,  $p \in (1, +\infty)$ . Moreover, assume that there  
177 exist  $\mathcal{Z} \in L^p(\Omega)^N$  and  $\widehat{v} \in L^p(\Omega; W_{per,0}^{1,p}(Y))^N$  such that

$$\begin{aligned} 178 \quad & \frac{1}{\varepsilon} (\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup \mathcal{Z} \quad \text{weakly in } L^p(\Omega)^N, \\ & \mathcal{T}_\varepsilon(\nabla v_\varepsilon) \rightharpoonup \nabla v + \nabla_y \widehat{v} \quad \text{weakly in } L^p(\Omega \times Y)^{N \times N}. \end{aligned}$$

179 Then,  $u$  belongs to  $W^{2,p}(\Omega)$ . Moreover, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  
 180  $\{\varepsilon\}$ , and  $\mathbf{u} \in L^p(\Omega; W_{per,0}^{1,p}(Y))$  such that

$$181 \quad \frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup \mathcal{Z} + \nabla_y \mathbf{u} + \hat{v} \quad \text{weakly in } L^p(\Omega \times Y)^N.$$

182 As an immediate consequence, one has the following.

183 **COROLLARY 5.2.** *Let  $\mathcal{O}$  be an open set in  $\mathbb{R}^k$ ,  $k \geq 1$ . Let  $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$  be a sequence  
 184 converging weakly to  $(u, v)$  in  $L^p(\mathcal{O}; W^{1,p}(\Omega)) \times L^p(\mathcal{O}; W^{1,p}(\Omega))^N$ ,  $p \in (1, +\infty)$ .  
 185 Moreover, assume that there exist  $\mathcal{Z} \in L^p(\mathcal{O} \times \Omega)^N$  and  $\hat{v} \in L^p(\mathcal{O} \times \Omega; W_{per,0}^{1,p}(Y))^N$   
 186 such that*

$$187 \quad \begin{aligned} \frac{1}{\varepsilon}(\nabla u_\varepsilon + v_\varepsilon) &\rightharpoonup \mathcal{Z} \quad \text{weakly in } L^p(\mathcal{O} \times \Omega)^N, \\ \mathcal{T}_\varepsilon(\nabla v_\varepsilon) &\rightharpoonup \nabla v + \nabla_y \hat{v} \quad \text{weakly in } L^p(\mathcal{O} \times \Omega \times Y)^{N \times N}. \end{aligned}$$

188 Then,  $u$  belongs to  $L^p(\mathcal{O}; W^{2,p}(\Omega))$ . Furthermore, there exist a subsequence of  $\{\varepsilon\}$ ,  
 189 still denoted  $\{\varepsilon\}$ , and  $\mathbf{u} \in L^p(\mathcal{O} \times \Omega; W_{per,0}^{1,p}(Y))$  such that:

$$190 \quad \frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup \mathcal{Z} + \nabla_y \mathbf{u} + \hat{v} \quad \text{weakly in } L^p(\mathcal{O} \times \Omega \times Y)^N.$$

191 Define the spaces

$$192 \quad \begin{aligned} L^p(\Omega \times Y'', D_{x'}^2) &\doteq \{\tilde{\phi} \text{ and } \nabla_{x'} \tilde{\phi} \in L^p(\Omega \times Y'', \nabla_{x'})^{N_1}\}, \\ 193 \quad L^p(\Omega, D_{x'}^2; W_{per}^{1,p}(Y'')) &\doteq \{\tilde{\phi} \text{ and } \nabla_{x'} \tilde{\phi} \in L^p(\Omega, \nabla_{x'}; W_{per}^{1,p}(Y''))^{N_1}\}, \end{aligned}$$

195 endowed with the respective norms:

$$196 \quad \begin{aligned} \|\cdot\|_{L^p(\Omega \times Y'', D_{x'}^2)} &\doteq \|\cdot\|_{L^p(\Omega \times Y'')} + \|\nabla_{x'}(\cdot)\|_{L^p(\Omega \times Y'')}^{N_1} + \|D_{x'}^2(\cdot)\|_{L^p(\Omega \times Y'')}^{N_1 \times N_1}, \\ 197 \quad \|\cdot\|_{L^p(\Omega, D_{x'}^2; W_{per}^{1,p}(Y''))} &\doteq \|\cdot\|_{L^p(\Omega \times Y'', D_{x'}^2)} + \|\nabla_{y''}(\cdot)\|_{L^p(\Omega \times Y'')}^{N_2}. \end{aligned}$$

199 We are ready to extend Lemma 5.1 to the class of anisotropically bounded sequences.

200 **LEMMA 5.3.** *Let  $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$  be a sequence in the space  $L^p(\Omega, \nabla_{x'}) \times L^p(\Omega, \nabla_{x'})^{N_1}$ ,  
 201  $p \in (1, +\infty)$ , satisfying*

$$202 \quad (5.1) \quad \|u_\varepsilon\|_{L^p(\Omega, \nabla_{x'})} \leq C, \quad \|v_\varepsilon\|_{L^p(\Omega, \nabla_{x'})} \leq C,$$

203 where the constant does not depend on  $\varepsilon$ .

204 Moreover, assume that there exist  $\mathcal{Z} \in L^p(\Omega)^{N_1}$  such that

$$205 \quad (5.2) \quad \frac{1}{\varepsilon}(\nabla_{x'} u_\varepsilon + v_\varepsilon) \rightharpoonup \mathcal{Z} \quad \text{weakly in } L^p(\Omega)^{N_1}.$$

206 Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and  $\tilde{\mathcal{Z}} \in L^p(\Omega \times Y'')^{N_1}$  with  
 207  $\mathcal{M}_{Y''}(\tilde{\mathcal{Z}}) = \mathcal{Z}$ ,  $\mathbf{u} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$ ,  $\tilde{u} \in L^p(\Omega \times Y'', D_{x'}^2)$  and a function  
 208  $\hat{v} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))^{N_1}$  such that

$$209 \quad (5.3) \quad \begin{aligned} \mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) &\rightharpoonup -D_{x'}^2 \tilde{u} + \nabla_{y'} \hat{v} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1 \times N_1}, \\ \frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon + v_\varepsilon) &\rightharpoonup \tilde{\mathcal{Z}} + \nabla_{y'} \mathbf{u} + \hat{v} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}. \end{aligned}$$

210 *Proof.* We first apply the unfolding operator  $\mathcal{T}_\varepsilon$  to both sequences  $\{u_\varepsilon\}$  and  $\{v_\varepsilon\}$ .  
 211 By Lemma 4.2 and estimates (5.1), there exist a subsequence of  $\{\varepsilon\}$ , still denoted  
 212  $\{\varepsilon\}$ ,  $\tilde{u} \in L^p(\Omega \times Y''; \nabla_{x'})$ ,  $\tilde{v} \in L^p(\Omega \times Y''; \nabla_{x'})^{N_1}$ ,  $\hat{u} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$ ,  
 213  $\hat{v} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))^{N_1}$  such that

$$\begin{aligned}
 & \mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup \tilde{u} \quad \text{weakly in } L^p(\Omega \times Y''; W^{1,p}(Y')), \\
 & \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon) \rightharpoonup \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}, \\
 & \mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup \tilde{v} \quad \text{weakly in } L^p(\Omega \times Y''; W^{1,p}(Y'))^{N_1}, \\
 & \mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) \rightharpoonup \nabla_{x'} \tilde{v} + \nabla_{y'} \hat{v} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1 \times N_1}.
 \end{aligned}
 \tag{5.4}$$

215 By convergence (5.2), there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and functions  
 216  $\hat{\mathcal{Z}} \in L^p(\Omega \times Y)^{N_1}$  with  $\mathcal{M}_Y(\hat{\mathcal{Z}}) = \mathcal{Z}$  such that

$$\frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon + v_\varepsilon) \rightharpoonup \hat{\mathcal{Z}} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}.
 \tag{5.5}$$

From convergences (5.4)<sub>2,3</sub> and (5.5) we get

$$\nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} + \tilde{v} = 0 \quad \text{a.e. in } \Omega \times Y.$$

Applying  $\mathcal{M}_{Y'}$  to the above equality and since  $\hat{u} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$ , while  
 $\tilde{u} \in L^p(\Omega \times Y''; \nabla_{x'})$ ,  $\tilde{v} \in L^p(\Omega \times Y''; \nabla_{x'})^{N_1}$ , we get that  $\nabla_{x'} \tilde{u} + \tilde{v} = 0$  a.e. in  $\Omega \times Y''$ .  
 Hence,  $\nabla_{y'} \hat{u} = 0$  and thus  $\hat{u} = 0$  because it belongs to  $L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$ . As a  
 consequence, one has

$$\tilde{u} \in L^p(\Omega \times Y'', D_{x'}^2).$$

218 Set  $U_\varepsilon = \mathcal{T}_\varepsilon''(u_\varepsilon)$ ,  $V_\varepsilon = \mathcal{T}_\varepsilon''(v_\varepsilon)$ . Again by convergence (5.2), there exist a subsequence  
 219 of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and  $\tilde{\mathcal{Z}} \in L^p(\Omega \times Y'')^{N_1}$  such that

$$\frac{1}{\varepsilon} \nabla_{x'} U_\varepsilon + V_\varepsilon \rightharpoonup \tilde{\mathcal{Z}} \quad \text{weakly in } L^p(\Omega \times Y'')^{N_1}.
 \tag{5.6}$$

221 Then, due to convergence (5.5) we have  $\tilde{\mathcal{Z}} = \mathcal{M}_{Y'}(\hat{\mathcal{Z}})$ .

222 Now, let  $\omega'$  and  $\omega''$  be two open sets such that

$$\omega' \subset \mathbb{R}^{N_1}, \quad \omega'' \subset \mathbb{R}^{N_2} \quad \text{and} \quad \overline{\omega' \times \omega''} \subset \Omega.$$

First, observe that

$$U_\varepsilon \in L^p(\omega'' \times Y''; W^{1,p}(\omega')), \quad V_\varepsilon \in L^p(\omega'' \times Y''; W^{1,p}(\omega'))^{N_1}.$$

By the above convergence and (5.4)<sub>4</sub>, one has

$$\begin{aligned}
 & \frac{1}{\varepsilon} \nabla_{x'} U_\varepsilon + V_\varepsilon \rightharpoonup \tilde{\mathcal{Z}} \quad \text{weakly in } L^p(\omega' \times \omega'' \times Y'')^{N_1}, \\
 & \mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) = \mathcal{T}_\varepsilon'(\nabla_{x'} V_\varepsilon) \rightharpoonup \nabla_{x'} \tilde{v} + \nabla_{y'} \hat{v} \quad \text{weakly in } L^p(\omega' \times \omega'' \times Y' \times Y'')^{N_1 \times N_1}.
 \end{aligned}$$

224 Lemma 5.2 claims that up to a subsequence, there exists  $\mathbf{u}_{\omega' \times \omega''}$ , which belongs to  
 225  $L^p(\omega' \times \omega'' \times Y''; W_{per,0}^{1,p}(Y'))$ , such that the following convergence holds:

$$\frac{1}{\varepsilon} \mathcal{T}_\varepsilon'(\nabla_{x'} U_\varepsilon + V_\varepsilon) \rightharpoonup \tilde{\mathcal{Z}} + \nabla_{y'} \mathbf{u}_{\omega' \times \omega''} + \hat{v} \quad \text{weakly in } L^p(\omega' \times \omega'' \times Y' \times Y'')^{N_1}.$$

Taking into account convergence (5.5) we get

$$\widehat{\mathbf{Z}} = \widetilde{\mathbf{Z}} + \nabla_{y'} \ddot{u}_{\omega' \times \omega''} + \widehat{v} \quad \text{in } \omega' \times \omega'' \times Y.$$

227 Since one can cover  $\Omega$  by a countable family of open subsets  $\omega' \times \omega''$  satisfying (5.6),  
 228 there exists  $\mathbf{u}$  in  $L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$  such that  $\widehat{\mathbf{Z}} - \widetilde{\mathbf{Z}} - \widehat{v} = \nabla_{y'} \mathbf{u}$ . This completes  
 229 the proof of (5.3).  $\square$

230 With some more assumptions, we can improve the regularity of the limit functions.

231 **LEMMA 5.4.** *Let  $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$  be a sequence in  $L^p(\Omega, \nabla_{x'}) \times L^p(\Omega, \nabla_{x'})^{N_1}$ , with*  
 232  *$p \in (1, +\infty)$ , satisfying the assumptions in Lemma 5.3. Moreover, assume that*

$$233 \quad (5.7) \quad \|\nabla_{x''}(\nabla_{x'} u_\varepsilon + v_\varepsilon)\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x''}(\nabla_{x'} v_\varepsilon)\|_{L^p(\Omega)} \leq C,$$

234 *where the constant does not depend on  $\varepsilon$ .*

235 *Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ ,  $\widetilde{\mathbf{Z}} \in L^p(\Omega; W_{per}^{1,p}(Y''))^{N_1}$ ,*  
 236  *$\widehat{\mathbf{v}} \in L^p(\Omega; W_{per,0}^{1,p}(Y))^{N_1}$ ,  $\mathfrak{u} \in L^p(\Omega; W_{per,0}^{1,p}(Y))$  and  $\widetilde{u} \in L^p(\Omega, D_{x'}^2; W_{per}^{1,p}(Y''))$  such*  
 237 *that*

$$238 \quad \begin{aligned} \mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) &\rightharpoonup -D_{x'}^2 \widetilde{u} + \nabla_{y'} \widehat{\mathbf{v}} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1 \times N_1}, \\ \frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon + v_\varepsilon) &\rightharpoonup \widetilde{\mathbf{Z}} + \nabla_{y'} \mathfrak{u} + \widehat{\mathbf{v}} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}. \end{aligned}$$

239 *Proof.* From Lemma 5.3, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and  
 240  $\widetilde{\mathbf{Z}} \in L^p(\Omega \times Y'')^{N_1}$ ,  $\mathbf{u} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$ ,  $\widetilde{u} \in L^p(\Omega \times Y'', D_{x'}^2)$  and a function  
 241  $\widehat{v} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))^{N_1}$  such that

$$242 \quad \begin{aligned} \mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) &\rightharpoonup -D_{x'}^2 \widetilde{u} + \nabla_{y'} \widehat{\mathbf{v}} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1 \times N_1}, \\ \frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon + v_\varepsilon) &\rightharpoonup \widetilde{\mathbf{Z}} + \nabla_{y'} \mathbf{u} + \widehat{\mathbf{v}} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}. \end{aligned}$$

By hypothesis (5.7), Lemma 4.1 (swapping  $Y'$  and  $Y''$ ) and the proof of Lemma 4.4 one has

$$\begin{aligned} \mathcal{T}_\varepsilon(\nabla_{x'} v_\varepsilon) &\rightharpoonup -D_{x'}^2 \widetilde{u} + \nabla_{y'} \widehat{\mathbf{v}} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1 \times N_1}, \\ \frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon + v_\varepsilon) &\rightharpoonup \widetilde{\mathbf{Z}} + \nabla_{y'} \mathbf{u} + \widehat{\mathbf{v}} \in L^p(\Omega \times Y'; W_{per}^{1,p}(Y''))^{N_1} \end{aligned}$$

with  $\widetilde{\mathbf{Z}} \in L^p(\Omega \times Y'')^{N_1}$ ,  $\widetilde{u} \in L^p(\Omega, D_{x'}^2; W_{per}^{1,p}(Y''))$  and  $\widehat{\mathbf{v}} \in L^p(\Omega; W_{per}^{1,p}(Y))^{N_1}$  satisfying  $\mathcal{M}_{Y'}(\widehat{\mathbf{v}}) = 0$  a.e. in  $\Omega \times Y''$ .

Since,  $\widehat{\mathbf{v}}$  satisfies  $\mathcal{M}_{Y'}(\widehat{\mathbf{v}}) = 0$  a.e. in  $\Omega \times Y''$  and  $\mathcal{M}_{Y'}(\nabla_{y'} \mathbf{u}) = 0$  a.e. in  $\Omega \times Y''$  by periodicity of  $\mathbf{u}$ , we obtain

$$\widetilde{\mathbf{Z}} = \mathcal{M}_{Y'}(\widetilde{\mathbf{Z}}) \in L^p(\Omega; W_{per}^{1,p}(Y''))^{N_1}.$$

243 Hence  $\nabla_{y'} \mathbf{u}$  lies in  $L^p(\Omega \times Y'; W_{per}^{1,p}(Y''))^{N_1}$ . Lemma 7.2 in Appendix gives a function  
 244  $\mathfrak{u} \in L^p(\Omega; W_{per,0}^{1,p}(Y))$  such that  $\nabla_{y'} \mathfrak{u} = \nabla_{y'} \mathbf{u}$ . The proof is complete.  $\square$

245 **6. Application: homogenization of a homogeneous Dirichlet problem.**  
 246 We want to give a direct application of the periodic unfolding for anisotropically  
 247 bounded sequences.

248 Let  $\mathcal{O}$  be an open subset of  $\mathbb{R}^N$  and let  $\alpha, \beta \in \mathbb{R}$  with  $0 < \alpha < \beta$ . Denote  $M(\alpha, \beta, \mathcal{O})$   
 249 the set of  $N \times N$  matrices  $A = (a_{ij})_{1 \leq i, j \leq N}$  with coefficients in  $L^\infty(\mathcal{O})$  such that for  
 250 every  $\lambda \in \mathbb{R}^N$  and for a.e.  $x \in \mathcal{O}$ , the following inequalities hold:

251 (i)  $(A(x)\lambda, \lambda) \geq \alpha|\lambda|^2$ ;

252 (ii)  $|A(x)\lambda|^2 \leq \beta(A(x)\lambda, \lambda)$ .

253 Let  $A$  be in  $\mathcal{M}(\alpha, \beta, Y)$  and let  $\{A_\varepsilon\}_\varepsilon$  be the sequence of matrices belonging to  
254  $\mathcal{M}(\alpha, \beta, \Omega)$  defined by

$$255 \quad A_\varepsilon \doteq A\left(\left\{\frac{x}{\varepsilon}\right\}_Y\right) \quad \text{a.e. } x \in \Omega.$$

256 Let  $f$  be a function in  $L^2(\Omega)$ .

257 Consider the homogeneous Dirichlet problem in variational formulation:

$$258 \quad (6.1) \quad \begin{cases} \text{Find } u_\varepsilon \in H_0^1(\Omega) \text{ such that:} \\ \int_\Omega A_\varepsilon \begin{pmatrix} \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'} \phi \\ \varepsilon \nabla_{x''} \phi \end{pmatrix} dx = \int_\Omega f \phi dx, \quad \forall \phi \in H_0^1(\Omega). \end{cases}$$

By the Poincaré inequality and the fact that  $u_\varepsilon \in H_0^1(\Omega)$ , we have that

$$\|u_\varepsilon\|_{L^2(\Omega)} \leq C \|\nabla_{x'} u_\varepsilon\|_{L^2(\Omega)}.$$

Thus, problem (6.1) admits a unique solution by the Lax–Milgram theorem and the following inequality holds:

$$\alpha(\|\nabla_{x'} u_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla_{x''} u_\varepsilon\|_{L^2(\Omega)}^2) \leq \|f\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \|\nabla_{x'} u_\varepsilon\|_{L^2(\Omega)}.$$

259 Hence

$$260 \quad (6.2) \quad \|u_\varepsilon\|_{L^2(\Omega)} + \|\nabla_{x'} u_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla_{x''} u_\varepsilon\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)},$$

where the constant does not depend on  $\varepsilon$ .

Set

$$\mathbf{H}_{0,per}^1(\Omega \times Y'') = \left\{ \phi \in H^1(\Omega \times Y'') \mid \phi(x, y'') = 0 \text{ for a.e. } (x, y'') \in \partial\Omega \times Y'' \right. \\ \left. \text{and } \phi(x, \cdot) \text{ is } Y'' \text{ periodic for a.e. } x \in \Omega \right\}.$$

261 Denote  $L_0^2(\Omega, \nabla_{x'})$  (resp.  $L_0^2(\Omega, \nabla_{x'}; H_{per}^1(Y''))$ ) the closure of  $H_0^1(\Omega)$  (resp. of  
262  $\mathbf{H}_{0,per}^1(\Omega \times Y'')$ ) in  $L^2(\Omega)$  (resp.  $L^2(\Omega \times Y'')$ ) for the norm of  $L^2(\Omega, \nabla_{x'})$  (resp.  
263  $L^2(\Omega, \nabla_{x'}; H_{per}^1(Y''))$ ), see (3.1)-(3.2).

264 Below, we give the periodic homogenization via unfolding.

265 **THEOREM 6.1.** *Let  $u_\varepsilon$  be the solution of problem (6.1).*

266 *There exist  $\tilde{u} \in L_0^2(\Omega, \nabla_{x'}; H_{per}^1(Y''))$  and  $\hat{u} \in L^2(\Omega \times Y''; H_{per,0}^1(Y'))$  such that*

$$267 \quad (6.3) \quad \begin{aligned} u_\varepsilon &\rightharpoonup \mathcal{M}_Y(\tilde{u}) && \text{weakly in } L_0^2(\Omega, \nabla_{x'}), \\ \mathcal{T}_\varepsilon(u_\varepsilon) &\rightharpoonup \tilde{u} && \text{weakly in } L^2(\Omega; H^1(Y)), \\ \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon) &\rightarrow \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} && \text{strongly in } L^2(\Omega \times Y)^{N_1}, \\ \varepsilon \mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon) &\rightarrow \nabla_{y''} \tilde{u} && \text{strongly in } L^2(\Omega \times Y)^{N_2}. \end{aligned}$$

268 *The couple  $(\tilde{u}, \hat{u})$  is the unique solution of problem*

$$269 \quad (6.4) \quad \begin{cases} \int_{\Omega \times Y} A(y) \begin{pmatrix} \nabla_{x'} \tilde{u}(x, y'') + \nabla_{y'} \hat{u}(x, y) \\ \nabla_{y''} \tilde{u}(x, y'') \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'} \tilde{\phi}(x, y'') + \nabla_{y'} \hat{\phi}(x, y) \\ \nabla_{y''} \tilde{\phi}(x, y'') \end{pmatrix} dx dy \\ = |Y'| \int_{\Omega \times Y''} f(x) \tilde{\phi}(x, y'') dx dy'', \\ \forall \tilde{\phi} \in L_0^2(\Omega, \nabla_{x'}; H_{per,0}^1(Y'')) \text{ and } \forall \hat{\phi} \in L^2(\Omega \times Y''; H_{per,0}^1(Y')). \end{cases}$$

270 *Proof. Step 1.* We show (6.4) and the weak convergences (6.3).

271 The solution  $u_\varepsilon$  of (6.1) satisfies (6.2). Hence, up to a subsequence of  $\{\varepsilon\}$ , still denoted  
 272  $\{\varepsilon\}$ , Lemma 4.3 gives  $\tilde{u} \in L_0^2(\Omega, \nabla_{x'}; H_{per}^1(Y''))$  and  $\hat{u} \in L^2(\Omega \times Y''; H_{per,0}^1(Y'))$  such  
 273 that

$$(6.5) \quad \begin{aligned} u_\varepsilon &\rightharpoonup \mathcal{M}_Y(\tilde{u}) && \text{weakly in } L_0^2(\Omega, \nabla_{x'}), \\ \mathcal{T}_\varepsilon(u_\varepsilon) &\rightharpoonup \tilde{u} && \text{weakly in } L^2(\Omega; H^1(Y)), \\ \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon) &\rightharpoonup \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} && \text{weakly in } L^2(\Omega \times Y)^{N_1}. \end{aligned}$$

Moreover, from convergence (6.5)<sub>2</sub> we also get that

$$\varepsilon \mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon) = \nabla_{y''} \mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup \nabla_{y''} \tilde{u} \quad \text{weakly in } L^2(\Omega \times Y'')^{N_2}.$$

275 Now, we choose the test functions

- 276 •  $\tilde{\phi}$  in  $\mathcal{C}^1(\bar{\Omega} \times Y'') \cap L_0^2(\Omega, \nabla_{x'}; H_{per}^1(Y''))$ ,
- 277 •  $\Phi$  in  $\mathcal{C}_c^1(\Omega \times Y'')$ ,
- 278 •  $\hat{\phi}$  in  $H_{per,0}^1(Y')$ .

Set

$$\phi_\varepsilon(x) \doteq \tilde{\phi}\left(x, \frac{x''}{\varepsilon}\right) + \varepsilon \Phi\left(x, \frac{x''}{\varepsilon}\right) \hat{\phi}\left(\frac{x'}{\varepsilon}\right), \quad \text{a.e. } x \in \Omega.$$

279 Applying the unfolding operator to the sequence  $\{\phi_\varepsilon\}_\varepsilon$ , we get that

$$\begin{aligned} \mathcal{T}_\varepsilon(\phi_\varepsilon) &\rightarrow \tilde{\phi} && \text{strongly in } L^2(\Omega; H^1(Y)), \\ \mathcal{T}_\varepsilon(\nabla_{x'} \phi_\varepsilon) &\rightarrow \nabla_{x'} \tilde{\phi} + \Phi \nabla_{y'} \hat{\phi} && \text{strongly in } L^2(\Omega \times Y)^{N_1}, \\ \varepsilon \mathcal{T}_\varepsilon(\nabla_{x''} \phi_\varepsilon) &\rightarrow \nabla_{y''} \tilde{\phi} && \text{strongly in } L^2(\Omega \times Y)^{N_2}. \end{aligned}$$

281 Taking  $\phi_\varepsilon$  as test function in (6.1), then transforming by unfolding and passing to the  
 282 limit give (6.4) with  $(\tilde{\phi}, \Phi \hat{\phi})$ . By density argumentation, we extend such results for  
 283 all  $\tilde{\phi} \in L_0^2(\Omega, \nabla_{x'}; H_{per}^1(Y''))$  and all  $\hat{\phi} \in L^2(\Omega \times Y''; H_{per,0}^1(Y'))$ . Since the solution  
 284 is unique the whole sequences converge to their limit.

285 *Step 2.* We prove that convergences (6.3)<sub>3,4</sub> are strong.

286 First, setting  $\phi = u_\varepsilon$  in (6.1), then transforming by unfolding and using the weak  
 287 lower semicontinuity yield

$$\begin{aligned} &\int_{\Omega \times Y} A \left( \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} \right) \cdot \left( \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} \right) dx dy \\ &\leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega \times Y} A \left( \frac{\mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon)}{\varepsilon \mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon)} \right) \cdot \left( \frac{\mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon)}{\varepsilon \mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon)} \right) dx dy \\ &= \liminf_{\varepsilon \rightarrow 0} \left( \int_{\Omega} A_\varepsilon \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) \cdot \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) dx dy - \int_{\Lambda_\varepsilon} A_\varepsilon \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) \cdot \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) dx dy \right) \\ 288 &\leq \limsup_{\varepsilon \rightarrow 0} \left( \int_{\Omega} A_\varepsilon \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) \cdot \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) dx dy - \int_{\Lambda_\varepsilon} A_\varepsilon \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) \cdot \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) dx dy \right) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) \cdot \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x''} u_\varepsilon} \right) dx dy = \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} f u_\varepsilon dx, \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(f) \mathcal{T}_\varepsilon(u_\varepsilon) dx = \int_{\Omega \times Y} f \tilde{u} dx dy \\ &= \int_{\Omega \times Y} A \left( \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} \right) \cdot \left( \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} \right) dx dy, \end{aligned}$$

289 from which it follows that

$$290 \quad \int_{\Lambda_\varepsilon} A_\varepsilon \begin{pmatrix} \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{pmatrix} dx dy = 0$$

291 and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times Y} A \begin{pmatrix} \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon) \\ \varepsilon \mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon) \\ \varepsilon \mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon) \end{pmatrix} dx dy \\ 292 \quad & = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_\varepsilon \begin{pmatrix} \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'} u_\varepsilon \\ \varepsilon \nabla_{x''} u_\varepsilon \end{pmatrix} dx dy \\ & = \int_{\Omega \times Y} A \begin{pmatrix} \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} \\ \nabla_{y''} \tilde{u} \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} \\ \nabla_{y''} \tilde{u} \end{pmatrix} dx dy. \end{aligned}$$

293 Since the map  $\Psi \in L^2(\Omega \times Y)^N \mapsto \sqrt{\int_{\Omega \times Y} A \Psi \cdot \Psi dx dy}$  is a norm equivalent to the  
294 usual norm of  $L^2(\Omega \times Y)^N$ , we get

$$295 \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times Y} \left| \begin{pmatrix} \mathcal{T}_\varepsilon(\nabla_{x'} u_\varepsilon) \\ \varepsilon \mathcal{T}_\varepsilon(\nabla_{x''} u_\varepsilon) \end{pmatrix} \right|^2 dx dy = \int_{\Omega \times Y} \left| \begin{pmatrix} \nabla_{x'} \tilde{u} + \nabla_{y'} \hat{u} \\ \nabla_{y''} \tilde{u} \end{pmatrix} \right|^2 dx dy.$$

296 This, together with the fact that (6.5)<sub>2,3</sub> already converge weakly, ensures the strong  
297 convergences (6.3)<sub>3,4</sub>. The proof is therefore complete.  $\square$

Set

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

298 where

- 299  $\bullet$   $A_1$  is a  $N_1 \times N_1$  matrix with entries in  $L^\infty(Y)$ ,
- 300  $\bullet$   $A_2$  is a  $N_1 \times N_2$  matrix with entries in  $L^\infty(Y)$ ,
- 301  $\bullet$   $A_3$  is a  $N_2 \times N_1$  matrix with entries in  $L^\infty(Y)$ ,
- 302  $\bullet$   $A_4$  is a  $N_2 \times N_2$  matrix with entries in  $L^\infty(Y)$ .

303 We define the correctors  $\hat{\chi}_k$ ,  $k \in \{1, \dots, N\}$ , as the unique solutions in the space  
304  $L^\infty(Y'', H_{per,0}^1(Y'))$  of the cell problems:

$$305 \quad (6.6) \quad \begin{cases} \int_{Y'} A_1(y', \cdot) \nabla_{y'} \hat{\chi}_k(y', \cdot) \cdot \nabla_{y'} \hat{w}(y') dy' = - \int_{Y'} A(y', \cdot) \mathbf{e}_k \cdot \begin{pmatrix} \nabla_{y'} \hat{w}(y') \\ 0 \end{pmatrix} dy', \\ \forall \hat{w} \in H_{per,0}^1(Y'). \end{cases}$$

306 By the Lax–Milgram theorem applied in Hilbert space  $L^2(Y'', H_{per,0}^1(Y'))$ , we obtain  
307 the existence and uniqueness of the solution of (6.6) for every  $k \in \{1, \dots, N\}$ .

308 Since  $A$  belongs to  $\mathcal{M}(\alpha, \beta, Y)$  we get for every  $k \in \{1, \dots, N\}$ :

$$309 \quad \|\nabla_{y'} \hat{\chi}_k(\cdot, y'')\|_{H^1(Y')} \leq \frac{\beta}{\alpha} \quad \text{for a.e. } y'' \in Y''.$$

As a consequence  $\hat{\chi}_k \in L^\infty(Y'', H_{per,0}^1(Y'))$ <sup>1</sup> for every  $k \in \{1, \dots, N\}$  and

$$\|\hat{\chi}_k\|_{L^\infty(Y'', H^1(Y'))} \leq C.$$

310

<sup>1</sup>One can prove that  $\hat{\chi}_k$  also belongs to  $L^\infty(Y)$ .

311 PROPOSITION 6.2. *The function  $\tilde{u}_0$  is the unique solution of the following homog-*  
 312 *enized problem:*

$$313 \quad (6.7) \quad \begin{cases} \int_{\Omega \times Y''} A^{hom} \begin{pmatrix} \nabla_{x'} \tilde{u}_0 \\ \nabla_{y''} \tilde{u}_0 \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'} \tilde{\phi} \\ \nabla_{y''} \tilde{\phi} \end{pmatrix} dx dy'' = \int_{\Omega \times Y''} f \tilde{\phi} dx dy'', \\ \forall \tilde{\phi} \in L^2_0(\Omega, \nabla_{x'}; H^1_{per}(Y'')). \end{cases}$$

314 *The homogenizing operator  $A^{hom} \in L^\infty(Y'')^{N \times N}$  is defined by*

$$315 \quad (6.8) \quad A^{hom} \doteq \frac{1}{|Y'|} \int_{Y'} \left( A + \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \nabla_{y'} \chi \right) (y', \cdot) dy',$$

316 *where  $\hat{\chi} = (\hat{\chi}_1 \ \hat{\chi}_2 \ \dots \ \hat{\chi}_{N_1} \ \hat{\chi}_{N_1+1} \ \dots \ \hat{\chi}_N)$  and thus  $\nabla_{y'} \hat{\chi}$  is the  $N_1 \times N$*   
 317 *matrix*

$$318 \quad \nabla_{y'} \hat{\chi} \doteq (\nabla_{y'} \hat{\chi}_1 \ \nabla_{y'} \hat{\chi}_2 \ \dots \ \nabla_{y'} \hat{\chi}_{N_1} \ \nabla_{y'} \hat{\chi}_{N_1+1} \ \dots \ \nabla_{y'} \hat{\chi}_N).$$

319 *Proof.* Equation (6.4) with  $\tilde{\phi} = 0$  leads to:

$$320 \quad (6.9) \quad \begin{cases} \int_{\Omega \times Y' \times Y''} A(y', y'') \begin{pmatrix} \nabla_{y'} \hat{u}(x, y', y'') \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla_{y'} \hat{\phi}(x, y', y'') \\ 0 \end{pmatrix} dx dy' dy'' \\ = - \int_{\Omega \times Y' \times Y''} A(y', y'') \begin{pmatrix} \nabla_{x'} \tilde{u}(x, y'') \\ \nabla_{y''} \tilde{u}(x, y'') \end{pmatrix} \cdot \begin{pmatrix} \nabla_{y'} \hat{\phi}(x, y', y'') \\ 0 \end{pmatrix} dx dy' dy'', \\ \forall \hat{\phi} \in L^2(\Omega \times Y''; H^1_{per,0}(Y'')), \end{cases}$$

from which the form of the cell problems (6.6) follows.

By (6.9), we can write  $\hat{u}$  as

$$\hat{u}(x, y', y'') = \sum_{k=1}^{N_1} \hat{\chi}_k(y', y'') \partial_k \tilde{u}(x, y'') + \sum_{k=N_1+1}^N \hat{\chi}_k(y', y'') \partial_{y_k} \tilde{u}(x, y'')$$

for a.e.  $(x, y', y'') \in \Omega \times Y' \times Y''$ .

321 Replacing  $\hat{u}$  by the above equality in (6.4) and gathering all the  $y'$  dependent terms,  
 322 we get the form (6.8) of the homogenizing operator  $A^{hom}$ .

323 Since  $A \in L^\infty(Y)^{N \times N}$  and the  $\hat{\chi}_k$ 's are in  $L^\infty(Y''; H^1(Y'))$ , it is clear that  $A^{hom}$   
 324 belongs to  $L^\infty(Y'')^{N \times N}$ .

325 We prove now that  $A^{hom}$  is coercive. Let  $\xi \doteq (\xi_1, \xi_2)$  be a vector with fixed entries  
 326 in  $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ . By construction of the homogenizing operator, we have

$$327 \quad \begin{aligned} A^{hom} \xi \cdot \xi &= A^{hom} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \\ 328 \quad &= \frac{1}{|Y'|} \int_{Y'} \left( A + \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \nabla_{y'} \chi(y', \cdot) \right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} dy' \\ 329 \quad &= \frac{1}{|Y'|} \int_{Y'} A \begin{pmatrix} \xi_1 + \nabla_{y'} \hat{\chi} \xi \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} dy' \\ 330 \quad &= \frac{1}{|Y'|} \int_{Y'} A \begin{pmatrix} \xi_1 + \nabla_{y'} \hat{\chi} \xi \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 + \nabla_{y'} \hat{\chi} \xi \\ \xi_2 \end{pmatrix} dy' \\ 331 \quad &- \frac{1}{|Y'|} \int_{Y'} A \begin{pmatrix} \xi_1 + \nabla_{y'} \hat{\chi} \xi \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \nabla_{y'} \hat{\chi} \xi \\ 0 \end{pmatrix} dy', \\ 332 \end{aligned}$$



333 where  $\widehat{\chi}_{\xi}(y', y'') \doteq \sum_{k=1}^N \widehat{\chi}_k(y', y'') \xi_k$ . Observe that by the cell problems (6.6), the  
 334 second term in the last equality is equal to zero.  
 335 Now, the coercivity of the matrix  $A$  and the fact that  $\widehat{\chi}_{\xi} \in L^{\infty}(Y'', H_{per,0}^1(Y'))$ , imply  
 336 that

$$\begin{aligned}
 337 \quad A^{hom}(y'') \xi \cdot \xi &= \frac{1}{|Y'|} \int_{Y'} A \begin{pmatrix} \xi_1 + \nabla_{y'} \widehat{\chi}_{\xi} \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 + \nabla_{y'} \widehat{\chi}_{\xi} \\ \xi_2 \end{pmatrix} dy' \\
 338 \quad &\geq \alpha (\|\xi_1 + \nabla_{y'} \widehat{\chi}_{\xi}(y'', \cdot)\|_{L^2(Y')}^2 + |\xi_2|^2) \\
 339 \quad &= \alpha (|\xi_1|^2 + |\xi_2|^2 + \|\nabla_{y'} \widehat{\chi}_{\xi}(y'', \cdot)\|_{L^2(Y')}^2) \geq \alpha |\xi|^2, \quad \text{for a.e. } y'' \in Y'', \\
 340
 \end{aligned}$$

341 which proves that  $A^{hom}$  is coercive.

342 Replacing the form of  $A^{hom}$  on the original problem (6.4), we get (6.7). By the  
 343 boundedness and coercivity of  $A^{hom}$  and by the fact that the function  $\tilde{u}$  belongs to  
 344  $L_0^2(\Omega, \nabla_{x'}; H_{per}^1(Y''))$ , the above problem admits a unique solution  $\tilde{u}_0$  by the Poincaré  
 345 inequality and the Lax–Milgram theorem.  $\square$

## 346 7. Appendix.

LEMMA 7.1. *Let  $u$  be in  $L^p(Y''; W^{1,p}(Y'))$  ( $p \in (1, +\infty)$ ) such that*

$$\nabla_{y'} u \in L^p(Y'; W^{1,p}(Y''))^{N_1}.$$

347 *Then  $\mathbf{u} = u - \mathcal{M}_{Y'}(u)$  belongs to  $W^{1,p}(Y)$ <sup>2</sup>. It satisfies*

$$348 \quad (7.1) \quad \nabla_{y'} \mathbf{u} = \nabla_{y'} u \quad \text{a.e. in } Y$$

349 *and*

$$350 \quad (7.2) \quad \|\mathbf{u}\|_{W^{1,p}(Y)} \leq C (\|\nabla_{y'} u\|_{L^p(Y' \times Y'')} + \|\nabla_{y''}(\nabla_{y'} u)\|_{L^p(Y' \times Y'')}).$$

351

352 *Proof. Step 1.* First, assume  $u \in C^2(\bar{Y})$ .

353 Set  $\mathbf{u} = u - \mathcal{M}_{Y'}(u)$ . It is clear that (7.1) is satisfied. We prove now the estimate  
 354 (7.2) of  $\mathbf{u}$ .

355 By definition of  $\mathbf{u}$ , (7.1) and the Poincaré–Wirtinger Inequality we have

$$\begin{aligned}
 356 \quad (7.3) \quad \|\mathbf{u}\|_{L^p(Y' \times Y'')} &= \|u - \mathcal{M}_{Y'}(u)\|_{L^p(Y' \times Y'')} \leq C \|\nabla_{y'} u\|_{L^p(Y' \times Y'')}, \\
 &\|\nabla_{y'} \mathbf{u}\|_{L^p(Y' \times Y'')} = \|\nabla_{y'} u\|_{L^p(Y' \times Y'')}.
 \end{aligned}$$

357 Observe that  $\mathcal{M}_{Y'}(\nabla_{y''} \mathbf{u}) = \nabla_{y''} \mathcal{M}_{Y'}(\mathbf{u}) = 0$ .

358 Then, again by equality (7.1) and the Poincaré–Wirtinger Inequality, we get

$$\begin{aligned}
 359 \quad (7.4) \quad \|\nabla_{y''} \mathbf{u}\|_{L^p(Y' \times Y'')} &= \|\nabla_{y''} \mathbf{u} - \mathcal{M}_{Y'}(\nabla_{y''} \mathbf{u})\|_{L^p(Y' \times Y'')} \\
 &\leq C \|\nabla_{y'}(\nabla_{y''} \mathbf{u})\|_{L^p(Y' \times Y'')} = \|\nabla_{y''}(\nabla_{y'} \mathbf{u})\|_{L^p(Y' \times Y'')} \\
 &= \|\nabla_{y''}(\nabla_{y'} u)\|_{L^p(Y' \times Y'')}.
 \end{aligned}$$

Hence, by estimates (7.3)–(7.4) we obtain (7.2).

*Step 2.* Now, suppose  $u$  in  $L^p(Y''; W^{1,p}(Y'))$  and  $\nabla_{y'} u$  in  $L^p(Y'; W^{1,p}(Y''))^{N_1}$ . Since

<sup>2</sup>It also belongs to  $W^{1,p}(Y'; W^{1,p}(Y''))$ .

$\mathcal{C}^2(\overline{Y})$  is dense in this subspace of  $L^p(Y''; W^{1,p}(Y'))$ , there exists a sequence of functions  $u_n \in \mathcal{C}^2(\overline{Y})$  such that

$$\begin{aligned} u_n &\rightarrow u && \text{strongly in } L^p(Y''; W^{1,p}(Y')), \\ \mathcal{M}_{Y'}(u_n) &\rightarrow \mathcal{M}_{Y'}(u) && \text{strongly in } L^p(Y''), \\ \nabla_{y'} u_n &\rightarrow \nabla_{y'} u && \text{strongly in } L^p(Y'; W^{1,p}(Y''))^{N_1}. \end{aligned}$$

The corresponding sequence  $\{u_n\}$  (given by Step 1) satisfies  $\nabla_{y'} u_n = \nabla_{y'} u_n$ , moreover it belongs to  $\mathcal{C}^2(\overline{Y})$  and is bounded in  $W^{1,p}(Y)$  (from (7.2)). Passing to the limit gives  $\mathbf{u} \in W^{1,p}(Y)$  such that

$$u_n \rightarrow \mathbf{u} \quad \text{strongly in } W^{1,p}(Y), \quad \nabla_{y'} \mathbf{u} = \nabla_{y'} u \quad \text{a.e. in } Y.$$

360 Finally, observe that  $\mathbf{u} = u - \mathcal{M}_{Y'}(u)$ . □

LEMMA 7.2. *Let  $u$  be in  $L^p(Y''; W_{per}^{1,p}(Y'))$  ( $p \in (1, +\infty)$ ) such that*

$$\nabla_{y'} u \in L^p(Y'; W_{per}^{1,p}(Y''))^{N_1}.$$

361 *Then, there exists  $\mathfrak{U} \in W_{per}^{1,p}(Y)$  such that*

$$362 \quad (7.5) \quad \nabla_{y'} \mathfrak{U} = \nabla_{y'} u \quad \text{a.e. in } Y.$$

*Proof.* Since  $u$  in  $L^p(Y''; W_{per}^{1,p}(Y'))$  and  $\nabla_{y'} u \in L^p(Y'; W_{per}^{1,p}(Y''))^{N_1}$ , the above Lemma 7.1 shows that the function  $\mathbf{u} = u - \mathcal{M}_{Y'}(u)$  belongs to  $W^{1,p}(Y)$ . It is obvious that  $\mathbf{u}$  is periodic with respect to the variables  $y_1, \dots, y_{N_1}$ . One also has  $\nabla_{y'} \mathbf{u} = \nabla_{y'} u \in L^p(Y'; W_{per}^{1,p}(Y''))^{N_1}$ . Denote

$$\begin{aligned} Y_i &= \{y \in \overline{Y} \mid y_i = 0, y_j \in (0, 1), j \in \{1, \dots, N\} \ j \neq i\}, \quad i \in \{1, \dots, N\}, \\ Y_i'' &= \{y \in \overline{Y}'' \mid y_i = 0, y_j \in (0, 1), j \in \{N_1 + 1, \dots, N\} \ j \neq i\}, \quad i \in \{N_1 + 1, \dots, N\}. \end{aligned}$$

Since  $\nabla_{y'} \mathbf{u} = \nabla_{y'} u$  and is  $y_j$  periodic,  $j \in \{N_1 + 1, \dots, N\}$ , one gets

$$\nabla_{y'} \mathbf{u}|_{Y_j + \mathbf{e}_j} - \nabla_{y'} \mathbf{u}|_{Y_j} = \nabla_{y'} u|_{Y_j + \mathbf{e}_j} - \nabla_{y'} u|_{Y_j} = 0 \quad \text{a.e. in } Y_j.$$

Hence

$$\mathbf{u}|_{Y_j + \mathbf{e}_j} - \mathbf{u}|_{Y_j} \in W^{1-1/p, p}(Y_j''), \quad j \in \{N_1 + 1, \dots, N\}.$$

Besides, one has

$$\mathbf{u}|_{Y_j + \mathbf{e}_j} - \mathbf{u}|_{Y_j} = 0, \quad j \in \{1, \dots, N_1\}.$$

Then, following the same lines of the proofs of [13, Proposition 13.34 and Lemmas 13.35-13.36], there exists  $\mathfrak{U} \in W_{per}^{1,p}(Y)$  such that

$$\mathfrak{U} - \mathbf{u} \in W^{1,p}(Y'')$$

363 and we have  
364

$$\begin{aligned} 365 \quad \|\mathfrak{U} - \mathbf{u}\|_{W^{1,p}(Y)} &\leq C \sum_{j=N_1+1}^N \|\mathbf{u}|_{Y_j + \mathbf{e}_j} - \mathbf{u}|_{Y_j}\|_{W^{1-1/p, p}(Y_j'')} \\ 366 \quad &\leq C (\|\nabla_{y'} u\|_{L^p(Y)} + \|\nabla_{y'} u\|_{L^p(Y'; W^{1,p}(Y''))}). \end{aligned}$$

368 The function  $\mathfrak{U}$  satisfies (7.5). □

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