PERIODIC UNFOLDING FOR ANISOTROPICALLY BOUNDED SEQUENCES

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Abstract. This paper is focused on the asymptotic behavior of sequences of functions, whose partial derivatives estimates in one or more directions are highly contrasted with respect to the periodic parameter \( \varepsilon \). In particular, a direct application for the homogenization of a homogeneous Dirichlet problem defined on an anisotropic structure is presented. In general, the obtained results can be applied to thin structures where the behavior is different according to the observed direction.

Key words. Periodic unfolding method, homogenization, anisotropic Sobolev spaces, Dirichlet problem

AMS subject classifications. 31C25, 35B27, 46E35, 49J45

1. Introduction. The aim of this paper is to extend some of the results already developed in [1, 6, 13] about the periodic unfolding method to new classes of functions, the so-called "anisotropically bounded", including a direct application of the obtained results.

Given a small parameter \( \varepsilon \) and a bounded domain \( \Omega \subset \mathbb{R}^N \) with Lipschitz boundary, we consider the periodic paving of \( \Omega \) made with cells of size \( \varepsilon \). In [13, Section 1.4], it is extensively investigated the asymptotic behavior of sequences \( \{\phi_\varepsilon\} \) such that \( \varepsilon \) is anisotropically bounded, that are, sequences whose gradient is bounded by \( \varepsilon \) regardless of the direction observed. In this sense, the asymptotic behavior of sequences \( \{\phi_\varepsilon\} \) such that

\[
\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla \phi_\varepsilon\|_{L^p(\Omega)} \leq C \quad \text{and} \quad \|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla \phi_\varepsilon\|_{L^p(\Omega)} \leq C
\]

are shown in detail. In the present, we focus on the periodic unfolding for sequences whose assumptions on the gradient bound are anisotropic, e.g. where at least one or more directions are privileged and have more information than the others. In particular, we define a domain partition for every \( x \in \Omega \) by setting \( x = (x', x'') \), where the variable \( x' \) corresponds to the first \( N_1 < N \) directions, and we show the asymptotic behavior of sequences \( \{\phi_\varepsilon\} \) such that

(i) \( \|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x'} \phi_\varepsilon\|_{L^p(\Omega)} \leq C; \)
(ii) \( \|\phi_\varepsilon\|_{L^p(\Omega)} + \|\nabla_{x'} \phi_\varepsilon\|_{L^p(\Omega)} \leq C; \)
(iii) \( \|\phi_{\varepsilon_{x'}}\|_{L^p(\Omega)} + \|\nabla_{x'} \phi_{\varepsilon_{x'}}\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x'^{\prime\prime}} \phi_{\varepsilon_{x'}}\|_{L^p(\Omega)} \leq C; \)
(iv) \( \|\phi_{\varepsilon_{x'}}\|_{L^p(\Omega)} + \|\nabla_{x'} \phi_{\varepsilon_{x'}}\|_{L^p(\Omega)} + \varepsilon\|\nabla_{x'^{\prime\prime}}(\nabla_{x'} \phi_{\varepsilon_{x'}})\|_{L^p(\Omega)} \leq C. \)

These results allow to extend some more complex lemmas concerning the periodic unfolding (see for an instance [13, Lemma 11.11]) to this new classes of functions. In the end, a direct application of such lemmas is done by homogenizing via unfolding the homogeneous Dirichlet problem

\[
\text{Find } u_\varepsilon \in H^1_0(\Omega) \text{ such that:}
\int_{\Omega} A_{\varepsilon_{x'}} \left( \frac{\nabla_{x'} u_\varepsilon}{\varepsilon \nabla_{x'^{\prime\prime}} u_\varepsilon} \right) \cdot \left( \frac{\nabla_{x'} \phi}{\varepsilon \nabla_{x'^{\prime\prime}} \phi} \right) \, dx = \int_{\Omega} f \, \phi \, dx, \quad \forall \phi \in H^1_0(\Omega),
\]

whose nature is anisotropic.

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The homogenization via unfolding method, an equivalent to the two-scale convergence, has been presented the first time by D. Cioranescu and al. in [1], with further development in [2, 3, 4, 6] and extensively in [13]. Such method has largely found application (see e.g. [5, 7, 8, 9, 10, 11, 12]) and also for thin periodic structures like periodically perforated shells (see [18]), textiles made of long curved beams (see [15, 16]) and stable lattice structures made of beams (see [14, 17]).

The present paper provides additional tools for upcoming papers dealing with a wider range of periodic thin structures. Among them, we cite [20], where the assumption of loose contact between fibers on a textile made of beams arises different gradient estimates with respect to the observed direction, and [19], where the same peculiarity applies but on periodic lattice structures. More generally, such tools can be applied to many other problems related to partial differential equations on domains involving periodic grids, lattices, thin frames and fiber structures.

The paper is organized as follows. In section 2, the standard notation for the classical homogenization via unfolding method on bounded domains Ω ∈ \(\mathbb{R}^N\) with Lipschitz boundary are listed. In section 3, we set a domain partition \(x = (x', x'')\) for every \(x \in \Omega\) and from [13, Chap. 7], we recall the unfolding with parameters and prove the main properties. In section 4, we give the asymptotic behavior for the classes of sequences bounded anisotropically (i)-(iv) described above. The aim of section 5 is to extend [13, Lemma 11.11] to the anisotropic case. Different anisotropic assumptions arise different regularity of the limit fields. Such results are fundamental if one considers the key role that such a Lemma has in the study of elasticity problem for thin structures (see [16, 18]). At last, in section 6, we consider the Dirichlet problem shown above. Using the results in the previous sections, existence and uniqueness of the limit problem are shown and through the homogenization via unfolding, the cell problems and the macroscopic limit problem are found.

2. Preliminaries and notation. Let \(\mathbb{R}^N\) be the euclidean space with usual basis \((e_1, \ldots, e_N)\) and \(Y = (0, 1)^N\) the open unit parallelepiped associated with this basis. For a.e. \(z \in \mathbb{R}^N\), we set the unique decomposition \(z = [z]_Y + \{z\}_Y\) such that

\[
[z]_Y := \sum_{i=1}^N k_i e_i, \quad k_i \in \mathbb{Z}^N \quad \text{and} \quad \{z\}_Y := z - [z]_Y \in Y.
\]

Let \(\{\varepsilon\}\) be a sequence of strictly positive parameters going to 0. We scale our paving by \(\varepsilon\) writing

\[
x = \varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon \left\{\frac{x}{\varepsilon}\right\}_Y \quad \text{for a.e. } x \in \mathbb{R}^N.
\]

Let now \(\Omega\) be a bounded domain in \(\mathbb{R}^N\) with Lipschitz boundary. We consider the covering

\[
\Xi_{\varepsilon} := \{\xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega\}
\]

and set

\[
\bar{\Omega}_{\varepsilon} := \text{int}\left\{ \bigcup_{\xi \in \Xi_{\varepsilon}} \varepsilon(\xi + Y) \right\}, \quad \Lambda_{\varepsilon} := \Omega \setminus \bar{\Omega}_{\varepsilon}.
\]

We recall the definitions of classical unfolding operator and mean value operator.
Definition 2.1. (see [13, Definition 1.2]) For every measurable function \( \phi \) on \( \Omega_\varepsilon \), the unfolding operator \( \mathcal{T}_\varepsilon \) is defined as follows:

\[
\mathcal{T}_\varepsilon(\phi) = \begin{cases} 
\phi \left( \frac{x}{\varepsilon} + \varepsilon y \right) & \text{for a.e. } (x, y) \in \Omega_\varepsilon \times Y, \\
0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y.
\end{cases}
\]

Note that such an operator acts on functions defined in \( \Omega \) by operating on their restriction to \( \Omega_\varepsilon \).

Definition 2.2. (see [13, Definition 1.10]) For every measurable function \( \hat{\phi} \) on \( L^1(\Omega \times Y) \), the mean value operator \( \mathcal{M}_Y \) is defined as follows:

\[
\mathcal{M}_Y(\hat{\phi})(x) = \frac{1}{|Y|} \int_Y \hat{\phi}(x, y) dy, \quad \text{for a.e. } x \in \Omega.
\]

Let \( p \in [1, +\infty] \). From [13, Propositions 1.8 and 1.11], we recall the properties of these operators:

\[
\|\mathcal{T}_\varepsilon(\phi)\|_{L^p(\Omega \times Y)} \leq |Y|^\frac{1}{p} \|\phi\|_{L^p(\Omega)} \quad \text{for every } \phi \in L^p(\Omega),
\]

\[
\|\mathcal{M}_Y(\hat{\phi})\|_{L^p(\Omega \times Y)} \leq |Y|^\frac{1}{p} \|\hat{\phi}\|_{L^p(\Omega \times Y)} \quad \text{for every } \hat{\phi} \in L^p(\Omega \times Y).
\]

Since we will deal with Sobolev spaces, we give hereafter some definitions:

\[
W^{1,p}_{\text{per},Y}(\Omega) \equiv \{ \phi \in W^{1,p}(Y) \mid \phi \text{ is periodic with respect to } y_i, i \in \{1, \ldots, N\} \},
\]

\[
W^{1,p}_{\text{per},0}(Y) \equiv \{ \phi \in W^{1,p}_{\text{per},Y}(\Omega) \mid \mathcal{M}_Y(\phi) = 0 \},
\]

\[
L^p(\Omega; W^{1,p}(Y)) \equiv \{ \phi \in L^p(\Omega \times Y) \mid \nabla_y \phi \in L^p(\Omega \times Y)^N \}.
\]

3. The unfolding with parameters. We start by giving a rigorous definition of the domain split. Let \( (N_1, N_2) \) be in \( \mathbb{N} \times \mathbb{N}^* \) and such that \( N = N_1 + N_2 \). Denote

\[
\mathbb{R}^{N_1} = \left\{ x' \in \mathbb{R}^N \mid x' = \sum_{i=1}^{N_1} x_i e_i, \ x_i \in \mathbb{R} \right\},
\]

\[
\mathbb{R}^{N_2} = \left\{ x'' \in \mathbb{R}^N \mid x'' = \sum_{i=N_1+1}^{N} x_i e_i, \ x_i \in \mathbb{R} \right\},
\]

\[
Y' = \left\{ y' \in \mathbb{R}^N \mid y' = \sum_{i=1}^{N_1} y_i e_i, \ y_i \in (0, 1) \right\},
\]

\[
Y'' = \left\{ y'' \in \mathbb{R}^N \mid y'' = \sum_{i=N_1+1}^{N} y_i e_i, \ y_i \in (0, 1) \right\}
\]

and

\[
Z^{N_1} = \mathbb{Z} e_1 \oplus \ldots \oplus \mathbb{Z} e_{N_1}, \quad Z^{N_2} = \mathbb{Z} e_{N_1+1} \oplus \ldots \oplus \mathbb{Z} e_N.
\]

One has

\[
\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \quad Y = Y' \oplus Y'', \quad Z^N = Z^{N_1} \oplus Z^{N_2}.
\]

For every \( x \in \mathbb{R}^N \) and \( y \in Y \), we write

\[
x = x' + x'' \in \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \quad y = y' + y'' \in Y' \oplus Y''.
\]

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From now on, however, we find easier to refer to such partition with the vectorial notation
\[ x = (x', x'') \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \quad y = (y', y'') \in Y' \times Y''. \]
Similarly to (2.1), we apply the paving to a.e. \( x' \in \mathbb{R}^{N_1} \) and \( x'' \in \mathbb{R}^{N_2} \) setting
\[
\begin{align*}
x' &= \varepsilon \left\lfloor \frac{x'}{\varepsilon} \right\rfloor_{Y'}, + \varepsilon \left\lfloor \frac{x'}{\varepsilon} \right\rfloor_{Y'}, \\
x'' &= \varepsilon \left\lfloor \frac{x''}{\varepsilon} \right\rfloor_{Y''}, + \varepsilon \left\lfloor \frac{x''}{\varepsilon} \right\rfloor_{Y''},
\end{align*}
\]
with \( \left\lfloor \frac{x'}{\varepsilon} \right\rfloor_{Y'}, \left\lfloor \frac{x'}{\varepsilon} \right\rfloor_{Y'} \in Y' \),
\( \left\lfloor \frac{x''}{\varepsilon} \right\rfloor_{Y''}, \left\lfloor \frac{x''}{\varepsilon} \right\rfloor_{Y''} \in Y'' \).

**Definition 3.1.** For every \( \hat{\phi} \in L^1(\Omega \times Y) \), the partial mean value operators are defined as follows:
\[
\begin{align*}
\mathcal{M}_{Y'}(\hat{\phi})(x, y') &= \frac{1}{|Y'|} \int_{Y'} \hat{\phi}(x, y', y') dy', & \text{for a.e. } (x, y') \in \Omega \times Y', \\
\mathcal{M}_{Y''}(\hat{\phi})(x, y') &= \frac{1}{|Y''|} \int_{Y''} \hat{\phi}(x, y', y') dy', & \text{for a.e. } (x, y') \in \Omega \times Y'.
\end{align*}
\]
Denote
\[
\begin{align*}
L^p(\Omega, \nabla_{x'}) &= \left\{ \phi \in L^p(\Omega) \mid \nabla_{x'} \phi \in L^p(\Omega)^{N_1} \right\}, \\
L^p(\Omega, \nabla_{x''}) &= \left\{ \phi \in L^p(\Omega) \mid \nabla_{x''} \phi \in L^p(\Omega)^{N_2} \right\}, \\
L^p(\Omega, \nabla_{x'}; W^{1,p}(Y')) &= \left\{ \tilde{\phi} \in L^p(\Omega \times Y') \mid \nabla_{x'} \tilde{\phi} \in L^p(\Omega \times Y')^{N_1} \right\}, \\
\nabla_{y'} \tilde{\phi} \in L^p(\Omega \times Y')^{N_1}, \\
L^p(\Omega, \nabla_{x''}; W^{1,p}(Y'')) &= \left\{ \tilde{\phi} \in L^p(\Omega \times Y'') \mid \nabla_{x''} \tilde{\phi} \in L^p(\Omega \times Y'')^{N_2} \right\}, \\
\nabla_{y''} \tilde{\phi} \in L^p(\Omega \times Y'')^{N_2},
\end{align*}
\]
\[ (3.1) \]
We endow these spaces with the respective norms:
\[
\begin{align*}
\| \cdot \|_{L^p(\Omega, \nabla_{x'})} &= \| \cdot \|_{L^p(\Omega)} + \| \nabla_{x'}(\cdot) \|_{L^p(\Omega)}^{N_1}, \\
\| \cdot \|_{L^p(\Omega, \nabla_{x''})} &= \| \cdot \|_{L^p(\Omega)} + \| \nabla_{x''}(\cdot) \|_{L^p(\Omega)}^{N_2}, \\
\| \cdot \|_{L^p(\Omega, \nabla_{x'}; W^{1,p}(Y'))} &= \| \cdot \|_{L^p(\Omega \times Y')} + \| \nabla_{x'}(\cdot) \|_{L^p(\Omega \times Y')}^{N_1} + \| \nabla_{y'}(\cdot) \|_{L^p(\Omega \times Y')}^{N_1}, \\
\| \cdot \|_{L^p(\Omega, \nabla_{x''}; W^{1,p}(Y''))} &= \| \cdot \|_{L^p(\Omega \times Y'')} + \| \nabla_{x''}(\cdot) \|_{L^p(\Omega \times Y'')}^{N_2} + \| \nabla_{y''}(\cdot) \|_{L^p(\Omega \times Y'')}^{N_2}.
\end{align*}
\]
We recall the unfolding with parameters tools already developed in [13, Chap. 7] and we define two partial unfolding operators.

**Definition 3.2.** For every measurable function \( \phi \) on \( \Omega \), the unfolding operator \( \mathcal{T}_\varepsilon \) is defined as follows:
\[
\mathcal{T}_\varepsilon(\phi)(x', x'', y'') = \begin{cases} 
\phi(x', y''), & \text{for a.e. } (x', x'', y'') \in \tilde{\Omega}_\varepsilon \times Y'', \\
0, & \text{for a.e. } (x', x'', y'') \in \Lambda_\varepsilon \times Y''.
\end{cases}
\]
For every measurable function $\psi$ on $\Omega \times Y''$, the unfolding operator $T_\epsilon'$ is defined as follows:

$$T_\epsilon'(\psi)(x',x'',y',y'') = \begin{cases} \psi\left(\varepsilon \left[\frac{x'}{\varepsilon}\right]_{Y'}, + \varepsilon y', x'', y''\right) & \text{for a.e. } (x',x'',y',y'') \in \hat{\Omega}_\epsilon \times Y, \\
0 & \text{for a.e. } (x',x'',y',y'') \in \Lambda_\epsilon \times Y. \end{cases}$$

Note that in the partial unfolding operator $T_\epsilon''(\phi)$ the variable $x'$ plays the role of a parameter, while in $T_\epsilon'(\psi)$ the role of parameters is played by the variables $(x'',y'')$.

**Lemma 3.3.** One has

$$T_\epsilon = T_\epsilon' \circ T_\epsilon'', \quad \mathcal{M}_Y = \mathcal{M}_{Y'} \circ \mathcal{M}_{Y''}.$$  

Moreover, for every $\phi \in L^1(\Omega, \nabla_{x'})$ one has

$$\nabla_{x'} T_\epsilon''(\phi) = T_\epsilon''(\nabla_{x'} \phi) \quad \text{a.e. in } \hat{\Omega}_\epsilon \times Y''.$$  

**Proof.** Let $\phi$ be measurable on $\Omega$. We have that

$$T_\epsilon' \circ T_\epsilon''(\phi)(x,y) = T_\epsilon' \left( \phi \left( x', \varepsilon \left[ \frac{x''}{\varepsilon} \right]_{Y''}, + \varepsilon y'' \right) \right) = \phi \left( \varepsilon \left[ \frac{x'}{\varepsilon} \right]_{Y'}, + \varepsilon y', \varepsilon \left[ \frac{x''}{\varepsilon} \right]_{Y''}, + \varepsilon y'' \right)$$

$$= \phi \left( \varepsilon \left[ \frac{x'}{\varepsilon} \right]_{Y'}, + \varepsilon y \right) = T_\epsilon(\phi)(x,y) \quad \text{for a.e. } (x,y) \in \hat{\Omega}_\epsilon \times Y.$$

For $(x,y) \in \Lambda_\epsilon \times Y$ the result is obvious.

Let $\hat{\phi}$ be in $L^1(\Omega \times Y')$. We have

$$\mathcal{M}_{Y'} \circ \mathcal{M}_{Y''}(\hat{\phi})(x) = \mathcal{M}_{Y''} \left( \frac{1}{|Y''|} \int_{Y''} \hat{\phi}(x,y',y'')dy'' \right)$$

$$= \frac{1}{|Y''| |Y'|} \int_{Y'} \int_{Y''} \hat{\phi}(x,y',y'')dy''dy' = \frac{1}{|Y'|} \int_{Y'} \hat{\phi}(x,y')dy$$

$$= \mathcal{M}_{Y'}(\hat{\phi})(x) \quad \text{for a.e. } x \in \Omega.$$

Let now $\phi$ be in $L^1(\Omega, \nabla_{x'})$. We have

$$\nabla_{x'} T_\epsilon''(\phi)(x,y') = \nabla_{x'} \left( \phi \left( x', \varepsilon \left[ \frac{x''}{\varepsilon} \right]_{Y''}, + \varepsilon y'' \right) \right) = \nabla_{x'} \phi \left( x', \varepsilon \left[ \frac{x''}{\varepsilon} \right]_{Y''}, + \varepsilon y'' \right)$$

$$= T_\epsilon''(\nabla_{x'} \phi)(x,y') \quad \text{for a.e. } (x,y') \in \hat{\Omega}_\epsilon \times Y''.$$  

**4. Asymptotic behavior of anisotropically bounded sequences.** We start
with sequences bounded in $L^p(\Omega, \nabla_{x'})$, $p \in [1, +\infty]$, whose gradient is bounded with
order $\varepsilon^{-1}$ in $L^p(\Omega)$ in the first $N_1$ directions.

**Lemma 4.1.** Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $L^p(\Omega, \nabla_{x'})$, $p \in (1, +\infty)$, satisfying

$$\|\phi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x'} \phi_\varepsilon\|_{L^p(\Omega)} \leq C,$$

where the constant does not depend on $\varepsilon$.

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and a function $\hat{\phi}$ in the space

$L^p(\Omega \times Y''; W^{1,p}(Y'))$ such that

$$\phi_\varepsilon \rightharpoonup \hat{\phi} \quad \text{weakly in } L^p(\Omega),$$

$$T_\epsilon(\phi_\varepsilon) \rightharpoonup \hat{\phi} \quad \text{weakly in } L^p(\Omega \times Y''; W^{1,p}(Y')).$$

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Then, there exist a subsequence of \( \{\varepsilon\} \), still denoted \( \{\varepsilon\} \), and \( \hat{\phi} \in L^p(\Omega \times Y'', \nabla x') \), \( \hat{\phi} \in L^p(\Omega \times Y''; W^{1,p}_{\text{per,0}}(Y')) \) such that
\[
\phi_{\varepsilon} \rightharpoonup \hat{\phi} \quad \text{weakly in } \quad L^p(\Omega, \nabla x'),
\]
\[
\mathcal{T}_\varepsilon(\phi_{\varepsilon}) \rightharpoonup \hat{\phi} \quad \text{weakly in } \quad L^p(\Omega \times Y''; W^{1,p}(Y')),
\]
\[
\mathcal{T}_\varepsilon(\nabla x'\phi_{\varepsilon}) \rightharpoonup \nabla x'\hat{\phi} + \nabla y'\hat{\phi} \quad \text{weakly in } \quad L^p(\Omega \times Y)^{N_1},
\]
\[
\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(\phi_{\varepsilon}) - \mathcal{M}_{Y'}(\hat{\phi})) \rightharpoonup \nabla x'\hat{\phi} \cdot y'^c + \hat{\phi} \quad \text{weakly in } \quad L^p(\Omega \times Y)^{N_1}
\]
where \( \phi = \mathcal{M}_{Y''}(\hat{\phi}) \) and \( y'^c \equiv y' - \mathcal{M}_{Y'}(y') \).

\[ \text{Proof.} \] The proof is similar to [13, Theorem 1.36]. \[ \square \]

An analogous result holds for sequences uniformly bounded in \( L^p(\Omega, \nabla x'), p \in [1, +\infty] \).

**Lemma 4.2.** Let \( \{\phi_{\varepsilon}\}_\varepsilon \) be a sequence in \( L^p(\Omega, \nabla x'), p \in (1, +\infty) \), such that
\[ \|\phi_{\varepsilon}\|_{L^p(\Omega, \nabla x')} \leq C. \]
Then, there exist a subsequence of \( \{\varepsilon\} \), still denoted \( \{\varepsilon\} \), and \( \tilde{\phi} \in L^p(\Omega \times Y'', \nabla x') \), \( \tilde{\phi} \in L^p(\Omega \times Y''; W^{1,p}_{\text{per,0}}(Y')) \) such that
\[ \phi_{\varepsilon} \rightharpoonup \tilde{\phi} \quad \text{weakly in } \quad L^p(\Omega, \nabla x'), \]
\[ \mathcal{T}_\varepsilon(\phi_{\varepsilon}) \rightharpoonup \tilde{\phi} \quad \text{weakly in } \quad L^p(\Omega \times Y''; W^{1,p}(Y')), \]
\[ \mathcal{T}_\varepsilon(\nabla x'\phi_{\varepsilon}) \rightharpoonup \nabla x'\tilde{\phi} + \nabla y'\tilde{\phi} \quad \text{weakly in } \quad L^p(\Omega \times Y)^{N_1}, \]
\[ \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(\phi_{\varepsilon}) - \mathcal{M}_{Y'}(\tilde{\phi})) \rightharpoonup \nabla x'\tilde{\phi} \cdot y'^c + \tilde{\phi} \quad \text{weakly in } \quad L^p(\Omega \times Y)^{N_1} \]
where \( \tilde{\phi} = \mathcal{M}_{Y''}(\tilde{\phi}) \) and \( y'^c \equiv y' - \mathcal{M}_{Y'}(y') \).

\[ \text{Proof.} \] The proof is similar to [13, Corollary 1.37] and [13, Theorem 1.41]. \[ \square \]

Now, we consider the sequences in \( W^{1,p}(\Omega) \), \( p \in [1, +\infty] \), whose gradient is estimated with different order according to the considered direction.

**Lemma 4.3.** Let \( \{\phi_{\varepsilon}\}_\varepsilon \) be a sequence in \( W^{1,p}(\Omega) \), \( p \in (1, +\infty) \), satisfying
\[ (4.1) \quad \|\phi_{\varepsilon}\|_{L^p(\Omega, \nabla x')} + \varepsilon\|\nabla x'\phi_{\varepsilon}\|_{L^p(\Omega)} \leq C, \]
where the constant does not depend on \( \varepsilon \).

Then, there exist a subsequence of \( \{\varepsilon\} \), still denoted \( \{\varepsilon\} \), and functions
\[ \tilde{\phi} \in L^p(\Omega, \nabla x'; W^{1,p}_{\text{per}}(Y'')) \] and \( \hat{\phi} \in L^p(\Omega \times Y''; W^{1,p}_{\text{per,0}}(Y')) \)
such that
\[ \phi_{\varepsilon} \rightharpoonup \tilde{\phi} \quad \text{weakly in } \quad L^p(\Omega, \nabla x'), \]
\[ \mathcal{T}_\varepsilon(\phi_{\varepsilon}) \rightharpoonup \tilde{\phi} \quad \text{weakly in } \quad L^p(\Omega; W^{1,p}(Y)), \]
\[ \mathcal{T}_\varepsilon(\nabla x'\phi_{\varepsilon}) \rightharpoonup \nabla x'\tilde{\phi} + \nabla y'\tilde{\phi} \quad \text{weakly in } \quad L^p(\Omega \times Y)^{N_1}, \]
\[ \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(\phi_{\varepsilon}) - \mathcal{M}_{Y'}(\tilde{\phi})) \rightharpoonup \nabla x'\tilde{\phi} \cdot y'^c + \tilde{\phi} \quad \text{weakly in } \quad L^p(\Omega \times Y)^{N_1} \]
where \( \tilde{\phi} = \mathcal{M}_{Y''}(\tilde{\phi}) \) and \( y'^c \equiv y' - \mathcal{M}_{Y'}(y') \).

The same results hold for \( p = +\infty \) with weak topology replaced by weak-* topology in the corresponding spaces.

**Proof.** From (4.1), up to a subsequence of \( \{\varepsilon\} \), still denoted \( \{\varepsilon\} \), one has the existence of \( \phi \in L^p(\Omega, \nabla x') \) such that (4.2) holds.
Set \( \{ \Phi_\varepsilon \}_\varepsilon = \{ T_\varepsilon''(\phi_\varepsilon) \}_\varepsilon \). This sequence belongs to \( L^p(\hat{\Omega}_\varepsilon, \nabla x'; W^{1,p}(Y'')) \) and from estimate (4.1) and equality (3.4), it satisfies

\[
\|\Phi_\varepsilon\|_{L^p(\hat{\Omega}_\varepsilon, \nabla x'; W^{1,p}(Y''))} \leq C,
\]

where the constant does not depend on \( \varepsilon \).

Up to a subsequence of \( \{ \varepsilon \} \), still denoted \( \{ \varepsilon \} \), there exists \( \bar{\phi} \in L^p(\Omega; W^{1,p}(Y'')) \) and \( \bar{\Phi} \in L^p(\Omega \times Y'')^N_1 \) (the periodicity of \( \bar{\phi} \) is proved as in [13, Theorem 1.36]) such that

\[
\Phi_\varepsilon 1_{\hat{\Omega}_\varepsilon \times Y''} \rightharpoonup \bar{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(Y'')),
\]

\[
\nabla x' \Phi_\varepsilon 1_{\hat{\Omega}_\varepsilon \times Y''} \rightharpoonup \bar{\Phi} \quad \text{weakly in} \quad L^p(\Omega \times Y''^N_1).
\]

Let \( g \) be in \( C^\infty_c(\Omega \times Y'')^N_1 \). For \( \varepsilon \) sufficiently small such that \( \text{supp}(g) \subset \hat{\Omega}_\varepsilon \times Y'' \), we have

\[
\int_{\Omega \times Y''} \nabla x' \Phi_\varepsilon 1_{\hat{\Omega}_\varepsilon \times Y''} \cdot g \, dx dy'' = \int_{\hat{\Omega}_\varepsilon \times Y''} \nabla x' \Phi_\varepsilon \cdot g \, dx dy''
\]

\[
= - \int_{\hat{\Omega}_\varepsilon \times Y''} \Phi_\varepsilon \nabla x' g \, dx dy'' = - \int_{\Omega \times Y''} \Phi_\varepsilon 1_{\hat{\Omega}_\varepsilon \times Y''} \nabla x' g \, dx dy''.
\]

Then, passing to the limit yields

\[
\int_{\Omega \times Y''} \bar{\Phi} \cdot g \, dx dy'' = - \int_{\Omega \times Y''} \bar{\phi} \cdot \nabla x' g \, dx dy'', \quad \forall g \in C^\infty_c(\Omega \times Y'')^N_1.
\]

This means that \( \bar{\Phi} = \nabla x' \bar{\phi} \) a.e. in \( \Omega \times Y'' \), thus \( \nabla x' \bar{\phi} \in L^p(\Omega \times Y'')^N_1 \) and therefore \( \bar{\phi} \in L^p(\Omega, \nabla x'; W^{1,p}(Y'')) \).

Now, we transform the sequence \( \{ \Phi_\varepsilon \}_\varepsilon \) using the unfolding operator \( T_\varepsilon' \), \( Y'' \) being a set of parameters.

From the above convergence and estimate (4.3), up to a subsequence of \( \{ \varepsilon \} \), still denoted \( \{ \varepsilon \} \), [13, Corollary 1.37] and [13, Theorem 1.41] give \( \bar{\phi} \in L^p(\Omega \times Y''^N_1, W^{1,p}_{\text{per,0}}(Y')) \) such that (using the rule (3.3)1)

\[
T_\varepsilon(\phi_\varepsilon) = T_\varepsilon'(\Phi_\varepsilon) \rightharpoonup \bar{\phi} \quad \text{weakly in} \quad L^p(\Omega; W^{1,p}(Y' \times Y'')),
\]

\[
T_\varepsilon'(\nabla x' \phi_\varepsilon) = T_\varepsilon''(\nabla x' \Phi_\varepsilon) \rightharpoonup \nabla x' \bar{\phi} + \nabla y' \bar{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y' \times Y''^N_1),
\]

\[
\frac{1}{\varepsilon}(T_\varepsilon'(\phi_\varepsilon) - MY'(T_\varepsilon'(\phi_\varepsilon))) \rightharpoonup \nabla x' \bar{\phi} \cdot y' + \bar{\phi} \quad \text{weakly in} \quad L^p(\Omega \times Y' \times Y'').
\]

This ends the proof of (4.2).

As the lemma below shows, an analogous asymptotic behavior is achieved starting from sequences uniformly bounded in \( L^p(\Omega, \nabla x') \), \( p \in [1, +\infty] \), with some assumptions on the gradient derivatives.

**Lemma 4.4.** Let \( \{ \phi_\varepsilon \}_\varepsilon \) be a sequence in \( L^p(\Omega, \nabla x') \), \( p \in (1, +\infty) \), satisfying

\[
\|\phi_\varepsilon\|_{L^p(\Omega, \nabla x')} + \varepsilon\|\nabla x''(\nabla x' \phi_\varepsilon)\|_{L^p(\Omega)} \leq C,
\]

where the constant does not depend on \( \varepsilon \).

Then, there exist a subsequence of \( \{ \varepsilon \} \), still denoted \( \{ \varepsilon \} \), functions

\[
\bar{\phi} \in L^p(\Omega, \nabla x'; W^{1,p}_{\text{per}}(Y'')) \quad \text{and} \quad \bar{\psi} \in L^p(\Omega; W^{1,p}_{\text{per}}(Y'))
\]
such that $\mathcal{M}_{Y'}(\hat{\psi}) = 0$ a.e. in $\Omega \times Y''$,
\[ \nabla_{x'} \hat{\phi} \in L^p(\Omega; W^{1,p}(Y''))^N, \quad \nabla_y \hat{\psi} \in L^p(\Omega \times Y'; W^{1,p}_{\text{per}}(Y''))^N, \]
and we have
\[ \phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega, \nabla_{x'}), \]
\[ T_\varepsilon(\phi_\varepsilon) \rightharpoonup \hat{\phi} \quad \text{weakly in } L^p(\Omega, W^{1,p}(Y')), \]
\[ T_\varepsilon(\nabla_{x'} \phi_\varepsilon) \rightharpoonup \nabla_{x'} \hat{\phi} + \nabla_y \hat{\psi} \quad \text{weakly in } L^p(\Omega \times Y'; W^{1,p}(Y''))^N, \]
where $\phi = \mathcal{M}_{Y'}(\hat{\phi})$.

The same results hold for $p = +\infty$ with weak topology replaced by weak-* topology in the corresponding spaces.

**Proof.** By estimate (4.4) and Lemma 4.2, there exists a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\hat{\phi} \in L^p(\Omega \times Y'', \nabla_{x'})$, $\phi \in L^p(\Omega \times Y'', W^{1,p}_{\text{per}}(Y'))$ such that
\[ \phi_\varepsilon \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega, \nabla_{x'}), \]
\[ T_\varepsilon(\phi_\varepsilon) \rightharpoonup \hat{\phi} \quad \text{weakly in } L^p(\Omega, W^{1,p}(Y')), \]
\[ T_\varepsilon(\nabla_{x'} \phi_\varepsilon) \rightharpoonup \nabla_{x'} \hat{\phi} + \nabla_y \hat{\psi} \quad \text{weakly in } L^p(\Omega \times Y'; W^{1,p}(Y''))^N. \]

Set $\{\psi_\varepsilon\}_\varepsilon = \{\nabla_{x'} \phi_\varepsilon\}_\varepsilon$. By estimate (4.4), this sequence satisfies
\[ \|\psi_\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla_{x''} \psi_\varepsilon\|_{L^p(\Omega)} \leq C, \]
where the constant does not depend on $\varepsilon$.

Hence, applying Lemma 4.1 to the above sequence (but swapping $Y'$ and $Y''$), there exists a function $\psi \in L^p(\Omega \times Y'; W^{1,p}_{\text{per}}(Y''))^N$ such that
\[ T_\varepsilon(\nabla_{x'} \phi_\varepsilon) = T_\varepsilon(\psi_\varepsilon) \rightharpoonup \hat{\psi} \quad \text{weakly in } L^p(\Omega \times Y'; W^{1,p}(Y''))^N. \]

This, together with convergence (4.6), implies that the quantity $\nabla_{x'} \hat{\phi} + \nabla_y \hat{\psi}$ belongs to $L^p(\Omega \times Y'; W^{1,p}_{\text{per}}(Y''))^N$. Since $\hat{\phi}$ does not depend on $y'$ and $\hat{\phi}$ is periodic with respect to $y'$, we have that
\[ \nabla_{x'} \hat{\phi} = \mathcal{M}_{Y'}(\nabla_{x'} \hat{\phi}) + \mathcal{M}_{Y'}(\nabla_{y'} \hat{\phi}) = \mathcal{M}_{Y'}(\hat{\psi}), \]
thus $\nabla_{x'} \hat{\phi} \in L^p(\Omega; W^{1,p}(Y''))^N$ and therefore $\hat{\phi} \in L^p(\Omega, \nabla_{x'}; W^{1,p}(Y''))$.

Moreover, the quantity $\nabla_{y'} \hat{\phi}$ belongs to $L^p(\Omega \times Y'; W^{1,p}_{\text{per}}(Y''))^N$, and thus Lemma 7.2 implies that there exists $\hat{\psi} \in L^p(\Omega; W^{1,p}_{\text{per}}(Y'))$ with $\nabla_{y'} \hat{\phi} = \nabla_{y'} \hat{\psi}$ such that (4.5) holds. The proof follows by replacing $\hat{\phi}$ by the function $\hat{\psi} = \hat{\phi} - \mathcal{M}_{Y'}(\hat{\psi})$, which belongs to the space $L^p(\Omega; W^{1,p}_{\text{per}}(Y'))$.

**5. Other unfolding results for anisotropically bounded sequences.** In this section we want to extend the lemma below to the anisotropic case.

**Lemma 5.1.** (see [13, Lemma 11.11]) Let $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$ be a sequence converging weakly to $(u, v)$ in $W^{1,p}(\Omega) \times W^{1,p}(\Omega)^N$, $p \in (1, +\infty)$. Moreover, assume that there exist $Z \in L^p(\Omega)^N$ and $\bar{v} \in L^p(\Omega; W^{1,p}_{\text{per}}(Y'))^N$ such that
\[ \frac{1}{\varepsilon}(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup Z \quad \text{weakly in } L^p(\Omega)^N, \]
\[ T_\varepsilon(\nabla v_\varepsilon) \rightharpoonup \nabla v + \nabla_y \bar{v} \quad \text{weakly in } L^p(\Omega \times Y')^{N \times N}. \]
Then, $u$ belongs to $W^{2,p}(\Omega)$. Moreover, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $u \in L^p(\Omega; W^{1,p}_{\text{per,0}}(Y))$ such that
\[
\frac{1}{\varepsilon} T_\varepsilon(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup Z + \nabla_y u + \hat{v} \quad \text{weakly in} \quad L^p(\Omega \times Y)^N.
\]
As an immediate consequence, one has the following.

**Corollary 5.2.** Let $\mathcal{O}$ be an open set in $\mathbb{R}^k$, $k \geq 1$. Let $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$ be a sequence converging weakly to $(u, v)$ in $L^p(\mathcal{O}; W^{1,p}(\Omega)) \times L^p(\mathcal{O}; W^{1,p}(\Omega))^N$, $p \in (1, +\infty)$.

Moreover, assume that there exist $Z \in L^p(\mathcal{O} \times \Omega)^N$ and $\hat{v} \in L^p(\mathcal{O} \times \Omega; W^{1,p}_{\text{per,0}}(Y))^N$ such that
\[
\frac{1}{\varepsilon} T_\varepsilon(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup Z \quad \text{weakly in} \quad L^p(\mathcal{O} \times \Omega)^N,
\]
\[
T_\varepsilon(\nabla v_\varepsilon) \rightharpoonup \nabla v + \nabla_y \hat{v} \quad \text{weakly in} \quad L^p(\mathcal{O} \times \Omega \times Y)^{N \times N}.
\]
Then, $u$ belongs to $L^p(\mathcal{O}; W^{2,p}(\Omega))$. Furthermore, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $u \in L^p(\mathcal{O} \times \Omega; W^{1,p}_{\text{per,0}}(Y))$ such that:
\[
\frac{1}{\varepsilon} T_\varepsilon(\nabla u_\varepsilon + v_\varepsilon) \rightharpoonup Z + \nabla_y u + \hat{v} \quad \text{weakly in} \quad L^p(\mathcal{O} \times \Omega \times Y)^N.
\]

Define the spaces
\[
L^p(\Omega \times Y'', D^2_y) = \{ \tilde{\phi} \text{ and } \nabla_y \tilde{\phi} \in L^p(\Omega \times Y'', \nabla_y) \}^N_1,
\]
\[
L^p(\Omega, D^2_y; W_{\text{per,0}}^{1,p}(Y'')) = \{ \tilde{\phi} \text{ and } \nabla_y \tilde{\phi} \in L^p(\Omega, \nabla_y; W^{1,p}_{\text{per,0}}(Y'')) \}^N_1,
\]
edowed with the respective norms:
\[
\| \cdot \|_{L^p(\Omega \times Y'', D^2_y)} \doteq \| \cdot \|_{L^p(\Omega \times Y'')} + \| \nabla_y \cdot (\cdot) \|_{L^p(\Omega \times Y'')}^N_1 + \| D^2_y \cdot (\cdot) \|_{L^p(\Omega \times Y'')}^N \times N_1,
\]
\[
\| \cdot \|_{L^p(\Omega, D^2_y; W^{1,p}_{\text{per,0}}(Y''))} \doteq \| \cdot \|_{L^p(\Omega \times Y'')} + \| \nabla_y \cdot (\cdot) \|_{L^p(\Omega \times Y'')}^N \times N_2.
\]
We are ready to extend Lemma 5.1 to the class of anisotropically bounded sequences.

**Lemma 5.3.** Let $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$ be a sequence in the space $L^p(\Omega, \nabla_y) \times L^p(\Omega, \nabla_y) \times N_1$, $p \in (1, +\infty)$, satisfying
\[
(1.1) \quad \| u_\varepsilon \|_{L^p(\Omega, \nabla_y)} \leq C, \quad \| v_\varepsilon \|_{L^p(\Omega, \nabla_y)} \leq C,
\]
where the constant does not depend on $\varepsilon$.

Moreover, assume that there exist $Z \in L^p(\Omega)^{N_1}$ such that
\[
(2.1) \quad \frac{1}{\varepsilon} (\nabla_y u_\varepsilon + v_\varepsilon) \rightharpoonup Z \quad \text{weakly in} \quad L^p(\Omega)^{N_1}.
\]

Then, there exists a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\tilde{Z} \in L^p(\Omega \times Y'')^N_1$ with
\[
M_{Y''}(\tilde{Z}) = Z, \quad u \in L^p(\Omega \times Y'', W^{1,p}_{\text{per,0}}(Y'')), \quad \tilde{u} \in L^p(\Omega \times Y''', D^2_y) \quad \text{and a function}
\]
\[
\tilde{v} \in L^p(\Omega \times Y''; W^{1,p}_{\text{per,0}}(Y''))^N_1 \quad \text{such that}
\]
\[
T_\varepsilon(\nabla_y v_\varepsilon) \rightharpoonup D^2_y \tilde{u} + \nabla_y \tilde{v} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1 \times N_1},
\]
\[
(3) \quad \frac{1}{\varepsilon} T_\varepsilon(\nabla_y u_\varepsilon + v_\varepsilon) \rightharpoonup \tilde{Z} + \nabla_y u + \tilde{v} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1}.
\]
Proof. We first apply the unfolding operator $\mathcal{T}_\varepsilon$ to both sequences $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$. By Lemma 4.2 and estimates (5.1), there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, $\tilde{u} \in L^p(\Omega \times Y''; \nabla_{x'})$, $\tilde{v} \in L^p(\Omega \times Y''; \nabla_{x'})^{N_1}$, $\tilde{u} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$, $\tilde{v} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))^{N_1}$ such that

$$
\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup \tilde{u} \quad \text{weakly in} \quad L^p(\Omega \times Y''; W^{1,p}(Y')),
$$

$$
\mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon) \rightharpoonup \nabla_{x'}\tilde{u} + \nabla_{y'}\tilde{u} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1},
$$

$$
\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup \tilde{v} \quad \text{weakly in} \quad L^p(\Omega \times Y''; W^{1,p}(Y'))^{N_1},
$$

$$
\mathcal{T}_\varepsilon(\nabla_{x'}v_\varepsilon) \rightharpoonup \nabla_{x'}\tilde{v} + \nabla_{y'}\tilde{v} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1 \times N_1}.
$$

(5.4)

By convergence (5.2), there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $\tilde{Z} \in L^p(\Omega \times Y)^{N_1}$ with $\mathcal{M}_Y(\tilde{Z}) = 2$ such that

$$
\frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla_{x'}u_\varepsilon + v_\varepsilon) \rightharpoonup \tilde{Z} \quad \text{weakly in} \quad L^p(\Omega \times Y)^{N_1}.
$$

(5.5)

From convergences (5.4) and (5.5) we get

$$
\nabla_{x'}\tilde{u} + \nabla_{y'}\tilde{u} + \tilde{v} = 0 \quad \text{a.e. in} \quad \Omega \times Y.
$$

Applying $\mathcal{M}_{Y'}$ to the above equality and since $\tilde{u} \in L^p(\Omega \times Y''; W_{per,0}^{1,p}(Y'))$, while $\tilde{u} \in L^p(\Omega \times Y''; \nabla_{x'})$, $\tilde{v} \in L^p(\Omega \times Y'', \nabla_{x'})$, we get that $\nabla_{x'}\tilde{u} + \tilde{v} = 0$ a.e. in $\Omega \times Y''$. Hence, $\nabla_{y'}\tilde{u} = 0$ and thus $\tilde{u} = 0$ because it belongs to $L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'))$. As a consequence, one has

$$
\tilde{u} \in L^p(\Omega \times Y'', D_{x'}^2).
$$

(5.6)

Then, $\omega'$ and $\omega''$ be two open sets such that $\omega' \subset \mathbb{R}^{N_1}$, $\omega'' \subset \mathbb{R}^{N_2}$ and $\overline{\omega' \times \omega''} \subset \Omega$.

First, observe that

$$
U_\varepsilon \in L^p(\omega' \times Y''; W^{1,p}(\omega')) \quad \text{and} \quad V_\varepsilon \in L^p(\omega'' \times Y''; W^{1,p}(\omega''))^{N_1}.
$$

By the above convergence and (5.4), one has

$$
\frac{1}{\varepsilon} \nabla_{x'}U_\varepsilon + V_\varepsilon \rightharpoonup \tilde{Z} \quad \text{weakly in} \quad L^p(\omega' \times \omega'' \times Y'')^{N_1},
$$

$$
\mathcal{T}_\varepsilon(\nabla_{x'}v_\varepsilon) = \mathcal{T}_\varepsilon(\nabla_{x'}V_\varepsilon) \rightharpoonup \nabla_{x'}\tilde{u} + \nabla_{y'}\tilde{u} \quad \text{weakly in} \quad L^p(\omega' \times \omega'' \times Y' \times Y'')^{N_1 \times N_1}.
$$

(5.7)

Lemma 5.2 claims that up to a subsequence, there exists $u_{\omega' \times \omega''}$, which belongs to

$$
L^p(\omega' \times Y''; W^{1,p}_{per,0}(Y')) \quad \text{and} \quad L^p(\omega'' \times Y''; W^{1,p}_{per,0}(Y'))^{N_1},
$$

such that the following convergence holds:

$$
\frac{1}{\varepsilon} \mathcal{T}_\varepsilon(\nabla_{x'}U_\varepsilon + V_\varepsilon) \rightharpoonup \tilde{Z} + \nabla_{y'}u_{\omega' \times \omega''} + \tilde{v} \quad \text{weakly in} \quad L^p(\omega' \times \omega'' \times Y' \times Y'')^{N_1}.
$$

(5.8)
Taking into account convergence (5.5) we get
\[ \tilde{Z} = \tilde{Z} + \nabla_y' \tilde{u} \times \omega'' + \tilde{v} \quad \text{in} \quad \omega' \times \omega'' \times Y. \]

Since one can cover \( \Omega \) by a countable family of open subsets \( \omega' \times \omega'' \) satisfying (5.6), there exists \( u \) in \( L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y')) \) such that \( \tilde{Z} - \tilde{Z} - \tilde{v} = \nabla_y u \). This completes the proof of (5.3).

With some more assumptions, we can improve the regularity of the limit functions.

**Lemma 5.4.** Let \( \{(u_\varepsilon, v_\varepsilon)\}_\varepsilon \) be a sequence in \( L^p(\Omega, \nabla_x') \times L^p(\Omega, \nabla_x')^N_1 \), with \( p \in (1, +\infty) \), satisfying the assumptions in Lemma 5.3. Moreover, assume that

\[ \|\nabla_x''(\nabla_x'u_\varepsilon + v_\varepsilon)\|_{L^p(\Omega)} + \varepsilon \|\nabla_x''(\nabla_x'u_\varepsilon)\|_{L^p(\Omega)} \leq C, \]

where the constant does not depend on \( \varepsilon \).

Then, there exist a subsequence of \( \{\varepsilon\} \), still denoted \( \{\varepsilon\} \), still denoted \( \{\varepsilon\} \), and \( \tilde{Z} \in L^p(\Omega \times Y'')^N_1 \), \( u \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y')) \), \( \tilde{v} \in L^p(\Omega; W^{1,p}_{per,0}(Y')) \), \( \tilde{u} \in L^p(\Omega; D^2_{per}) \) and a function \( \tilde{\nu} \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'')) \) such that

\[ \frac{1}{\varepsilon} T_\varepsilon(\nabla_x'u_\varepsilon + v_\varepsilon) \rightarrow \tilde{Z} + \nabla_y' \tilde{\nu} \quad \text{weakly in} \quad L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y')) \]

**Proof.** From Lemma 5.3, there exist a subsequence of \( \{\varepsilon\} \), still denoted \( \{\varepsilon\} \), and

\[ \tilde{Z} \in L^p(\Omega \times Y'')^N_1 \], \( u \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y')) \), \( \tilde{v} \in L^p(\Omega \times Y'', D^2_{per}) \) and a function

\[ \tilde{u} \in L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'')) \]

such that

\[ \frac{1}{\varepsilon} T_\varepsilon(\nabla_x'u_\varepsilon + v_\varepsilon) \rightarrow \tilde{Z} + \nabla_y' \tilde{\nu} \quad \text{weakly in} \quad L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'')) \]

By hypothesis (5.7), Lemma 4.1 (swapping \( Y' \) and \( Y'' \)) and the proof of Lemma 4.4 one has

\[ \frac{1}{\varepsilon} T_\varepsilon(\nabla_x'u_\varepsilon + v_\varepsilon) \rightarrow \tilde{Z} + \nabla_y' \tilde{\nu} \quad \text{weakly in} \quad L^p(\Omega \times Y''; W^{1,p}_{per,0}(Y'')) \]

with \( \tilde{Z} \in L^p(\Omega \times Y'')^N_1 \), \( \tilde{u} \in L^p(\Omega, D^2_{per}; W^{1,p}_{per}(Y'')) \) and \( \tilde{\nu} \in L^p(\Omega; W^{1,p}_{per}(Y'')) \) satisfying \( M_Y(\tilde{\nu}) = 0 \) a.e. in \( \Omega \times Y'' \).

Since, \( \tilde{\nu} \) satisfies \( M_Y(\tilde{\nu}) = 0 \) a.e. in \( \Omega \times Y'' \) and \( M_Y(\nabla_y' u) = 0 \) a.e. in \( \Omega \times Y'' \) by periodicity of \( u \), we obtain

\[ \tilde{Z} = M_Y(\tilde{Z}) \in L^p(\Omega; W^{1,p}_{per}(Y''))^N_1. \]

Hence \( \nabla_y' u \) lies in \( L^p(\Omega \times Y''; W^{1,p}_{per}(Y''))^N_1 \). Lemma 7.2 in Appendix gives a function

\( u \in L^p(\Omega; W^{1,p}_{per,0}(Y'')) \) such that \( \nabla_y' u = \nabla_y' u \). The proof is complete.

**6. Application: homogenization of a homogeneous Dirichlet problem.**

We want to give a direct application of the periodic unfolding for anisotropically bounded sequences.

Let \( \mathcal{O} \) be an open subset of \( \mathbb{R}^N \) and let \( \alpha, \beta \in \mathbb{R} \) with \( 0 < \alpha < \beta \). Denote \( M(\alpha, \beta, \mathcal{O}) \)

the set of \( N \times N \) matrices \( A = (a_{ij})_{1 \leq i, j \leq N} \) with coefficients in \( L^\infty(\mathcal{O}) \) such that for every \( \lambda \in \mathbb{R}^N \) and for a.e. \( x \in \mathcal{O} \), the following inequalities hold:
There exist \( \tilde{L} \) \( (6.4) \) where the constant does not depend on \( (6.2) \)

\[ \| H \| \leq C \| \nabla^2 u \| L^2(\Omega). \]

Thus, problem (6.1) admits a unique solution by the Lax–Milgram theorem and the following inequality holds:

\[ \alpha \left( \| \nabla u \|_{L^2(\Omega)}^2 + \epsilon^2 \| \nabla u \|_{L^2(\Omega)}^2 \right) \leq \| f \|_{L^2(\Omega)} \| \nabla u \|_{L^2(\Omega)} \leq C \| f \|_{L^2(\Omega)} \| \nabla^2 u \|_{L^2(\Omega)}. \]

Hence

\[ (6.2) \quad \| u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\Omega)} + \epsilon \| \nabla^2 u \|_{L^2(\Omega)} \leq C \| f \|_{L^2(\Omega)}, \]

where the constant does not depend on \( \epsilon \).

Set

\[ H_{0,\text{per}}(\Omega \times Y'') = \{ \phi \in H^1(\Omega \times Y'') \mid \phi(x, y) = 0 \text{ for a.e. } (x, y') \in \partial \Omega \times Y'' \text{ and } \phi(x, \cdot) \text{ is } Y'' \text{ periodic for a.e. } x \in \Omega \}. \]

Denote \( L_0^0(\Omega, \nabla x') \) (resp. \( L_0^0(\Omega, \nabla x'; H^1_{\text{per}}(Y'')) \)) the closure of \( H_0^1(\Omega) \) (resp. of \( H_{0,\text{per}}(\Omega \times Y'') \)) in \( L^2(\Omega) \) (resp. \( L^2(\Omega \times Y'') \)) for the norm of \( L^2(\Omega, \nabla x') \) (resp. \( L^2(\Omega, \nabla x'; H^1_{\text{per}}(Y'')) \)), see (3.1)-(3.2).

Below, we give the periodic homogenization via unfolding.

**Theorem 6.1.** Let \( u_\epsilon \) be the solution of problem (6.1).

There exist \( \tilde{u} \in L_0^0(\Omega, \nabla x'; H^1_{\text{per}}(Y'')) \) and \( \tilde{u} \in L^2(\Omega \times Y''; H^1_{\text{per,0}}(Y'')) \) such that

\[ u_\epsilon \rightharpoonup \mathcal{M}(\tilde{u}) \quad \text{weakly in } L^2(\Omega, \nabla x'), \]

\[ T_\epsilon(u_\epsilon) \rightharpoonup \tilde{u} \quad \text{weakly in } L^2(\Omega; H^1(Y)), \]

\[ \epsilon T_\epsilon(\nabla x'u_\epsilon) \rightharpoonup \nabla x'\tilde{u} + \nabla y'\tilde{u} \quad \text{strongly in } L^2(\Omega \times Y)^{N_1}, \]

\[ \epsilon T_\epsilon(\nabla x'u_\epsilon) \rightharpoonup \nabla y'\tilde{u} \quad \text{strongly in } L^2(\Omega \times Y)^{N_2}. \]

The couple \( (\tilde{u}, \hat{u}) \) is the unique solution of problem

\[ (6.4) \quad \left\{ \begin{array}{l}
\int_{\Omega \times Y} A(y) \left( \nabla x'\tilde{u}(x, y') + \nabla y'\tilde{u}(x, y) \right) \cdot \left( \nabla x'\hat{\phi}(x, y') + \nabla y'\hat{\phi}(x, y) \right) \, dx \, dy \\
\quad = \left| Y' \right| \int_{\Omega \times Y''} f(x) \hat{\phi}(x, y') \, dx \, dy',
\end{array} \right. \]

\[ \forall \hat{\phi} \in L_0^0(\Omega, \nabla x'; H^1_{\text{per,0}}(Y'')) \quad \text{and} \quad \forall \hat{\phi} \in L^2(\Omega \times Y''; H^1_{\text{per,0}}(Y'')). \]
Proof. Step 1. We show (6.4) and the weak convergences (6.3).

The solution \( u_\varepsilon \) of (6.1) satisfies (6.2). Hence, up to a subsequence of \( \{ \varepsilon \} \), still denoted \( \{ \varepsilon \} \), Lemma 4.3 gives \( \tilde{u} \in L^2_0(\Omega, \nabla_{x'}; H^1_{\text{per}}(Y')) \) and \( \tilde{u} \in L^2(\Omega \times Y''; H^1_{\text{per},0}(Y')) \) such that

\[
\begin{align*}
    u_\varepsilon & \rightharpoonup M_Y(\tilde{u}) \quad \text{weakly in} \quad L^2_0(\Omega, \nabla_{x'}), \\
    T_\varepsilon(u_\varepsilon) & \rightharpoonup \tilde{u} \quad \text{weakly in} \quad L^2(\Omega; H^1(Y)), \\
    T_\varepsilon(\nabla_{x'} u_\varepsilon) & \rightharpoonup \nabla_{x'} \tilde{u} + \nabla_{y'} \tilde{u} \quad \text{weakly in} \quad L^2(\Omega \times Y)^{N_1}.
\end{align*}
\]

Moreover, from convergence (6.5) we also get that

\[
\varepsilon T_\varepsilon(\nabla_{x''} u_\varepsilon) = \nabla_{y''} T_\varepsilon(u_\varepsilon) \rightharpoonup \nabla_{y''} \tilde{u} \quad \text{weakly in} \quad L^2(\Omega \times Y'')^{N_2}.
\]

Now, we choose the test functions

- \( \tilde{\phi} \in C^1(\Omega \times Y) \cap L^2_0(\Omega, \nabla_{x'}; H^1_{\text{per}}(Y')) \),
- \( \Phi \in C^1(\Omega \times Y''), \)
- \( \tilde{\phi} \in H^1_{\text{per},0}(Y') \).

Set

\[
    \phi_\varepsilon(x) = \tilde{\phi}(x, x''/\varepsilon) + \varepsilon \Phi(x, x''/\varepsilon) \tilde{\phi}(x'/\varepsilon), \quad \text{a.e.} \ x \in \Omega.
\]

Applying the unfolding operator to the sequence \( \{ \phi_\varepsilon \} \), we get that

\[
    T_\varepsilon(\phi_\varepsilon) \rightharpoonup \tilde{\phi} \quad \text{strongly in} \quad L^2(\Omega; H^1(Y)),
\]

\[
    T_\varepsilon(\nabla_{x'} \phi_\varepsilon) \rightharpoonup \nabla_{x'} \tilde{\phi} + \Phi \nabla_{y'} \tilde{\phi} \quad \text{strongly in} \quad L^2(\Omega \times Y)^{N_1},
\]

\[
    \varepsilon T_\varepsilon(\nabla_{x''} \phi_\varepsilon) \rightharpoonup \nabla_{y''} \tilde{\phi} \quad \text{strongly in} \quad L^2(\Omega \times Y'')^{N_2}.
\]

Taking \( \phi_\varepsilon \) as test function in (6.1), then transforming by unfolding and passing to the limit give (6.4) with \( (\tilde{\phi}, \Phi \tilde{\phi}) \). By density argumentation, we extend such results for all \( \phi \in L^2_0(\Omega, \nabla_{x'}; H^1_{\text{per}}(Y')) \) and all \( \tilde{\phi} \in L^2(\Omega \times Y''; H^1_{\text{per},0}(Y')) \). Since the solution is unique the whole sequences converge to their limit.

Step 2. We prove that convergences (6.3) are strong.

First, setting \( \phi = u_\varepsilon \) in (6.1), then transforming by unfolding and using the weak lower semicontinuity yield

\[
\int_{\Omega \times Y} A \left( \nabla_{x'} \tilde{u} + \nabla_{y'} \tilde{u} \right) \cdot \left( \nabla_{x'} u_\varepsilon + \nabla_{y'} u_\varepsilon \right) \, dx \, dy
\]

\[
\leq \liminf_{\varepsilon \to 0} \int_{\Omega \times Y} A \left( \varepsilon T_\varepsilon(\nabla_{x''} u_\varepsilon) \right) \cdot \left( T_\varepsilon(\nabla_{x''} u_\varepsilon) \right) \, dx \, dy
\]

\[
= \liminf_{\varepsilon \to 0} \left( \int_{\Omega} A_{\varepsilon} \left( \nabla_{x''} u_\varepsilon \right) \cdot \left( \nabla_{x''} u_\varepsilon \right) \, dx - \int_{\Omega} A_{\varepsilon} \left( \varepsilon \nabla_{x''} u_\varepsilon \right) \cdot \left( \varepsilon \nabla_{x''} u_\varepsilon \right) \, dx \right)
\]

\[
\leq \limsup_{\varepsilon \to 0} \left( \int_{\Omega} A_{\varepsilon} \left( \nabla_{x''} u_\varepsilon \right) \cdot \left( \nabla_{x''} u_\varepsilon \right) \, dx - \int_{\Omega} A_{\varepsilon} \left( \varepsilon \nabla_{x''} u_\varepsilon \right) \cdot \left( \varepsilon \nabla_{x''} u_\varepsilon \right) \, dx \right)
\]

\[
= \limsup_{\varepsilon \to 0} \int_{\Omega} A \left( \nabla_{x''} \tilde{u} + \nabla_{y''} \tilde{u} \right) \cdot \left( \nabla_{x''} u_\varepsilon + \nabla_{y''} u_\varepsilon \right) \, dx \, dy = \limsup_{\varepsilon \to 0} \int_{\Omega} f u_\varepsilon \, dx,
\]

\[
= \lim_{\varepsilon \to 0} \int_{\Omega \times Y} T_\varepsilon(f) T_\varepsilon(u_\varepsilon) \, dx = \int_{\Omega \times Y} f \tilde{u} \, dx \, dy
\]

\[
= \int_{\Omega \times Y} A \left( \nabla_{x'} \tilde{u} + \nabla_{y'} \tilde{u} \right) \cdot \left( \nabla_{x'} \tilde{u} + \nabla_{y'} \tilde{u} \right) \, dx \, dy.
\]
Since the map \( \Psi \in L^2(\Omega \times Y)^N \mapsto \sqrt{\int_{\Omega \times Y} A \Psi \cdot \Psi \, dx dy} \) is a norm equivalent to the usual norm of \( L^2(\Omega \times Y)^N \), we get

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times Y} \left| \left( T_{y}(u_{x}) \right)_{\varepsilon} \right|^2 \, dx dy = \int_{\Omega \times Y} \left| \left( \nabla_{x} \tilde{u} + \nabla_{y} \tilde{u} \right) \right|^2 \, dx dy.
\]

This, together with the fact that (6.5) already converge weakly, ensures the strong convergences (6.3). The proof is therefore complete. \( \square \)

Set

\[
A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}
\]

where

- \( A_1 \) is a \( N_1 \times N_1 \) matrix with entries in \( L^\infty(Y) \),
- \( A_2 \) is a \( N_1 \times N_2 \) matrix with entries in \( L^\infty(Y) \),
- \( A_3 \) is a \( N_2 \times N_1 \) matrix with entries in \( L^\infty(Y) \),
- \( A_4 \) is a \( N_2 \times N_2 \) matrix with entries in \( L^\infty(Y) \).

We define the correctors \( \hat{x}_k \), \( k \in \{1, \ldots, N\} \), as the unique solutions in the space \( L^\infty(Y'', H^1_{\text{per},0}(Y')) \) of the cell problems:

\[
(6.6) \quad \left\{ \begin{array}{l}
\int_{Y'} A_1(y', \cdot) \nabla_{y'} \hat{x}_k(y', \cdot) \cdot \nabla_{y'} \hat{w}(y') \, dy' = - \int_{Y'} A(y', \cdot) \mathbf{e}_k \cdot \left( \nabla_{y'} \hat{w}(y') \right)_0 \, dy', \\
\forall \hat{w} \in H^1_{\text{per},0}(Y').
\end{array} \right.
\]

By the Lax–Milgram theorem applied in Hilbert space \( L^2(Y'', H^1_{\text{per},0}(Y')) \), we obtain the existence and uniqueness of the solution of (6.6) for every \( k \in \{1, \ldots, N\} \).

Since \( A \) belongs to \( M(\alpha, \beta, Y) \) we get for every \( k \in \{1, \ldots, N\} \):

\[ \| \nabla_{y'} \hat{x}_k(\cdot, y'') \|_{H^1(Y')} \leq \frac{\beta}{\alpha} \text{ for a.e. } y'' \in Y''. \]

As a consequence \( \hat{x}_k \in L^\infty(Y'', H^1_{\text{per},0}(Y')) \) for every \( k \in \{1, \ldots, N\} \) and

\[ \| \hat{x}_k \|_{L^\infty(Y'', H^1(Y'))} \leq C. \]

---

1One can prove that \( \hat{x}_k \) also belongs to \( L^\infty(Y) \).
We prove now that $L$ belongs to $A$. Replacing $\hat{\phi}$ in (6.8) we get the form (6.8) of the homogenizing operator $A$. By (6.9), we can write
\[
\begin{aligned}
A_{\text{hom}} \hat{\phi} &= \frac{1}{|Y'|} \int_{Y'} \left( A + \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \nabla y' \hat{x} \right) \cdot \begin{pmatrix} \hat{\phi} \\ 0 \end{pmatrix} dy', \\
\forall \hat{\phi} &\in L^2(\Omega, \nabla x'; H^1_{\text{per}}(Y')).
\end{aligned}
\]

The homogenizing operator $A_{\text{hom}} \in L^\infty(Y')^{N \times N}$ is defined by
\[
A_{\text{hom}} = \frac{1}{|Y'|} \int_{Y'} \left( A + \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \nabla y' \hat{x} \right) \cdot \begin{pmatrix} \hat{\phi} \\ 0 \end{pmatrix} dy',
\]

where $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{N_1}, \hat{x}_{N_1+1}, \ldots, \hat{x}_N)$ and thus $\nabla y' \hat{x}$ is the $N_1 \times N$ matrix
\[
\nabla y' \hat{x} = \begin{pmatrix} \nabla y' \hat{x}_1 \\ \nabla y' \hat{x}_2 \\ \vdots \\ \nabla y' \hat{x}_{N_1} \\ \nabla y' \hat{x}_{N_1+1} \\ \vdots \\ \nabla y' \hat{x}_N \end{pmatrix}.
\]

Proof. Equation (6.4) with $\hat{\phi} = 0$ leads to:
\[
\begin{aligned}
\int_{\Omega \times Y''} A(y', y'') \nabla y' \hat{u}(x, y', y'') \cdot \nabla y' \hat{\phi}(x, y', y'') \, dxdy'dy'' \\
&= - \int_{\Omega \times Y''} A(y', y'') \nabla x' \hat{u}(x, y'') \cdot \nabla y' \hat{\phi}(x, y', y'') \, dxdy'dy'', \\
\forall \hat{\phi} &\in L^2(\Omega \times Y''; H^1_{\text{per}}(Y')).
\end{aligned}
\]

from which the form of the cell problems (6.6) follows.

By (6.9), we can write $\tilde{u}$ as
\[
\tilde{u}(x, y', y'') = \sum_{k=1}^{N_1} \hat{x}_k(y', y'') \partial_k \hat{u}(x, y'') + \sum_{k=N_{1+1}}^{N} \hat{x}_k(y', y'') \partial_k \hat{u}(x, y'')
\]

for a.e. $(x, y', y'') \in \Omega \times Y' \times Y''$.

Replacing $\tilde{u}$ by the above equality in (6.4) and gathering all the $y'$ dependent terms, we get the form (6.8) of the homogenizing operator $A_{\text{hom}}$.

Since $A \in L^\infty(Y')^N \times N$ and the $\hat{x}_k$'s are in $L^\infty(Y''; H^1(Y'))$, it is clear that $A_{\text{hom}}$ belongs to $L^\infty(Y'')^N \times N$.

We prove now that $A_{\text{hom}}$ is coercive. Let $\xi = (\xi_1, \xi_2)$ be a vector with fixed entries in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. By construction of the homogenizing operator, we have
\[
A_{\text{hom}} \xi \cdot \xi = A_{\text{hom}} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} =
\]
\[
= \frac{1}{|Y'|} \int_{Y'} A \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} dy',
\]
\[
= \frac{1}{|Y'|} \int_{Y'} \begin{pmatrix} \xi_1 + \nabla y' \hat{x} \xi \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} dy',
\]
\[
= \frac{1}{|Y'|} \int_{Y'} A \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} dy',
\]

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Now, suppose \( H_{\text{per}, 0}(Y') \). Hence, by estimates (7.3)-(7.4) we obtain (7.2).

Then, again by equality (7.1) and the Poincaré-Wirtinger Inequality, we get

\[
A_{\text{hom}}(y'') \xi \cdot \xi = \frac{1}{|Y'|} \int_{Y'} A \left( \frac{\xi_1 + \nabla y' \hat{\chi}_\xi}{\xi_2} \right) \frac{\xi_1 + \nabla y' \hat{\chi}_\xi}{\xi_2} \ dy'
\]

\[
\geq \alpha (\|\xi_1 + \nabla y' \hat{\chi}_\xi(y'', \cdot)\|_{L^2(Y')}^2 + |\xi_2|^2)
\]

which proves that \( A_{\text{hom}}(y'') \) is coercive.

Replacing the form of \( A_{\text{hom}} \) on the original problem (6.4), we get (6.7). By the boundedness and coercivity of \( A_{\text{hom}} \) and by the fact that the function \( \tilde{u} \) belongs to \( L^2(\Omega, \nabla x'; H^{\text{per}}_Y(Y')) \), the above problem admits a unique solution \( \tilde{u}_0 \) by the Poincaré inequality and the Lax–Milgram theorem.

7. Appendix.

**Lemma 7.1.** Let \( u \) be in \( L^p(Y''; W^{1, p}(Y')) \) \( (p \in (1, +\infty)) \) such that

\[
\nabla y'' u \in L^p(Y'; W^{1, p}(Y'))^N.
\]

Then \( u = u - M_{Y'}(u) \) belongs to \( W^{1, p}(Y') \). It satisfies

(7.1)

\[
\nabla y'' u = \nabla y'' u \quad \text{a.e. in } Y
\]

and

(7.2)

\[
\|u\|_{W^{1, p}(Y)} \leq C \left( \|\nabla y'' u\|_{L^p(Y''; Y')} + \|\nabla y''(\nabla y'' u)\|_{L^p(Y''; Y'')} \right).
\]

**Proof.** Step 1. First, assume \( u \in \mathcal{C}^2(\overline{Y}) \).

Set \( u = u - M_{Y'}(u) \). It is clear that (7.1) is satisfied. We prove now the estimate (7.2) of \( u \).

By definition of \( u \), (7.1) and the Poincaré-Wirtinger Inequality we have

(7.3)

\[
\|u\|_{L^p(Y''; Y')} = \|u - M_{Y'}(u)\|_{L^p(Y''; Y')} \leq C \|\nabla y'' u\|_{L^p(Y''; Y')},
\]

\[
\|\nabla y'' u\|_{L^p(Y''; Y')} = \|\nabla y'' u\|_{L^p(Y''; Y')}.
\]

Observe that \( M_{Y'}(\nabla y'' u) = \nabla y'' M_{Y'}(u) = 0 \).

Then, again by equality (7.1) and the Poincaré-Wirtinger Inequality, we get

(7.4)

\[
\|\nabla y'' u\|_{L^p(Y''; Y')} \leq C \|\nabla y''(\nabla y'' u)\|_{L^p(Y''; Y')} = \|\nabla y''(\nabla y'' u)\|_{L^p(Y''; Y')} = \|\nabla y''(\nabla y'' u)\|_{L^p(Y''; Y')}.
\]

Hence, by estimates (7.3)-(7.4) we obtain (7.2).

**Step 2.** Now, suppose \( u \) in \( L^p(Y''; W^{1, p}(Y')) \) and \( \nabla y'' u \) in \( L^p(Y'; W^{1, p}(Y')) \). Since

\[\text{It also belongs to } W^{1, p}(Y''; W^{1, p}(Y')).\]
$C^2(Y)$ is dense in this subspace of $L^p(Y'''; W^{1,p}(Y'))$, there exists a sequence of functions $u_n \in C^2(Y)$ such that

$$u_n \to u \quad \text{strongly in} \quad L^p(Y'''; W^{1,p}(Y')),$$

$$M_{Y'}(u_n) \to M_{Y'}(u) \quad \text{strongly in} \quad L^p(Y'''),$$

$$\nabla_{y'}u_n \to \nabla_{y'}u \quad \text{strongly in} \quad L^p(Y'''; W^{1,p}(Y'))^N_1.$$

The corresponding sequence $\{u_n\}$ (given by Step 1) satisfies $\nabla_{y'}u_n = \nabla_{y'}u_n$, moreover it belongs to $C^2(Y)$ and is bounded in $W^{1,p}(Y)$ (from (7.2)). Passing to the limit gives $u \in W^{1,p}(Y)$ such that

$$u_n \to u \quad \text{strongly in} \quad W^{1,p}(Y), \quad \nabla_{y'}u = \nabla_{y'}u \quad \text{a.e. in} \ Y.$$

Finally, observe that $u = u - M_{Y'}(u)$.

**Lemma 7.2.** Let $u$ be in $L^p(Y'''; W^{1,p}_{per}(Y'))$ ($p \in (1, +\infty)$) such that

$$\nabla_{y'}u \in L^p(Y'; W^{1,p}_{per}(Y''))^N_1.$$

Then, there exists $\Omega \in W^{1,p}_{per}(Y)$ such that

$$\nabla_{y'}\Omega = \nabla_{y'}u \quad \text{a.e. in} \ Y.$$

**Proof.** Since $u$ in $L^p(Y'''; W^{1,p}_{per}(Y'))$ and $\nabla_{y'}u \in L^p(Y'; W^{1,p}_{per}(Y''))^N_1$, the above Lemma 7.1 shows that the function $u = u - M_{Y'}(u)$ belongs to $W^{1,p}(Y)$. It is obvious that $u$ is periodic with respect to the variables $y_1, \ldots, y_N$. One also has $\nabla_{y'}u = \nabla_{y'}u \in L^p(Y'; W^{1,p}_{per}(Y''))^N_1$. Denote

$$Y_i = \{y \in Y \mid y_i = 0, y_j \in (0, 1), j \in \{1, \ldots, N\}, j \neq i, i \in \{1, \ldots, N\},$$

$$Y_i'' = \{y \in Y'' \mid y_i = 0, y_j \in (0, 1), j \in \{N_1 + 1, \ldots, N\}, j \neq i, i \in \{N_1 + 1, \ldots, N\}.$$ 

$$\nabla_{y'}u|_{Y_i + e_j} - \nabla_{y'}u|_{Y_i} = \nabla_{y'}u|_{Y_i + e_j} - \nabla_{y'}u|_{Y_i} = 0 \quad \text{a.e. in} \ Y_j.$$ 

Hence

$$u|_{Y_i + e_j} - u|_{Y_i} \in W^{1-1/p,p}(Y_i''), \quad j \in \{N_1 + 1, \ldots, N\}. $$

Besides, one has

$$u|_{Y_j + e_j} - u|_{Y_j} = 0, \quad j \in \{1, \ldots, N_1\}.$$ 

Then, following the same lines of the proofs of [13, Proposition 13.34 and Lemmas 13.35-13.36], there exits $\Omega \in W^{1,p}_{per}(Y)$ such that

$$\Omega - u \in W^{1,p}(Y').$$

and we have

$$\|\Omega - u\|_{W^{1,p}(Y)} \leq C \sum_{j=N_1+1}^N \|u|_{Y_j + e_j} - u|_{Y_j}\|_{W^{1-1/p,p}(Y_j'')},$$

$$\leq C(\|\nabla_{y'}u\|_{L^p(Y)} + \|\nabla_{y'}u\|_{L^p(Y'; W^{1,p}(Y'''))}).$$

The function $\Omega$ satisfies (7.5). 

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