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# PERIODIC UNFOLDING FOR LATTICE STRUCTURES

RICCARDO FALCONI\*, GEORGES GRISO†, AND JULIA ORLIK‡

**Abstract.** This paper deals with the periodic unfolding for sequences defined on one dimensional lattices in  $\mathbb{R}^N$ . In order to port the known results of the periodic unfolding in  $\mathbb{R}^N$  to lattices, the investigation of functions defined as interpolation on lattice nodes play the main role. The asymptotic behavior for sequences defined on periodic lattices with information until the first and until the second order derivatives are shown. In the end, a direct application of the results is given by homogenizing a 4th order Dirichlet problem defined on a periodic lattice.

**Key words.** Periodic Unfolding Method, homogenization, lattice graphs, anisotropic Sobolev spaces, thin structures

**AMS subject classifications.** 31C25, 35B27, 46E35, 49J45

**1. Introduction.** In the present, starting from the results obtained in [8, 14] about the periodic unfolding method for sequences defined on bounded domains in  $\mathbb{R}^N$ , we show in detail how to port such results to one-dimensional periodic lattice structures, spotting the obstacles we encountered and the tools we came up with to overcome them.

Given a small parameter  $\varepsilon$  and a bounded domain  $\Omega \subset \mathbb{R}^N$  with Lipschitz boundary, we consider the periodic paving of  $\Omega$  made with cells of size  $\varepsilon$ . In [8, Section 1.4] it is extensively investigated the asymptotic behavior of sequences  $\{\phi_\varepsilon\}_\varepsilon$  uniformly bounded in  $W^{1,p}(\Omega)$  and in  $W^{2,p}(\Omega)$ , while the entirety of [14] is devoted to sequences anisotropically bounded on  $W^{1,p}(\Omega)$ . In this paper, we first introduce the periodic lattice  $\mathcal{S}_\varepsilon \subset \mathbb{R}^N$  as one-dimensional arbitrary grids defined on the  $\varepsilon$  cells and periodically repeated for each cell of  $\Omega$ . The main idea to port the periodic unfolding results from  $\Omega \subset \mathbb{R}^N$  to  $\mathcal{S}_\varepsilon$  is based on extending the sequences bounded on  $\mathcal{S}_\varepsilon$  by different interpolation on the lattice nodes, applying the unfolding results in  $\mathbb{R}^N$  and then restricting the convergences to the lattice itself. Specifically, the  $Q_1$  interpolation on lattice nodes (already introduced in [12, 13]) allows to show the asymptotic behavior of sequences uniformly bounded in  $W^{1,p}(\mathcal{S}_\varepsilon)$  and anisotropically on  $W^{1,p}(\mathcal{S}_\varepsilon)$ , while for sequences bounded in  $W^{2,p}(\mathcal{S}_\varepsilon)$  some more work is involved due to the lack of mixed derivatives. Starting from different assumption strengths and leading to different regularity of the unfolded limit fields, two methods are developed (one involving extensions by a special  $Q_3$  interpolation and another involving the obtained results for sequences bounded in  $W^{1,p}(\mathcal{S}_\varepsilon)$ ). The sufficient assumptions on the sequences to ensure weak convergence in the space, as well as the rescaling factors for the unfolding operator for lattices according to the space dimension  $N$  and the  $L^p$  norm are proved. In the end, a direct application of such lemmas is done by homogenizing via unfolding the fourth order homogeneous Dirichlet problem defined on a lattice structure

$$\left\{ \begin{array}{l} \text{Find } u_\varepsilon \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon) \text{ such that:} \\ \int_{\mathcal{S}_\varepsilon} A_\varepsilon \partial_s^2 u_\varepsilon \partial_s^2 \phi \, ds = \int_{\mathcal{S}_\varepsilon} g_\varepsilon \partial_s \phi \, ds + \int_{\mathcal{S}_\varepsilon} f_\varepsilon \phi \, ds, \quad \forall \phi \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon). \end{array} \right.$$

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39 The homogenization via unfolding method, an equivalent to the two-scale convergence,  
 40 has been exhaustively explained in [8] and it is a constant reference throughout this  
 41 work. The method itself has, among many others, found application in the homog-  
 42 enization for thin periodic structures like periodically perforated shells (see [9]) and  
 43 textiles made of long curved beams (see [10, 11]). About the homogenization in the  
 44 frame of lattice structures one can look, for an instance, into [1, 2, 4, 5, 6, 7].

45 The present provides the main tools concerning the unfolding for lattice structures  
 46 and gives a rigorous base for up-coming papers dealing with thin structures made  
 47 from lattices. Among them, we would like to cite the homogenization via unfolding  
 48 for stable lattice structures made of beams (see [12, 13]) and the upcoming unstable  
 49 case [15], where it is additionally taken into consideration the problem of an aniso-  
 50 tropically bounded sequence. More generally, such tools can be applied to many other  
 51 problems related to partial differential equations on domains involving periodic grids,  
 52 lattices, thin frames and glued fiber structures.

53 The paper is organized as follows. In section 2, the standard notation and tools for  
 54 the classical homogenization via unfolding method in periodic domains  $\Omega \subset \mathbb{R}^N$  are  
 55 listed. In section 3, we recall the main results concerning the periodic unfolding for  
 56 sequences defined as  $Q_1$  interpolated on the vertexes of the  $\varepsilon$  cells paving  $\Omega$  and  
 57 bounded uniformly and anisotropically on  $W^{1,p}(\Omega)$ , whose properties will be needed  
 58 in the next sections. In section 4, we give a rigorous definition of one-dimensional  
 59 lattice structure  $\mathcal{S}_\varepsilon \subset \Omega$ , build the unfolding operator for lattices and give its main  
 60 properties. In section 5, we show the asymptotic behavior of sequences asymptot-  
 61 ically and uniformly bounded in  $W^{1,p}(\mathcal{S}_\varepsilon)$ . We first do it for functions defined as  
 62  $Q_1$  interpolated on lattice nodes, showing that for such sequences, the unfolding for  
 63 lattices is the mere extension of the functions from  $\mathcal{S}_\varepsilon$  to  $\mathbb{R}^N$  by  $Q_1$  interpolation,  
 64 application of the known results in section 3 and then restriction of the convergences  
 65 to the lattice itself. Later, we extend the results to Sobolev spaces by first decom-  
 66 posing them into  $Q_1$  interpolated part and reminder term. In section 6, we show the  
 67 asymptotic behavior of sequences asymptotically and uniformly bounded in  $W^{2,p}(\mathcal{S}_\varepsilon)$ .  
 68 The nature of a sequence bounded on a lattice leads to the lack of mixed derivatives,  
 69 since the derivation only makes sense in the lattice directions. To overcome such  
 70 deficiency, two approach are considered, one by a procedure analogous to section 5  
 71 but with a decomposition on  $Q_3$  interpolation on lattice nodes and reminder term,  
 72 and one by using twice (on the sequence and on the sequence gradient) the proved  
 73 for functions bounded in  $W^{1,p}(\mathcal{S}_\varepsilon)$ . At last, in section 7 we consider the fourth order  
 74 Dirichlet problem shown above. Using the results in the previous sections, existence  
 75 and uniqueness of the limit problem are shown and through the homogenization via  
 76 unfolding, the cell problems and the macroscopic limit problem are found.

**2. Preliminaries and notation.** Let  $\mathbb{R}^N$  be the euclidean space with usual  
 basis  $(\mathbf{e}_1, \dots, \mathbf{e}_N)$  and  $Y = (0, 1)^N$  the open unit parallelotope associated with this  
 basis. For a.e.  $z \in \mathbb{R}^N$ , we set the unique decomposition  $z = [z]_Y + \{z\}_Y$  such that

$$[z]_Y \doteq \sum_{i=1}^N k_i \mathbf{e}_i, \quad k = (k_1, \dots, k_N) \in \mathbb{Z}^N \quad \text{and} \quad \{z\}_Y \doteq z - [z]_Y \in Y.$$

77 Let  $\{\varepsilon\}$  be a sequence of strictly positive parameters going to 0. We scale our paving  
 78 by  $\varepsilon$  writing

$$79 \quad (2.1) \quad x = \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon \left\{ \frac{x}{\varepsilon} \right\}_Y \quad \text{for a.e. } x \in \mathbb{R}^N.$$

80 Let now  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary. We consider

$$81 \quad \Xi_\varepsilon \doteq \left\{ \xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega \right\}$$

82 and set

$$83 \quad (2.2) \quad \widehat{\Omega}_\varepsilon \doteq \text{int} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \bar{Y}) \right\}, \quad \Lambda_\varepsilon \doteq \Omega \setminus \widehat{\Omega}_\varepsilon.$$

84 We recall the definitions of classical unfolding operator and mean value operator.

DEFINITION 2.1. (see [8, Definition 1.2]) For every measurable function  $\phi$  on  $\Omega$ , the unfolding operator  $\mathcal{T}_\varepsilon$  is defined as follows:

$$\mathcal{T}_\varepsilon(\phi) \doteq \begin{cases} \phi \left( \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

85 Note that such an operator acts on functions defined in  $\Omega$  by operating on their  
86 restriction to  $\widehat{\Omega}_\varepsilon$ .

DEFINITION 2.2. (see [8, Definition 1.10]) For every measurable function  $\widehat{\phi}$  on  $L^1(\Omega \times Y)$ , the mean value operator  $\mathcal{M}_Y$  is defined as follows:

$$\mathcal{M}_Y(\widehat{\phi})(x) \doteq \frac{1}{|Y|} \int_Y \widehat{\phi}(x, y) dy, \quad \text{for a.e. } x \in \Omega.$$

87 Let  $p \in [1, +\infty]$ . From [8, Propositions 1.8 and 1.11], we recall the properties of these  
88 operators:

$$89 \quad \begin{aligned} \|\mathcal{T}_\varepsilon(\phi)\|_{L^p(\Omega \times Y)} &\leq |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Omega)} \quad \text{for every } \phi \in L^p(\Omega), \\ \|\mathcal{M}_Y(\widehat{\phi})\|_{L^p(\Omega)} &\leq |Y|^{-\frac{1}{p}} \|\widehat{\phi}\|_{L^p(\Omega \times Y)} \quad \text{for every } \widehat{\phi} \in L^p(\Omega \times Y). \end{aligned}$$

Since we will deal with Sobolev spaces, we give hereafter some definitions:

$$\begin{aligned} W_{per}^{1,p}(Y) &\doteq \left\{ \phi \in W^{1,p}(Y) \mid \phi \text{ is periodic with respect to } y_i, i \in \{1, \dots, N\} \right\}, \\ W_{per,0}^{1,p}(Y) &\doteq \left\{ \phi \in W_{per}^{1,p}(Y) \mid \mathcal{M}_Y(\phi) = 0 \right\}, \\ L^p(\Omega; W^{1,p}(Y)) &\doteq \left\{ \phi \in L^p(\Omega \times Y) \mid \nabla_y \phi \in L^p(\Omega \times Y)^N \right\}. \end{aligned}$$

90 Now, let  $(N_1, N_2)$  be in  $\mathbb{N} \times \mathbb{N}^*$  such that  $N = N_1 + N_2$ . We split the space by setting

$$\begin{aligned} 91 \quad \mathbb{R}^{N_1} &= \left\{ x' \in \mathbb{R}^N \mid x' = \sum_{i=1}^{N_1} x_i \mathbf{e}_i, \ x_i \in \mathbb{R} \right\}, \\ 92 \quad \mathbb{R}^{N_2} &= \left\{ x'' \in \mathbb{R}^N \mid x'' = \sum_{i=N_1+1}^N x_i \mathbf{e}_i, \ x_i \in \mathbb{R} \right\}, \\ 93 \quad Y' &= \left\{ y' \in \mathbb{R}^N \mid y' = \sum_{i=1}^{N_1} y_i \mathbf{e}_i, \ y_i \in (0, 1) \right\}, \\ 94 \quad Y'' &= \left\{ y'' \in \mathbb{R}^N \mid y'' = \sum_{i=N_1+1}^N y_i \mathbf{e}_i, \ y_i \in (0, 1) \right\} \\ 95 \end{aligned}$$

and

$$\mathbb{Z}^{N_1} = \mathbb{Z}\mathbf{e}_1 \oplus \dots \oplus \mathbb{Z}\mathbf{e}_{N_1}, \quad \mathbb{Z}^{N_2} = \mathbb{Z}\mathbf{e}_{N_1+1} \oplus \dots \oplus \mathbb{Z}\mathbf{e}_N.$$

One has

$$\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \quad Y = Y' \oplus Y'', \quad \mathbb{Z}^N = \mathbb{Z}^{N_1} \oplus \mathbb{Z}^{N_2}.$$

For every  $x \in \mathbb{R}^N$  and  $y \in Y$ , we write

$$x = x' + x'' \in \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}, \quad y = y' + y'' \in Y' \oplus Y''.$$

From now on, however, we find easier to refer to such partition with the vectorial notation

$$x = (x', x'') \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \quad y = (y', y'') \in Y' \times Y''.$$

Similarly to (2.1), we apply the paving to a.e.  $x' \in \mathbb{R}^{N_1}$  and  $x'' \in \mathbb{R}^{N_2}$  setting

$$\begin{aligned} x' &= \varepsilon \left[ \frac{x'}{\varepsilon} \right]_{Y'} + \varepsilon \left\{ \frac{x'}{\varepsilon} \right\}_{Y'}, & \text{with } \left[ \frac{x'}{\varepsilon} \right]_{Y'} &\in \mathbb{Z}^{N_1}, \quad \left\{ \frac{x'}{\varepsilon} \right\}_{Y'} \in Y', \\ x'' &= \varepsilon \left[ \frac{x''}{\varepsilon} \right]_{Y''} + \varepsilon \left\{ \frac{x''}{\varepsilon} \right\}_{Y''}, & \text{with } \left[ \frac{x''}{\varepsilon} \right]_{Y''} &\in \mathbb{Z}^{N_2}, \quad \left\{ \frac{x''}{\varepsilon} \right\}_{Y''} \in Y''. \end{aligned}$$

96

97 **DEFINITION 2.3.** For every  $\widehat{\phi} \in L^1(\Omega \times Y)$ , the partial mean value operators are  
98 defined as follows:

$$99 \quad \mathcal{M}_{Y'}(\widehat{\phi})(x, y'') \doteq \frac{1}{|Y''|} \int_{Y''} \widehat{\phi}(x, y', y'') dy'', \quad \text{for a.e. } (x, y'') \in \Omega \times Y'',$$

$$100 \quad \mathcal{M}_{Y''}(\widehat{\phi})(x, y') \doteq \frac{1}{|Y''|} \int_{Y''} \widehat{\phi}(x, y', y'') dy'', \quad \text{for a.e. } (x, y') \in \Omega \times Y'.$$

101

102 Denote

$$\begin{aligned} L^p(\Omega, \nabla_{x'}) &\doteq \{ \phi \in L^p(\Omega) \mid \nabla_{x'} \phi \in L^p(\Omega)^{N_1} \}, \\ L^p(\Omega, \nabla_{x''}) &\doteq \{ \phi \in L^p(\Omega) \mid \nabla_{x''} \phi \in L^p(\Omega)^{N_2} \}, \\ L^p(\Omega, \nabla_{x'}; W^{1,p}(Y'')) &\doteq \{ \widetilde{\phi} \in L^p(\Omega \times Y'') \mid \nabla_{x'} \widetilde{\phi} \in L^p(\Omega \times Y'')^{N_1}, \\ &\quad \nabla_{y''} \widetilde{\phi} \in L^p(\Omega \times Y'')^{N_2} \}, \\ 103 \quad L^p(\Omega, \nabla_{x''}; W^{1,p}(Y')) &\doteq \{ \widetilde{\phi} \in L^p(\Omega \times Y') \mid \nabla_{x''} \widetilde{\phi} \in L^p(\Omega \times Y')^{N_2}, \\ &\quad \nabla_{y'} \widetilde{\phi} \in L^p(\Omega \times Y')^{N_1} \}, \\ L^p(\Omega \times Y''; W^{1,p}(Y')) &\doteq \{ \widehat{\phi} \in L^p(\Omega \times Y) \mid \nabla_{y'} \widehat{\phi} \in L^p(\Omega \times Y)^{N_1} \}, \\ L^p(\Omega \times Y'; W^{1,p}(Y'')) &\doteq \{ \widehat{\phi} \in L^p(\Omega \times Y) \mid \nabla_{y''} \widehat{\phi} \in L^p(\Omega \times Y)^{N_2} \}. \end{aligned}$$

104 We endow these spaces with the respective norms:

$$\begin{aligned} \|\cdot\|_{L^p(\Omega, \nabla_{x'})} &\doteq \|\cdot\|_{L^p(\Omega)} + \|\nabla_{x'}(\cdot)\|_{L^p(\Omega)^{N_1}}, \\ \|\cdot\|_{L^p(\Omega, \nabla_{x''})} &\doteq \|\cdot\|_{L^p(\Omega)} + \|\nabla_{x''}(\cdot)\|_{L^p(\Omega)^{N_2}}, \\ 105 \quad \|\cdot\|_{L^p(\Omega, \nabla_{x'}; W^{1,p}(Y''))} &\doteq \|\cdot\|_{L^p(\Omega \times Y'')} + \|\nabla_{x'}(\cdot)\|_{L^p(\Omega \times Y'')^{N_1}} + \|\nabla_{y''}(\cdot)\|_{L^p(\Omega \times Y'')^{N_2}}, \\ \|\cdot\|_{L^p(\Omega, \nabla_{x''}; W^{1,p}(Y'))} &\doteq \|\cdot\|_{L^p(\Omega \times Y')} + \|\nabla_{x''}(\cdot)\|_{L^p(\Omega \times Y')^{N_2}} + \|\nabla_{y'}(\cdot)\|_{L^p(\Omega \times Y')^{N_1}}, \\ \|\cdot\|_{L^p(\Omega \times Y''; W^{1,p}(Y'))} &\doteq \|\cdot\|_{L^p(\Omega \times Y)} + \|\nabla_{y'}(\cdot)\|_{L^p(\Omega \times Y)^{N_1}}, \\ \|\cdot\|_{L^p(\Omega \times Y'; W^{1,p}(Y''))} &\doteq \|\cdot\|_{L^p(\Omega \times Y)} + \|\nabla_{y''}(\cdot)\|_{L^p(\Omega \times Y)^{N_2}}. \end{aligned}$$

106 **3. Periodic unfolding in  $\mathbb{R}^N$  for sequences defined as  $Q_1$  interpolates.**

107 The periodic unfolding for this class of functions has two main advantages. The first  
 108 is that less hypothesis are required for the sequences to ensure weak convergence. The  
 109 second is that the convergences can be restricted to subspaces with lower dimension  
 110 and it will be fundamental in the next sections, where lattice structures are taken into  
 111 account.

112 Define the spaces

113  $Q^1(Y) \doteq \{\phi \in W^{1,\infty}(Y) \mid \phi \text{ is the } Q_1 \text{ interpolation of its values on the vertexes of } \overline{Y}\},$

114  $Q_{per}^1(Y) \doteq \{\phi \in Q^1(Y) \mid \phi \text{ is periodic with respect to } y_i, i \in \{1, \dots, N\}\},$

115  $Q_{per,0}^1(Y) \doteq \{\phi \in Q_{per}^1(Y) \mid \mathcal{M}_Y(\phi) = 0\}.$

117 Denote

118 (3.1)  $\tilde{\Omega}_\varepsilon \doteq \text{int} \left\{ \bigcup_{\xi \in \tilde{\Xi}_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\}, \quad \tilde{\Xi}_\varepsilon \doteq \left\{ \xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \cap \Omega \neq \emptyset \right\}.$

Note that the covering  $\tilde{\Omega}_\varepsilon$  is now a connected open set and from (2.2) we have

$$\widehat{\Omega}_\varepsilon \subset \Omega \subset \tilde{\Omega}_\varepsilon.$$

Hence, we need to extend the definition of the classical unfolding operator (2.1) to functions defined in the following neighborhood of  $\Omega$ :

$$\{x \in \mathbb{R}^N \mid \text{dist}(x, \Omega) < \varepsilon \text{diam}(Y)\}.$$

119

120 **DEFINITION 3.1.** *For every measurable function  $\phi$  on  $\tilde{\Omega}_\varepsilon$ , the unfolding operator*  
 121  *$\mathcal{T}_\varepsilon^{ext}$  is defined as follows:*

122 
$$\mathcal{T}_\varepsilon^{ext}(\phi) \doteq \phi \left( \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) \quad \text{for a.e. } (x, y) \in \tilde{\Omega}_\varepsilon \times Y.$$

Every measurable function defined in  $\Omega$  can be extend to  $\tilde{\Omega}_\varepsilon$  by setting it to 0 in  $\tilde{\Omega}_\varepsilon \cap (\mathbb{R}^N \setminus \overline{\Omega})$ . Now, assume  $\{\Phi_\varepsilon\}_\varepsilon$  to be a sequence uniformly bounded in  $L^p(\tilde{\Omega}_\varepsilon)$ ,  $p \in (1, +\infty)$ . Then, the sequence  $\{\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)\}_\varepsilon$  is uniformly bounded in  $L^p(\tilde{\Omega}_\varepsilon \times Y)$  and thus in  $L^p(\Omega \times Y)$ . Hence, there exists a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and  $\widehat{\Phi} \in L^p(\Omega \times Y)$  such that

$$\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)|_{\Omega \times Y} \rightharpoonup \widehat{\Phi} \quad \text{weakly in } L^p(\Omega \times Y).$$

For simplicity, we will omit the restriction and always write the above convergence as

$$\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon) \rightharpoonup \widehat{\Phi} \quad \text{weakly in } L^p(\Omega \times Y).$$

123 In this sense, all the results obtained in [8, 14] are easily transposed to this operator.

124 Define the space of  $Q_1$  interpolated functions on  $\tilde{\Omega}_\varepsilon$  by

125 
$$Q_\varepsilon^1(\tilde{\Omega}_\varepsilon) \doteq \left\{ \Phi \in W^{1,\infty}(\tilde{\Omega}_\varepsilon) \mid \Phi|_{\varepsilon\xi + \varepsilon Y} \in Q^1(\varepsilon\xi + \varepsilon Y) \text{ for every } \xi \in \tilde{\Xi}_\varepsilon \right\}.$$

126

127 Due to the  $Q_1$  interpolation character, for every function  $\Phi \in Q_\varepsilon^1(\tilde{\Omega}_\varepsilon)$  we remind that  
 128 there exist a constant depending only on  $p$  such that

$$129 \quad (3.2) \quad \|\nabla \Phi\|_{L^p(\tilde{\Omega}_\varepsilon)} \leq \frac{C}{\varepsilon} \|\Phi\|_{L^p(\tilde{\Omega}_\varepsilon)}.$$

130 We have the following.

LEMMA 3.2. *Let  $\{\Phi_\varepsilon\}_\varepsilon$  be a sequence in  $Q_\varepsilon^1(\tilde{\Omega}_\varepsilon)$  that satisfies*

$$\|\Phi_\varepsilon\|_{L^p(\tilde{\Omega}_\varepsilon)} + \|\nabla_{x'} \Phi_\varepsilon\|_{L^p(\tilde{\Omega}_\varepsilon)} \leq C,$$

131 *where the constant does not depend on  $\varepsilon$ .*

132 *Then, there exist a subsequence of  $\{\varepsilon\}$ , denoted  $\{\varepsilon\}$ , and  $\tilde{\Phi} \in L^p(\Omega, \nabla_{x'}; Q_{per}^1(Y''))$ ,  
 133  $\hat{\Phi} \in L^p(\Omega \times Y''; Q_{per}^1(Y')) \cap L^p(\Omega; Q^1(Y))$ , satisfying  $\mathcal{M}_{Y'}(\hat{\Phi}) = 0$  a.e. in  $\Omega \times Y''$ ,  
 134 such that*

$$\begin{aligned} & \Phi_{\varepsilon|\Omega} \rightharpoonup \Phi \quad \text{weakly in } L^p(\Omega, \nabla_{x'}), \\ & \mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon) \rightharpoonup \tilde{\Phi} \quad \text{weakly in } L^p(\Omega; Q^1(Y)), \\ 135 \quad & \mathcal{T}_\varepsilon^{ext}(\nabla_{x'} \Phi_\varepsilon) \rightharpoonup \nabla_{x'} \tilde{\Phi} + \nabla_{y'} \hat{\Phi} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}, \\ & \frac{1}{\varepsilon} (\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon) - \mathcal{M}_{Y'} \circ \mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)) \rightharpoonup \nabla_{x'} \tilde{\Phi} \cdot y'^c + \hat{\Phi} \quad \text{weakly in } L^p(\Omega \times Y), \end{aligned}$$

136 *where  $\Phi = \mathcal{M}_{Y''}(\tilde{\Phi})$  and  $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$ .*

137 *The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
 138 the corresponding spaces.*

*Proof.* First, since the sequence  $\{\Phi_\varepsilon\}_\varepsilon$  belongs to  $Q_\varepsilon^1(\tilde{\Omega}_\varepsilon)$  we get (see (3.2))

$$\|\Phi_\varepsilon\|_{L^p(\tilde{\Omega}_\varepsilon)} + \|\nabla_{x'} \Phi_\varepsilon\|_{L^p(\tilde{\Omega}_\varepsilon)} + \varepsilon \|\nabla_{x''} \Phi_\varepsilon\|_{L^p(\tilde{\Omega}_\varepsilon)} \leq C.$$

139 The constant does not depend on  $\varepsilon$ . The statement follows by [14, Lemma 4.3] and  
 140 the fact that  $\{\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)\}_\varepsilon \subset L^p(\tilde{\Omega}_\varepsilon; Q^1(Y))$ .  $\square$

141 As a direct consequence, we have the following corollary.

COROLLARY 3.3. *Let  $\{\Phi_\varepsilon\}_\varepsilon$  be a sequence in  $Q_\varepsilon^1(\tilde{\Omega}_\varepsilon)$  satisfying*

$$\|\Phi_\varepsilon\|_{W^{1,p}(\tilde{\Omega}_\varepsilon)} \leq C,$$

142 *where the constant does not depend on  $\varepsilon$ .*

143 *Then, there exist a subsequence of  $\{\varepsilon\}$ , denoted  $\{\varepsilon\}$ , and functions  $\Phi \in W^{1,p}(\Omega)$ ,  
 144  $\hat{\Phi} \in L^p(\Omega; Q_{per,0}^1(Y))$  such that*

$$\begin{aligned} & \Phi_{\varepsilon|\Omega} \rightharpoonup \Phi \quad \text{weakly in } W^{1,p}(\Omega), \\ & \mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon) \rightharpoonup \Phi \quad \text{weakly in } L^p(\Omega; Q^1(Y)), \\ 145 \quad & \mathcal{T}_\varepsilon^{ext}(\nabla \Phi_\varepsilon) \rightharpoonup \nabla \Phi + \nabla_y \hat{\Phi} \quad \text{weakly in } L^p(\Omega \times Y)^N, \\ & \frac{1}{\varepsilon} (\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon) - \mathcal{M}_Y \circ \mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)) \rightharpoonup \nabla \Phi \cdot y'^c + \hat{\Phi} \quad \text{weakly in } L^p(\Omega \times Y), \end{aligned}$$

146 *where  $y'^c \doteq y' - \mathcal{M}_{Y'}(y')$ .*

147 *The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
 148 the corresponding spaces.*

149 *Proof.* The proof directly follow from Lemma 3.2 in the particular case  $N_1 = N$   
 150 and  $N_2 = 0$ . As an equivalent proof, the statement follows by [8, Corollary 1.37 and  
 151 Theorem 1.41] and the fact that  $\{\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)\}_\varepsilon \in L^p(\tilde{\Omega}_\varepsilon; Q^1(Y))$ .  $\square$

**4. The periodic lattice structure.** We start by giving a rigorous definition of 1-dimensional periodic lattice structure in  $\mathbb{R}^N$ .

Let  $i \in \{1, \dots, N\}$  and let  $K_1, \dots, K_N \in \mathbb{N}^*$ . Set

$$\begin{aligned} \mathbf{K} &\doteq \prod_{i=1}^N \{0, \dots, K_i\} \subset \mathbb{N}^N, & \mathbf{K}_i &\doteq \{k \in \mathbf{K} \mid k_i = 0\}, \\ \widehat{\mathbf{K}} &\doteq \prod_{i=1}^N \{0, \dots, K_i - 1\}, & \widehat{\mathbf{K}}_i &\doteq \{k \in \widehat{\mathbf{K}} \mid k_i = 0\}. \end{aligned}$$

We denote  $\mathcal{K}$  the set of points in the closure of the unit cell  $\overline{Y}$  by

$$\mathcal{K} \doteq \left\{ A(k) \in \mathbb{R}^N \mid A(k) = \sum_{i=1}^N \frac{k_i}{K_i} \mathbf{e}_i, \quad k \in \mathbf{K} \right\} \subset \overline{Y}.$$

In this sense, the whole unit cell  $\overline{Y}$  has the following split

$$\overline{Y} = \sum_{k \in \widehat{\mathbf{K}}} A(k) + \overline{Y}_K,$$

where  $Y_K$  is the cell defined by

$$Y_K \doteq \prod_{i=1}^N (0, l_i), \quad l_i = \frac{1}{K_i}.$$

We denote  $\mathcal{S}^{(i)}$  the set of all segments whose direction is  $\mathbf{e}_i$  by

$$\mathcal{S}_c^{(i)} \doteq \bigcup_{k \in \mathbf{K}_i} [A(k), A(k) + \mathbf{e}_i], \quad \mathcal{S}^{(i)} \doteq \bigcup_{k \in \widehat{\mathbf{K}}_i} [A(k), A(k) + \mathbf{e}_i]$$

152 Hence, the lattice structure in the unit cell  $\overline{Y}$  is defined by

$$153 \quad \mathcal{S}_c \doteq \bigcup_{i=1}^N \mathcal{S}_c^{(i)} \subset \overline{Y}, \quad \mathcal{S} \doteq \bigcup_{i=1}^N \mathcal{S}^{(i)} \subset \overline{Y}.$$

Given  $\Omega \subset \mathbb{R}^N$ , we cover it as in (3.1) by a union of  $\varepsilon$  cells. The periodic lattice structure is therefore defined by

$$\begin{aligned} \mathcal{S}_\varepsilon &\doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{S}_c) \subset \widetilde{\Omega}_\varepsilon, & \mathcal{K}_\varepsilon &\doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{K}), \\ \mathcal{S}_\varepsilon^{(i)} &\doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{S}_c^{(i)}). \end{aligned}$$

154 Denote  $\mathbf{S}$  the running point of  $\mathcal{S}$  and  $\mathbf{s}$  that of  $\mathcal{S}_\varepsilon$ . That gives ( $i \in \{1, \dots, N\}$ )

$$\begin{aligned} 155 \quad \mathbf{S} &= A(k) + t\mathbf{e}_i \quad \text{in } \mathcal{S}^{(i)}, \quad t \in [0, 1], \quad k \in \widehat{\mathbf{K}}_i, \\ \mathbf{s} &= \varepsilon\xi + \varepsilon A(k) + \varepsilon t\mathbf{e}_i \quad \text{in } \mathcal{S}_\varepsilon^{(i)}, \quad t \in [0, 1], \quad k \in \widehat{\mathbf{K}}_i, \quad \xi \in \widetilde{\Xi}_\varepsilon. \end{aligned}$$



156 Let  $\mathcal{C}(\mathcal{S})$  and  $\mathcal{C}(\mathcal{S}_\varepsilon)$  be the spaces of continuous functions defined on  $\mathcal{S}$  and  $\mathcal{S}_\varepsilon$  respec-  
 157 tively. For  $p \in [1, +\infty]$ , we denote the spaces of functions defined on the lattice by  
 158 ( $i \in \{1, \dots, N\}$ )

$$\begin{aligned} W^{1,p}(\mathcal{S}^{(i)}) &\doteq \{\phi \in L^p(\mathcal{S}^{(i)}) \mid \partial_{\mathbf{s}}\phi \in L^p(\mathcal{S}^{(i)})\}, \\ W^{1,p}(\mathcal{S}_\varepsilon^{(i)}) &\doteq \{\phi \in L^p(\mathcal{S}_\varepsilon^{(i)}) \mid \partial_{\mathbf{s}}\phi \in L^p(\mathcal{S}_\varepsilon^{(i)})\}, \\ W^{1,p}(\mathcal{S}) &\doteq \{\phi \in \mathcal{C}(\mathcal{S}) \mid \partial_{\mathbf{s}}\phi \in L^p(\mathcal{S})\}, \\ W^{1,p}(\mathcal{S}_\varepsilon) &\doteq \{\phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \mid \partial_{\mathbf{s}}\phi \in L^p(\mathcal{S}_\varepsilon)\} \end{aligned}$$

160 and for  $k \in \mathbb{N} \setminus \{0, 1\}$

$$\begin{aligned} W^{k,p}(\mathcal{S}^{(i)}) &\doteq \{\phi \in W^{k-1,p}(\mathcal{S}^{(i)}) \mid \partial_{\mathbf{s}}\phi \in W^{k-1,p}(\mathcal{S}^{(i)})\}, \\ W^{k,p}(\mathcal{S}_\varepsilon^{(i)}) &\doteq \{\phi \in W^{k-1,p}(\mathcal{S}_\varepsilon^{(i)}) \mid \partial_{\mathbf{s}}\phi \in W^{k-1,p}(\mathcal{S}_\varepsilon^{(i)})\}, \\ W^{k,p}(\mathcal{S}) &\doteq \{\phi \in \mathcal{C}(\mathcal{S}) \mid \partial_{\mathbf{s}}\phi|_{\mathcal{S}^{(j)}} \in W^{k-1,p}(\mathcal{S}^{(j)}), j \in \{1, \dots, N\}\}, \\ W^{k,p}(\mathcal{S}_\varepsilon) &\doteq \{\phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \mid \partial_{\mathbf{s}}\phi|_{\mathcal{S}_\varepsilon^{(j)}} \in W^{k-1,p}(\mathcal{S}_\varepsilon^{(j)}), j \in \{1, \dots, N\}\}. \end{aligned}$$

162 **4.1. The unfolding operator for periodic lattices.** We are now in the po-  
 163 sition to define an equivalent formulation of the unfolding operator and mean value  
 164 operator (see Definition 2.1 and 2.2) for lattice structures.

DEFINITION 4.1. *For every measurable function  $\phi$  on  $\mathcal{S}_\varepsilon$ , the unfolding operator  $\mathcal{T}_\varepsilon^{\mathcal{S}}$  is defined as follows:*

$$\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)(x, \mathbf{S}) = \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon\mathbf{S}\right) \quad \text{for a.e. } (x, \mathbf{S}) \in \tilde{\Omega}_\varepsilon \times \mathcal{S}.$$

For every function  $\hat{\phi}$  on  $L^1(\mathcal{S}^{(i)})$ ,  $i \in \{1, \dots, N\}$ , the mean value operator  $\mathcal{M}_{\mathcal{S}^{(i)}}$  on direction  $\mathbf{e}_i$  is defined as follows:

$$\mathcal{M}_{\mathcal{S}^{(i)}}(\hat{\phi})(\mathbf{S}) \doteq \int_{A(k)}^{A(k)+\mathbf{e}_i} \hat{\phi}(x, \mathbf{S}') d\mathbf{S}', \quad \forall \mathbf{S} \in [A(k), A(k) + \mathbf{e}_i], \quad \forall k \in \hat{\mathbf{K}}_i.$$

165 Observe that in the above definition of  $\mathcal{T}_\varepsilon^{\mathcal{S}}$ , the map  $(x, \mathbf{S}) \mapsto \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon\mathbf{S}$  from  $\tilde{\Omega}_\varepsilon \times \mathcal{S}$   
 166 into  $\mathcal{S}_\varepsilon$  is almost everywhere one to one. This is not the case if we replace  $\mathcal{S}$  by  $\mathcal{S}_\varepsilon$ .  
 167 Below, we give the main property of  $\mathcal{T}_\varepsilon^{\mathcal{S}}$ .

168 PROPOSITION 4.2. *For every  $\phi \in L^p(\mathcal{S}_\varepsilon)$ ,  $p \in [1, +\infty]$ , one has*

$$\|\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)\|_{L^p(\tilde{\Omega}_\varepsilon \times \mathcal{S})} \leq \varepsilon^{\frac{N-1}{p}} |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\mathcal{S}_\varepsilon)}.$$

169 *Proof.* We start with  $p = 1$ . Let  $\phi$  be in  $L^1(\mathcal{S}_\varepsilon)$ . We have

$$\begin{aligned} \int_{\tilde{\Omega}_\varepsilon \times \mathcal{S}} |\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)(x, \mathbf{S})| dx d\mathbf{S} &= \int_{\tilde{\Omega}_\varepsilon} \sum_{i=1}^N \int_{\mathcal{S}^{(i)}} |\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi)(x, \mathbf{S})| dx d\mathbf{S} \\ &= \sum_{\xi \in \tilde{\Xi}_\varepsilon} |\varepsilon\xi + \varepsilon Y| \sum_{i=1}^N \sum_{k \in \hat{\mathbf{K}}_i} \int_0^1 |\phi(\varepsilon\xi + \varepsilon A(k) + \varepsilon t)| dt \\ &= \varepsilon^N |Y| \sum_{i=1}^N \sum_{k \in \hat{\mathbf{K}}_i} \int_0^1 |\phi(\varepsilon\xi + \varepsilon A(k) + \varepsilon t)| dt \\ &\leq \varepsilon^{N-1} |Y| \int_{\mathcal{S}_\varepsilon} |\phi(\mathbf{s})| d\mathbf{s}. \end{aligned}$$

174  
 175

176 The case  $p \in (1, +\infty)$  follows by definition of  $L^p$  norm. The case  $p = +\infty$  is trivial.  $\square$

177 **5. Periodic unfolding for sequences defined on lattices with information**  
 178 **on the first order derivatives.**

**5.1. Asymptotic behavior of bounded sequences defined as  $Q_1$  interpolated on lattice nodes.** On  $\mathcal{S}_\varepsilon$  (resp.  $\mathcal{S}$ ) we define the space  $Q^1(\mathcal{S}_\varepsilon)$  (resp.  $Q^1(\mathcal{S})$ ) by

$$Q^1(\mathcal{S}_\varepsilon) \doteq \left\{ \phi \in \mathcal{C}(\mathcal{S}_\varepsilon) \mid \phi \text{ is affine between two contiguous points of } \mathcal{K}_\varepsilon \right\},$$

$$\text{(resp. } Q^1(\mathcal{S}) \doteq \left\{ \phi \in \mathcal{C}(\mathcal{S}) \mid \phi \text{ is affine between two contiguous points of } \mathcal{K} \right\}.$$

179 Similarly we define the spaces  $Q^1(\mathcal{S}'_\varepsilon)$ ,  $Q^1(\mathcal{S}''_\varepsilon)$  and  $Q^1(\mathcal{S}')$ ,  $Q^1(\mathcal{S}'')$ ,  $Q^1_{per}(\mathcal{S})$ ,  $Q^1_{per}(\mathcal{S}')$ ,  
 180  $Q^1_{per}(\mathcal{S}'')$  (see (5.5)).

181 A function belonging to  $Q^1(\mathcal{S}_\varepsilon)$  is determined only by its values on the set of  
 182 nodes  $\mathcal{K}_\varepsilon$  and thus can be naturally extended to a function defined in  $\tilde{\Omega}_\varepsilon$ .

183 **DEFINITION 5.1.** For every function  $\psi \in Q^1(\mathcal{S}_\varepsilon)$ , its extension  $\mathfrak{Q}_\varepsilon(\psi)$  belonging  
 184 to  $W^{1,\infty}(\tilde{\Omega}_\varepsilon)$  is defined by  $Q_1$  interpolation on each paralleloptope  $\varepsilon\xi + \varepsilon A(k) + \varepsilon \overline{Y_K}$   
 185 belonging to  $\varepsilon\xi + \varepsilon \overline{Y}$  for every  $\xi \in \tilde{\Xi}_\varepsilon$  and  $k \in \widehat{\mathbf{K}}$ .

Define the spaces

$$Q^1_{\mathcal{K}_\varepsilon}(\tilde{\Omega}_\varepsilon) \doteq \left\{ \Psi \in W^{1,\infty}(\tilde{\Omega}_\varepsilon) \mid \Psi|_{\varepsilon\xi + \varepsilon A(k) + \varepsilon \overline{Y_K}} \text{ is the } Q_1 \text{ interpolate of its values} \right.$$

$$\left. \text{on the vertexes of } \varepsilon\xi + \varepsilon A(k) + \varepsilon \overline{Y_K}, \forall k \in \widehat{\mathbf{K}}, \forall \xi \in \tilde{\Xi}_\varepsilon \right\},$$

$$Q^1_{\mathcal{K}}(Y) \doteq \left\{ \Psi \in W^{1,\infty}(Y) \mid \Psi|_{A(k) + \overline{Y_K}} \text{ is the } Q_1 \text{ interpolate of its values} \right.$$

$$\left. \text{on the vertexes of } A(k) + \overline{Y_K}, \forall k \in \widehat{\mathbf{K}} \right\}.$$

186 Similarly we define the spaces  $Q^1_{\mathcal{K}}(Y')$ ,  $Q^1_{\mathcal{K}}(Y'')$ ,  $Q^1_{\mathcal{K},per}(Y)$ ,  $Q^1_{\mathcal{K},per}(Y')$ ,  $Q^1_{\mathcal{K},per}(Y'')$ .  
 187 By definition, the extension operator  $\mathfrak{Q}_\varepsilon$  is both one to one and onto from  $Q^1(\mathcal{S}_\varepsilon)$  to  
 188  $Q^1_{\mathcal{K}_\varepsilon}(\tilde{\Omega}_\varepsilon)$ . Its inverse is given by the restriction  $|_{\mathcal{S}_\varepsilon}$  from  $Q^1_{\mathcal{K}_\varepsilon}(\tilde{\Omega}_\varepsilon)$  to  $Q^1(\mathcal{S}_\varepsilon)$ .

189 Below, we show the main properties of this operator.

190 **LEMMA 5.2.** For every  $\psi \in Q^1(\mathcal{S}_\varepsilon)$ , one has ( $p \in [1, +\infty]$ ,  $i \in \{1, \dots, N\}$ )

191 (5.1)  $\|\mathfrak{Q}_\varepsilon(\psi)\|_{L^p(\tilde{\Omega}_\varepsilon)} \leq C\varepsilon^{\frac{N-1}{p}} \|\psi\|_{L^p(\mathcal{S}_\varepsilon)}, \quad \|\partial_i \mathfrak{Q}_\varepsilon(\psi)\|_{L^p(\tilde{\Omega}_\varepsilon)} \leq C\varepsilon^{\frac{N-1}{p}} \|\partial_{\mathbf{s}} \psi\|_{L^p(\mathcal{S}_\varepsilon^{(i)})},$   
 192

193 where the constants do not depend on  $\varepsilon$ .

194 *Proof.* We will only consider the case  $p \in [1, +\infty)$ , since the case  $p = +\infty$  is  
 195 trivial. First, remind that for every function  $\phi$  defined as  $Q_1$  interpolate of its values  
 196 on the vertexes of the nodes in  $\mathcal{K}$ , we have ( $i \in \{1, \dots, N\}$ )

197 (5.2) 
$$c\|\phi\|_{L^p(Y)} \leq \left( \sum_{k \in \mathbf{K}} |\phi(A(k))|^p \right)^{1/p} \leq C\|\phi\|_{L^p(\mathcal{S})},$$

$$c\|\partial_{y_i} \phi\|_{L^p(Y)} \leq \|\partial_{\mathbf{s}} \phi\|_{L^p(\mathcal{S}^{(i)})},$$

198 where the constants do not depend on  $p$ .

199 We now prove (5.1)<sub>1</sub>. For every  $\psi \in Q^1(\mathcal{S}_\varepsilon)$ , set  $\Psi = \mathfrak{Q}_\varepsilon(\psi)$ . From (5.2)<sub>1</sub> and an  
200 affine change of variables, we easily get

$$\begin{aligned} 201 \quad \int_{\tilde{\Omega}_\varepsilon} |\Psi(x)|^p dx &= \sum_{\xi \in \tilde{\Xi}_\varepsilon} \int_{\varepsilon\xi + \varepsilon Y} |\Psi(x)|^p dx = \varepsilon^N \sum_{\xi \in \tilde{\Xi}_\varepsilon} \int_Y |\Psi(\varepsilon\xi + \varepsilon y)|^p dy \\ 202 \quad &\leq \varepsilon^N \sum_{\xi \in \tilde{\Xi}_\varepsilon} \int_{\mathcal{S}} |\Psi(\varepsilon\xi + \varepsilon \mathbf{S})|^p d\mathbf{S} \leq \varepsilon^{N-1} \int_{\mathcal{S}_\varepsilon} |\Psi(\mathbf{s})|^p d\mathbf{s} \end{aligned}$$

204 and thus (5.1)<sub>1</sub> holds since  $\Psi|_{\mathcal{S}_\varepsilon} = \psi$ .

205 We prove now (5.1)<sub>2</sub>. Let  $i$  be in  $\{1, \dots, N\}$ . From (5.2)<sub>2</sub> and an affine change of  
206 variables, we have

$$\begin{aligned} 207 \quad \int_{\tilde{\Omega}_\varepsilon} |\partial_i \Psi(x)|^p dx &= \sum_{\xi \in \tilde{\Xi}_\varepsilon} \int_{\varepsilon\xi + \varepsilon Y} \left| \frac{\partial}{\partial x_i} \Psi(x) \right|^p dx = \varepsilon^{N-p} \sum_{\xi \in \tilde{\Xi}_\varepsilon} \int_Y \left| \frac{\partial}{\partial y_i} \Psi(\varepsilon\xi + \varepsilon y) \right|^p dy \\ 208 \quad &\leq \varepsilon^{N-p} \sum_{\xi \in \tilde{\Xi}_\varepsilon} \int_{\mathcal{S}^{(i)}} |\partial_{\mathbf{s}} \Psi(\varepsilon\xi + \varepsilon \mathbf{S})|^p d\mathbf{S} \leq \varepsilon^{N-1} \int_{\mathcal{S}_\varepsilon^{(i)}} |\partial_{\mathbf{s}} \Psi(\mathbf{s})|^p d\mathbf{s}. \end{aligned}$$

210 And thus (5.1)<sub>2</sub> holds since  $\Psi|_{\mathcal{S}_\varepsilon^{(i)}} = \psi|_{\mathcal{S}_\varepsilon^{(i)}}$ .  $\square$

211 Note now that for every  $\psi \in Q^1(\mathcal{S}_\varepsilon)$ , the unfolding on the lattice is equivalent to first  
212 extending  $\psi$  to  $\Psi = \mathfrak{Q}_\varepsilon(\psi)$  (see Definition 5.1), then applying the unfolding results  
213 in  $\mathbb{R}^N$  and lastly restricting the convergences to the lattice again, as the following  
214 commutative diagrams show ( $i \in \{1, \dots, N\}$ ):

$$215 \quad (5.3) \quad \begin{cases} \mathcal{T}_\varepsilon^{\mathcal{S}}(\psi) = \mathcal{T}_\varepsilon^{\mathcal{S}}(\Psi|_{\mathcal{S}_\varepsilon}) = \mathcal{T}_\varepsilon^{ext}(\Psi)|_{\tilde{\Omega}_\varepsilon \times \mathcal{S}}, \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}} \psi) = \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}} \Psi|_{\mathcal{S}_\varepsilon^{(i)}}) = \mathcal{T}_\varepsilon^{ext}(\partial_i \Psi)|_{\tilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}}. \end{cases}$$

216 We can finally show the asymptotic behavior of sequences which belong to  $Q^1(\mathcal{S}_\varepsilon)$   
217 and we start with the following.

218 **LEMMA 5.3.** *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $W^{1,p}(\mathcal{S}_\varepsilon)$  satisfying ( $p \in (1, +\infty)$ )*

$$219 \quad \|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon \|\partial_{\mathbf{s}} \phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C \varepsilon^{\frac{1-N}{p}},$$

220 where the constant does not depend on  $\varepsilon$ .

221 Then, there exist a subsequence of  $\{\varepsilon\}$ , denoted  $\{\varepsilon\}$ , and  $\hat{\phi} \in L^p(\Omega; W_{per}^{1,p}(\mathcal{S}))$  such  
222 that

$$223 \quad (5.4) \quad \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightharpoonup \hat{\phi} \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{S})).^1$$

224 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
225 the corresponding spaces.

226 *Proof.* The sequence  $\{\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\}_\varepsilon$  satisfies

$$227 \quad \|\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\|_{L^p(\tilde{\Omega}_\varepsilon; W^{1,p}(\mathcal{S}))} \leq C \quad \implies \quad \|\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)\|_{L^p(\Omega; W^{1,p}(\mathcal{S}))} \leq C.$$

<sup>1</sup>As for  $\mathcal{T}_\varepsilon^{ext}$ , this convergence must be understood

$$\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)|_{\Omega \times \mathcal{S}} \rightharpoonup \hat{\phi} \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{S})).$$

It will be the same for all convergences involving the unfolding operator  $\mathcal{T}_\varepsilon^{\mathcal{S}}$ .

228 The constant does not depend on  $\varepsilon$ .

229 Hence, there exist a subsequence of  $\{\varepsilon\}$ , denoted  $\{\varepsilon\}$ , and  $\widehat{\phi} \in L^p(\Omega; W_{per}^{1,p}(\mathcal{S}))$  such  
 230 that convergence (5.4) holds. The periodicity of  $\widehat{\phi}$  is proved as in [8, Theorem 1.36]. $\square$

231 We consider now sequences whose gradient is anisotropically bounded on the lattice.

232 Accordingly to Section 2, we apply the decomposition of the space  $\mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$   
 233 and define

$$(5.5) \quad \begin{aligned} \mathcal{S}' &\doteq \bigcup_{i=1}^{N_1} \mathcal{S}^{(i)}, & \mathcal{S}'_c &\doteq \bigcup_{i=1}^{N_1} \mathcal{S}_c^{(i)}, & \mathcal{S}'_\varepsilon &\doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{S}'_c), \\ \mathcal{S}'' &\doteq \bigcup_{i=N_1+1}^N \mathcal{S}^{(i)}, & \mathcal{S}''_c &\doteq \bigcup_{i=N_1+1}^N \mathcal{S}_c^{(i)}, & \mathcal{S}''_\varepsilon &\doteq \bigcup_{\xi \in \widetilde{\Xi}_\varepsilon} (\varepsilon\xi + \varepsilon\mathcal{S}''_c). \end{aligned}$$

235 We have the following.

236 LEMMA 5.4. *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $Q^1(\mathcal{S}_\varepsilon)$  satisfying ( $p \in (1, +\infty)$ )*

$$237 \quad \|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{S}}\phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}},$$

238 where the constant does not depend on  $\varepsilon$ .

239 Then, there exist a subsequence of  $\{\varepsilon\}$ , denoted  $\{\varepsilon\}$ , and  $\widetilde{\phi} \in L^p(\Omega, \nabla_{x'}; Q^1_{per}(\mathcal{S}''))$ ,

240  $\widehat{\phi} \in L^p(\Omega; Q^1_{per}(\mathcal{S}))$ , such that ( $i \in \{1, \dots, N_1\}$ )

$$(5.6) \quad \begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\rightharpoonup \widetilde{\phi} \quad \text{weakly in } L^p(\Omega; Q^1(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}}\phi_\varepsilon) &\rightharpoonup \partial_i\widetilde{\phi} + \partial_{\mathbf{S}}\widehat{\phi} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}), \\ \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)) &\rightharpoonup \partial_i\widetilde{\phi} \mathbf{S}^c + \widehat{\phi} - \mathcal{M}_{\mathcal{S}^{(i)}}(\widehat{\phi}) \\ &\text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}), \end{aligned}$$

242 where  $\mathbf{S}^c \doteq (\mathbf{S} - \mathcal{M}_{\mathcal{S}^{(i)}}(\mathbf{S})) \cdot \mathbf{e}_i^2$ .

243 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
 244 the corresponding spaces.

245 *Proof.* We extend the sequence  $\{\phi_\varepsilon\}_\varepsilon$  to the sequence  $\{\Phi_\varepsilon\}_\varepsilon = \{\mathfrak{Q}_\varepsilon(\phi_\varepsilon)\}_\varepsilon$  belong-  
 246 ing to  $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$ . By Lemma 5.2 and the  $Q_1$  property (3.2), we get

$$247 \quad \|\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \|\nabla_{x'}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} + \varepsilon\|\nabla_{x''}\Phi_\varepsilon\|_{L^p(\widetilde{\Omega}_\varepsilon)} \leq C,$$

248 where the constant does not depend on  $\varepsilon$ .

249 By construction, the sequence  $\{\Phi_\varepsilon\}_\varepsilon$  belongs to  $Q^1_{\mathcal{K}_\varepsilon}(\widetilde{\Omega}_\varepsilon)$  and thus  $\{\mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon)\}_\varepsilon$  be-  
 250 longs to  $L^p(\widetilde{\Omega}_\varepsilon; Q^1(Y))$ .

251 Hence, Lemma 3.2 imply that there exist functions  $\widetilde{\Phi} \in L^p(\Omega, \nabla_{x'}; Q^1_{\mathcal{K},per}(Y''))$  and

252  $\widehat{\Phi} \in L^p(\Omega \times Y''; Q^1_{\mathcal{K},per}(Y')) \cap L^p(\Omega; Q^1_{\mathcal{K}}(Y))$  satisfying  $\mathcal{M}_{Y'}(\widehat{\Phi}) = 0$  a.e. in  $\Omega \times Y''$ ,

253 such that

$$(5.6) \quad \begin{aligned} \Phi_{\varepsilon|\Omega} &\rightharpoonup \Phi \quad \text{weakly in } L^p(\Omega, \nabla_{x'}), \\ \mathcal{T}_\varepsilon^{ext}(\Phi_\varepsilon) &\rightharpoonup \widetilde{\Phi} \quad \text{weakly in } L^p(\Omega; Q^1_{\mathcal{K}}(Y)), \\ \mathcal{T}_\varepsilon^{ext}(\nabla_{x'}\Phi_\varepsilon) &\rightharpoonup \nabla_{x'}\widetilde{\Phi} + \nabla_{y'}\widehat{\Phi} \quad \text{weakly in } L^p(\Omega \times Y)^{N_1}, \end{aligned}$$

<sup>2</sup>One has  $\mathbf{S} = A(k) + t\mathbf{e}_i$  in the line  $[A(k), A(k) + t\mathbf{e}_i]$ ,  $t \in [0, 1]$ ,  $k \in \widehat{\mathbf{K}}_i$ . Hence  $\mathbf{S}^c = t - 1/2$ .

255 where  $\Phi = \mathcal{M}_{Y''}(\tilde{\Phi})$ .

256 Using the relations (5.3), we can restrict the above convergences from  $\Omega \times Y$  to  
 257 the subset  $\Omega \times \mathcal{S}$  (and from  $\Omega \times Y'$ ,  $\Omega \times Y''$  to  $\Omega \times \mathcal{S}'$ ,  $\Omega \times \mathcal{S}''$  respectively). Hence,  
 258  $\tilde{\phi} = \tilde{\Phi}|_{\Omega \times \mathcal{S}}$  and thus  $\tilde{\phi} \in L^p(\Omega, \nabla_{x'}; Q_{per}^1(\mathcal{S}''))$ . Now, let us consider  $\hat{\Phi}|_{\Omega \times \mathcal{S}'}$ , we extend  
 259 this function as an affine function between two contiguous nodes in  $\mathcal{S}''$ , this gives  
 260 a function  $\hat{\phi}$  belonging to  $L^p(\Omega; Q_{per}^1(\mathcal{S}))$  (see Figure 1). This proves convergences  
 261 (5.6)<sub>1,2</sub>, while (5.6)<sub>3</sub> is an immediate consequence of the Poincaré-Wirtinger inequality  
 262 and (5.6)<sub>2</sub>.  $\square$

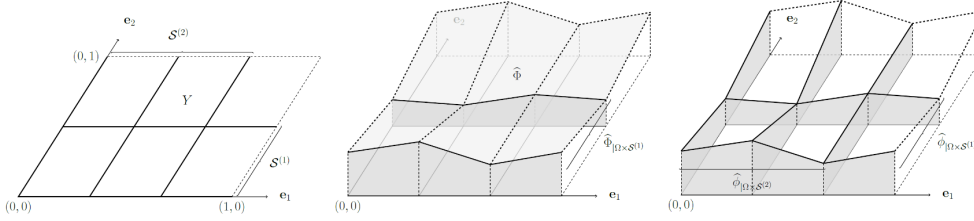


FIG. 1. Construction of the periodic function  $\hat{\phi}$  for  $N = 2$  and  $(K_1, K_2) = (3, 2)$ . On the left, the reference cell and the lattice  $\mathcal{S} \doteq \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$  and the nodes  $A(k)$ , where  $k$  belongs to  $\mathbf{K} \doteq \{0, 1, 2, 3\} \times \{0, 1, 2\}$ . On the center, the  $Q_1$  interpolated on the lattice nodes  $\hat{\Phi}$  and its restriction to  $\mathcal{S}^{(1)}$  (horizontal lines). On the right, the function  $\hat{\phi}$  given by  $\hat{\Phi}|_{\Omega \times \mathcal{S}^{(1)}}$  and the  $Q_1$  interpolation along the segments in  $\mathcal{S}^{(2)}$  (vertical lines).

263 Now, we show the asymptotic behavior of sequences in  $Q^1(\mathcal{S}_\varepsilon)$  which are uniformly  
 264 bounded in  $W^{1,p}(\mathcal{S}_\varepsilon)$ .

265 COROLLARY 5.5. Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $Q^1(\mathcal{S}_\varepsilon)$  satisfying ( $p \in (1, +\infty)$ )

$$266 \quad \|\phi_\varepsilon\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}},$$

267 where the constant does not depend on  $\varepsilon$ .

268 Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and functions  $\phi \in W^{1,p}(\Omega)$   
 269 and  $\hat{\phi} \in L^p(\Omega; Q_{per,0}^1(\mathcal{S}))$  such that ( $i \in \{1, \dots, N\}$ )

$$270 \quad \begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\rightharpoonup \phi \quad \text{weakly in } L^p(\Omega; Q^1(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &\rightharpoonup \partial_i\phi + \partial_{\mathbf{S}}\hat{\phi} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

271 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
 272 the corresponding spaces.

273 *Proof.* The proof directly follows from Lemma 5.4 in the particular case  $\mathcal{S}' = \mathcal{S}$   
 274 and  $\mathcal{S}'' = \emptyset$ .  $\square$

**5.2. Asymptotic behavior of sequences bounded anisotropically and uniformly in  $W^{1,p}$ .** Denote ( $p \in [1, +\infty]$ ,  $i \in \{1, \dots, N\}$ )

$$\begin{aligned} \mathcal{W}_{0,\mathcal{K}}^{1,p}(\mathcal{S}) &= \{\phi \in W^{1,p}(\mathcal{S}) \mid \phi = 0 \text{ on } \mathcal{K}\}, \\ \mathcal{W}_{0,\mathcal{K}_\varepsilon}^{1,p}(\mathcal{S}_\varepsilon) &= \{\phi \in W^{1,p}(\mathcal{S}_\varepsilon) \mid \phi = 0 \text{ on } \mathcal{K}_\varepsilon\}. \end{aligned}$$

275 Every function  $\phi$  in  $W^{1,p}(\mathcal{S})$  (resp.  $\psi \in W^{1,p}(\mathcal{S}_\varepsilon)$ ) is defined on the set of nodes  $\mathcal{K}$   
 276 (resp.  $\mathcal{K}_\varepsilon$ ) and therefore can be decomposed as

$$277 \quad (5.7) \quad \begin{aligned} \phi &= \phi_a + \phi_0, & \phi_a &\in Q^1(\mathcal{S}), & \phi_0 &\in \mathcal{W}_{0,\mathcal{K}}^{1,p}(\mathcal{S}), \\ \text{(resp. } \psi &= \psi_a + \psi_0, & \psi_a &\in Q^1(\mathcal{S}_\varepsilon), & \psi_0 &\in \mathcal{W}_{0,\mathcal{K}_\varepsilon}^{1,p}(\mathcal{S}_\varepsilon)), \end{aligned}$$

278 where  $\phi_a, \psi_a$  are the affine function defined as  $Q_1$  interpolation on the nodes, and  $\phi_0,$   
 279  $\psi_0$  the reminder term which is zero on every node.

280 LEMMA 5.6. *There exists a constant  $C > 0$ , which does not depend on  $\varepsilon$ , such*  
 281 *that ( $i \in \{1, \dots, N\}$ )*

$$282 \quad (5.8) \quad \begin{aligned} \forall \phi &\in W^{1,p}(\mathcal{S}), & \|\partial_{\mathbf{s}}\phi_a\|_{L^p(\mathcal{S}^{(i)})} + \|\partial_{\mathbf{s}}\phi_0\|_{L^p(\mathcal{S}^{(i)})} &\leq C\|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)})}, \\ \forall \psi &\in W^{1,p}(\mathcal{S}_\varepsilon), & \|\partial_{\mathbf{s}}\psi_a\|_{L^p(\mathcal{S}_\varepsilon^{(i)})} + \|\partial_{\mathbf{s}}\psi_0\|_{L^p(\mathcal{S}_\varepsilon^{(i)})} &\leq C\|\partial_{\mathbf{s}}\psi\|_{L^p(\mathcal{S}_\varepsilon^{(i)})}, \\ & & \|\psi_0\|_{L^p(\mathcal{S}_\varepsilon^{(i)})} &\leq C\varepsilon\|\partial_{\mathbf{s}}\psi\|_{L^p(\mathcal{S}_\varepsilon^{(i)})}. \end{aligned}$$

*Proof. Step 1.* First, we recall a simple result. Let  $\psi$  be in the space  $W^{1,p}(0,1)$   
 ( $p \in [1, +\infty]$ ). Denote  $\psi_a$  the affine function

$$\psi_a(t) \doteq \psi(0) + t(\psi(1) - \psi(0)), \quad t \in [0, 1].$$

283 One has

$$284 \quad (5.9) \quad \|\psi'_a\|_{L^p(0,1)} \leq \|\psi'\|_{L^p(0,1)}, \quad \|\psi - \psi_a\|_{L^p(0,1)} \leq 2\|\psi'\|_{L^p(0,1)}.$$

285 *Step 2.* We prove the statements of the Lemma.

286 We start with (5.8)<sub>1</sub>. By construction,  $\mathcal{S}^{(i)}$  is the union of a finite number of segments  
 287 whose extremities belong to  $\mathbf{K}$ . Hence, inequality (5.9)<sub>1</sub> and an affine change of  
 288 variables leads to ( $i \in \{1, \dots, N\}$ )

$$289 \quad \|\partial_{\mathbf{s}}\phi_a\|_{L^p(\mathcal{S}^{(i)})} \leq \|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)})},$$

$$290 \quad \|\partial_{\mathbf{s}}\phi_0\|_{L^p(\mathcal{S}^{(i)})} \leq \|\partial_{\mathbf{s}}\phi_a\|_{L^p(\mathcal{S}^{(i)})} + \|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)})} \leq 2\|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S}^{(i)})}$$

292 and thus (5.8)<sub>1</sub> is proved. Estimate (5.8)<sub>2</sub> follows by (5.8)<sub>1</sub> and an affine change of  
 293 variables, while (5.8)<sub>3</sub> follows by (5.8)<sub>2</sub> and again a change of variables. The constant  
 294 does not depend on  $\varepsilon$  since  $\mathcal{S}^{(i)}$  has a finite number of segments.  $\square$

295 We show now the asymptotic behavior of sequences that are anisotropically bounded.

296 LEMMA 5.7. *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $W^{1,p}(\mathcal{S}_\varepsilon)$  satisfying ( $p \in (1, +\infty)$ )*

$$297 \quad (5.10) \quad \|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}''_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}},$$

298 where the constant does not depend on  $\varepsilon$ .

299 Then, there exist a subsequence of  $\{\varepsilon\}$ , denoted  $\{\varepsilon\}$ , and  $\tilde{\phi} \in L^p(\Omega, \nabla_{x'}; W_{per}^{1,p}(\mathcal{S}''))$ ,  
 300  $\hat{\phi} \in L^p(\Omega; W_{per}^{1,p}(\mathcal{S}))$ , such that ( $i \in \{1, \dots, N_1\}$ )

$$301 \quad (5.11) \quad \begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\rightharpoonup \tilde{\phi} \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &\rightharpoonup \partial_i\tilde{\phi} + \partial_{\mathbf{s}}\hat{\phi} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}), \\ \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \right) &\rightharpoonup \partial_i\tilde{\phi} \mathbf{S}^c + \hat{\phi} - \mathcal{M}_{\mathcal{S}^{(i)}}(\hat{\phi}) \\ &\quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}), \end{aligned}$$

302 where  $\mathbf{S}^c \doteq (\mathbf{S} - \mathcal{M}_{\mathcal{S}^{(i)}}(\mathbf{S})) \cdot \mathbf{e}_i$ .

303 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
304 the corresponding spaces.

305 *Proof.* Given  $\{\phi_\varepsilon\}_\varepsilon \subset W^{1,p}(\mathcal{S}_\varepsilon)$ , we decompose  $\phi_\varepsilon$  as in (5.7) and get

$$306 \quad \phi_\varepsilon = \phi_{a,\varepsilon} + \phi_{0,\varepsilon}, \quad \phi_{a,\varepsilon} \in Q^1(\mathcal{S}_\varepsilon), \quad \phi_{0,\varepsilon} \in \mathcal{W}_{0,\mathcal{K}_\varepsilon}^{1,p}(\mathcal{S}_\varepsilon).$$

307 By Lemma 5.6 and hypothesis (5.10) we have

$$308 \quad (5.12) \quad \begin{aligned} & \|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} + \varepsilon \|\partial_{\mathbf{S}} \phi_{0,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} \leq C\varepsilon \|\partial_{\mathbf{S}} \phi_\varepsilon\|_{L^p(\mathcal{S}'_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}+1}, \\ & \|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}''_\varepsilon)} + \varepsilon \|\partial_{\mathbf{S}} \phi_{0,\varepsilon}\|_{L^p(\mathcal{S}''_\varepsilon)} \leq C\varepsilon \|\partial_{\mathbf{S}} \phi_\varepsilon\|_{L^p(\mathcal{S}''_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}, \\ & \|\phi_{a,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{S}} \phi_{a,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon \|\partial_{\mathbf{S}} \phi_{a,\varepsilon}\|_{L^p(\mathcal{S}'_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}. \end{aligned}$$

309 where the constant does not depend on  $\varepsilon$ .

310 By estimates (5.12)<sub>1,2</sub> and [8, Theorem 1.36] applied on each line of  $\mathcal{S}_\varepsilon$ , there ex-

311 ist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and functions  $\widehat{\phi}'_0 \in L^p(\Omega; \mathcal{W}_{0,\mathcal{K},per}^{1,p}(\mathcal{S}'))$

312 ( $\mathcal{W}_{0,\mathcal{K},per}^{1,p}(\mathcal{S}') \doteq \mathcal{W}_{0,\mathcal{K}}^{1,p}(\mathcal{S}') \cap \mathcal{W}_{per}^{1,p}(\mathcal{S}')$ ) and  $\widehat{\phi}''_0 \in L^p(\Omega; \mathcal{W}_{0,\mathcal{K},per}^{1,p}(\mathcal{S}''))$  such that

$$313 \quad \begin{aligned} & \frac{1}{\varepsilon} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_{0,\varepsilon}) \rightharpoonup \widehat{\phi}'_0 \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{S}')), \\ & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_{0,\varepsilon}) \rightharpoonup \widehat{\phi}''_0 \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{S}'')). \end{aligned}$$

314 By estimates (5.12)<sub>3</sub> and Lemma 5.4, there exist a subsequence, still denoted  $\{\varepsilon\}$ , and

315 functions  $\widetilde{\phi}_a \in L^p(\Omega, \nabla_{x'}; Q_{per}^1(\mathcal{S}''))$ ,  $\widehat{\phi}_a \in L^p(\Omega; Q_{per}^1(\mathcal{S}))$  such that ( $i \in \{1, \dots, N_1\}$ )

$$316 \quad \begin{aligned} & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_{a,\varepsilon}) \rightarrow \widetilde{\phi}_a \quad \text{strongly in } L^p(\Omega; Q^1(\mathcal{S})), \\ & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_{a,\varepsilon}) \rightarrow \partial_i \widetilde{\phi}_a + \partial_{\mathbf{S}} \widehat{\phi}_a \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

317 Hence ( $i \in \{1, \dots, N_1\}, j \in \{N_1 + 1, \dots, N\}$ )

$$318 \quad \begin{aligned} & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightarrow \widetilde{\phi}_a \quad \text{strongly in } L^p(\Omega; W^{1,p}(\mathcal{S}')), \\ & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightarrow \widetilde{\phi}_a + \widehat{\phi}''_0 \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{S}')), \\ & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_\varepsilon) \rightarrow \partial_i \widetilde{\phi}_a + \partial_{\mathbf{S}}(\widehat{\phi}_a + \widehat{\phi}'_0) \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

319 Setting  $\widetilde{\phi} \doteq \widetilde{\phi}_a + \widehat{\phi}''_0$ , we get that  $\widetilde{\phi}$  belongs to  $L^p(\Omega, \nabla_{x'}; W_{per}^{1,p}(\mathcal{S}''))$ . Then, we set

320  $\widehat{\phi} \doteq \widehat{\phi}_a + \widehat{\phi}'_0$ , this function belongs to  $L^p(\Omega; W_{per}^{1,p}(\mathcal{S}))$ . Convergence (5.11)<sub>3</sub> is an

321 immediate consequence of (5.11)<sub>2</sub>. The proof is complete.  $\square$

322 As a direct consequence, it follows the asymptotic behavior of the uniformly bounded  
323 sequences.

324 **COROLLARY 5.8.** *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $W^{1,p}(\mathcal{S}_\varepsilon)$  satisfying ( $p \in (1, +\infty)$ )*

$$325 \quad \|\phi_\varepsilon\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}},$$

326 where the constant does not depend on  $\varepsilon$ .

327 Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and functions  $\phi \in W^{1,p}(\Omega)$

328 and  $\widehat{\phi} \in L^p(\Omega; W_{per,0}^{1,p}(\mathcal{S}))$  such that ( $i \in \{1, \dots, N\}$ )

$$329 \quad \begin{aligned} & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^p(\Omega; W^{1,p}(\mathcal{S})), \\ & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_\varepsilon) \rightarrow \partial_i \phi + \partial_{\mathbf{S}} \widehat{\phi} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

330 *The same results hold for  $p = +\infty$  with weak topology replaced by weak- $*$  topology in*  
 331 *the corresponding spaces.*

332 *Proof.* The proof directly follows from Lemma 5.7 in the particular case  $\mathcal{S}' = \mathcal{S}$   
 333 and  $\mathcal{S}'' = \emptyset$ . □

334 **6. Periodic unfolding for sequences defined on lattices with information**  
 335 **until the second order derivatives.** The main problem that arises for functions  
 336 in  $W^{2,p}(\mathcal{S}_\varepsilon)$  is the lack of mixed derivatives. This comes from the fact that a function  
 337 defined on the lattice segments can be derived twice, only in the segment directions.  
 338 We overcome the problem in two different ways.

339 **6.1. Unfolding via special  $Q_3$  interpolation.** Analogously to the previous  
 340 section, we decompose a function into a reminder term and a cubic polynomial, this  
 341 latter is extended to a special  $Q_3$  interpolation to the whole space. Then, we use  
 342 the periodic unfolding results for open subset in  $\mathbb{R}^N$  and finally restrict these results  
 343 to the lattice. However, to bound the extension, further assumptions on the original  
 344 function must be applied.

345 First, we recall a basic result concerning the functions in  $W^{2,p}(0, 1)$ .

LEMMA 6.1. *Let  $\phi$  be in  $W^{2,p}(0, 1)$ . There exist a unique decomposition*

$$\phi = \phi_p + \phi_0, \quad (\phi_p, \phi_0) \in W^{2,p}(0, 1)^2,$$

where  $\phi_p$  is the cubic polynomial defined by ( $t \in [0, 1]$ )

$$\phi_p(t) = \phi(0)(2t + 1)(t - 1)^2 + \phi(1)t^2(3 - 2t) + \phi'(0)t(t - 1)^2 + \phi'(1)t^2(t - 1)$$

346 and  $\phi_0$  is the reminder term satisfying

347 (6.1) 
$$\phi_0(0) = \phi_0(1) = \phi_0'(0) = \phi_0'(1) = 0.$$

348 Moreover, there exists a constant  $C > 0$ , such that

349 (6.2) 
$$\begin{aligned} \forall \phi \in W^{2,p}(0, 1), \quad & \|\phi_p''\|_{L^p(0,1)} \leq C\|\phi''\|_{L^p(0,1)}, \\ & \|\phi_p'\|_{L^p(0,1)} \leq C\|\phi'\|_{W^{1,p}(0,1)}, \\ & \|\phi_p\|_{L^p(0,1)} \leq C\|\phi\|_{W^{2,p}(0,1)}, \\ & \|\phi_0\|_{W^{2,p}(0,1)} \leq C\|\phi''\|_{L^p(0,1)}. \end{aligned}$$

*Proof.* Given  $\phi$  be in  $W^{2,p}(0, 1)$ , it is clear that the decomposition is unique. Indeed, condition (6.1) implies that the function  $\phi_p$  must satisfy

$$\phi_p(0) = \phi(0), \quad \phi_p(1) = \phi(1), \quad \phi_p'(0) = \phi'(0), \quad \phi_p'(1) = \phi'(1)$$

and therefore the 4 coefficients of the cubic polynomial are uniquely determined. Now, we observe that

$$\begin{aligned} \phi_p'(t) &= \left(\phi(1) - \phi(0) - \frac{1}{2}(\phi'(0) + \phi'(1))\right)6t(1 - t) + (\phi'(1) - \phi'(0))t + \phi'(0), \\ \phi_p''(t) &= \left(\phi(1) - \phi(0) - \frac{1}{2}(\phi'(0) + \phi'(1))\right)6(1 - 2t) + (\phi'(1) - \phi'(0)). \end{aligned}$$

350 Then, we easily obtain the estimates (6.2)<sub>1,2,3</sub>. Estimate (6.2)<sub>4</sub> follows by assumption  
 351 (6.1), the Poincaré inequality applied twice and estimate (6.2)<sub>1</sub>. □



Define the spaces ( $p \in [1, +\infty]$ )

$$\begin{aligned} \mathcal{W}_{0,\mathcal{K}}^{2,p}(\mathcal{S}) &= \{\phi \in W^{2,p}(\mathcal{S}) \mid \phi = \partial_{\mathbf{s}}\phi = 0 \text{ on } \mathcal{K}\}, \\ \mathcal{W}_{0,\mathcal{K}_\varepsilon}^{2,p}(\mathcal{S}_\varepsilon) &= \{\psi \in W^{2,p}(\mathcal{S}_\varepsilon) \mid \psi = \partial_{\mathbf{s}}\psi = 0 \text{ on } \mathcal{K}_\varepsilon\}. \end{aligned}$$

352 Remind that for any  $\phi \in W^{2,p}(\mathcal{S})$  (resp.  $\psi \in W^{2,p}(\mathcal{S}_\varepsilon)$ ), its derivatives  $\partial_{\mathbf{s}}\phi$  (resp.  
353  $\partial_{\mathbf{s}}\psi$ ) in direction  $\mathbf{e}_i$  are functions belonging to  $W^{1,p}(\mathcal{S}^{(i)})$  (resp.  $W^{1,p}(\mathcal{S}_\varepsilon^{(i)})$ ), for  
354 every  $i \in \{1, \dots, N\}$  and therefore defined on every node of the structure  $\mathcal{S}$  (resp.  
355  $\mathcal{S}_\varepsilon$ ). Hence, they can be extended by  $Q_1$  interpolation on the small segments of  $\mathcal{S}^{(j)}$   
356 (resp.  $\mathcal{S}_\varepsilon^{(j)}$ ) for every  $j \in \{1, \dots, N\}$ ,  $j \neq i$ . We denote these extensions by  $\partial_i\phi$  (resp.  
357  $\partial_i\psi$ ), for every  $i \in \{1, \dots, N\}$ .

358 LEMMA 6.2. *For every  $\phi \in W^{2,p}(\mathcal{S})$ , there exist two functions  $\Phi_p \in W^{2,p}(Y)$  and*  
359  *$\phi_0 \in \mathcal{W}_{0,\mathcal{K}}^{2,p}(\mathcal{S})$  such that*

$$360 \quad (6.3) \quad \phi = \Phi_p + \phi_0 \quad \text{a.e. in } \mathcal{S},$$

361 *where  $\Phi_p|_{\mathcal{S}}$  is a cubic polynomial on every small segment of  $\mathcal{S}$ .*

362 *Moreover, there exists a constant  $C > 0$  such that*

$$\begin{aligned} 363 \quad (6.4) \quad \|D^2\Phi_p\|_{L^p(Y)} &\leq C \sum_{i=1}^N \|\partial_{\mathbf{s}}(\partial_i\phi)\|_{L^p(\mathcal{S})}, \\ \|\nabla\Phi_p\|_{L^p(Y)} &\leq C \left( \|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S})} + \sum_{i=1}^N \|\partial_{\mathbf{s}}(\partial_i\phi)\|_{L^p(\mathcal{S})} \right), \\ \|\Phi_p\|_{L^p(Y)} &\leq C \left( \|\phi\|_{L^p(\mathcal{S})} + \|\partial_{\mathbf{s}}\phi\|_{L^p(\mathcal{S})} + \sum_{i=1}^N \|\partial_{\mathbf{s}}(\partial_i\phi)\|_{L^p(\mathcal{S})} \right) \end{aligned}$$

364 *and that*

$$365 \quad (6.5) \quad \|\phi_0\|_{L^2(\mathcal{S})} + \|\partial_{\mathbf{s}}\phi_0\|_{L^2(\mathcal{S})} + \|\partial_{\mathbf{s}}^2\phi_0\|_{L^2(\mathcal{S})} \leq C \|\partial_{\mathbf{s}}^2\phi\|_{L^p(\mathcal{S})}.$$

*Proof.* We will only prove the case  $N = 2$ , since the extension to higher dimension is done by an analogous argumentation.

*Step 1.* A first result.

Denote  $Q_0, Q_1, dQ_0$  and  $dQ_1$  the following polynomial functions ( $t \in [0, 1]$ )

$$Q_0(t) = (2t+1)(t-1)^2, \quad Q_1(t) = t^2(3-2t), \quad dQ_0(t) = t(t-1)^2, \quad dQ_1(t) = t^2(t-1).$$

366 Let  $\phi$  be a function continuous on  $\partial Z$ ,  $Z = (0, 1)^2$ , and of class  $W^{2,p}$  on every edge  
367 of  $\bar{Z}$ . We define the polynomial function  $\Phi \in W^{2,\infty}(Z)$  by

$$\begin{aligned} 368 \quad \Phi(t) &= \phi(0,0)P_{00}(t) + \phi(0,1)P_{0,1}(t) + \phi(1,0)P_{1,0}(t) + \phi(1,1)P_{1,1}(t) \\ &+ \partial_1\phi(0,0)d_1P_{00}(t) + \partial_1\phi(0,1)d_1P_{10}(t) + \partial_1\phi(0,1)d_1P_{01}(t) + \partial_1\phi(1,1)d_1P_{11}(t) \\ &+ \partial_2\phi(0,0)d_2P_{00}(t) + \partial_2\phi(0,1)d_2P_{01}(t) + \partial_2\phi(1,0)d_2P_{10}(t) + \partial_2\phi(1,1)d_2P_{11}(t) \end{aligned}$$

where for all  $t = (t_1, t_2) \in [0, 1]^2$

$$\begin{aligned} P_{00}(t) &= Q_0(t_1)Q_0(t_2), & P_{01}(t) &= Q_0(t_1)Q_1(t_2), \\ P_{10}(t) &= Q_1(t_1)Q_0(t_2), & P_{11}(t) &= Q_1(t_1)Q_1(t_2), \\ d_1 P_{00} &= dQ_0(t_1)Q_0(t_2), & d_1 P_{10} &= dQ_1(t_1)Q_0(t_2), \\ d_1 P_{01} &= dQ_0(t_1)Q_1(t_2), & d_1 P_{11} &= dQ_1(t_1)Q_1(t_2), \\ d_2 P_{00} &= Q_0(t_1)dQ_0(t_2), & d_2 P_{01} &= Q_0(t_1)dQ_1(t_2), \\ d_2 P_{10} &= Q_1(t_1)dQ_0(t_2), & d_2 P_{11} &= Q_1(t_1)dQ_1(t_2). \end{aligned}$$

First, observe that the polynomial  $\Phi$  can be rewritten as

$$\begin{aligned} \Phi(t) &= (\phi(0, 0)Q_0(t_1) + \phi(1, 0)Q_1(t_1) + \partial_1\phi(0, 0)dQ_0(t_1) + \partial_1\phi(1, 0)dQ_1(t_1))Q_0(t_2) \\ &\quad + (\phi(0, 1)Q_0(t_1) + \phi(1, 1)Q_1(t_1) + \partial_1\phi(0, 1)dQ_0(t_1) + \partial_1\phi(1, 1)dQ_1(t_1))Q_1(t_2) \\ &\quad + (\partial_2\phi(0, 0)dQ_0(t_2) + \partial_2\phi(0, 1)dQ_1(t_2))Q_0(t_1) \\ &\quad + (\partial_2\phi(1, 0)dQ_0(t_2) + \partial_2\phi(1, 1)dQ_1(t_2))Q_1(t_1). \end{aligned}$$

369 A straightforward calculation and Lemma (6.1) lead to

$$\begin{aligned} &\|D^2\Phi\|_{L^p(Z)} \\ &\leq C \left( \sum_{i=1}^2 \|\partial_{ii}^2\phi\|_{L^p((\partial Z)_i)} + |\partial_2\phi(1, 0) - \partial_2\phi(0, 0)| + |\partial_2\phi(1, 1) - \partial_2\phi(0, 1)| \right. \\ 370 &\quad \left. + |\partial_1\phi(0, 1) - \partial_1\phi(0, 0)| + |\partial_1\phi(1, 1) - \partial_1\phi(1, 0)| \right) \\ &\leq C \sum_{i=1}^2 \|\partial_{\mathbf{S}}(\partial_i\phi)\|_{L^p((\partial Z)_i)}. \end{aligned}$$

371 where  $(\partial Z)_1 = (0, 1) \times \{0, 1\}$  and  $(\partial Z)_2 = \{0, 1\} \times (0, 1)$ .

372 Observe also that ( $i \in \{1, 2\}$ )

$$373 \quad (6.6) \quad \|\partial_{ii}^2\Phi\|_{L^p((\partial Z)_i)} \leq C \|\partial_{\mathbf{S}}^2\phi\|_{L^p((\partial Z)_i)}.$$

Then, we obtain

$$\|\nabla\Phi\|_{L^p(Z)} \leq C(\|\partial_{\mathbf{S}}\phi\|_{L^p(\partial Z)} + \|D^2\Phi\|_{L^p(Z)}).$$

374 and thus

$$375 \quad \|\Phi\|_{L^p(Z)} \leq C(\|\phi\|_{L^p(\partial Z)} + \|\nabla\Phi\|_{L^p(Z)}).$$

*Step 2.* We prove the estimates (6.4) for  $N = 2$ .

In every small rectangle build on the nodes of  $\mathcal{S}$  we extend  $\phi$  as described in Step 1. That gives a function  $\Phi_p \in W^{2,p}(Y)$  satisfying (6.4) for  $N = 2$ . Estimate (6.5) follows by applying the Poincaré inequality twice and the fact that (see (6.6))

$$\|\partial_{\mathbf{S}}^2\phi_0\|_{L^p(\mathcal{S})} \leq \sum_{i=1}^2 \|\partial_{ii}^2\Phi\|_{L^p((\partial Z)_i)} + \|\partial_{\mathbf{S}}^2\phi\|_{L^p(\mathcal{S})} \leq C \|\partial_{\mathbf{S}}^2\phi\|_{L^p(\mathcal{S})}.$$

376 The proof is complete. □

377 We can finally show the asymptotic behavior of sequences bounded in  $W^{2,p}(\mathcal{S}_\varepsilon)$ ,  
 378 whose derivatives of the gradient extension from the lattice to the whole space are  
 379 also bounded.

380 **THEOREM 6.3.** *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $W^{2,p}(\mathcal{S}_\varepsilon)$ ,  $p \in (1, +\infty)$ , satisfying*

$$381 \quad (6.7) \quad \|\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \sum_{i=1}^N \|\partial_{\mathbf{s}}(\partial_i\phi_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}.$$

382 *Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and functions  $\phi \in W^{2,p}(\Omega)$   
 383 and  $\widehat{\phi} \in L^p(\Omega; W_{per}^{2,p}(\mathcal{S}))$  such that ( $i \in \{1, \dots, N\}$ )*

$$384 \quad \begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\rightarrow \phi && \text{strongly in } L^p(\Omega; W^{2,p}(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) &\rightarrow \partial_i\phi && \text{strongly in } L^p(\Omega; W^{1,p}(\mathcal{S}^{(i)})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2\phi_\varepsilon) &\rightarrow \partial_{ii}^2\phi + \partial_{\mathcal{S}}^2\widehat{\phi} && \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

385 *Proof.* Given  $\{\phi_\varepsilon\}_\varepsilon \subset W^{2,p}(\mathcal{S}_\varepsilon)$ , we decompose  $\phi_\varepsilon(\varepsilon\xi + \varepsilon\mathbf{S})$ ,  $\xi \in \widetilde{\Xi}_\varepsilon$ ,  $\mathbf{S} \in \mathcal{S}$ , as in  
 386 (6.3) and get

$$387 \quad \phi_\varepsilon = \Phi_{p,\varepsilon} + \phi_{0,\varepsilon}, \quad \Phi_{p,\varepsilon} \in W^{2,p}(\widetilde{\Omega}_\varepsilon), \quad \phi_{0,\varepsilon} \in \mathcal{W}_{0,\mathcal{K}_\varepsilon}^{2,p}(\mathcal{S}_\varepsilon).$$

388 We first consider the sequence  $\{\phi_{0,\varepsilon}\}_\varepsilon$  belonging to  $\mathcal{W}_{0,\mathcal{K}_\varepsilon}^{2,p}(\mathcal{S}_\varepsilon)$ . By estimate (6.5)  
 389 together with an affine change of variables and (6.7), we have

$$390 \quad \|\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon\|\partial_{\mathbf{s}}\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} + \varepsilon^2\|\partial_{\mathbf{s}}^2\phi_{0,\varepsilon}\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^2\|\partial_{\mathbf{s}}^2\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}+2},$$

391 where the constant does not depend on  $\varepsilon$ . Hence, there exist a subsequence, still  
 392 denoted  $\{\varepsilon\}$ , and a function  $\widehat{\phi}_0 \in L^p(\Omega; \mathcal{W}_{0,\mathcal{K},per}^{2,p}(\mathcal{S}))$  such that

$$393 \quad (6.8) \quad \frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_{0,\varepsilon}) \rightharpoonup \widehat{\phi}_0 \quad \text{weakly in } L^2(\Omega; W^{2,p}(\mathcal{S})).$$

394 Now we consider the sequence  $\{\Phi_{p,\varepsilon}\}_\varepsilon$ . By estimates (6.4) together with an affine  
 395 change of variables and hypothesis (6.7) we have

$$396 \quad \|\Phi_{p,\varepsilon}\|_{W^{2,p}(\widetilde{\Omega}_\varepsilon)} \leq C\varepsilon^{\frac{N-1}{p}} \left( \|\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \sum_{i=1}^N \|\partial_{\mathbf{s}}(\partial_i\phi_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \right) \leq C.$$

397 Hence, by [8, Theorem 1.47], there exist a subsequence, still denoted  $\{\varepsilon\}$ , and functions  
 398  $\phi \in W^{2,p}(\Omega)$  and  $\widehat{\Phi}_p \in L^p(\Omega; W_{per}^{2,p}(Y))$  such that

$$399 \quad \begin{aligned} \Phi_{p,\varepsilon|\Omega} &\rightharpoonup \phi && \text{weakly in } W^{2,p}(\Omega), \\ \mathcal{T}_\varepsilon(\Phi_{p,\varepsilon}) &\rightarrow \phi && \text{strongly in } L^p(\Omega; W^{2,p}(Y)), \\ \mathcal{T}_\varepsilon(\nabla\Phi_{p,\varepsilon}) &\rightarrow \nabla\phi && \text{strongly in } L^p(\Omega; W^{1,p}(Y))^N, \\ \mathcal{T}_\varepsilon(D^2\Phi_{p,\varepsilon}) &\rightharpoonup D^2\phi + D_y^2\widehat{\Phi}_p && \text{weakly in } L^p(\Omega \times Y)^{N \times N}. \end{aligned}$$

400 Note that the following relations hold ( $i \in \{1, \dots, N\}$ ):

$$401 \quad \begin{cases} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_{p,\varepsilon}) = \mathcal{T}_\varepsilon^{\mathcal{S}}(\Phi_{p,\varepsilon|\mathcal{S}_\varepsilon}) = \mathcal{T}_\varepsilon^{ext}(\Phi_{p,\varepsilon})_{|\widetilde{\Omega}_\varepsilon \times \mathcal{S}}, \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_{p,\varepsilon})_{|\widetilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}} = \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\Phi_{p,\varepsilon|\mathcal{S}_\varepsilon^{(i)}}) = \mathcal{T}_\varepsilon^{ext}(\partial_i\Phi_{p,\varepsilon})_{|\widetilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}}, \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2\phi_{p,\varepsilon})_{|\widetilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}} = \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2\Phi_{p,\varepsilon|\mathcal{S}_\varepsilon^{(i)}}) = \mathcal{T}_\varepsilon^{ext}(\partial_i^2\Phi_{p,\varepsilon})_{|\widetilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}}. \end{cases}$$

402 Hence, we can restrict the above convergences from  $\Omega \times Y$  to the subsets  $\Omega \times \mathcal{S}$  and  
 403  $\Omega \times \mathcal{S}^{(i)}$ , for every  $i \in \{1, \dots, N\}$ . Hence, there exists  $\widehat{\phi}_p = \widehat{\Phi}_{p|\Omega \times \mathcal{S}} \in L^p(\Omega; W_{per}^{2,p}(\mathcal{S}))$   
 404 such that ( $i \in \{1, \dots, N\}$ )

$$\begin{aligned} & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_{p,\varepsilon}) \rightarrow \phi \quad \text{strongly in } L^p(\Omega; W^{2,p}(\mathcal{S})), \\ 405 & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_{p,\varepsilon}) \rightarrow \partial_i\phi \quad \text{strongly in } L^p(\Omega; W^{1,p}(\mathcal{S}^{(i)})), \\ & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2\phi_{p,\varepsilon}) \rightharpoonup \partial_{ii}^2\phi + \partial_{\mathbf{s}}^2\widehat{\phi}_p \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}), \end{aligned}$$

406 where the strong convergences are preserved due to the polynomial character of the  
 407 function  $\mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_{p,\varepsilon})$  with respect to the second variable.

408 Hence, by the above convergences and (6.8) we get ( $i \in \{1, \dots, N\}$ )

$$\begin{aligned} & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^2(\Omega; W^{2,p}(\mathcal{S})), \\ 409 & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \rightarrow \partial_i\phi \quad \text{strongly in } L^2(\Omega; W^{1,p}(\mathcal{S}^{(i)})), \\ & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2\phi_\varepsilon) \rightharpoonup \partial_{ii}^2\phi + \partial_{\mathbf{s}}^2(\widehat{\phi}_p + \widehat{\phi}_0) \quad \text{weakly in } L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

410 Hence, the proof follows by setting  $\widehat{\phi} \doteq \widehat{\phi}_p + \widehat{\phi}_0$ , which belongs to  $L^2(\Omega; W_{per}^{2,p}(\mathcal{S}))$ .  $\square$

411 **6.2. Unfolding via known results for sequences of functions uniformly**  
 412 **bounded in  $W^{1,p}$ .** We consider the sequences in  $W^{2,p}(\mathcal{S}_\varepsilon)$  as sequences in  $W^{1,p}(\mathcal{S}_\varepsilon)$   
 413 with partial derivatives belonging to  $W^{1,p}(\mathcal{S}_\varepsilon^{(i)})$ , for each  $i \in \{1, \dots, N\}$ . In this sense,  
 414 we can apply the results obtained in section 5. Even though no gradient extension is  
 415 needed, the additional work must be done to show that the  $N$  different limit functions,  
 416 one for each partial derivative, are in fact a unique function restricted to each line.

417 From [3, Chapter 9], we recall that ( $p \in (1, +\infty)$ ):

- 418
- 419 (i) if  $u \in W^{1,p}(\Omega)$  satisfies  $\Delta u \in L^p(\Omega)$  then  $u \in W^{1,p}(\Omega) \cap W_{loc}^{2,p}(\Omega)$ <sup>3</sup>;
  - 420 (ii) if  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with a  $\mathcal{C}^{1,1}$  boundary and if  $u \in W_0^{1,p}(\Omega)$   
 421 satisfies  $\Delta u \in L^p(\Omega)$  then  $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ .
- 422

Denote ( $p \in [1, +\infty]$ )

$$\mathcal{W}^{2,p}(\Omega) \doteq \{\phi \in W^{1,p}(\Omega) \cap W_{loc}^{2,p}(\Omega) \mid \partial_{ii}^2\phi \in L^p(\Omega) \text{ for every } i \in \{1, \dots, N\}\}.$$

We endow  $\mathcal{W}^{2,p}(\Omega)$  with the following norm

$$\|\phi\|_{\mathcal{W}^{2,p}(\Omega)} \doteq \|\phi\|_{W^{1,p}(\Omega)} + \sum_{i=1}^N \|\partial_{ii}^2\phi\|_{L^p(\Omega)}.$$

423

424 **THEOREM 6.4.** *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $W^{2,p}(\mathcal{S}_\varepsilon)$ ,  $p \in (1, +\infty)$ , satisfying*

$$425 \quad (6.9) \quad \|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}^2\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}.$$

426 *Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and functions  $\phi \in \mathcal{W}^{2,p}(\Omega)$*

<sup>3</sup>In fact, we have  $\rho D^2 u \in L^p(\Omega)^{N \times N}$  where  $\rho(x) = \text{dist}(x, \partial\Omega)$  for all  $x \in \mathbb{R}^N$ .

427 and  $\widehat{\phi} \in L^p(\Omega; W_{per,0}^{2,p}(\mathcal{S}))$  such that ( $i \in \{1, \dots, N\}$ )

$$\begin{aligned}
& \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^p(\Omega; W^{1,p}(\mathcal{S})), \\
428 \quad (6.10) \quad & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\phi \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}), \\
& \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2\phi_\varepsilon) \rightharpoonup \partial_{ii}^2\phi + \partial_{\mathbf{S}}^2\widehat{\phi} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}).
\end{aligned}$$

429 The same results hold for  $p = +\infty$  with weak topology replaced by weak-\* topology in  
430 the corresponding spaces.

431 *Proof. Step 1.* We prove convergences (6.10)<sub>1,2</sub>.

432 By estimate (6.9), the sequence  $\{\phi_\varepsilon\}_\varepsilon$  satisfies

$$433 \quad \|\phi_\varepsilon\|_{W^{1,p}(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{p}}$$

434 and thus by Corollary 5.8, there exist  $\phi \in W^{1,p}(\Omega)$  and  $\widehat{\phi} \in L^p(\Omega; W_{per,0}^{1,p}(\mathcal{S}))$  such  
435 that

$$\begin{aligned}
436 \quad (6.11) \quad & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^p(\Omega; W^{1,p}(\mathcal{S})), \\
& \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \rightharpoonup \partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}), \quad i \in \{1, \dots, N\}.
\end{aligned}$$

Now, we consider the sequences  $\{\psi_\varepsilon^{(i)}\}_\varepsilon = \{\partial_{\mathbf{s}}\phi_{\varepsilon|\mathcal{S}_\varepsilon^{(i)}}\}_\varepsilon$ ,  $i \in \{1, \dots, N\}$ . From estimate (6.9) we have

$$\|\psi_\varepsilon^{(i)}\|_{W^{1,p}(\mathcal{S}_\varepsilon^{(i)})} \leq C\varepsilon^{\frac{1-N}{p}}.$$

Since  $\psi_\varepsilon^{(i)}$ ,  $i \in \{1, \dots, N\}$ , is defined on every node of  $\mathcal{S}_\varepsilon$ , we extend it as a function affine on every small segments in  $\mathcal{S}_\varepsilon^{(j)}$ ,  $j \in \{1, \dots, N\} \setminus \{i\}$ . We still denote this extension  $\psi_\varepsilon^{(i)}$ . It satisfies

$$\|\psi_\varepsilon^{(i)}\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}\psi_\varepsilon^{(i)}\|_{L^p(\mathcal{S}_\varepsilon^{(i)})} + \varepsilon\|\partial_{\mathbf{s}}\psi_\varepsilon^{(i)}\|_{L^p(\mathcal{S}_\varepsilon^{[i]})} \leq C\varepsilon^{\frac{1-N}{p}}, \quad \text{where } \mathcal{S}_\varepsilon^{[i]} = \bigcup_{j=1, j \neq i}^N \mathcal{S}_\varepsilon^{(j)}.$$

437 Observe that a function defined and constant on every line of  $\mathcal{S}^{(i)}$  can be extended  
438 to a function periodic on  $\mathcal{S}$  and affine between two contiguous nodes of  $\mathcal{S}^{(j)}$ , where  
439  $j \in \{1, \dots, N\} \setminus \{i\}$ . Lemma 5.7 gives a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and  
440 functions  $\widetilde{\psi}^{(i)} \in L^p(\Omega, \partial_i; W_{per}^{1,p}(\mathcal{S}^{[i]}))$ ,  $\widehat{\psi}^{(i)} \in L^p(\Omega; W_{per}^{1,p}(\mathcal{S}))$ . Here, due to the above  
441 remark, we assume that  $\mathcal{M}_{\mathcal{S}^{(i)}}(\widehat{\psi}^{(i)}) = 0$  a.e. in  $\Omega \times \mathcal{S}^{(i)}$ .

442 Thus, one has ( $i \in \{1, \dots, N\}$ )

$$\begin{aligned}
443 \quad & \mathcal{T}_\varepsilon^{\mathcal{S}}(\psi_\varepsilon^{(i)}) \rightharpoonup \widetilde{\psi}^{(i)} \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{S})), \\
& \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\psi_\varepsilon^{(i)}) \rightharpoonup \partial_i\widetilde{\psi}^{(i)} + \partial_{\mathbf{S}}\widehat{\psi}^{(i)} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}).
\end{aligned}$$

The above second convergence and (6.11)<sub>2</sub> yield

$$\partial_i\phi + \partial_{\mathbf{S}}\widehat{\phi} = \widetilde{\psi}^{(i)} \quad \text{a.e. in } \Omega \times \mathcal{S}^{(i)}, \quad i \in \{1, \dots, N\}.$$

444 Since  $\widetilde{\psi}^{(i)}$  does not depend on  $\mathbf{S}$  in  $\mathcal{S}^{(i)}$  and  $\widehat{\phi}$  is periodic with respect to  $\mathbf{S}$  in  $\mathcal{S}^{(i)}$  we  
445 have  $\partial_i\phi = \widetilde{\psi}^{(i)}$  and  $\partial_{\mathbf{S}}\widehat{\phi} = 0$  a.e.  $\Omega \times \mathcal{S}^{(i)}$  for every  $i \in \{1, \dots, N\}$ .

446 Hence,  $\tilde{\psi}^{(i)}$  belongs to  $L^p(\Omega, \partial_i)$  and thus  $\partial_i \phi \in L^p(\Omega, \partial_i)$ . Now, since  $\Delta \phi \in L^p(\Omega)$   
 447 we have  $\phi \in \mathbf{W}^{2,p}(\Omega)$ . Therefore, the following convergences hold:

$$\begin{aligned} & \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) \rightarrow \phi \quad \text{strongly in } L^p(\Omega; W^{2,p}(\mathcal{S})), \\ 448 \quad & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_\varepsilon) \rightarrow \partial_i \phi \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}), \\ & \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}}^2 \phi_\varepsilon) \rightarrow \partial_{ii}^2 \phi + \partial_{\mathbf{S}} \widehat{\psi}^{(i)} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

449 Moreover, we also have that, for each  $i \in \{1, \dots, N\}$ :

$$450 \quad (6.12) \quad \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_\varepsilon) \right) \rightarrow \partial_{ii}^2 \phi \mathbf{S}^c + \widehat{\psi}^{(i)} \quad \text{weakly in } L^p(\Omega \times \mathcal{S}^{(i)}).$$

451 *Step 2.* We prove the convergence (6.10)<sub>3</sub>.

452 We have to prove the existence of  $\widehat{\phi} \in L^p(\Omega; W_{per,0}^{2,p}(\mathcal{S}))$  such that

$$453 \quad \begin{cases} \partial_{\mathbf{S}} \widehat{\phi} = \widehat{\psi}^{(1)} & \text{a.e. in } \Omega \times \mathcal{S}^{(1)}, \\ \vdots \\ \partial_{\mathbf{S}} \widehat{\phi} = \widehat{\psi}^{(N)} & \text{a.e. in } \Omega \times \mathcal{S}^{(N)}. \end{cases}$$

454 A necessary and sufficient condition to get existence of the function  $\widehat{\phi}$  is (remind that  
 455  $A(k + \mathbf{e}_i) = A(k) + l_i \mathbf{e}_i$ )

$$456 \quad (6.13) \quad \forall k \in \widehat{\mathbf{K}}, \quad \begin{aligned} & \int_{A(k)}^{A(k+\mathbf{e}_i)} \widehat{\psi}^{(i)}(\cdot, \mathbf{S}) d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \widehat{\psi}^{(j)}(\cdot, \mathbf{S}) d\mathbf{S} \\ & = \int_{A(k)}^{A(k+\mathbf{e}_j)} \widehat{\psi}^{(j)}(\cdot, \mathbf{S}) d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \widehat{\psi}^{(i)}(\cdot, \mathbf{S}) d\mathbf{S} \end{aligned}$$

457 a.e. in  $\Omega$ .

458 Since on a line belonging to  $\mathcal{S}^{(i)}$ , one has (see Lemma 5.4)  $\mathbf{S}^c = t - \frac{1}{2}$ ,  $t \in [0, 1]$ , the

459 above equality (6.13) is equivalent to  $\forall k \in \widehat{\mathbf{K}}$ ,

$$460 \quad (6.14) \quad \begin{aligned} & \int_{A(k)}^{A(k+\mathbf{e}_i)} (\partial_{ii}^2 \phi \mathbf{S}^c + \widehat{\psi}^{(i)}(\cdot, \mathbf{S})) d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} (\partial_{jj}^2 \phi \mathbf{S}^c + \widehat{\psi}^{(j)}(\cdot, \mathbf{S})) d\mathbf{S} \\ & = \int_{A(k)}^{A(k+\mathbf{e}_j)} (\partial_{jj}^2 \phi \mathbf{S}^c + \widehat{\psi}^{(j)}(\cdot, \mathbf{S})) d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} (\partial_{ii}^2 \phi \mathbf{S}^c + \widehat{\psi}^{(i)}(\cdot, \mathbf{S})) d\mathbf{S} \end{aligned}$$

461 a.e. in  $\Omega$ .

462 Convergence (6.12) gives (remind that  $\partial_{ii}^2 \phi$  does not depends on  $\mathbf{S}$ )

$$463 \quad \forall k \in \widehat{\mathbf{K}} \quad \begin{aligned} & \int_{A(k)}^{A(k+\mathbf{e}_i)} \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_\varepsilon) \right) d\mathbf{S} \\ & \rightarrow \int_{A(k)}^{A(k+\mathbf{e}_i)} (\partial_{ii}^2 \phi \mathbf{S}^c + \widehat{\psi}^{(i)}) d\mathbf{S} \\ & = \partial_{ii}^2 \phi \int_{k_i l_i}^{(k_i+1)l_i} \left( t - \frac{1}{2} \right) dt + \int_{A(k)}^{A(k+\mathbf{e}_i)} \widehat{\psi}^{(i)}(x, \mathbf{S}) d\mathbf{S}.. \end{aligned}$$

464 Similarly, one has ( $j \neq i$ )

$$465 \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \right) d\mathbf{S} \\ \rightarrow \partial_{ii}^2 \phi \int_{k_i l_i}^{(k_i+1)l_i} \left( t - \frac{1}{2} \right) dt + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \widehat{\psi}^{(i)}(x, \mathbf{S}) d\mathbf{S}$$

466 and same kind of results for the other two quantities.

467 Hence, to get (6.13), we have to prove that both quantities

$$468 \quad (6.15) \quad \int_{A(k)}^{A(k+\mathbf{e}_i)} \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \right) d\mathbf{S} \\ + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(j)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \right) d\mathbf{S}$$

469 and

$$470 \quad (6.16) \quad \int_{A(k)}^{A(k+\mathbf{e}_j)} \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(j)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \right) d\mathbf{S} \\ + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) - \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) \right) d\mathbf{S}.$$

admit the same limit or equivalently that the limit of their difference is 0.

First we note that

$$\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} = \frac{1}{\varepsilon} \int_{A(k)}^{A(k+\mathbf{e}_i)} \partial_{\mathbf{S}} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) d\mathbf{S} \\ = \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k+\mathbf{e}_i)) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k)) \right) \text{ a.e. in } \widetilde{\Omega}_\varepsilon.$$

Hence,

$$\frac{1}{\varepsilon} \left( \int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} + \int_{A(k+\mathbf{e}_i)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} \right) \\ = \frac{1}{\varepsilon} \left( \int_{A(k)}^{A(k+\mathbf{e}_j)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} + \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_i+\mathbf{e}_j)} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} \right) \text{ a.e. in } \widetilde{\Omega}_\varepsilon.$$

Now, recall that the function  $\mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon)$  is defined on  $\widetilde{\Omega}_\varepsilon \times \mathcal{S}^{(i)}$  and is constant on every line of  $\mathcal{S}^{(i)}$ . One has a.e. in  $\widetilde{\Omega}_\varepsilon$

$$\mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) = \int_{A(k')}^{A(k')+\mathbf{e}_i} \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} = \frac{1}{\varepsilon} \int_{A(k')}^{A(k')+\mathbf{e}_i} \partial_{\mathbf{S}} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) d\mathbf{S} \\ = \frac{1}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k')+\mathbf{e}_i) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k')) \right)$$

on  $\widetilde{\Omega}_\varepsilon \times [A(k'), A(k')+\mathbf{e}_i]$ ,  $k' \in \widehat{\mathbf{K}}_i$ .

Hence

$$\int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}\phi_\varepsilon) d\mathbf{S} \\ = \frac{l_i}{\varepsilon} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k')+\mathbf{e}_i) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k')) \right) \text{ a.e. in } \widetilde{\Omega}_\varepsilon,$$

471 where  $k' \in \widehat{\mathbf{K}}_i$  is such that  $k = k' + k_i \mathbf{e}_i$ . Hence, we get

$$\begin{aligned}
 & \frac{1}{\varepsilon} \left( \int_{A(k)}^{A(k+\mathbf{e}_i)} \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}} \phi_\varepsilon) d\mathbf{S} - \int_{A(k+\mathbf{e}_j)}^{A(k+\mathbf{e}_j+\mathbf{e}_i)} \mathcal{M}_{\mathcal{S}^{(i)}} \circ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}} \phi_\varepsilon) d\mathbf{S} \right) \\
 472 &= \frac{l_i}{\varepsilon^2} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k') + \mathbf{e}_i) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k')) \right. \\
 & \quad \left. - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k' + \mathbf{e}_j) + \mathbf{e}_i) + \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k' + \mathbf{e}_j)) \right) \text{ a.e. in } \widetilde{\Omega}_\varepsilon
 \end{aligned}$$

473 where  $k' \in \widehat{\mathbf{K}}_i$  is such that  $k = k' + k_i \mathbf{e}_i$ .

474 Now, we can apply Lemma 8.1 and claim that the limit of the difference of the  
 475 quantities in (6.15) and (6.16) is equal to 0. This proves (6.14) for every  $k \in \widehat{\mathbf{K}}$ .

476 As a consequence, there exists a unique  $\widehat{\phi} \in L^p(\Omega; W_{per,0}^{2,p}(\mathcal{S}))$  such that convergence  
 477 (6.10)<sub>3</sub> holds.  $\square$

**7. Application: homogenization of a fourth order homogeneous Dirichlet problem on a periodic lattice structure.** We can now give a direct application of the periodic unfolding for sequences in  $H^2(\mathcal{S}_\varepsilon)$ .

From now on, let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with a  $C^{1,1}$  boundary. Let  $\{A_\varepsilon\}_\varepsilon$  be the sequence of functions belonging to  $L^\infty(\mathcal{S}_\varepsilon)$  defined by

$$A_\varepsilon(\mathbf{s}) \doteq A\left(\left\{\frac{\mathbf{s}}{\varepsilon}\right\}\right) \quad \text{for a.e. } \mathbf{s} \in \mathcal{S}_\varepsilon,$$

478 where  $A$  belongs to  $L^\infty(\mathcal{S})$  satisfies

$$479 \quad (7.1) \quad \exists(c, C) \in (0, +\infty)^2 \quad \text{such that} \quad c \leq A(\mathbf{S}) \leq C \quad \text{for a.e. } \mathbf{S} \in \mathcal{S}$$

and let  $\{g_\varepsilon\}_\varepsilon$  and  $\{f_\varepsilon\}_\varepsilon$  be sequences in  $L^2(\mathcal{S}_\varepsilon)$ .

Set the space

$$H_0^1(\mathcal{S}_\varepsilon) \doteq \{\phi \in H^1(\mathcal{S}_\varepsilon) \mid \phi = 0 \text{ a.e. on } \partial\widetilde{\Omega}_\varepsilon \cap \mathcal{S}_\varepsilon\}.$$

By the Poincaré and Poincaré–Wirtinger inequalities, we have

$$\forall \phi \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon), \quad \|\phi\|_{L^2(\mathcal{S}_\varepsilon)} \leq C \|\partial_{\mathbf{s}} \phi\|_{L^2(\mathcal{S}_\varepsilon)} \leq C \|\partial_{\mathbf{s}}^2 \phi\|_{L^2(\mathcal{S}_\varepsilon)},$$

480 where the constants do not depend on the parameter  $\varepsilon$  (note that  $\mathcal{M}_{\mathcal{S}^{(i)}}(\partial_{\mathbf{s}} \phi) = 0$  for

481 every  $i \in \{1, \dots, N\}$ ).

482 Consider the 4th order homogeneous Dirichlet problem in variational formulation:

$$483 \quad (7.2) \quad \begin{cases} \text{Find } u_\varepsilon \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon) \text{ such that:} \\ \int_{\mathcal{S}_\varepsilon} A_\varepsilon \partial_{\mathbf{s}}^2 u_\varepsilon \partial_{\mathbf{s}}^2 \phi \, d\mathbf{s} = \int_{\mathcal{S}_\varepsilon} g_\varepsilon \partial_{\mathbf{s}} \phi \, d\mathbf{s} + \int_{\mathcal{S}_\varepsilon} f_\varepsilon \phi \, d\mathbf{s}, \quad \forall \phi \in H_0^1(\mathcal{S}_\varepsilon) \cap H^2(\mathcal{S}_\varepsilon). \end{cases}$$

484 The Lax–Milgram theorem implies that the problem (7.2) has a unique solution.

485 Moreover, one has

$$\begin{aligned}
 486 \quad c \|\partial_{\mathbf{s}}^2 u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}^2 &\leq \|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \\
 487 &\leq C (\|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}) \|\partial_{\mathbf{s}}^2 u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}.
 \end{aligned}$$

489 Hence

$$490 \quad (7.3) \quad \|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}^2 u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \leq C (\|g_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|f_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)}).$$

491 The constant does not depend on  $\varepsilon$ .

492 Below, we give the periodic homogenization via unfolding.



493 THEOREM 7.1. Let  $u_\varepsilon$  be the solution of problem (7.2) and  $\{g_\varepsilon\}_\varepsilon, \{f_\varepsilon\}_\varepsilon$  satisfying

$$494 \quad (7.4) \quad \begin{aligned} \varepsilon^{\frac{1-N}{2}} \mathcal{T}_\varepsilon^{\mathcal{S}}(g_\varepsilon) &\rightarrow g \quad \text{strongly in } L^2(\Omega \times \mathcal{S}), \\ \varepsilon^{\frac{1-N}{2}} \mathcal{T}_\varepsilon^{\mathcal{S}}(f_\varepsilon) &\rightarrow f \quad \text{strongly in } L^2(\Omega \times \mathcal{S}). \end{aligned}$$

495 Then, there exist functions  $u \in H_0^1(\Omega) \cap H^2(\Omega)$  and  $\hat{u} \in L^2(\Omega; H_{per,0}^2(\mathcal{S}))$  such that  
496 ( $i \in \{1, \dots, N\}$ )

$$497 \quad (7.5) \quad \begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(u_\varepsilon) &\rightarrow u \quad \text{strongly in } L^2(\Omega; H^2(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}} u_\varepsilon) &\rightharpoonup \partial_i u \quad \text{weakly in } L^2(\Omega; H^1(\mathcal{S}^{(i)})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2 u_\varepsilon) &\rightharpoonup \partial_{ii}^2 u + \partial_{\mathbf{S}}^2 \hat{u} \quad \text{strongly in } L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

498 The couple  $(u, \hat{u})$  is the unique solution of problem

$$499 \quad (7.6) \quad \begin{cases} \sum_{i=1}^N \frac{1}{|\mathcal{S}|} \int_{\Omega \times \mathcal{S}^{(i)}} A(\partial_{ii}^2 u + \partial_{\mathbf{S}}^2 \hat{u})(\partial_{ii}^2 \phi + \partial_{\mathbf{S}}^2 \hat{\phi}) dx d\mathbf{S} \\ = \int_{\Omega} G \cdot \nabla \phi dx + \int_{\Omega} F \phi dx, \\ \forall \phi \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{and} \quad \forall \hat{\phi} \in L^2(\Omega; H_{per,0}^2(\mathcal{S})) \end{cases}$$

where

$$G \doteq \sum_{i=1}^N \frac{1}{|\mathcal{S}|} \left( \int_{\mathcal{S}^{(i)}} g(\cdot, \mathbf{S}) d\mathbf{S} \right) \mathbf{e}_i, \quad F \doteq \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} f(\cdot, \mathbf{S}) d\mathbf{S}.$$

500 *Proof. Step 1.* We show (7.6).

501 The solution  $u_\varepsilon$  of (7.2) satisfies (7.3). Due to the convergences (7.4) we have that

$$502 \quad \|u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}} u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{s}}^2 u_\varepsilon\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\frac{1-N}{2}}.$$

503 The constant does not depend on  $\varepsilon$ .

504 Hence, up to a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , Theorem 6.4 gives functions  
505  $u \in H_0^1(\Omega) \cap H^2(\Omega)$  and  $\hat{u} \in L^2(\Omega; H_{per,0}^2(\mathcal{S}))$  such that the following convergences  
506 hold ( $i \in \{1, \dots, N\}$ ):

$$507 \quad \begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(u_\varepsilon) &\rightarrow u \quad \text{strongly in } L^2(\Omega; H^2(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}} u_\varepsilon) &\rightharpoonup \partial_i u \quad \text{weakly in } L^2(\Omega; H^1(\mathcal{S}^{(i)})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2 u_\varepsilon) &\rightharpoonup \partial_{ii}^2 u + \partial_{\mathbf{S}}^2 \hat{u} \quad \text{weakly in } L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

508 Now, we choose the test functions

- 509 •  $\phi$  in  $C^\infty(\bar{\Omega}) \cap H_0^1(\Omega)$ ,
- 510 •  $\Phi$  in  $\mathcal{D}(\Omega)$ ,
- 511 •  $\hat{\phi}$  in  $H_{per,0}^2(\mathcal{S})$ .

Set

$$\phi_\varepsilon(x) \doteq \varepsilon^{\frac{1-N}{2}} \left( \phi(\mathbf{s}) + \varepsilon^2 \Phi(\mathbf{s}) \hat{\phi}\left(\frac{\mathbf{s}}{\varepsilon}\right) \right), \quad \text{a.e. } \mathbf{s} \in \mathcal{S}_\varepsilon.$$

512 Applying the unfolding operator to the sequence  $\{\phi_\varepsilon\}_\varepsilon$ , we get that ( $i \in \{1, \dots, N\}$ )

$$513 \quad \begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\rightarrow \phi \quad \text{strongly in } L^2(\Omega; H^2(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}} \phi_\varepsilon) &\rightarrow \partial_i \phi \quad \text{strongly in } L^2(\Omega; H^1(\mathcal{S}^{(i)})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{s}}^2 \phi_\varepsilon) &\rightarrow \partial_{ii}^2 \phi + \Phi \partial_{\mathbf{S}}^2 \hat{\phi} \quad \text{strongly in } L^2(\Omega \times \mathcal{S}^{(i)}). \end{aligned}$$

514 Taking  $\phi_\varepsilon$  as test function in (7.2), then transforming by unfolding and passing to  
 515 the limit give (7.6) with  $(\phi, \Phi\widehat{\phi})$ . By density argumentation, we extend such results  
 516 to all  $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$  and  $\widehat{\phi} \in L^2(\Omega; H_{per,0}^2(\mathcal{S}))$ . Since the solution is unique, the  
 517 whole sequences converge to their limit.

518 *Step 2.* We show that convergence (7.5)<sub>3</sub> is strong.

519 Taking  $\phi = u_\varepsilon$  in (7.2), then transforming by unfolding and using the weak lower  
 520 semicontinuity yield

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega \times \mathcal{S}^{(i)}} A |\partial_{ii}^2 u + \partial_{\mathbf{S}}^2 \widehat{u}|^2 dx d\mathbf{S} \\
 & \leq \liminf_{\varepsilon \rightarrow 0} \sum_{i=1}^N \int_{\Omega \times \mathcal{S}} \mathcal{T}_\varepsilon(A_\varepsilon) |\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}}^2 u_\varepsilon)|^2 dx d\mathbf{S} \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{N-1} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} A_\varepsilon |\partial_{\mathbf{S}}^2 u_\varepsilon|^2 ds \\
 521 & \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{N-1} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} A_\varepsilon |\partial_{\mathbf{S}}^2 u_\varepsilon|^2 ds = \limsup_{\varepsilon \rightarrow 0} \varepsilon^{N-1} \left( \int_{\mathcal{S}_\varepsilon} g_\varepsilon \partial_{\mathbf{S}} u_\varepsilon ds + \int_{\mathcal{S}_\varepsilon} f_\varepsilon u_\varepsilon ds \right) \\
 & = |\mathcal{S}| \left( \int_{\Omega} G \cdot \nabla \phi dx + \int_{\Omega} F \phi dx \right) = \sum_{i=1}^N \int_{\Omega \times \mathcal{S}^{(i)}} A |\partial_{ii}^2 u + \partial_{\mathbf{S}}^2 \widehat{u}|^2 dx d\mathbf{S}.
 \end{aligned}$$

Also observe that

$$\begin{aligned}
 & \liminf_{\varepsilon \rightarrow 0} \sum_{i=1}^N \int_{\Omega \times \mathcal{S}} \mathcal{T}_\varepsilon(A_\varepsilon) |\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}}^2 u_\varepsilon)|^2 dx d\mathbf{S} \leq \limsup_{\varepsilon \rightarrow 0} \sum_{i=1}^N \int_{\Omega \times \mathcal{S}} \mathcal{T}_\varepsilon(A_\varepsilon) |\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}}^2 u_\varepsilon)|^2 dx d\mathbf{S} \\
 & \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{N-1} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} A_\varepsilon |\partial_{\mathbf{S}}^2 u_\varepsilon|^2 ds
 \end{aligned}$$

522 From the above inequalities it follows that

$$\begin{aligned}
 & \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} \mathcal{T}_\varepsilon^{\mathcal{S}}(A_\varepsilon) |\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}}^2 u_\varepsilon)|^2 dx d\mathbf{S} \\
 523 & = \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^N \int_{\mathcal{S}_\varepsilon} A_\varepsilon |\partial_{\mathbf{S}}^2 u_\varepsilon|^2 ds = \sum_{i=1}^N \int_{\Omega \times \mathcal{S}^{(i)}} A |\partial_{ii}^2 u + \partial_{\mathbf{S}}^2 \widehat{u}|^2 dx d\mathbf{S}.
 \end{aligned}$$

524 Since the map  $\Psi \in L^2(\Omega \times \mathcal{S}) \mapsto \sqrt{\int_{\Omega \times \mathcal{S}} A |\Psi|^2 dx d\mathbf{S}}$  is a norm equivalent to the  
 525 usual norm of  $L^2(\Omega \times \mathcal{S})$ , we get

$$526 \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times \mathcal{S}} |\mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}}^2 u_\varepsilon)|^2 dx d\mathbf{S} = \int_{\Omega \times \mathcal{S}} |\partial_{ii}^2 u + \partial_{\mathbf{S}}^2 \widehat{u}|^2 dx d\mathbf{S}.$$

527 This, together with the fact that (7.5)<sub>3</sub> already converge weakly, ensures the strong  
 528 convergence.  $\square$

529 We define the corrector  $\widehat{\chi}_k$ ,  $k \in \{1, \dots, N\}$ , as the unique solution in  $H_{per,0}^2(\mathcal{S})$  of the  
 530 cell problem

$$531 \quad (7.7) \quad \int_{\mathcal{S}} A (\mathbf{1}_{\mathcal{S}^{(k)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_k) \partial_{\mathbf{S}}^2 \widehat{w} d\mathbf{S} = 0, \quad \forall \widehat{w} \in H_{per,0}^2(\mathcal{S}).$$

532

533 THEOREM 7.2. *The function  $u \in H_0^1(\Omega) \cap H^2(\Omega)$  is the unique solution of the*  
 534 *following homogenized problem:*

$$535 \quad (7.8) \quad \begin{cases} \int_{\Omega} A^{hom} \partial^2 u \cdot \partial^2 \phi \, dx = \int_{\Omega} G \cdot \nabla \phi \, dx + \int_{\Omega} F \phi \, dx, \\ \forall \phi \in H_0^1(\Omega) \cap H^2(\Omega), \end{cases}$$

$$536 \quad \text{where } \partial^2 u \doteq \begin{pmatrix} \partial_{11}^2 u \\ \vdots \\ \partial_{NN}^2 u \end{pmatrix} \text{ and } \partial^2 \phi \doteq \begin{pmatrix} \partial_{11}^2 \phi \\ \vdots \\ \partial_{NN}^2 \phi \end{pmatrix}.$$

537 *The homogenized matrix  $A^{hom}$  is given by  $((i, j) \in \{1, \dots, N\}^2)$*

$$538 \quad (7.9) \quad A_{ij}^{hom} \doteq \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} A(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_i)(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_j) \, d\mathbf{S}.$$

539 *Proof.* Equation (7.6) with  $\phi = 0$  leads to

$$540 \quad \begin{cases} \sum_{i=1}^N \int_{\Omega \times \mathcal{S}^{(i)}} A(\partial_{ii}^2 u + \partial_{\mathbf{S}}^2 \widehat{u}) \partial_{\mathbf{S}}^2 \widehat{\phi} \, dx d\mathbf{S} = 0, \\ \forall \widehat{\phi} \in L^2(\Omega; H_{per,0}^2(\mathcal{S})), \end{cases}$$

541 from which we obtain the form of the cell problems (7.7) and thus the representation  
 542 of  $\widehat{u}$

$$543 \quad \widehat{u}(x, \mathbf{S}) = \sum_{k=1}^N \partial_{kk}^2 u(x) \widehat{\chi}_k(\mathbf{S}), \quad \text{for a.e. } (x, \mathbf{S}) \in \Omega \times \mathcal{S}.$$

Replacing the above expression of  $\widehat{u}$  in (7.6) and choosing

$$\widehat{\phi}(x, \mathbf{S}) = \sum_{k=1}^N \partial_{kk}^2 \phi(x) \widehat{\chi}_k(\mathbf{S}), \quad \text{for a.e. } (x, \mathbf{S}) \in \Omega \times \mathcal{S}$$

lead to the following left hand side of (7.6):

$$\begin{aligned} & \frac{1}{|\mathcal{S}|} \int_{\Omega \times \mathcal{S}} A \left( \sum_{i=1}^N (\mathbf{1}_{\mathcal{S}^{(i)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_i) \partial_{ii}^2 u \right) \left( \sum_{j=1}^N (\mathbf{1}_{\mathcal{S}^{(j)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_j) \partial_{jj}^2 \phi \right) \, dx d\mathbf{S} \\ &= \int_{\Omega} \sum_{i,j=1}^N \left( \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} A(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_i)(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_j) \, d\mathbf{S} \right) \partial_{ii}^2 u \partial_{jj}^2 \phi \, dx. \end{aligned}$$

544 Taking into account (7.7), the above expression becomes  $\int_{\Omega} A^{hom} \partial^2 u \cdot \partial^2 \phi \, dx$  with

545 the matrix  $A^{hom}$  given by (7.9).

546 We prove now that  $A^{hom}$  is coercive. Let  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$  be a vector with  
 547 fixed entries. From (7.9) we first have

$$\begin{aligned} 548 \quad A^{hom} \boldsymbol{\xi} \cdot \boldsymbol{\xi} &= \frac{1}{|\mathcal{S}|} \sum_{i,j=1}^N \int_{\mathcal{S}} A(\mathbf{1}_{\mathcal{S}^{(i)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_i)(\mathbf{1}_{\mathcal{S}^{(j)}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_j) \, d\mathbf{S} \xi_i \xi_j \\ 549 \quad &= \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} A(\widetilde{\boldsymbol{\xi}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_{\boldsymbol{\xi}})^2 \, d\mathbf{S} \\ 550 \end{aligned}$$

where

$$\tilde{\boldsymbol{\xi}} \doteq \sum_{i=1}^N \xi_i \mathbf{1}_{\mathcal{S}^{(i)}}, \quad \widehat{\chi}_{\boldsymbol{\xi}} = \sum_{k=1}^N \xi_k \widehat{\chi}_k, \quad \text{a.e. in } \mathcal{S} \text{ and for all } \boldsymbol{\xi} \in \mathbb{R}^N.$$

551 Then, by hypothesis (7.1) on  $A$ , we get

$$552 \quad A^{hom} \boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \frac{c}{|\mathcal{S}|} \|\tilde{\boldsymbol{\xi}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_{\boldsymbol{\xi}}\|_{L^2(\mathcal{S})}^2.$$

554 By the periodicity of  $\partial_{\mathbf{S}} \widehat{\chi}_{\boldsymbol{\xi}}$ , for every  $\boldsymbol{\xi} \in \mathbb{R}^N$  we get that

$$555 \quad \begin{aligned} \|\tilde{\boldsymbol{\xi}} + \partial_{\mathbf{S}}^2 \widehat{\chi}_{\boldsymbol{\xi}}\|_{L^2(\mathcal{S})}^2 &= \|\tilde{\boldsymbol{\xi}}\|_{L^2(\mathcal{S})}^2 + \|\partial_{\mathbf{S}}^2 \widehat{\chi}_{\boldsymbol{\xi}}\|_{L^2(\mathcal{S})}^2 \geq \|\tilde{\boldsymbol{\xi}}\|_{L^2(\mathcal{S})}^2 \\ &= \sum_{i=1}^N |\mathcal{S}^{(i)}| |\xi_i|^2 \geq \min_k |\mathcal{S}^{(k)}| \sum_{i=1}^N |\xi_i|^2 = (\min_k |\mathcal{S}^{(k)}|) |\boldsymbol{\xi}|^2. \end{aligned}$$

556 Thus the coercivity of  $A^{hom}$  is proved since

$$557 \quad A^{hom} \boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq c |\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^N.$$

558 By the coercivity of  $A^{hom}$  and the fact that  $u \in H_0^1(\Omega) \cap H^2(\Omega)$ , problem (7.8) admits  
559 a unique solution.  $\square$

## 560 8. Appendix.

561 LEMMA 8.1. *Let  $\{\phi_\varepsilon\}_\varepsilon$  be a sequence in  $W^{2,p}(\mathcal{S}_\varepsilon)$ ,  $p \in (1, +\infty)$ , satisfying*

$$562 \quad \|\phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{S}} \phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} + \|\partial_{\mathbf{S}}^2 \phi_\varepsilon\|_{L^p(\mathcal{S}_\varepsilon)} \leq C \varepsilon^{\frac{1-N}{p}}.$$

For every  $k' \in \widehat{\mathbf{K}}$  we define in  $\widetilde{\Omega}_\varepsilon \times \widehat{\mathbf{K}}_i$  the piecewise constant function  $\Phi_\varepsilon^{(i,j)}$ , where  $(i,j) \in \{1, \dots, N\}^2$ ,  $i \neq j$ , by

$$\Phi_\varepsilon^{(i,j)}(\cdot, k') \doteq \begin{cases} \frac{l_i}{\varepsilon^2} \left( \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k') + \mathbf{e}_i) - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k')) \right. \\ \quad \left. - \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k' + \mathbf{e}_j) + \mathbf{e}_i) + \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon)(\cdot, A(k' + \mathbf{e}_j)) \right) \\ \quad \quad \quad \text{a.e. in } \widetilde{\Omega}_\varepsilon \times \widehat{\mathbf{K}}_i, \\ 0 \quad \quad \text{a.e. in } (\mathbb{R}^N \setminus \widetilde{\Omega}_\varepsilon) \times \widehat{\mathbf{K}}_i. \end{cases}$$

563 Then, there exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and  $\phi \in W^{1,p}(\Omega) \cap W_{loc}^{2,p}(\Omega)$   
564 such that  $((i,j) \in \{1, \dots, N\}^2, i \neq j, k' \in \widehat{\mathbf{K}}_i)$

$$565 \quad (8.1) \quad \begin{aligned} \mathcal{T}_\varepsilon^{\mathcal{S}}(\phi_\varepsilon) &\rightarrow \phi \quad \text{strongly in } L^p(\Omega; W^{2,p}(\mathcal{S})), \\ \mathcal{T}_\varepsilon^{\mathcal{S}}(\partial_{\mathbf{S}} \phi_\varepsilon) &\rightharpoonup \partial_j \phi \quad \text{weakly in } L^p(\Omega; W^{1,p}(\mathcal{S}^{(j)})), \\ \Phi_\varepsilon^{(i,j)}(\cdot, k') &\rightharpoonup -l_i l_j \partial_{ij}^2 \phi \quad \text{weakly in } W^{-1,p}(\mathbb{R}^N). \end{aligned}$$

*Proof.* There exist a subsequence of  $\{\varepsilon\}$ , still denoted  $\{\varepsilon\}$ , and a function  $\phi$  in the space  $W^{1,p}(\Omega) \cap W_{loc}^{2,p}(\Omega)$  such that convergences (8.1)<sub>1,2</sub> hold (see Theorem 6.4).

Now, let  $\psi$  be in  $W^{1,p'}(\mathbb{R}^N)$ , one has

$$\begin{aligned} & \int_{\Omega} \psi(x) \Phi_{\varepsilon}^{(i,j)}(x, k') dx \\ &= \varepsilon^N \sum_{\xi \in \mathbb{Z}^N} \mathcal{M}_Y(\psi)(\varepsilon\xi) \frac{l_i}{\varepsilon^2} \left( \phi_{\varepsilon}(\varepsilon\xi + \varepsilon A(k') + \varepsilon \mathbf{e}_i) - \phi_{\varepsilon}(\varepsilon\xi + \varepsilon A(k')) \right. \\ & \quad \left. - \phi_{\varepsilon}(\varepsilon\xi + \varepsilon A(k' + \varepsilon \mathbf{e}_j) + \varepsilon \mathbf{e}_i) + \phi_{\varepsilon}(\varepsilon\xi + \varepsilon A(k' + \mathbf{e}_j)) \right) \\ &= \varepsilon^N l_i \sum_{\xi \in \mathbb{Z}^N} \frac{\mathcal{M}_Y(\psi)(\varepsilon\xi - \varepsilon \mathbf{e}_i) - \mathcal{M}_Y(\psi)(\varepsilon\xi)}{\varepsilon} \frac{\phi_{\varepsilon}(\varepsilon\xi + \varepsilon A(k')) - \phi_{\varepsilon}(\varepsilon\xi + \varepsilon A(k' + \mathbf{e}_j))}{\varepsilon} \\ &= l_i \int_{\Omega} \frac{\psi - \psi(\cdot - \varepsilon \mathbf{e}_i)}{\varepsilon} \left( \int_{A(k')}^{A(k'+\mathbf{e}_j)} \mathcal{T}_{\varepsilon}^S(\partial_{\mathbf{s}} \phi_{\varepsilon}) d\mathbf{S} \right) dx. \end{aligned}$$

Then, due to convergences (8.1)<sub>2</sub>, we get

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \psi(x) \Phi_{\varepsilon}^{(i,j)}(x, k') dx = l_i \int_{\Omega} \partial_i \psi \left( \int_{A(k')}^{A(k'+\mathbf{e}_j)} \partial_j \phi d\mathbf{S} \right) dx = l_i l_j \int_{\Omega} \partial_i \psi \partial_j \phi dx.$$

Hence, (8.1)<sub>3</sub> is proved.  $\square$

567

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