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Algebraic invariants of orbit configuration spaces in genus zero associated to finite groups

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Abstract

We consider orbit configuration spaces associated to finite groups acting freely by orientation preserving homeomorphisms on the 2-sphere minus a finite number of points (eventually none). We compute the cohomology ring and the Poincaré series of these spaces. This generalizes the work of V. Arnold for "classical" configuration spaces of points of the plane. The results imply that the spaces we consider are formal in the sense of rational homotopy theory. We also prove the existence of an LCS formula relating the Poincaré series of such spaces to the

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Introduction

Context

For M a topological space and $n \geq 1$ an integer, the configuration space of n ordered points of M is the space:

$$C_n(M) = \{(p_1, \ldots, p_n) \in M | p_i \neq p_j, \text{for } i \neq j\}.$$

Configuration spaces appear naturally in mathematics. For instance, they are encountered in the study braid groups, Knizhnik-Zamolodchikov connections and manifold embeddings. The topology of these spaces and their algebraic invariants are widely studied.

Let S be an orientable surface. In general, for S with compact boundary (eventually empty), the space $C_n(S)$ is aspheric, except for S homeomorphic to the 2-sphere S^2. The fundamental group $\pi_1 C_n(S)$ of $C_n(S)$ is known as Artin pure braid group on n strands for $S = \mathbb{C}$ ([Art47]) or generally as the surface pure braid group on n strands ([GG04]) .

In [Arn69], the author computes the cohomology ring with integer coefficients and the Poincaré series of

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$C_n(\mathbb{C})$. The cohomology ring of $C_n(S^2)$ was computed in [FZ00] (also considered in [FH01]). The spaces $C_n(\mathbb{C} \setminus X)$ where X has cardinal 1 or n were considered in [FRV07] and [LV12]. In a more general context, models for $C_n(M)$ with M a smooth compact complex projective variety, are known ([FM94]). These models were simplified in ([Kri94]) and then used in ([Bez94]) to compute the Malcev Lie algebra Lie$\pi_1 C_n(S)$ of $\pi_1 C_n(S)$ for S closed (see also [Enr14], and [Koh83] for the case $S = \mathbb{C}$ where the Lie algebra is the Kohno-Drinfeld algebra). We will be interested in variants of configurations spaces (introduced in [Xic97]), known as orbit configuration spaces:

$$C_n^H(M) = \{(p_1, \ldots, p_n) \in M | p_i \neq h \cdot p_j, \text{for } i \neq j \text{ and } h \in H\},$$

where H is a group acting (continuously) on M. As for classical configuration spaces the projections $C_n^H(M) \to C_k^H(M)$ on the first k coordinates ($k \leq n$) are locally trivial fibrations under reasonable assumptions ([Xic97],[Xic14]). In [Coh01] and [Enr07] the authors considered the orbit configuration space $C_n^{\mu_p}(\mathbb{C}^\times)$ where μ_p is the group of p-roots of the unity acting by multiplication on $\mathbb{C}^\times$. Using different approaches they studied $\pi_1 C_n^{\mu_p}(\mathbb{C}^\times)$ and computed its Malcev Lie algebra. In [CX02], the orbit configuration spaces $C_n^H(\mathbb{C})$ where $H = \mathbb{Z} + i\mathbb{Z}$ acts additively is studied. In [CKX09], the case of orbit configuration spaces obtained out of a surface subgroup (of genus $g \geq 2$) of $\mathrm{PSL}(\mathbb{R}^2)$ acting freely on the upper half plane $\{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$ is considered. Cohomology of orbit configuration spaces of spheres with respect to the antipodal action were studied in [Xic00] and [FZ02] (see also [GGSX15]). Cohomology and homotopy groups of orbit configuration spaces are analyzed in [Cas16] using the notion of FL_G-modules. The "duality" and "abelian duality" properties for orbit configuration spaces of surfaces is considered in [DS18]. In [BG18], the authors use a spectral sequence related to posets to study the cohomology (with compact support) of orbit configuration spaces (see also [BG19]).

Here, we consider the orbit configuration spaces $C_n^H(S^2 \setminus Y)$ where Y is a finite set and H is a finite group acting freely by orientation preserving homeomorphisms on $S^2 \setminus Y$. Under these assumptions, the action of H on $S^2 \setminus Y$ is in fact equivalent to the natural action of a finite homography group $G \subset \mathrm{PGL}(\mathbb{C}^2)$ (isomorphic to H) on $\mathbb{P}^1 \setminus Z$, where $\mathbb{P}^1 \simeq S^2$ is the complex projective line and Z is a finite G-stable set containing the irregular points of G (points with non-trivial stabilizer). One can give a complete classification of such actions (Cf. Subsection 1.4) and give a biholomorphic equivalence between some orbit configuration spaces associated to isomorphic groups (Cf. Subsection 2.1).

In [Maa19], we have computed the Malcev Lie algebra of $C_n^G(\mathbb{P}^1 \setminus Z)$ for Z equal to the set of irregular points of G. In particular, we recover from [Maa19] the Lie algebras computed by [Coh01] and [Enr07] for $(G, Z) = (\{z \mapsto \zeta z | \zeta \in \mu_p\}, \{0, \infty\})$ and the Kohno-Drinfeld Lie algebras for $(G, Z) = (\{1\}, \infty)$. The work implies that $C_n^G(\mathbb{P}^1 \setminus Z)$ is 1-formal in the sense of rational homotopy theory. For $(G, Z) \neq (\{1\}, \emptyset)$, the space $C_n^G(\mathbb{P}^1 \setminus Z)$ is biholomorphic to the complement of a hypersurface in $\mathbb{C}^n$ and hence in $\mathbb{P}^n(\mathbb{C})$ (Cf. Subsection 2.2) and hence the 1-formality for $(G, Z) \neq \{1, \emptyset\}$ follows also from [Koh83].

Main results of the paper

Fix $R \subset \mathbb{C}$ a unital ring and set $X_n := C_n^G(\mathbb{P}^1 \setminus Z)$.

For $(G, Z) \neq (\{1\}, \emptyset)$ we prove (Cf. Subsections 4.1 and 5.1) that:

1) The singular cohomology ring $H^*(X_n, R)$ is isomorphic to the R-subalgebra $\Omega_D^*(X_n)_R$ of holomorphic (in fact algebraic) closed forms generated by logarithmic 1-forms $\{\omega_a\}_a$ (having integer periods) corresponding to the irreducible components $\{D_a\}_a$ of $(\mathbb{P}^1)^n \setminus X_n$. The isomorphism is induced by integration of forms on homology classes.

2) We find relations between the elements $\{\omega_a\}_a$ of $\Omega_D^*(X_n)_R \simeq H^*(X_n, R)$ wich together with the antisymmetry relations give a presentation of the algebra.

3) The homology and cohomology groups of X_n with coefficientss in R are free R-modules of finite type and the Poincaré series P_{X_n} of X_n is given by:

$$P_{X_n}(t) = \prod_{k=1}^{n} (1 + \alpha_k t),$$

where $\alpha_k = |G|(k-1) + |Z| - 1$.

4) The space X_n is formal in the sense of rational homotopy (Cf. Subsection 1.1, for the definition).

5) The space X_n is a $K(\pi, 1)$ space.

6) The fundamental group $\pi_1(X_n)$ of X_n is an iterated almost direct product of free groups in the sense of [CS98] (Cf. Subsection 1.3 for the definition) and the ranks of the abelian groups $\Gamma_i \pi_1 X_n / \Gamma_{i+1} \pi_1 X_n$ corresponding to the lower central series filtration $\{\Gamma_i \pi X_n\}$ of $\pi_1 X_n$ can be related to the Poincaré series of X_n by the "LCS formula":

$$P_{X_n}(-t) = \prod_{i \geq 1} (1 - t^i)^{\phi_i(\pi_1 X_n)},$$

where $\phi_i(\pi_1 X_n)$ is the rank $\Gamma_i \pi_1 X_n / \Gamma_{i+1} \pi_1 X_n$, and for which we give an explicit formula.

For the case $(G, Z) = (\{1\}, \emptyset)$, i.e. $X_n = C_n(\mathbb{P}^1) \simeq C_n(S^2)$, the cohomology ring was computed in [FZ00]. Here we show (Cf. Subsections 4.2 and 5.2) that:

7) The space $C_n(\mathbb{P}^1)$ is formal and construct a subalgebra of closed differential forms isomorphic via integration to $H^*(C_n(\mathbb{P}^1), R)$ for $\frac{1}{2} \in R$ and R a principal ideal domain.

8) We have an LCS formula relating the Poincaré series (that factors into a product of linear terms) to the ranks $\phi_i(\pi_1 X_n)$ of the abelian groups $\Gamma_i \pi_1 X_n / \Gamma_{i+1} \pi_1 X_n$ for which we give an explicit formula.

The constants $\phi_i(\pi_1 X_n)$ appearing in (6) and (8) correspond to the dimension of homogenous elements of graded Lie algebras introduced in [Maa19] (Cf. Remark 5.8).

Results $(1), (2)$ and (3) are obtained as a generalization of the work of [Arn69]. Result (4) follows from (1) using standard facts. Result (5) is obtained by classical means using homotopy long exact sequences of fibrations associated to orbit configuration spaces. Claim (7) follows from (1) and a known decomposition of $C_n(\mathbb{P}^1)$. The proof of the existence of the LCS formula (6) (and roughly speaking (8)) is similar to the one in [FR85] (generalized to groups in [CS98]).

In the case G cyclic generated by $z \mapsto \zeta z$ where ζ is a root of the unity and $Z = \{0, \infty\}$ or $G = \{1\}$ and $|Z| = 1$, $C_n^G(\mathbb{P}^1 \setminus Z)$ is the complement in $\mathbb{C}^n$ of a central hyperplane arrangement. The corresponding algebras $\Omega_D^*(X_n)_{\mathbb{Z}}$ are those of [Bri73], [Arn69] and the presentations are equivalent to those given in [OS80] (Cf. last paragraph of Subsection 3.2). As seen previously, $\pi_1 C_n(\mathbb{C})$ is the Artin pure braid group on n strands and the LCS formula is known (see for instance [CS98]).

One can obtain the Poincaré polynomial of X_n using (5) from the work of [CS98] (Cf. Section 5). We note that a "collapsing" result for spectral sequences related to Borel-Moore homology groups of orbit configuration spaces ([BG19], Theorem A) applies to the spaces we study when $(G, Z) \neq (\{1\}, \emptyset)$.

Outline of the paper

Section 1 consists of reminders on: formality, De Rham theorem, Iterated almost direct products, their LCS formula, the classification of finite homography groups actions on $\mathbb{P}^1$ and the correspondence between these actions and orientation preserving finite group actions on a 2-sphere with finite punctures.

In section 2, we recall the definition of orbit configuration spaces and some of their classical properties, establish the correspondence between the spaces $C_n^H(S^2 \setminus Y)$ and the spaces $C_n^G(\mathbb{P}^1 \setminus Z)$, exhibit $C_n^G(\mathbb{P}^1 \setminus Z)$ as a hypersurface complement in $\mathbb{C}^n$ for $(G, Z) \neq (\{1\}, \emptyset)$, and construct generators of $\pi_1 C_n^G(\mathbb{P}^1 \setminus Z)$ in the general case.

Section 3 contains the definition of the holomorphic 1-forms $\{\omega_a\}_a$ (for $(G, Z) \neq (1, \emptyset)$), their periods (values of their integrals over homology classes, used in section 4), relations between the 1-forms (the relations mentioned in (2)), the definition of the R-algebra $\Omega_D^*(X_n)_R$ generated by these 1-forms and a family of forms spanning $\Omega_D^*(X_n)_R$. We prove in section 4 that the family is a basis.

In section 4, we prove the main results $(1), (2), (3), (4), (7)$ and compute the Poincaré polynomials of the spaces $C_n^G(\mathbb{P}^1 \setminus Z)$. The proofs of $(5), (6)$ and (8) are given in section 5.

In the appendix, we consider a closed oriented surface S. We prove that if $Y \subset S$ is finite (eventually empty) and H is a finite group acting by orientation preserving homeomorphisms on $S \setminus Y$, then the action of H extends to an action on S and there exists a complex structure on S in which H acts holomorphically. The result follows from the case $Y = \emptyset$ and this seems to be known, but the author was not able to track a proof. We use this in section 1 to link finite group actions on $S^2 \setminus Y$ to homography actions on $\mathbb{P}^1$.

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1. Reminders

In subsection 1.1, we recall the definition of formal spaces and some related facts for smooth manifolds. We go rapidly through the construction of the De Rham isomorphism in subsection 1.2. Subsection 1.3

is devoted to almost product of group and the LCS formula of an iterated almost product of free groups. The last subsection (1.4) contains reminders on homogarphies of $\mathbb{P}^1$ including the classification of finite homography groups of $\mathrm{PGL}(\mathbb{C}^2)$. We also relate continuous orientation preserving actions of finite groups on S^2 with a finite number of punctures, to actions of finite homography groups on $\mathbb{P}^1$.

1.1. Cochain algebras and formality of topological spaces

Most the material in this section can be found in [FHT01]. For R a ring an R-cochain algebra (dga for short) is a unital associative graded R-algebra $A^* = \oplus_{i\geq 0} A^i$, equipped with a differential $d : A^* \to A^*$ mapping $A^i \to A^{i+1}$, such that: $d(xy) = d(x)y + (-1)^i x d(y)$ and $d^2 = 0$, for $x \in A^i$ and $y \in A^j$. We say that A^* is commutative (A^* is a cdga) if $xy = (-1)^{ij} yx$, for x and y as before. The cohomology of the dga A^* is $H^*(A^*) := \mathrm{Ker}(d)/\mathrm{Im}(d)$ that inherits a structure of graded algebra from A. Throughout the rest of the text, a graded map of degree 0 is simply called graded map or map of graded modules-vector spaces-cdga's.

We say that two cdga's A^* and B^* are weakly equivalent if there exists a sequence of cdga morphisms (morphisms of algebras respecting the differentials) connecting A^* and B^*:

$$A^* \to C(1)^* \leftarrow C(2)^* \to \cdots \to C(n)^* \leftarrow B^*,$$

and inducing isomorphisms in cohomology (quasi-isomorphisms).

Let F be a field of characteristic zero and X a topological space. One assigns functorially to X the cdga $A^*_{PL}(X)_F$ of polynomial differential forms on X with coefficientss in F. The algebra $H^*(A^*_{PL}(X)_F)$ is naturally isomorphic to the singular cohomology algebra of X with coefficientss in F. For F' a field containing F, one has a natural quasi-isomorphism $A^*_{PL}(X)_F \otimes F' \to A^*_{PL}(X)_{F'}$. When X is a smooth manifold, $A^*_{PL}(X)_{\mathbb{R}}$ is weakly equivalent to the cdga $\Omega^*(X)_{\mathbb{R}}$ of real valued smooth differential forms on X with product the wedge product and differential the exterior differential. It follows from the last two statements that:

Proposition 1.1. *If X is a smooth manifold, then $A^*_{PL}(X)_{\mathbb{C}}$ is weakly equivalent to the cdga $\Omega^*(X)_{\mathbb{C}}$ of complex valued smooth differential forms on X with product the wedge product and differential the exterior differential.*

A path connected topological space X is called formal over F (usually over $\mathbb{Q}$) if $A^*_{PL}(X)_F$ is weakly equivalent to its cohomology (equivalently the singular cohomology of X with coefficientss in F) with zero differential. It turns out that: if the rational cohomology of X is finite dimensional, then X is formal over F if and only if X is formal over $\mathbb{Q}$.

1.2. De Rham theorem

Let X be a smooth manifold and F be the field of real numbers or complex numbers. A smooth singular k-simplex on X, is a singular k-simplex on X admitting a smooth extension to a neighborhood of the

standard k-simplex Δ^k. One can therefore define the integral of a differential form ω over a smooth simplex σ^s by $\int_{\sigma^s}\omega := \int_{\Delta_k}(\sigma^s)^*\omega$, where Δ^k is endowed with the standard orientation. We denote by $H^*_{DR}(X,F)$ the De Rham cohomology of smooth F-valued differential forms on X (corresponding to the cohomology of the cdga $\Omega^*(X)_F$ seen in subsection 1.1). One has an integration morphism ([Lee13], [Bre97]) of graded F-vector spaces:

$$\int : H^*_{DR}(X,F) \to \mathrm{Hom}_{\mathbb{Z}}(H_*(X,\mathbb{Z}),F),$$
$$\omega \mapsto ([\sigma] \mapsto \int_{\sigma^s}\omega)$$

where $H_*(X,F)$ is the singular homology group of X with coefficientss in F, $\mathrm{Hom}_{\mathbb{Z}}$ stands for morphisms of abelian groups, $[\sigma]$ is a homology class and σ^s is a smooth representative of $[\sigma]$ (a representative which is a sum of smooth simplices). The natural map $N^F_{X,*} : H^*(X,F) \to \mathrm{Hom}_{\mathbb{Z}}(H_*(X,\mathbb{Z}),F)$, where $H^*(X,F)$ is the singular cohomology of X with coefficientss in F, is an isomorphism, since F is a field. The De Rham theorem states that:

$$(N^F_{X,*})^{-1}\circ\int : H^*_{DR}(X,F) \to H^*(X,F) \text{ is an isomorphism of algebras,}$$

with $H^*_{DR}(X,F)$ equipped with the wedge product. The theorem is usually stated for $F=\mathbb{R}$. It also holds for $F=\mathbb{C}$, since $H^*_{DR}(X,\mathbb{C}) \simeq H^*_{DR}(X,\mathbb{R})\otimes\mathbb{C}$. One can find the a proof in [War83] (p. 205-207) or [Whi57] (p. 142).

1.3. Almost direct products and LCS formula

Let H be a group. We denote by the $\{\Gamma_i H\}_{i\geq 1}$ the lower central series of H: $\Gamma_1 H = H$ and $\Gamma_{i+1}H = (H,\Gamma_i H)$, for $i\geq 1$, where (A,B) is the subgroup of H generated by the commutators $(a,b) = aba^{-1}b^{-1}$ for $(a,b)\in A\times B$. Let H^{ab} be the abelianization of H (i.e. $H/\Gamma_2 H$). An IA-automorphism of H is an automorphism inducing the identity on H^{ab}. A semi-direct product of two groups $H_1 \rtimes_\beta H_2$ is called almost direct product if for all $h_2\in H_2$, $\beta(h_2)$ is IA. For k a positive integer, we denote by $F(k)$ the free group on k generators.

Proposition 1.2 ([FR85],[CS98]). *Let*

$$F := F(c_n)\rtimes_{\alpha_n}(F(c_{n-1})\rtimes_{\alpha_{n-1}}(F(c_{n-2})\rtimes_{\alpha_{n-1}}(\cdots\rtimes_{\alpha_3}(F(c_2)\rtimes_{\alpha_2}F(c_1))\cdots))),$$

be an iterated almost direct product, i.e. the image of α_i consists of IA-automorphisms, for $i\in[1,n-1]$. We have:

$$\prod_{i\geq 1}(1-t^i)^{\phi_i(F)} = \prod_{k=1}^{n}(1-c_k t),$$

where $\phi_i(F)$ is the rank of the abelian group $\Gamma_i F/\Gamma_{i+1}F$ for $i\geq 1$ and $\Gamma_k F$ is the k-th term of the lower central series of F.

The formula follows from the fact that $\Gamma_i F/\Gamma_{i+1}F = \oplus_{l=1}^m \Gamma_i F(c_l)/\Gamma_{i+1}F(c_l)$ ([FR85], Theorem 3.1) and the formula ([MKS66], p. 330) known for free groups $(1-ct) = \prod_{i\geq 1}(1-t^i)^{\phi_i(F(c))}$. An explicit formula for $\phi_i(F(c))$ is known ([MKS66]):

$$\phi_i(F(c)) = \frac{1}{i}\sum_{j|i}\mu(j)c^{i/j},$$

where μ is the Möbius function. Therefore, the constants $\phi_i(F)$ in the proposition are given by:

$$\phi_i(F) = \sum_{l=1}^n \phi_i(F(c_l)) = \frac{1}{i}\sum_{l=1}^n\sum_{j|i}\mu(j)c_l^{i/j}, \tag{1}$$

where μ is the Möbius function.

Remark 1.3. *In [CS98], it is shown that the homology with integer coefficients of F as in the proposition, is free as an abelian group and that the Poincaré series of F is given by $\prod_{k=1}^n(1+c_k t)$.*

1.4. Orientation preserving group actions in genus 0

Let $a_H : H \times X_H \to X_H$ and $a_{H'} : H' \times X_{H'} \to X_{H'}$ be actions by homeomorphisms of finite groups H and H' on topological spaces X_H and $X_{H'}$. We say that the actions a_H and $a_{H'}$ are equivalent if there exists an isomorphism $f : H \to H'$ and a homeomorphism $g : X_H \to X_{H'}$ such that $ga_{H'}(f\times g) = a_H$. The irregular points of H (with respect to a_H) are the elements of X_H with non-trivial stabilizer.

Recall that the action of $\mathrm{GL}(\mathbb{C}^2)$ on $\mathbb{C}^2$ is compatible to the projection $\mathbb{C}^2\setminus 0 \to \mathbb{P}^1, (z,w)\mapsto [z:w]$ and therefore we have an induced action of $\mathrm{GL}(\mathbb{C}^2)$ on $\mathbb{P}^1$ given for $A\in\mathrm{GL}(\mathbb{C}^2)$ by:

$$A\cdot[z:w] = [A_1(z,w):A_2(z,w)] \quad \text{where } A(z,w) = (A_1(z,w), A_2(z,w)),$$

The mapping $[z:w]\mapsto A\cdot[z:w]$ is called a homography. It is common to only give the homography in the chart $[z:1]$, i.e. $z\mapsto \frac{A_1(z,1)}{A_2(z,1)}$. The kernel of this action (a) is the center C of $\mathrm{GL}(\mathbb{C}^2)$ consisting of homotheties and hence the induced action of the projective linear group $\mathrm{PGL}(\mathbb{C}^2) = \mathrm{GL}(\mathbb{C}^2)/C$ on $\mathbb{P}^1$ is faithful. The group $\mathrm{PGL}(\mathbb{C}^2)$ is also called the homography group of $\mathbb{P}^1$.

The following facts are well known. We give proofs or references for completeness.

Proposition 1.4. *The group of automorphisms (biholomorphic self maps) of $\mathbb{P}^1$ is $\mathrm{PGL}(\mathbb{C}^2)$ acting by homorgraphy and every element of $\mathrm{PGL}(\mathbb{C}^2)$ fixes at least a point in $\mathbb{P}^1$.*

Proof. For the first statement one can find a proof in [FL12] (Cf. p. 83). The second statement follows from the fact that an element of $\mathrm{GL}(\mathbb{C}^2)$ have at least one eigenvector.

Proposition 1.5. *Let G be a finite subgroup of $\mathrm{PGL}(\mathbb{C}^2)$.*

1) *The group G is either cyclic, diherdral or isomorphic to $\mathfrak{A}_4, \mathfrak{S}_4$ or $\mathfrak{A}_5$, and all these groups occur.*

2) Two isomorphic finite subgroups of $\mathrm{PGL}(\mathbb{C}^2)$ *are conjugate.*

3) The action of G *on* $\mathbb{P}^1$ *is equivalent to one of these actions:*

- *The action of a group* G_R *generated by a rotation of finite order on the* 2*-sphere.*
- *The action of* $\langle G_R, r\rangle$ *on* S^2*, where* G_R *is as above and* r *is a rotation of order* 2 *with axis orthogonal to the axis of rotation of* G_R *(different choices of* r *give equivalent actions).*
- *The action of the isometry group of a Platonic solid (*$\mathfrak{A}_4, \mathfrak{S}_4$ *or* $\mathfrak{A}_5$*) on the surface of the corresponding solid which is a topological* 2*-sphere.*

4) Every non-trivial element of G *fixes exactly two points.*

5) If G *is not trivial, then the number of points with non-trivial stabilizer (irregular points) is* 2 *if* G *is cyclic or* $2 + |G|$ *otherwise. These points form* 2 *orbits if* G *is cyclic and* 3 *orbits otherwise.*

Proof. The diffeomorphism $S^2 \to \mathbb{P}^1$ obtained from stereographic projections gives an identification between the groups $\mathrm{SO}(\mathbb{R}^3)$ and $\mathrm{PSU}(\mathbb{C}^2)$ ([Bea95], p.63). The facts above hold for $\mathrm{SO}(\mathbb{R}^3)$ ([Art91], p.184) and one only needs to show that every finite subgroup of $\mathrm{PGL}(\mathbb{C}^2)$ is conjugate to a subgroup of $\mathrm{PSU}(\mathbb{C}^2)$. The inclusion $\mathrm{PSL}(\mathbb{C}^2) \to \mathrm{PGL}(\mathbb{C}^2)$ is an isomorphism and the preimage of a finite subgroup of $\mathrm{PSL}(\mathbb{C}^2)$ in $\mathrm{SL}(\mathbb{C}^2)$ is a finite subgroup. Let G be a finite subgroup of $\mathrm{SL}(\mathbb{C}^2)$ and f be the standard Hermitian form on $\mathbb{C}^2$. The group G is an isometry group for the non-degenerate Hermitian form $f_G := \sum_{g\in G} g^*f$, where g^*f is the pullback of f by g. Both f and f_G are non-degenerate. Hence, we have isomorphism of Hermitian spaces $(\mathbb{C}^2, f) \xrightarrow{T} (\mathbb{C}^2, f_G)$ and the group TGT^{-1} is a subgroup of $\mathrm{SU}(\mathbb{C}^2)$. We have proved the proposition.

The following proposition is a special case of proposition 6.8 proved in the appendix.

Proposition 1.6. *Let* H *be a finite group acting by orientation preserving homeomorphisms on* $S^2 \setminus Y$*, where* Y *is a finite subset of* S^2*. There exists a finite subgroup* G *of* $\mathrm{PGL}(\mathbb{C}^2)$ *such that the action of* G *by homography on* $\mathbb{P}^1 \setminus Z$*, for a given finite* $Z \subset \mathbb{P}^1$ *stable under the action of* G*, is equivalent to the action of* H *on* $S^2 \setminus Y$*.*

2. Orbit configuration spaces in genus 0 associated to finite groups

In the first subsection, we recall the definition of orbit configuration spaces, a fibration theorem for these spaces, and use the results of subsection 1.4 to show that the orbit configuration space $C_n^H(S^2 \setminus Y)$ where H is a finite group acting freely by orientation preserving homeomorphisms on S^2 minus a finite set Y, is homeomorphic to $C_n^G(\mathbb{P}^1 \setminus Z)$, where G is a homography group and Z is a given finite set. In the second subsection, we prove that the orbit configuration spaces $C_n^G(\mathbb{P}^1 \setminus Z)$ correspond to the complement of a singular hypersurface in $\mathbb{C}^n$ (except when $(G, Z) = (\{1\}, \emptyset)$). We then define loops generating the fundamental group of $C_n^G(\mathbb{P}^1 \setminus Z)$, in subsection 2.3.

2.1. Orbit configuration spaces of the 2-sphere

Let H be a group acting by homeomorphisms on a topological manifold M. The orbit configuration space of (ordered) n-points of M with respect to H is the topological subspace of M^n:

$$C_n^H(M) = \{(p_1, \ldots, p_n) \in M^n | p_i \notin H \cdot p_j, \text{ for } 1 \leq i \neq j \leq n\},$$

for $n \geq 1$, and by convention $C_0^H(M)$ is a point. When H is trivial, the space is the classical configuration space of ordered n-points of M.

The space $C_n^H(M)$ is naturally equipped with a topological action of the semidirect product $H^n \rtimes \mathfrak{S}_n$. The action is given by the datum:

$$g_i \cdot (p_1, \ldots, p_n) = (p_1, \ldots, p_{i-1}, g \cdot p_i, p_{i+1}, \ldots, p_n)$$

$$\text{and} \quad \sigma \cdot (p_1, \ldots, p_n) = (p_{\sigma^{-1}(1)}, \ldots, p_{\sigma^{-1}(n)}),$$

for $i \in [1, n], g \in G, \sigma \in \mathfrak{S}_n$ and where $g_i = (1, \ldots, 1, g, 1, \ldots, 1)$ with g at the i-th position.

Theorem 2.1 ([Xic97]). *If H is a finite group acting freely by homeomorphism on a boundaryless manifold M. For $n \geq k \geq 0$, the projection $C_n^H(M) \to C_k^H(M)$ on the first k coordinates is a locally trivial fibration.*

Here we study orbit configuration spaces associated to a finite group H acting freely by orientation preserving homeomorphisms on the 2-sphere $S^2 \simeq \mathbb{P}^1$ minus a finite number of points:

$$C_n^H(S^2 \setminus Y) = \{(p_1, \ldots, p_n) \in (S^2 \setminus Y)^n | p_i \notin H \cdot p_j, \text{ for } 1 \leq i \neq j \leq n\},$$

where $Y \subset S^2$ is finite. It follows from proposition 1.5 and proposition 1.6, that:

Proposition 2.2. *The space $C_n^H(S^2 \setminus Y)$ for n, H and Y as in the previous paragraph, is homeomorphic to $C_n^G(\mathbb{P}^1 \setminus Z)$, where $G \simeq H$ is a finite subgroup of $\mathrm{PGL}(\mathbb{C}^2)$ acting naturally (by homographies) on $\mathbb{P}^1$ and $Z \subset \mathbb{P}^1$ is finite, stable under the action of G, and contains the irregular points of G. Moreover, $Z \neq \emptyset$ if $G \neq \{1\}$.*

In the sequel G and Z are as in the above proposition.

Remark 2.3. *Let G be a subgroup of $\mathrm{PGL}(\mathbb{C}^2)$ and h a homography. The map $h^{\times n}$ induces a biholomorphism $C_n^G(\mathbb{P}^1 \setminus Z) \to C_n^{hGh^{-1}}(\mathbb{P}^1 \setminus h(Z))$. In particular, for $(G, Z) \neq (\{1\}, \emptyset)$ we can assume up to applying a biholomorphism that $\infty \in Z$ (as an irregular point if $G \neq \{1\}$). Moreover, two configuration spaces of the form $C_n^G(\mathbb{P}^1 \setminus Z)$ associated to two isomorphic subgroups of $\mathrm{PGL}(\mathbb{C}^2)$ can be biholomorphic depending on the choice of the stable sets, since two isomorphic subgroups of $\mathrm{PGL}(\mathbb{C}^2)$ are conjugate.*

If $G = \{1\}$, the spaces $C_n^G(\mathbb{P}^1 \setminus Z)$ correspond to classical configuration space : the configuration space of the sphere if Z is empty, the configuration space of the plane if $|Z| = 1$, the configuration space of the plane

minus 1 point if $|Z| = 1$ etc..

The space $\mathbb{P}^1 \setminus Z$ and the action of G satisfy the conditions of proposition 2.1. Hence, for $n \geq 1$, we have a fiber bundle (the base is paracompact):

$$F_n \to C_n^G(\mathbb{P}^1 \setminus Z) \overset{\pi_n}{\to} C_{n-1}^G(\mathbb{P}^1 \setminus Z)$$

where π_n is the projection on the first $n-1$ coordinates and the fiber F_n is a 2-sphere minus $|Z| + (n-1)|G|$ points. The orbit configuration space $C_n^G(\mathbb{P}^1 \setminus Z)$ is path connected.

2.2. Orbit configuration spaces of the 2-sphere as complements of hypersurfaces

Take G and Z as in the previous subsection. The space $C_n^G(\mathbb{P}^1 \setminus Z)$ is the complement in $(\mathbb{P}^1)^n$ of the hypersurfaces:

$$D_{ij}(g) = \{(p_1, \ldots, p_n) \in (\mathbb{P}^1)^n | p_i = g \cdot p_j\} \quad D_{kk}(p) = \{(p_1, \ldots, p_n) \in (\mathbb{P}^1)^n | p_k = p\}$$

for $i, j, k \in [1, n], g \in G$ and $p \in Z$ with $i < j$ (to avoid repetitions). Putting coordinates $([z_1 : w_1], \ldots, [z_n : w_n])$ on $(\mathbb{P}^1)^n$, setting $g([z : w]) = [g_1(z, w) : g_2(z, w)]$ for $g \in G$, and $p = [p_z : p_w]$ for $p \in Z$, we get that $D_{ij}(g)$ and $D_{ii}(p)$ are respectively the zero loci of the polynomials :

$$w_i g_1(z_j, w_j) - z_i g_2(z_j, w_j) \quad \text{and} \quad w_i p_z - z_i p_w.$$

For $Z \neq \emptyset$, up to a biholomorphism, we can assume that (Cf. Remark 2.3): $\infty \in Z$ and that the other elements of Z lie in $\mathbb{C}$. Under these assumptions $C_n^G(\mathbb{P}^1 \setminus Z)$ is the complement in $\mathbb{C}^n$ of the hypersurface of equation:

$$f(z_1, \ldots, z_n) = \prod_{1 \leq k \leq n, q \in Z \setminus \{\infty\}} P_{ii}^q(z_1, \ldots, z_n) \prod_{1 \leq i < j \leq n, g \in G} P_{ij}^\alpha(z_1, \ldots, z_n) = 0,$$

where $P_{ii}^q = z_i - q \quad \text{and} \quad P_{ij}^g = g_1(z_j, 1) - z_i g_2(z_j, 1)$.

If G is cyclic of order $m \geq 1$. We can assume, up to conjugacy-biholomorphism (Cf. Remark 2.3), that G is generated by $z \mapsto \zeta z$ where ζ is a m-th primitive root of the unity. In that case, the hypersurface is given by the equation:

$$f(z_1, \ldots, z_{n+1}) = \prod_{\substack{1 \leq i \leq n \\ q \in (Z \setminus \{\infty\})}} (z_i - q) \prod_{\substack{1 \leq i < j \leq n \\ 0 \leq k < m}} (z_i - \zeta^k z_j) = 0.$$

This arrangement is central (the intersection of all the hyperplanes is not empty) if and only if $Z = \{0, \infty\}$. In the case G not cyclic, the "collection" of the irreducible components of the hypersurface $f = 0$ can not be mapped homeomorphically to an arrangement of hyperplanes. Indeed, if G is not cyclic, there exists a $g \in G$ not stabilizing the infinity. The intersection $\{P_{ij}^1 = 0\} \cap \{P_{ij}^g = 0\}$ for such a g contains exactly two connected components corresponding to the 2 points fixed by g. Hence, $\{P_{ij}^1 = 0\} \cap \{P_{ij}^g = 0\}$ is not homeomorphic to the intersection of 2 hyperplanes, since the latter is always connected.

2.3. Generators of the fundamental group

We will construct loops of $C_n^G(\mathbb{P}^1 \setminus Z)$, for $n \geq 1$. We fix a base point $P_n = (p_1, \ldots, p_n) \in C_n^G(\mathbb{P}^1 \setminus Z)$ and we set for $i \in [1, n]$:

$$Y_i = Z \cup Y_{i,G}, \text{ where } Y_{i,G} = \cup_{j \neq i \in [1,n],} G \cdot p_j.$$

The space $\mathbb{P}^1 \setminus Y_i$ is naturally homeomorphic to the fiber of $\pi_n : C_{n+1}^G(\mathbb{P}^1 \setminus Z) \to C_n^G(\mathbb{P}^1 \setminus Z)$ over P_n. We endow $\mathbb{P}^1$ with its natural orientation. For $q \in Y_i$, let $\gamma_i(q)$ be a smooth simple anticlockwise oriented loop of $\mathbb{P}^1$ based at p_i, avoiding q and bounding a closed disc $D(q)$ such that $D(q) \cap Y_i = q$.

We choose the loops $\gamma_i(q)$ so that they generate the fundamental group of $\mathbb{P}^1 \setminus Y_i$ based at p_i.

Definition 2.4. *For $i, j \in [1, n], g \in G$ and $p \in Z$ we define the smooth loops x_{ij}^g (for $i \neq j$) and x_{ii}^p of $C_n^G(\mathbb{P}_*^1)$ based at P_n by the following:*

$$x_{ij}^g(t) = (p_1, \ldots, p_{i-1}, \gamma_i(g \cdot p_j)(t), p_{i+1}, \ldots, p_n),$$

$$x_{ii}^p(t) = (p_1, \ldots, p_{i-1}, \gamma_i(p)(t), p_{i+1}, \ldots, p_n),$$

for $t \in [0, 1]$.

Proposition 2.5. *For any $p_\infty \in Z$, the loops x_{ij}^g and x_{kk}^p for $g \in G, p \in Z$ and $i, j, k \in [1, n]$ such that $i < j$ and $p \neq p_\infty$, generate $\pi_1(C_n^G(\mathbb{P}^1 \setminus Z), P_n)$.*

Proof. The result can be obtained by induction using the long exact sequences of the fibrations $C_k^G(\mathbb{P}^1 \setminus Z) \to C_{k-1}^G(\mathbb{P}^1 \setminus Z)$ (see for instance proposition 3.3 of [Maa19]).

We have seen that $C_n^G(\mathbb{P}^1 \setminus Z)$ is biholomorphic to $\mathbb{C}^n \setminus \{f = 0\}$, where $\{f = 0\}$ is a hypersurface. The loops in the proposition are meridian loops with respect to irreducible components of the hypersurface $\{f = 0\}$ and hence generate $H^1(C_n^G(\mathbb{P}^1 \setminus Z), \mathbb{Z})$ ([GH94], p. 455). Moreover, they generate $H^1(C_n^G(\mathbb{P}^1 \setminus Z), \mathbb{Z})$ freely ([Dim92], p. 102). This will also follow from other elements in the next sections.

3. Differential forms on the orbit configuration spaces

In subsection 3.1, we introduce, for $(G, Z) \neq (\{1\}, \emptyset)$, closed holomorphic 1-forms on $X_n := C_n^G(\mathbb{P}^1 \setminus Z)$ and consider, for $R \subset \mathbb{C}$ an unital ring, the R-algebra $\Omega_D^*(X_n)_R$ of differential forms on X_n generated by these forms. We also study the action of $G^n \rtimes \mathfrak{S}_n$ on $\Omega_D^*(X_n)_R$. The action of $G^n \rtimes \mathfrak{S}_n$ is used in subsection 3.2 to deduce relations (defined over $\mathbb{Z}$) in $\Omega_D^2(X_n)_R$. The relations allow us to give a family of forms spanning $\Omega_D^*(X_n)_R$ as an R-module and to define, by generators and relations, an R-algebra $A_n(R)$ equipped with a natural surjective map to $\Omega_D^*(X_n)_R$. In the last subsection, we study the periods of $\Omega_D^1(X_n)_R$ (values of integrals of elements of $\Omega_D^1(X_n)_R$ on 1-homology classes of X_n). The results on periods will be used in

the next section.

Till the end of this section, we assume that (G, Z) is different from $(\{1\}, \emptyset)$.

3.1. Differential forms and the algebra $\Omega^*_D(C^G_n(\mathbb{P}^1 \setminus Z))_R$

For $A \in \mathrm{PGL}(\mathbb{C}^2)$ with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the canonical basis of $\mathbb{C}^2$, set:

$$P_A(x, y) = (cy + d)(x - A \cdot y) = x(cy + d) - (ay + b),$$

where $A \cdot y = \frac{ay+b}{cy+d}$. We assign to $h \in \mathrm{PGL}_2(\mathbb{C})$ a 1-form

$$\omega^h(x, y) := d\log(P_A(x, y)) = \frac{dP_A(x, y)}{P_A(x, y)}, \tag{2}$$

where A is a lift of h to $\mathrm{PGL}(\mathbb{C}^2)$. The definition of $\omega^h(x, y)$ do not depend on the choice of the lift, since $P_{\lambda A} = \lambda P_A$ for λ a scalar. The form $\omega^h(x, y)$ is holomorphic over $\mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 | x = h(y)\}$. Indeed, the set of zeros of the polynomial P_A is exactly the set $\{(x, y) \in \mathbb{C}^2 | x = h(y)\}$. We also define for $p \in \mathbb{C}$ the holomorphic 1-form on $\mathbb{C} \setminus \{p\}$:

$$\omega^p(x) := d\log(x - p). \tag{3}$$

Both $\omega^p(x)$ and $\omega^h(x, y)$ are closed forms.

We pick a $p_\infty \in Z$ and chose a homography $h_Z \in \mathrm{PGL}(\mathbb{C}^2)$ mapping p_∞ to ∞. If Z contains ∞ we take $p_\infty = \infty$ and $h_Z = \mathrm{id}$. The map $h_Z^{\times n}$ induces a biholomorphism $C^G_n(\mathbb{P}^1 \setminus Z) \to C^{h_Z G h_Z^{-1}}_n(\mathbb{P}^1 \setminus h_Z(Z))$ and $\mathbb{P}^1 \setminus h_Z(Z)$ is $\mathbb{C} \setminus (h_Z(Z) \setminus \{\infty\})$. We will introduce differential forms on $C^G_n(\mathbb{P}^1 \setminus Z)$, using the coordinates $([z_1 : 1], \ldots, [z_n : 1])$. The forms will depend on the choice of p_∞ and h_Z, but as we will see later on the algebra of complex valued forms generated by these forms is independent of these choices.

Definition 3.1. *1) For $g \in G, p \in Z \setminus \{p_\infty\}$ and $i, j, k \in [1, n]$, with $i \neq j$, we define the holomorphic closed forms ω^g_{ij} and ω^p_{kk} of $C^G_n(\mathbb{P}^1 \setminus Z)$, by:*

$$\omega^p_{kk} = \frac{1}{2i\pi}(h_Z^{\times n})^* \omega^{h_Z(p)}(z_k) \quad \text{and} \quad \omega^g_{ij} = \frac{1}{2i\pi}(h_Z^{\times n})^* \omega^{h_Z g h_Z^{-1}}(z_i, z_j),$$

where the star $*$ is for pullback, $\omega^{h_Z(p)}$ and $\omega^{h_Z g h_Z^{-1}}$ are as in (3) and (2), and $\omega^\infty = 0$ (and hence $\omega^{p_\infty}_{kk} = 0$) by convention.

2) For R an unital subring of $\mathbb{C}$, we define $\Omega^*_D(C^G_n(\mathbb{P}^1 \setminus Z))_R$ to be the R-subalgebra of complex valued forms on $C^G_n(\mathbb{P}^1 \setminus Z)$, generated by the 1-forms defined in (1) with grading induced by the degree of forms.

For instance $\omega^1_{ij} = \frac{1}{2i\pi} d\log(z_i - z_j)$ if $h_Z = \mathrm{id}$.

Lemme 3.2. *For A and $B \in \mathrm{GL}(\mathbb{C}^2)$. We have:*

$$P_A(x, B\cdot y) = \frac{P_{AB}(x,y)}{D_B(y)} \quad \text{and} \quad P_A(B\cdot x, y) = \det(B)\frac{P_{B^{-1}A}(x,y)}{D_B(x)},$$

for $B\cdot z = \frac{b_{1,1}z+b_{1,2}}{b_{2,1}z+b_{2,2}}$ and $D_B(z) = b_{2,1}z + b_{2,2}$, where $b_{i,j}$ $(i,j \in [1,2])$ is the i,j entry of the matrix of B in the canonical basis of $\mathbb{C}^2$.

Proof. To prove the lemma one can assume that the matrices of A and B in the canonical basis are $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, then compute both sides of each equation.

Recall that the group $G^n \rtimes \mathfrak{S}_n$ acts on $C_n^G(\mathbb{P}^1 \setminus Z)$. The action is holomorphic and hence the group acts on the right (by pullback) on complex valued differential forms on $C_n^G(\mathbb{P}^1 \setminus Z)$. For h a homography of $\mathbb{P}^1$ and $i \in [1,n]$, we denote by h_i the bijection $(\mathbb{P}^1)^n \to (\mathbb{P}^1)^n$ acting by h on the i-th coordinate and acting trivially on the other coordinates.

Proposition 3.3. *For $g,h,f \in G, p \in (Z \setminus \{p_\infty\}), \sigma \in \mathfrak{S}_n$ and $i,j,k,l \in [1,n]$ with $i \neq j$, we have:*

$$(h_i f_j)^*\omega_{ij}^g = \omega_{ij}^{h^{-1}gf} - \omega_{ii}^{h^{-1}(p_\infty)} - \omega_{jj}^{f^{-1}(p_\infty)}, \quad h_k^*\omega_{kk}^p = \omega_{kk}^{h^{-1}(p)} - \omega_{kk}^{h^{-1}(p_\infty)},$$

$$h_k^*\omega_{ij}^g = \omega_{ij}^g \quad \text{if } k \neq i,j, \quad h_l^*\omega_{kk}^p = \omega_{kk}^p,$$

if $l \neq k$, and

$$\sigma^*\omega_{ij}^g = \omega_{\sigma^{-1}(i)\sigma^{-1}(j)}^g, \quad \sigma^*\omega_{kk}^p = \omega_{\sigma^{-1}(k)\sigma^{-1}(k)}^p.$$

Proof. One deduces the proposition from the case $\infty \in Z$ (i.e. $p_\infty = \infty$ and $h_Z = \mathrm{id}$), by pulling back the equations. We hence only prove the proposition for $\infty \in Z$. For $v \in \{f,g,h\}$, we chose a lift $\tilde{v}$ of v to $\mathrm{GL}(\mathbb{C}^2)$. Using the definition of ω_{ij}^g, the definition of a pullback, then by applying lemma 3.2 twice and identifying the terms we get:

$$\begin{aligned}(2i\pi)(h_i f_j)^*\omega_{ij}^g &= d\log(P_{\tilde{g}}(\tilde{h}\cdot z_i, \tilde{f}\cdot z_j)) \\ &= d\log(P_{\tilde{h}^{-1}\tilde{g}\tilde{f}}(z_i,z_j)) - d\log(D_{\tilde{f}}(z_j)) - d\log(D_{\tilde{h}}(z_i)) \\ &= (2i\pi)(\omega_{ij}^{h^{-1}gf} - \omega_{ii}^{h^{-1}\cdot\infty} - \omega_{jj}^{f^{-1}\cdot\infty}).\end{aligned}$$

This proves the first equation of the proposition. We now prove the second equation of the proposition. The equation is true for $p = \infty$. Assume $p \neq \infty$. Setting $f = g = 1$ in the first equation of the proposition, we get $h_i^*\omega_{ij}^1 = \omega_{ij}^{h^{-1}} - \omega_{ii}^{h^{-1}\cdot\infty}$ and then by setting $z_j = p$, we find $h_i^*\omega_{ii}^p = \frac{1}{2i\pi}d\log(P_{\tilde{h}^{-1}}(z_i,p)) - \omega_{ii}^{h^{-1}\cdot\infty}$. The second equation of the proposition (for $p \neq \infty$) follows, since

$$P_{\tilde{h}^{-1}}(z_i,p) = \begin{cases}(cp+d)(z_i - h^{-1}\cdot p) & \text{if } h^{-1}\cdot p \neq \infty \\ ap+b & \text{otherwise}\end{cases} = (2i\pi)\omega_{ii}^{h^{-1}\cdot p},$$

if $\tilde{h}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have proved the second equation of the proposition. The remaining equations are straightforward. We have proved the proposition.

Corollary 3.4. *The algebra $\Omega^*_D(C^G_n(\mathbb{P}^1 \setminus Z))_R$ is stable under the action of $G^n \rtimes \mathfrak{S}_n$.*

Proposition 3.5. *Algebras $\Omega^*_D(C^G_n(\mathbb{P}^1 \setminus Z))_R$ obtained for different choices of p_∞ and h_Z are equal.*

Proof. Recall that if $\infty \in Z$, then we take $h_Z = \mathrm{id}$. Assume $\infty \notin Z$ and fix a choice of p_∞ and h_Z. Adapting the computations used to prove the first two equations of the previous proposition, by taking $h = f = h_Z$ and $p = h_Z(q)$, we get:

$$\omega^g_{ij} = \omega^g(z_i, z_j) - \omega^{p_\infty}(z_i) - \omega^{p_\infty}(z_j) \quad \text{and} \quad \omega^q_{kk} = \omega^q(z_k) - \omega^{p_\infty}(z_k),$$

where $\omega^g, \omega^{p_\infty}$ and ω^q are as in (2) and (3). Given a $q_0 \in Z$, we can add to the generators above $-\omega^{q_0}_{ll}$ for an (or two) appropriate l('s) (we add nothing if $q_0 = p_\infty$), to deduce that the algebra $\Omega^*_D(C^G_n(\mathbb{P}^1 \setminus Z))_R$ is generated by the forms:

$$\omega^g(z_i, z_j) - \omega^{q_0}(z_i) - \omega^{q_0}(z_j) \quad \text{and} \quad \omega^q(z_k) - \omega^{q_0}(z_k),$$

for $q \in Z, g \in G$ and $i, j, k \in [1, n]$, with $i \neq j$. This proves the proposition.

We see (from the proof above) that we can give a definition of forms generating $\Omega^*_D(C^G_n(\mathbb{P}^1 \setminus Z))_R$ without using h_Z. Anyway, the pullback formula by h_Z seems to be easier to manipulate.

3.2. Relations in $\Omega^*_D(C^G_n(\mathbb{P}^1 \setminus Z))_R$ and the algebra $A_n(R)$

Recall that for i, j, k three distinct integers in $[1, n]$, we have the following relation ([Arn69]):

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0,$$

where $\omega_{st} = d\log(z_s - z_t)$.

Proposition 3.6. *For $1 \leq i \neq j \leq n, h, g \in G$ and $p, q \in Z \setminus \{\infty\}$, we have:*

$$\omega^p_{ii} \wedge \omega^q_{ii} = 0 \quad , \quad \omega^g_{ij} = \omega^{g^{-1}}_{ji}, \tag{4}$$

$$\omega^h_{ij} \wedge \omega^p_{jj} = \omega^h_{ij} \wedge \omega^{h\cdot p}_{ii} + \omega^{h\cdot p}_{ii} \wedge \omega^p_{jj} + \omega^{h\cdot p_\infty}_{ii} \wedge \omega^h_{ij}, \tag{5}$$

$$\omega^h_{ik} \wedge \omega^g_{jk} = \omega^{hg^{-1}}_{ij} \wedge (\omega^g_{jk} - \omega^h_{ik}) + \omega^{g\cdot p_\infty}_{jj} \wedge (\omega^{h^{-1}g}_{ij} - \omega^g_{jk}) + \omega^{h\cdot p_\infty}_{ii} \wedge (\omega^h_{ik} - \omega^{hg^{-1}}_{ij}) + \omega^{h\cdot p_\infty}_{ii} \wedge \omega^{g\cdot p_\infty}_{jj}, \tag{6}$$

for $k \in [1, n] \setminus \{i, j\}$, and

$$\omega^h_{ij} \wedge \omega^g_{ij} = (\omega^{p_1}_{ii} + \omega^{p_2}_{ii}) \wedge (\omega^g_{ij} - \omega^h_{ij}) - \omega^{g\cdot p_\infty}_{ii} \wedge \omega^g_{ij} + \omega^{h\cdot p_\infty}_{ii} \wedge \omega^h_{ij}, \tag{7}$$

for $h \neq g$, with p_1 and p_2 the two points fixed by hg^{-1}.

Proof. The first equation in (4) is straightforward. We prove the second one. Using the definition of the polynomial P_A, we get:

$$P_A(y,x) = y(cx+d) - (ax+b) = -(x(-cy+a) - (dy-b)) = -\det(A)P_{A^{-1}}(x,y).$$

This proves that $\omega_{ij}^g = \omega_{ji}^{g^{-1}}$. We now prove (5). Pulling back the relation given before the proposition by $h_Z^{\times n}$, we obtain:

$$\omega_{ij}^1 \wedge \omega_{jk}^1 + \omega_{jk}^1 \wedge \omega_{ik}^1 + \omega_{ik}^1 \wedge \omega_{ij}^1 = 0.$$

Setting $z_k = p$, we get:

$$\omega_{ij}^1 \wedge \omega_{jj}^p + \omega_{jj}^p \wedge \omega_{ii}^p + \omega_{ii}^p \wedge \omega_{ij}^1 = 0,$$

which is true for $p = p_\infty$. Pulling back the last equation by h_i^{-1}, we find using proposition 3.3:

$$(\omega_{ij}^h - \omega_{ii}^{h\cdot p_\infty}) \wedge \omega_{jj}^p + \omega_{jj}^p \wedge (\omega_{ii}^{h\cdot p} - \omega_{ii}^{h\cdot p_\infty}) + (\omega_{ii}^{h\cdot p} - \omega_{ii}^{h\cdot p_\infty}) \wedge (\omega_{ij}^h - \omega_{ii}^{h\cdot p_\infty}) = 0.$$

This proves (5). Applying proposition 3.3 to the pullback by h_i^{-1} of the equation $\omega_{ij}^1 \wedge \omega_{jk}^1 + \omega_{jk}^1 \wedge \omega_{ik}^1 + \omega_{ik}^1 \wedge \omega_{ij}^1 = 0$, we find:

$$\omega_{ik}^h \wedge \omega_{jk}^1 = \omega_{ij}^h \wedge \omega_{jk}^1 + \omega_{ik}^h \wedge \omega_{ij}^h + \omega_{ii}^{h\cdot p_\infty} \wedge (\omega_{ik}^h - \omega_{ij}^h).$$

Pulling this equation by g_j^{-1}, we get:

$$\omega_{ik}^h \wedge (\omega_{jk}^g - \omega_{jj}^{g\cdot\infty}) = \begin{array}{l} (\omega_{ij}^{hg^{-1}} - \omega_{jj}^{g\cdot p_\infty}) \wedge (\omega_{jk}^g - \omega_{jj}^{g\cdot p_\infty}) + \omega_{ik}^h \wedge (\omega_{ij}^{hg^{-1}} - \omega_{jj}^{g\cdot p_\infty}) \\ + \omega_{ii}^{h\cdot p_\infty} \wedge (\omega_{ik}^h - \omega_{ij}^{hg^{-1}} + \omega_{jj}^{g\cdot p_\infty}). \end{array}$$

Simplifying this equation gives (6). Replacing z_j with z_i in (6), for $h \neq g$, we get:

$$\omega_{ik}^h \wedge \omega_{ik}^g = \omega_{ii}^{hg^{-1}} \wedge (\omega_{ik}^g - \omega_{ik}^h) - \omega_{ii}^{g\cdot p_\infty} \wedge \omega_{ik}^g + \omega_{ii}^{h\cdot p_\infty} \wedge \omega_{ik}^h,$$

where $\omega_{ii}^{hg^{-1}}$ is the form obtained by replacing z_j with z_i in $\omega_{ij}^{hg^{-1}}$. For $a \in \mathrm{PGL}(\mathbb{C}^2)$ of finite order and A a lift of a to $\mathrm{GL}(\mathbb{C}^2)$ the polynomial $P_A(x,x)$ admits simple roots corresponding to the elements $x \in \mathbb{C}$ fixed by a. Hence, using the convention $\omega^\infty(x) = 0$, we have:

$$d\log(P_A(x,x)) = \omega^{x_1}(x) + \omega^{x_2}(x),$$

where $x_1, x_2 \in \mathbb{P}^1$ are the two fixed points of $a \in \mathrm{PGL}(\mathbb{C}^2)$ and $\omega^{x_i}(x) = d\log(x - x_i)$ as in equation (3). From this, and the definition of ω_{ii}^p (for $p \in Z$) we deduce that:

$$\omega_{ii}^{hg^{-1}} = \omega_{ii}^{p_1} + \omega_{ii}^{p_2},$$

where p_1 and p_2 are the fixed points of hg^{-1} (and $\omega_{ii}^{p_\infty} = 0$ by convention). Replacing $\omega_{ii}^{hg^{-1}}$ by $\omega_{ii}^{p_1} + \omega_{ii}^{p_2}$ in the equation obtained previously, we get:

$$\omega_{ik}^h \wedge \omega_{ik}^g = (\omega_{ii}^{p_1} + \omega_{ii}^{p_2}) \wedge (\omega_{ik}^g - \omega_{ik}^h) - \omega_{ii}^{g\cdot p_\infty} \wedge \omega_{ik}^g + \omega_{ii}^{h\cdot p_\infty} \wedge \omega_{ik}^h.$$

We obtain (7) by replacing k with j in the last equation. We have proved the proposition.

Corollary 3.7. *Let ω_{ij}^{α} and ω_{kl}^{β} be forms as in definition 3.1. The product $\omega_{ij}^{\alpha} \wedge \omega_{kl}^{\beta}$, is equal to the sum of products $\varepsilon\omega_{rs}^{\alpha'} \wedge \omega_{tu}^{\beta'}$, where $\varepsilon \in \{1, -1\}$, $\omega_{rs}^{\alpha'}$ and $\omega_{tu}^{\beta'}$ are as in definition 3.1, $r \leq s, t \leq u$ and $s < u \leq \max(i, j, k, l)$.*

Corollary 3.8. *The algebra $\Omega_D^*(C_n^G(\mathbb{P}^1 \setminus Z))_R$ is generated as an R-module by the forms:*

$$\omega_{i_1 j_1}^{\beta_1} \wedge \cdots \wedge \omega_{i_k j_k}^{\beta_k}, \quad 0 \leq k \leq n, \quad 1 \leq i_k \leq j_k \leq n, \quad j_1 < j_2 < \cdots < j_k$$

with $\beta_k \in G$ if $i_k \neq j_k$ and $\beta_k \in Z \setminus \{p_\infty\}$ if $i_k = j_k$ (for $k = 0$ the product is equal to 1).

Definition 3.9. *Define $A_n(R)$ as the quotient of the R-exterior algebra generated by the elements $\tilde{\omega}_{ij}^{g}$ and $\tilde{\omega}_{kk}^{p}$ for $i, j, k \in [1, n]$, with $i \neq j$, $g \in G$ and $p \in Z \setminus \{p_\infty\}$, by the ideal corresponding to relations analogue to those of proposition 3.6.*

Proposition 3.10. *1) We have a surjective morphism of graded R-algebra $\Psi_R : A_n(R) \to \Omega_D^*(X_n)_R$ given by $\tilde{\omega}_{ij}^{g} \mapsto \omega_{ij}^{g}$ and $\tilde{\omega}_{kk}^{p} \mapsto \omega_{kk}^{p}$, for $i, j, k \in [1, n], g \in G$ and $p \in Z \setminus \{p_\infty\}$.*

2) The analogue of corollary 3.8 holds for $A_n(R)$.

For $(G, Z) = (\{1\}, \{\infty\})$, or $(G, Z) = (\langle \zeta z \rangle, \{0, \infty\})$ the space X_n is the complement in $\mathbb{C}^n$ of a central hyperplane arrangement (see subsection 2.2) and its cohomology ring with integral coefficients is isomorphic to $\Omega_D^*(X_n)_{\mathbb{Z}}$ ([Arn69], [Bri73] lemma 5) and a presentation (definition by generators and relations) is known ([Arn69], [OS80]). For $(G, Z) = (\{1\}, \{\infty\})$, the algebra $A_n(\mathbb{Z})$ is the algebra A_n of [Arn69]. As we will see later, for all X_n, $A_n(\mathbb{Z})$ is isomorphic to $\Omega_D^*(X_n)_{\mathbb{Z}}$. It follows easly that for (G, Z) as in the beginning of the paragraph, the relations defining $A_n(\mathbb{Z})$ are alternatives (or equal) to those in [OS80] for the corresponding central hyperplane arrangement.

3.3. Periods of $\Omega_D^1(C_n^G(\mathbb{P}^1 \setminus Z))_R$ and $H^1(C_n^G(\mathbb{P}^1 \setminus Z), R)$

We recall that for $P_n \in C_n^G(\mathbb{P}^1 \setminus Z)$, we have defined loops x_{ij}^{α} based at P_n, for $i \leq j \in [1, n]$ with $\alpha \in G$ if $i \neq j$ and $\alpha \in Z$ if $i = j$.

Proposition 3.11. *For $\alpha, \beta \in (G \cup Z \setminus \{p_\infty\})$ and $i \leq j, k \leq l \in [1, n]$, such that x_{ij}^{α} and ω_{kl}^{β} are well-defined, we have:*

$$\int_{x_{ij}^{\alpha}} \omega_{kl}^{\beta} = \delta_{ik}\delta_{jl}\delta_{\alpha\beta},$$

where $\delta_{ab} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise.

Proof. The formula follows from the case $h_Z = \mathrm{id}$ using the formula $\int_{f(\gamma)} \omega = \int_\gamma f^*\omega$, since $h_Z^{-1}(x_{ij}^{\alpha})$ has same homology of $x_{ij}^{\alpha^h}$, where α^h is $h\alpha h^{-1}$ or $h(\alpha)$ and $x_{ij}^{\alpha^h}$ is the analogue in the corresponding space of x_{ij}^{α}. Hence, we assume that $h_Z = \mathrm{id}$ ($\infty \in Z$). We only prove the case $k < l$ ($\beta \in G$). The easier case $k = l$

can be treated similarly. Assume that $k < l$. The s-th component $(x_{ij}^{\alpha})_s$ of x_{ij}^{α} is constant for $s \neq i$ and $(x_{ij}^{\alpha})_i = \gamma_i(p_\alpha)$ (see subsection 2.3). Hence:

$$\int_{x_{ij}^{\alpha}} \omega_{kl}^{\beta} = \frac{\delta_{ik}}{2i\pi} \int_{t\in[0,1]} d\log Q_\beta(\gamma_i(p_\alpha)(t), p_l) + \frac{\delta_{il}}{2i\pi} \int_{t\in[0,1]} d\log Q_\beta(p_k, \gamma_i(p_\alpha)(t)),$$

where $p_\alpha \in (Z \setminus \{\infty\})$ if $i = j$ or $p_\alpha = \alpha \cdot p_j$ if $i < j$, and Q_β is equal to $P_{\tilde{\beta}}$ for a given lift $\tilde{\beta} \in \mathrm{PSL}_2(\mathbb{C})$ of β. The polynomials $Q_\beta(x, p_l)$ and $Q_\beta(p_k, y)$ are degree 1 polynomials with respective zeros $x = \beta \cdot p_l$ and $y = \beta^{-1} \cdot p_k$. The equations therefore reduces to:

$$\int_{x_{ij}^{\alpha}} \omega_{kl}^{\beta} = \frac{\delta_{ik}}{2i\pi} \int_{t\in[0,1]} \frac{\gamma_i(p_\alpha)'(t)}{\gamma_i(p_\alpha)(t) - \beta \cdot p_l} + \frac{\delta_{il}}{2i\pi} \int_{t\in[0,1]} \frac{\gamma_i(p_\alpha)'(t)}{\gamma_i(p_\alpha)(t) - \beta^{-1} \cdot p_k},$$

The loops $\gamma_i(p_\alpha)$ are oriented clockwise. One deduce from the definition of $\gamma_i(p_\alpha)$ and the residue theorem that:

$$\int_{x_{ij}^{\alpha}} \omega_{kl}^{\beta} = \delta_{ik}\delta_{p_\alpha(\beta\cdot p_l)} + \delta_{il}\delta_{p_\alpha(\beta^{-1}\cdot p_k)}.$$

The proposition follows for $k < l$, since $\delta_{p_\alpha(\beta\cdot p_l)} = \delta_{jl}\delta_{\alpha\beta}$ and $\delta_{p_\alpha(\beta^{-1}\cdot p_k)} = 0$ if $i = l$ ($i \leq j$ by hypothesis, hence $k < j$).

Corollary 3.12. *The first singular homology group of $C_n^G(\mathbb{P}^1 \setminus Z)$ with coefficients in $\mathbb{Z}$, is freely generated by the cohomology classes of the loops, x_{ij}^g and x_{kk}^p, for $i, j, k \in [1, n], g \in G, p \in Z \setminus \{p_\infty\}$ with $i < j$.*

Proof. We combine the previous proposition with proposition 2.5.

4. The cohomology ring of the orbit configuration spaces and their homology

We keep the notation of the previous section $X_n = C_n^G(\mathbb{P}^1 \setminus Z)$. In the first subsection, we assume that $(G, Z) \neq (\{1\}, \emptyset)$. We show using results from the previous section, that for $R \subset \mathbb{C}$ an unital ring, we have isomorphism $H^*(X_n, R) \simeq H^*(X_{n-1}, R) \otimes W_n^*$ and $H_*(X_n, R) \simeq H^*(X_{n-1}, R) \otimes W_n^*$, where $W_n^* \subset \Omega_D^{1-}(X_n)_R$ is a space of differential forms identified to $H^*(F_n, R)$ (F_n the fiber of $X_n \to X_{n-1}$) via integration and $H^*(-, R), H_*(-, R)$ correspond to singular cohomology and homology with coefficientss in R respectively. It follows by induction that: the groups $H^k(X_n, R), H_k(X_n, R)$ are free R-modules, $H^*(X_n, R)$ admits a basis obtained out of classes corresponding to differential forms, and that the Poincaré series of X_n factors into a product of linear terms. We then show that: integration induces a isomorphism of graded R-algebras $\Phi_R : \Omega_D^*(X_n)_R \to H^*(X_n, R)$, that $\Psi_R : A_n(R) \to \Omega_D^*(X_n)_R$ of the previous section is an isomorphism, and that the products (defined in the previous section) spanning $A_n(R)$ and $\Omega_D^*(X_n)_R$ form in fact a basis of the corresponding modules. In particular, we get a description by generators and relations for both rings $H^*(X_n, R)$ and $\Omega_D^*(X_n)_R$. At the end of the section, we prove that X_n is formal is the sense of rational homotopy theory. In the second subsection, we consider the case $(G, Z) = (\{1\}, \emptyset)$. We show that the cohomology ring of X_n with coefficients in R corresponds to a subring of differential forms (for R a principal ideal domain containing $\frac{1}{2}$, if $n \geq 3$) and that the space X_n is formal. We also give the Poincaré series of

X_n.

As mentioned in the introduction in this section: $R \subset \mathbb{C}$ is a unital ring, $H^*(X,R)$ and $H_*(X,R)$ denote the singular cohomology and homology of X with coefficients in R and $X_n := C_n^G(\mathbb{P}^1 \setminus Z)$.

For $A^* = \oplus_{k\in\mathbb{N}} A_k, B^* = \oplus_{k\in\mathbb{N}} B_k$ graded R-modules and $f^* : A^* \to B^*$ a map of graded R-modules, we denote by A^{l^-} the graded R-module $\oplus_{0\leq k\leq l} A_k$ and by f^{l^-} (resp. f^l) the map of graded modules $A^{l^-} \xrightarrow{f^*} B^{l^-}$ (resp. $A^l \xrightarrow{f^*} B^l$).

4.1. The case $(G,Z) \neq (\{1\},\emptyset)$

In this subsection we assume that $(G,Z) \neq (\{1\},\emptyset)$. Recall that we have an integration isomorphism of graded $\mathbb{C}$-algebras (Cf. Subsection 1.2), $\int : H^*_{DR}(X_n,\mathbb{C}) \to \mathrm{Hom}_{\mathbb{Z}}(H_*(X_n,\mathbb{Z}),\mathbb{C})$. Since $\Omega^*_D(X_n)_R$ consists of closed forms, we have a morphism of graded R-algebras $T_R : \Omega^*_D(X_n)_R \to H^*_{DR}(X_n,\mathbb{C}), \omega \to [\omega]$. Set:

$$\phi_* := \int \circ T_R : \Omega^*_D(X_n)_R \to \mathrm{Hom}_{\mathbb{Z}}(H_*(X_n,\mathbb{Z}),\mathbb{C}),$$

given by $\phi_*(\omega)([\sigma]) = \int_{\sigma^s} \omega$, where σ_s is a smooth singular chain representing the homology class $[\sigma]$.

Proposition 4.1. *1) For $\omega \in \Omega^{1-}_D(X_n)_R$, the image of $\phi_{1-}(\omega)$ lies in R.*

2) We have a well-defined isomorphism ϕ^R_{1-} ("restriction" of ϕ_{1-}) of R-modules:

$$\Omega^{1-}_D(X_n)_R \to \mathrm{Hom}_{\mathbb{Z}}(H_{1-}(X_n,\mathbb{Z}),R)$$
$$\omega \mapsto ([\sigma] \mapsto \int_{\sigma^s} \omega),$$

Proof. The proposition follows from proposition 3.11, corollary 3.12 and the fact that X_n is path connected.

We recall that, for $n \geq 1$, we have a fiber bundle $F_n \to X_n \xrightarrow{\pi_n} X_{n-1}$, with fiber F_n homeomorphic to $\mathbb{P}^1$ minus $|G|(n-1)+|Z|$ points. In particular,

$$H_*(F_n,R) \simeq H^*(F_n,R) \simeq (R)_0 \oplus (R^{\alpha_n})_1,$$

where $(-)_i$ stands for the degree i component and $\alpha_n = |G|(n-1)+|Z|-1$. The fiber over a point $P = (p_1,\ldots,p_{n-1}) \in X_{n-1}$ is equal to:

$$F_{n,P} = \{(p_1,\ldots,p_{n-1},x) \in (\mathbb{P}^1 \setminus Z)^n | x \notin G\cdot p_i \text{ for } i \in [1,n-1]\}.$$

For X, a topological space, one has a natural morphism of R-modules:

$$N^R_{X,*} : H^*(X,R) \to \mathrm{Hom}_{\mathbb{Z}}(H_*(X,\mathbb{Z}),R),$$

The maps $N^R_{X_n,1-}$ and $N^R_{F_{n,p},*}$ are isomorphisms.

Proposition 4.2. *Let W_n^* be the R-submodule of $\Omega_D^{1-}(X_n)_R$ spanned by 1 and the 1-forms ω_{in}^g and ω_{nn}^p for $i < n, g \in G$ and $p \in Z \setminus \{p_\infty\}$. For $P \in X_{n-1}$, the composite of the following maps:*

$$W_n^* \xrightarrow{\phi_{1-}^R} \mathrm{Hom}_{\mathbb{Z}}(H_{1-}(X_n, \mathbb{Z}), R) \xrightarrow{(N_{X_n,1-}^R)^{-1}} H^*(X_n, R) \xrightarrow{i_n^*} H^*(F_{n,P}, R),$$

where i_n^* is the map induced by the inclusion i_n of the fiber $F_{n,P} \to X_n$, is an isomorphism of graded R-modules.

Proof. The fact that the composition is an isomorphism in degree 0 is clear. Since $H^*(F_{n,p}, R)$ is concentrated in degree 0 and 1 we still have to prove the assertion in degree 1. We have the following commutative diagram:

$$\begin{tikzcd}
W_n^* \arrow[r, "\phi_1^R"] \arrow[dr, "r_1"'] & \mathrm{Hom}_{\mathbb{Z}}(H_1(X_n, \mathbb{Z}), R) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}((i_n)_*, R)"] & H^1(X_n, R) \arrow[l, "N_{X_n,1}^R"'] \arrow[d, "i_n^*"] \\
& \mathrm{Hom}_{\mathbb{Z}}(H_1(F_{n,P}, \mathbb{Z}), R) & H^1(F_{n,P}, R) \arrow[l, "N_{F_{n,P},1}^R"]
\end{tikzcd},$$

where r_1 is $\mathrm{Hom}_{\mathbb{Z}}((i_n)_*, R) \circ \phi_1^R$. Since $N_{X_n,1}^R$ and $N_{F_{n,P},1}^R$ are isomorphisms, we only need to prove that r_1 is an isomorphism. For any point $Q \in F_{n,p}$, the group $H_1(F_{n,P}, R)$ is generated by the homology classes of the loops x_{in}^α and x_{nn}^β, associated to the base point Q, for $i \in [1, n-1], \alpha \in G$ and $\beta \in Z \setminus \{p_\infty\}$. Moreover, $\int_{x_{in}^\alpha} \omega_{kn}^\beta = \delta_{ik}\delta_{\alpha\beta}$, where $\delta_{ab} = 1$ if $a = b$ and $\delta_{ab} = 0$ otherwise (proposition 3.11). This proves that r_1 given by $\omega_{kn}^\beta \mapsto ([\gamma] \mapsto \int_{(i_n)_*([\gamma])} \omega_{kn}^\beta)$ is an isomorphism.

We deduce from the proposition that W_n^* is isomorphic to $H^*(F_{n,p}, R)$ and that the restriction of

$$\theta_{n,*} := (N_{X_n,1-}^R)^{-1} \circ \phi_{1-}^R : \Omega_D^{1-}(X_n)_R \to H^*(X_n, R)$$

to W_n^* corresponds to a cohomology extension of the fiber for the fiber bundle $F_n \to X_n \xrightarrow{\pi_n} X_{n-1}$ (in the sense of [Spa95], p. 256). Since $H_*(F_n, R)$ is a finitely generated free R-module, we can apply the Leray-Hirsch theorem (version in [Spa95], p. 259, theorem 9) stating in the case of π_n that:

Proposition 4.3. *We have isomorphisms of R-graded modules:*

$$\begin{aligned} H^*(X_{n-1}, R) \otimes_R W_n^* &\to H^*(X_n, R) \\ a \otimes b &\mapsto \pi_n^*(a) \smile \theta_{n,*}(b), \end{aligned}$$

where π_n^* is the map induced by $\pi_n : X_n \to X_{n-1}$ and $\smile$ is the cup product; and

$$\begin{aligned} H_*(X_n, R) &\to H_*(X_{n-1}, R) \otimes_R W_n^* \\ a &\mapsto \sum_\alpha (\pi_n)_*(\theta_{n,*}(\omega_\alpha) \frown a) \otimes \omega_\alpha), \end{aligned}$$

where $(\pi_n)_*$ is induced by π_n, $\frown$ is the cap product and the sum runs over the elements

$$\omega_\alpha \in \{1\} \cup \{\omega_{kn}^c | k \in [1, n], c \in G \text{ if } k < n \text{ and } p \in Z \setminus \{p_\infty\} \text{ if } k = n\}.$$

Using induction, we deduce the following corollaries:

Corollary 4.4. *For $k \geq 0$, the R-modules $H^k(X_n, R)$ and $H_k(X_n, R)$ are finitely generated free R-modules.*

Corollary 4.5. *The natural map $N^R_{X_n,*} : H^*(X_n, R) \to \mathrm{Hom}_{\mathbb{Z}}(H_*(X_n, \mathbb{Z}), R)$ is an isomorphism.*

Corollary 4.6. *The family of products:*

$$\theta_{n,*}(\omega^{\beta_1}_{i_1 j_1}) \smile \cdots \smile \theta_{n,*}(\omega^{\beta_k}_{i_k j_k}), \quad 0 \leq k \leq n, \quad 1 \leq i_k \leq j_k \leq n, \quad j_1 < j_2 < \cdots < j_k$$

with $\beta_k \in G$ if $i_k \neq j_k$ and $\beta_k \in Z \setminus \{p_\infty\}$ if $i_k = j_k$ (for $k = 0$ the product is equal to 1), forms a basis of the R-module $H^*(X_n, R)$.

Corollary 4.7. *For $n \geq 2$, $P_{X_n}(t) = P_{F_n}(t) P_{X_{n-1}}(t)$, where P_{X_k} and P_{F_k} are the Poincaré series of X_k and F_k respectively.*

Since the Poincaré series of $P_{F_k} = (1 + \alpha_k t)$, where $\alpha_k = |G|(k-1) + |Z| - 1$,

Corollary 4.8. *For $n \geq 1$:*

$$P_{X_n}(t) = \prod_{k=1}^{n} (1 + \alpha_k t),$$

where $\alpha_k = |G|(k-1) + |Z| - 1$.

Remark 4.9. *The existence of a cohomology extension implies that $\pi_1(X_{n-1}, P)$ acts trivially in the cohomology of the fiber $H^*(F_n, R)$ (also in homology, since it is free), and that the cohomology and homology spectral sequences of the fibration $F_n \to X_n \to X_{n-1}$ collapse at the second page.*

Proposition 4.10. *1) For $\omega \in \Omega^*_D(X_n)_R$, the image of $\phi_*(\omega)$ lies in R and the map ϕ_* induces a well-defined morphism of graded R-modules:*

$$\phi^R_* : \Omega^*_D(X_n)_R \to \mathrm{Hom}_{\mathbb{Z}}(H_*(X_n, \mathbb{Z}), R), \omega \mapsto ([\sigma] \mapsto \int_{\sigma^s} \omega).$$

2) The map $\Phi_R := (N^R_{X_n,*})^{-1} \circ \phi^R_* : (\Omega^*_D(X_n)_R, \wedge) \to (H^*(X_n, R), \smile)$ is a morphism of R-algebras.

Proof. Recall that $H^*(X_n, K)$ for K an abelian group is the cohomology of the cochain complex $\mathrm{Hom}_Z(C_*(X), K)$, where $C_*(X)$ is the singular chain complex of X (over $\mathbb{Z}$). The inclusion $i : R \to \mathbb{C}$ induces a natural map of cochain complexes $i' : \mathrm{Hom}_Z(C_*(X), R) \to \mathrm{Hom}_Z(C_*(X), \mathbb{C})$ compatible with the cup product, and we have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}}(H_*(X_n, \mathbb{Z}), \mathbb{C}) & \xleftarrow{N^{\mathbb{C}}_{X_n,*}} & H^*(X_n, \mathbb{C}) \\
{\scriptstyle \mathrm{Hom}_{\mathbb{Z}}(H_*(X_n,\mathbb{Z}),i)}\uparrow & & \uparrow{\scriptstyle H^*(i')} \\
\mathrm{Hom}_{\mathbb{Z}}(H_*(X_n, \mathbb{Z}), R) & \xleftarrow[N^R_{X_n,*}]{} & H^*(X_n, R)
\end{array}$$

where $H^*(i')$ is a morphism of algebras. In particular, if $[\sigma_i]_{\mathbb{C}} \in H^*(X_n, \mathbb{C})$ (for $i \in [1,k]$) is equal to $H^*(i')[\sigma_i]_R$ for $[\sigma_i]_R \in H^*(X_n, R)$, then:

$$N^{\mathbb{C}}_{X_n,*}([\sigma_1]_{\mathbb{C}} \smile \cdots \smile [\sigma_k]_{\mathbb{C}}) = \mathrm{Hom}_{\mathbb{Z}}(H_*(X_n,\mathbb{Z}), i)(N^R_{X_n,*})^{-1}([\sigma_1]_R \smile \cdots \smile [\sigma_k]_R).$$

Recall that $\phi_* = \int \circ T_R$. Hence, $(N^{\mathbb{C}}_{X,*})^{-1} \circ \phi^* = (N^{\mathbb{C}}_{X,*})^{-1} \circ \int \circ T_R$ is a map of algebras. Indeed, $(N^{\mathbb{C}}_{X,*})^{-1} \circ \int = \int'$ (Cf. Subsection 1.2) and T_R are both morphisms of algebras. From this and the last equation containing cohomology classes, we deduce (since ϕ^R_1 is the "restriction" of ϕ_*) that:

$$\phi_*(\omega_1 \wedge \cdots \wedge \omega_k) = \mathrm{Hom}_{\mathbb{Z}}(H_*(X_n,\mathbb{Z}), i)(N^R_{X_n,*})^{-1}(\Phi_{1,R}(\omega_1) \smile \cdots \smile \Phi_{1,R}(\omega_k)),$$

for $\omega_1, \ldots, \omega_k \in \Omega^1_D(X_n)_R$ and where $\Phi_{1,R} := (N^R_{X_n,1})^{-1}\phi^R_1$. In particular, the image of $\phi_*(\omega_1 \wedge \cdots \wedge \omega_k)$, with $\omega_i \in \Omega^1_D(X_n)_R$, lies in R and hence ϕ_* induces a well-defined integration morphism $\phi^R_* : \Omega^*_D(X_n)_R \to \mathrm{Hom}_{\mathbb{Z}}(H_*(X_n,\mathbb{Z}), R), \omega \mapsto ([\sigma] \mapsto \int_{\sigma^s} \omega)$. This proves point (1) of the proposition. Point (2) also follows from the last equation. We have proved the proposition.

Proposition 4.11. *1) The maps $\Psi_R : A_n(R) \to \Omega^*_D(X_n)_R$ and $\Phi_R : \Omega^*_D(X_n)_R \to H^*(X_n, R)$, of propositions 3.10 and 4.10, are isomorphisms of graded R-algebras.*

2) The family of products in corollary 3.8 and their analogues for $A_n(R)$ form basis of $\Omega^*_D(X_n)_R$ and $A_n(R)$ respectively.

Proof. The proposition follows from the fact that Φ_R and Ψ_R are algebra morphisms (propositions 3.10 and 4.10), corollary 3.8, proposition 3.10 and corollary 4.6.

Denote by $\Omega^*(X_n)_{\mathbb{C}}$ the commutative cochain algebra of complex valued differential forms on X_n, with differential the exterior differential and product the wedge product.

Corollary 4.12. *The commutative cochain algebras $\Omega^*_D(X_n)_{\mathbb{C}}$, $H^*(X_n, \mathbb{C})$ and $\Omega^*(X_n)_{\mathbb{C}}$, where the first two algebras are equipped with a zero differential, are weakly equivalent.*

Proof. The previous proposition states that $\Omega^*_D(X_n)_{\mathbb{C}}$ and $H^*(X_n, \mathbb{C})$ are isomorphic. Since the forms in $\Omega^*_D(X_n)_{\mathbb{C}}$ are closed the inclusion $\tilde{T}_{\mathbb{C}} : \Omega^*_D(X_n)_{\mathbb{C}} \to \Omega^*(X_n)_{\mathbb{C}}$ is a map of cochain algebras. Since the map induced by $\tilde{T}_{\mathbb{C}}$ in cohomology is the map $T_{\mathbb{C}}$ and corresponds to $(\int')^{-1} \circ \Phi_{\mathbb{C}}$, where $\int'$ is the De Rham isomorphism, we deduce that $T_{\mathbb{C}}$ is an isomorphism and $\tilde{T}_{\mathbb{C}}$ is a quasi-isomorphism.

Corollary 4.13. *The space X_n is formal in the sense of rational homotopy theory, i.e. the commutative cochain algebra $H^*(X_n, \mathbb{Q})$ with zero differential is weakly equivalent to the cochain algebra $A^*_{PL}(X_n)_{\mathbb{Q}}$ of polynomial differential forms on X with coefficients in $\mathbb{Q}$.*

Proof. Since the rational cohomology of X_n is of finite type, the formality of $A^*_{PL}(X_n)_F$ for any field F containing $\mathbb{Q}$ will imply the formality for $\mathbb{Q}$ and $A^*_{PL}(X_n)_{\mathbb{C}}$ will be weakly equivalent to $\Omega^*(X_n)_{\mathbb{C}}$ (Cf. Subsection 1.1). Hence, the above corollary becomes a consequence of the previous one.

Recall that $\Gamma := G^n \rtimes \mathfrak{S}_n$ acts on $C_n^G(\mathbb{P}^1 \setminus Z)$. The quotient map $\pi : X_n \to X_n/\Gamma$ is a covering map and hence π induces a cdga isomorphism $\pi^* : \Omega^*(X_n/\Gamma)_{\mathbb{C}} \to \Omega^*(X_n)_{\mathbb{C}}^{\Gamma}$ between complexes of complex valued forms (the superscript Γ is (and will be) used for Γ invariants). We know that $\Omega_D^*(X_n)_{\mathbb{C}}$ is stable under Γ (Cf. Section 3.1) and that the inclusion $\Omega_D^*(X_n)_{\mathbb{C}} \to \Omega^*(X_n)_{\mathbb{C}}$ is a quasi-isomorphism. It follows that the last inclusion also induces a quasi-isomorphism $\Omega_D^*(X_n)_{\mathbb{C}}^{\Gamma} \to \Omega^*(X_n)_{\mathbb{C}}^{\Gamma}$. In particular, $\Omega^*(X_n/\Gamma)_{\mathbb{C}}$ is quasi-isomorphic to its cohomology (and $\Omega_D^*(X_n)_{\mathbb{C}}^{\Gamma}$). One can derive from this that X_n/Γ is formal.

4.2. The case $(G, Z) = (1, \emptyset)$, i.e. $X_n = C_n(\mathbb{P}^1)$

Let $R \subset \mathbb{C}$ be an unital ring. The space $\mathbb{P}^1$ is homeomorphic to $S^2 \subset \mathbb{R}^3$. We will sometimes switch spaces implicitly for convenience. One can derive the formality of the upcoming spaces by theoretic arguments. Here, we derive this from the description of the cohomology using differential forms, since we are interested in the description also.

By definition $C_1(\mathbb{P}^1)$ is equal to $\mathbb{P}^1$. The singular cohomology algebra $H^*(\mathbb{P}^1, R)$ of $\mathbb{P}^1$ is isomorphic to the R-subalgebra of differential forms generated by the volume form $\omega = \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$ (the integral of ω over $\mathbb{P}^1$ with its canonical orientation is equal to 1).

We consider the case $n = 2$. It is known that the projection $\pi_2 : C_2(S^2) \to C_1(S^2)$ is a homotopy equivalence with homotopy inverse s given by $s(p) = (p, -p)$. Indeed, $\pi_2 \circ s = \mathrm{id}$, and

$$H((x_1, x_2), t) = (x_1, \frac{(1-t)x_2 - tx_1}{||(1-t)x_2 - tx_1||}),$$

where $||\cdot||$ is the Euclidean norm in $\mathbb{R}^3$, is a homotopy between $s \circ \pi_2$ and id. Hence, $H^*(C_2(\mathbb{P}^1), R)$ correspond to the R-subalgebra of differential forms generated by $\pi_2^*\omega$ and $C_2(\mathbb{P}^1)$ is formal .

We examine $C_n(\mathbb{P}^1)$, for $n \geq 3$. We assume that R is a principle ideal domain. For $\underline{p} = (p_1, p_2, p_3) \in (\mathbb{P}^1)^3$ where the coordinates are pairwise distinct (i.e. a projective basis of $\mathbb{P}^1$), there exists a unique $h_{\underline{p}} \in \mathrm{PGL}(\mathbb{C}^2)$ mapping p_1, p_2, p_3 to $0, 1, \infty$ respectively. The homography $h_{\underline{p}}$ is given by:

$$h_{\underline{p}}(z) = \frac{(z - p_1)(p_2 - p_3)}{(p_2 - p_1)(z - p_3)}.$$

This proves that $\mathrm{PGL}(\mathbb{C}^2)$ acts freely transitively on $C_3(\mathbb{P}^1)$. Hence, $C_3(\mathbb{P}^1)$ is diffeomorphic to $\mathrm{PGL}(\mathbb{C}^2)$ and for $n \geq 3$, we have an homeomorphism:

$$\begin{array}{c} C_n(\mathbb{P}^1) \to C_3(\mathbb{P}^1) \times C_{n-3}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \\ (p_1, \ldots, p_n) \mapsto ((p_1, p_2, p_3), (h_{\underline{p}}(p_4), \ldots, h_{\underline{p}}(p_n))) \end{array}, \tag{8}$$

where $\underline{p} = (p_1, p_2, p_3)$. This decomposition was used in [FZ00] to compute the cohomology of $C_n(\mathbb{P}^1)$ for $n \geq 3$.

Using the QR decomposition one obtains that $C_3(\mathbb{P}^1) \simeq \mathrm{PGL}(\mathbb{C}^2)$ is diffeomorphic to $\mathrm{PSU}(\mathbb{C}^2) \times \mathbb{R}^3$. We have $\mathrm{PSU}(\mathbb{C}^2) \simeq \mathrm{SO}(\mathbb{R}^3)$ and $\mathrm{SO}(\mathbb{R}^3)$ is homeomorphic to the three-dimensional projective real space $\mathbb{P}^3(\mathbb{R})$. In particular, $C_3(\mathbb{P}^1)$ is homotopy equivalent to $\mathbb{P}^3(\mathbb{R})$ (see also remark 5.12 section 5) and for $\frac{1}{2} \notin R$, the ring $H^*(C_n(\mathbb{P}^1), R)$ can not be described as an algebra of closed differential forms, since it contains torsion. Now take a volume form ω_V on $\mathrm{PSU}(\mathbb{C}^2)$ for which $\mathrm{PSU}(\mathbb{C}^2)$ have volume 1 (one can normalize any volume form) and denote by ω'_V the pullback of ω_V by the smooth map $C_3(\mathbb{P}^1) \simeq \mathrm{PGL}(\mathbb{C}_2) \simeq \mathrm{PSU}(\mathbb{C}^2) \times \mathbb{R}^3 \to \mathrm{PSU}(\mathbb{C}^2)$, where the last map is the projection. Using the cross product one can check that, for R containing $\frac{1}{2}$, the ring $H^*(C_n(\mathbb{P}^1), R))$ is isomorphic via integration to the subalgebra of closed differential forms on $C_3(\mathbb{P}^1) \times C_{n-3}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ given by $\Omega^*_{3,R} \otimes \Omega^*_D(C_n(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ (tensor with Koszul sign convention), where $\Omega_{3,R} = (R)_0 \oplus (R\omega'_V)_3$ (indices for the degree) is the subalgebra generated by ω'_V. This gives a description of $H^*(C_n(\mathbb{P}^1), R)$ as an algebra of closed differential forms (for $\frac{1}{2} \in R$) via integration and proves that $C_n(\mathbb{P}^1)$ is formal.

We have shown that $C_n(\mathbb{P}^1)$ is formal for $n \geq 1$, and described the singular cohomology ring of $C_n(\mathbb{P}^1)$ with coefficients in an unital ring $R \subset \mathbb{C}$, using differential forms with constraints on R for $n \geq 3$. One can easily derive presentations of the cohomology ring from what we have seen above and (for $n \geq 3$) the presentation of the cohomology ring of $C_n(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ from the previous section (see also [FZ00]).

Proposition 4.14. *The Poincaré series P_n of $C_n(\mathbb{P}^1)$ is given by:*

$$P_1(t) = P_2(t) = 1 + t^2, \quad \textit{and} \quad P_n(t) = (1 + t^3) \prod_{k=1}^{n-3} (1 + \beta_k t),$$

for $n \geq$ and $\beta_k = 1 + k$.

Proof. $C_1(\mathbb{P}^1)$ is a 2-sphere. Hence, $P_1(t) = 1 + t^2$. The space $C_2(\mathbb{P}^1)$ is homotopy equivalent to $\mathbb{P}^1$ as seen previously in this subsection and hence $P_2 = P_1$. For $n \geq 3$, $C_n(\mathbb{P}^1) \simeq C_3(\mathbb{P}^1) \times C_{n-3}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. Therefore, for $n \geq 3$, P_n is the product of the Poincaré series of $C_3(\mathbb{P}^1)$ and $C_{n-3}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. We have seen in this subsection that $C_3(\mathbb{P}^1)$ is homotopy equivalent to $\mathbb{P}^3(\mathbb{R})$ and hence its Poincaré series is given by $1 + t^3$. The Poincaré series of $C_{n-3}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ is equal to the other factor in the formula above by corollary 4.8.

5. Homotopy groups and LSC formula

In the first subsection, we consider the case $(G, Z) \neq (\{1\}, \emptyset)$. We prove that $X_n := C^G_n(\mathbb{P}^1 \setminus Z)$ is a $K(\pi, 1)$ and that the fibration $X_{n+1} \to X_n$ admits a cross section. The cross section is used to prove that the first homotopy group $\pi_1 X_n$ of X_n is an iterated almost direct product of free groups (Cf. Section 1.3 for the definition). This gives an LCS formula relating the Poincaré series of X_n to the rank of quotients of successive terms of the lower central series of $\pi_1 X_n$. In the second subsection, we study the remaining case $X_n = C_n(\mathbb{P}^1)$. We give the higher homotopy groups of X_n known in the literature, describe the structure of

the fundamental group of X_n (the structure is also known, in fact a presentation is known) and show that we have an LCS formula.

Let H be a group. We denote by $\{\Gamma_i H\}_{i\geq 1}$ the lower central series of H and by H^{ab} the abelianization of H (i.e. $H/\Gamma_2 H$).

5.1. The case $(G, Z) \neq (\{1\}, \emptyset)$

In this subsection we assume that $(G, Z) \neq (\{1\}, \emptyset)$.

Proposition 5.1. *The space X_n is aspheric, i.e. a $K(\pi, 1)$ space.*

PROOF. The fiber F_{n+1} of $X_{n+1} \to X_n$ is a 2-sphere with finite number (nonzero) of punctures and hence aspheric. Using this, we deduce from the long exact sequence of the fibration the following exact sequences:

$$0 \to \pi_k(X_{n+1}) \to \pi_k(X_n) \to 0 \quad \text{for } k \geq 3 \qquad \text{and} \quad 0 \to \pi_2(X_{n+1}) \to \pi_2(X_n) \to \pi_1 F_{n+1}.$$

Hence, $\pi_k(X_{n+1}) = \pi_k(X_1) = \pi_k(\mathbb{P}^1 \setminus Z)$ for $k \geq 3$, and $\pi_2(X_{n+1})$ injects into $\pi_2(X_1) = \pi_2(\mathbb{P}^1 \setminus Z)$. This shows that X_n is aspheric, since $\mathbb{P}^1 \setminus Z$ is aspheric.

We now study the fundamental group of X_n. Since F_{n+1}, X_{n+1} and X_n are $K(\pi, 1)$ spaces and are path connected. The long exact sequence of the fibration $X_{n+1} \to X_n$ gives the exact sequence:

$$1 \to \pi_1(F_{P,n+1}, \tilde{P}) \to \pi_1(X_{n+1}, \tilde{P}) \to \pi_1(X_n, P) \to 1, \tag{9}$$

where $\tilde{P} \in X_{n+1}$ is a preimage of $P \in X_n$, $F_{P,n+1}$ is the fiber over P and the maps are induced by the inclusion $F_{P,n+1} \to X_{n+1}$ and the projection $X_{n+1} \to X_n$.

Proposition 5.2. *The fibration $X_{n+1} \to X_n$ admits a continuous cross-section $s_n : X_n \to X_{n+1}$.*

PROOF. Fix a $p_\infty \in Z$ and a Riemannian metric on $\mathbb{P}^1$. Let d be the distance function associated to the Riemannian metric. One can find a real number $\eta > 0$ and an isometry $f : [0, \eta] \to \mathbb{P}^1$ such that $\alpha(0) = p_\infty$ and $\alpha(]0, \eta]) \subset \mathbb{P}^1 \setminus Z$. For $P = (p_1, \ldots, p_n) \in X_n$, set:

$$C_P = \frac{1}{2} \operatorname*{Min}_{(i,g)\in[1,n]\times G} d(p_\infty, g \cdot p_i) \qquad \text{and} \qquad C_P^\eta = \operatorname{Min}(C_P, \eta).$$

For $(i, g) \in [1, n] \times G$, we have $d(p_\infty, \alpha(C_P^\eta)) < d(p_\infty, g \cdot p_i)$. Therefore, the mapping $X_n \to X_{n+1}, (p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_n, \alpha(C_P^\eta))$ defines a continuous cross section.

Corollary 5.3. *The exact sequence (9) splits and $\pi_k(X_{n+1}, s_n(P))$ is isomorphic to*

$$\pi_1(F_{P,n+1}, s_n(P)) \rtimes_{\beta_n} \pi_1(X_n, P),$$

where $\beta_n([a])([b]) = [s_n(a) b s_n(a)^{-1}]$ for a, b loops of X_n and $F_{P,n+1}$ based at P and $s_n(P)$ respectively.

Let $1 \to N \to H \to K \to 1$ be an exact sequence of groups and $s : K \to H$ a section (map of sets). We have a morphism $\alpha_s : K \to \mathrm{Aut}(N)$ mapping $a \in K$ to the conjugacy by $s(a)$ in N. The map α_s induces a morphism $\beta : K \to \mathrm{Aut}(N^{ab})$, independent of the choice of s.

Lemme 5.4. *Let N, H, K and β be as in the previous paragraph. If the natural map $N^{ab} \to H^{ab}$ is injective then the image of β consists of IA-automorphisms.*

PROOF. For L a group, denote by (x, y) the commutator of $x, y \in L$ and by (L, L) the commutator subgroup of L. The injectiveness condition means that $N \cap (H, H) = (N, N)$. Therefore, under the injectiveness assumption, for $c \in K$: $s(c)bs(c)^{-1}b^{-1} \in N \cap (H, H) = (N, N)$ ($s : K \to H$ is the set section), and $\beta(c)(b)$ and b are equal in N^{ab}. We have proved the proposition.

Lemme 5.5. *For $P \in X_n$, the inclusion of the fiber $F_{P,n+1}$ over P of $X_{n+1} \to X_n$ into X_{n+1}, induces an injective morphism in homology.*

PROOF. We can use corollary 3.12 and the fact that $H_1(F_{P,n+1}, \mathbb{Z})$ is generated by the loops x^g_{in+1}, x^p_{n+1n+1} (with base point a preimage of P) for $i \in [1, n], g \in G$ and $p \in Z \setminus \{p_\infty\}$ (for any $p_\infty \in Z$). Another possibility is to use the fact that $H^1(X_{n+1})$ is freely generated by the loops x^g_{ij} and x^p_{kk} for $i, j, k \in [1, n+1], g \in G$ and $p \in Z \setminus \{p_\infty\}$ as discussed in the paragraph after proposition 2.5.

Proposition 5.6. *The semidirect product $\pi_1(F_{P,n+1}, s_n(P)) \rtimes_{\beta_n} \pi_1(X_n, P)$ of corollary 5.3 is an almost direct product, i.e. the image of β_n consists of IA-automorphisms.*

PROOF. The proposition follows from the last two lemmas.

As we have already seen $F_{P,n+1}$ is a 2-sphere minus $\alpha_{n+1} + 1 = |G|n + |Z|$ points. Hence, $\pi_1(F_{P,n+1}, s_n(P))$ is isomorphic to the free group $F(\alpha_{n+1})$ on α_{n+1} generators.

Corollary 5.7. *1) The fundamental group of X_n is isomorphic to an iterated almost direct product of free groups:*

$$F(\alpha_n) \rtimes_{\gamma_n} (F(\alpha_{n-1}) \rtimes_{\gamma_{n-1}} (F(\alpha_{n-2}) \rtimes_{\gamma_{n-1}} (\cdots \rtimes_{\gamma_3} (F(\alpha_2) \rtimes_{\gamma_2} F(\alpha_1)) \cdots))),$$

where $\alpha_k = |G|(n-1) + |Z| - 1$, for $k \in [1, n]$.

2) Let P_{X_n} be the Poincaré series of X_n:

$$P_{X_n}(t) = \prod_{k=1}^{n} (1 + \alpha_k t) = \prod_{i \geq 1} (1 - t^i)^{\phi_i(\pi_1 X_n)},$$

where $\alpha_k = |G|(n-1) + |Z| - 1$, $\phi_i(\pi_1 X_n)$ is the rank of the quotient $\Gamma_i \pi_1 X_n / \Gamma_{i+1} \pi_1 X_n$, with $\Gamma_l \pi_1 X_n$ the l-th term of the lower central series of $\pi_1 X_n$ and $\phi_i(\pi_1 X_n)$ is equal to:

$$\phi_i(\pi_1 X_n) = \sum_{k=1}^{n} \phi_i(F(\alpha_k)) = \frac{1}{i} \sum_{k=1}^{n} \sum_{j|i} \mu(j) \alpha_k^{i/j},$$

with μ the Möbius function.

Proof. Point (1) is obtained by induction on n using the previous proposition and the fact that the fundamental group of the fiber F_k of $X_{k+1} \to X_k$ is $F(\alpha_k)$. We have already proved that the Poincaré series of X_n is given by $P_{X_n}(t) = \prod_{k=1}^{n}(1+\alpha_k t)$ (corollary 4.8). By applying the LCS formula of proposition 1.2 and equation (1) (Subsection 1.3), we get the formula relating P_{X_n} to the infinite product and the formula for the constants $\phi_i(\pi_1 X_n)$.

Remark 5.8. *In an earlier work, we have introduced a graded Lie algebra $\mathfrak{p}(X_n)$ (denoted by $\mathfrak{p}_n(G)(\mathbb{Q})$ in [Maa19]) whose degree completion correspond to the Malcev Lie algebra ([Qui69]) $\mathrm{Lie}_{\mathbb{Q}}\pi_1 X_n$ of $\pi_1 X_n$ over $\mathbb{Q}$, for Z equal to the set of irregular points of G. Since the associated graded of $\mathrm{Lie}_{\mathbb{Q}}\pi_1 X_n$ is isomorphic to $(\oplus_{i\geq 1}\Gamma_i\pi_1 X_n/\Gamma_{i+1}\pi_1 X_n)\otimes\mathbb{Q}$, the constant $\phi_i(\pi_1 X_n)$ above (and in proposition 5.13) is the dimension of the degree i part of $\mathfrak{p}(X_n)$.*

Remark 5.9. *Knowing that $\pi_1(X_n)$ is an iterated almost direct product of free groups, one can deduce the Poincaré series of X_n from the work of [CS98] (Cf. Remark 1.3, Subsection 1.3), since X_n is a $K(\pi,1)$.*

Remark 5.10. *An iterated almost direct product of free groups is residually torsion free nilpotent ([BB09], Cf. end of Section 2). In particular, $\pi_1 X_n$ is residually torsion free nilpotent.*

5.2. The case $(G,Z) = (\{1\},\emptyset)$

In this subsection we consider the case $X_n = C_n(\mathbb{P}^1)$. We recall (Cf. Subsection 3.2) that $C_2(\mathbb{P}^2)$ is homotopy equivalent to $C_1(\mathbb{P}^1) = \mathbb{P}^1 \simeq S^2$, $C_3(\mathbb{P}^1)$ is homotopy equivalent to the 3-dimensional real projective plane $\mathbb{P}^3(\mathbb{R})$, and that $C_n(\mathbb{P}^1)$ is homeomorphic to $C_3(\mathbb{P}^1)\times C_n(\mathbb{P}^1\setminus\{0,1,\infty\})$ for $n\geq 3$.

Proposition 5.11. *The higher homotopy groups of $C_n(\mathbb{P}^1)$ are given by,*

$$\pi_2 C_n(\mathbb{P}^1) = \begin{cases} \mathbb{Z}, & \text{if } n\leq 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \pi_k C_n(\mathbb{P}^1) = \pi_k S^2,$$

for $k\geq 3$. The space $C_n(\mathbb{P}^1)$ is simply connected if $n\leq 2$, $\pi_1 C_3(\mathbb{P}^1)\simeq\mathbb{Z}/2\mathbb{Z}$ and for $n>3$, $\pi_1 C_n(\mathbb{P}^1)\simeq \mathbb{Z}/2\mathbb{Z}\times\pi_1 C_{n-3}(\mathbb{P}^1\setminus\{0,1,\infty\})$.

Proof. The proposition follows from the facts reminded before it, and the fact that $C_{n-3}(\mathbb{P}^1\setminus\{0,1,\infty\})$ is aspheric as we have seen in the previous subsection.

Remark 5.12. *A homotopy equivalence between $C_3(\mathbb{P}^1)$ and the Stiefel manifold $V_2(\mathbb{R}^3)\simeq \mathrm{SO}(\mathbb{R}^3)\simeq\mathbb{P}^3(\mathbb{R})$ appears in [FVB61], and the higher homotopy groups of $C_n(S^r)$ (S^r the r-sphere) for $r\geq 2$ were computed in in term of homotopy groups of Stiefel manifolds of orthogonal frames in [Fad62].*

Proposition 5.13. *1) For $n>3$, $\pi_1 C_n(\mathbb{P}^1)$ is the product of $\mathbb{Z}/2\mathbb{Z}$ with the iterated almost direct product:*

$$F(\beta_{n-3})\rtimes_{\varepsilon_{n-3}}(F(\beta_{n-2})\rtimes_{\varepsilon_{n-2}}(F(\beta_{n-3})\rtimes_{\varepsilon_{n-3}}(\cdots\rtimes_{\varepsilon_3}(F(\beta_2)\rtimes_{\varepsilon_2}F(\beta_1))\cdots))),$$

where $\beta_k = k+1$.

2) Denote by P_n the Poincaré series of $C_n(\mathbb{P}^1)$. For $n > 3$:

$$\frac{1}{1+t^3}P_n(t) = \prod_{k=1}^{n-3}(1+\beta_k t) = \prod_{i\geq 1}(1-t^i)^{\phi_i(\pi_1 C_n(\mathbb{P}^1))},$$

where β_k is as in (1), $\phi_i(\pi_1 C_n(\mathbb{P}^1))$ is the rank of the quotient $\Gamma_i\pi_1 C_n(\mathbb{P}^1)/\Gamma_{i+1}\pi_1 C_n(\mathbb{P}^1)$, with $\Gamma_l\pi_1 C_n(\mathbb{P}^1)$ the l-th term of the lower central series of $\pi_1 C_n(\mathbb{P}^1)$ and

$$\phi_i(\pi_1 C_n(\mathbb{P}^1)) = \sum_{k=1}^{n}\phi_i(F(\beta_k)) = \frac{1}{i}\sum_{k=1}^{n}\sum_{j|i}\mu(j)\beta_k^{i/j},$$

with μ the Möbius function.

Proof. Point (1) follows from the isomorphism concerning $\pi_1 C_n(\mathbb{P}^1)$ in the previous proposition and (1) of corollary 5.7 giving the structure of $\pi_1 C_n(\mathbb{P}^1 \setminus \{0,1,\infty\})$. We prove (2), $\pi_1 C_n(\mathbb{P}^1) \simeq \mathbb{Z}/2\mathbb{Z} \times K$, where K is the iterated almost direct product in the proposition. Since the factor $\mathbb{Z}/2\mathbb{Z}$ is central in $\pi_1 C_n(\mathbb{P}^1)$, $\Gamma_l\pi_1 C_n(\mathbb{P}^1) = \Gamma_l K$. Therefore, the equation $\prod_{k=1}^{n-3}(1+\beta_k t) = \prod_{i\geq 1}(1-t^i)^{\phi_i(\pi_1 C_n(\mathbb{P}^1))}$ and the formula for $\phi_i(\pi_1 C_n(\mathbb{P}^1))$ can be obtained by applying the LCS formula (Cf. Proposition 1.2) and equation (1) of subsection 1.3 to K. Finally, $\frac{1}{1+t^3}P_n(t) = \prod_{k=1}^{n-3}(1+\beta_k t)$ by proposition 4.14 of subsection 4.2.

Remark 5.14. *The iterated almost direct product in the previous proposition is residually nilpotent (Cf. Remark 5.10) and hence $\pi_1 C_n(\mathbb{P}^1)$ is residually nilpotent.*

6. Appendix: finite actions on surfaces with punctures

Let S be a compact boundaryless, oriented surface and Y a finite subset (eventually empty) of S. In this appendix, we show that if a finite group H acts by orientation preserving homeomorphisms on $S \setminus Y$, then the action is the restriction of an action of H on S stabilizing Y and that S admits a complex structure in which H acts holomorphically. We have applied this result in section 1.4 for the case $S = S^2$. The result seems to be known, but the author was not able to track a proof in the literature.

We implicitly assume that a surface is second-countable, Hausdorff and connected. A closed surface is a compact surface with no boundary. By a disc we mean a topological disk.

Proposition 6.1. *Let H be a finite group acting by orientation preserving homeomorphisms on an oriented connected surface S. For p in the interior of S (not on the boundary), and any neighborhood U_p of p, there exists a closed disc $D_p \subset U_p$ containing p in its interior such that:*

1) The stabilizer H_p of p stabilizes D_p.

2) The action of H_p on p is equivalent to the action of the cyclic subgroup of order $|H_p|$ of $\mathrm{SO}(\mathbb{R}^2)$ on the closed unit disk in $\mathbb{R}^2$. In particular, H_p is cyclic.

3) The intersection $h \cdot D_p \cap D_p$, for $h \in H$, is non-empty if and only if $h \in H_p$.

4) The set of points with a non-trivial stabilizer in D_p is either empty, either reduced to p.

PROOF. In this proof, we mean by "a disk D around p" a closed disk D containing p in its interior $\mathring{D}$. Note that (3) and (2) imply (4) and that the condition $D_p \subset U_p$ follows from (2), since (2) implies that the disc can be taken as small as needed. Therefore, we will only prove the existence of a disk satisfying (1), (2) and (3). We first show that a disk around p satisfying (3) exists. Take a neighborhood V_q for each $q \in H \cdot p$, such that V_q and $V_{q'}$ are disjoint for $q \neq q'$, and set $V'_p = \cap_{h \in H} h^{-1} V_{h \cdot p}$. The set V'_p is a neighborhood of p such that $h' \cdot V'_p \subset V_{h' \cdot p}$, for $h' \in H$. Hence, any disk around p contained in V'_p satisfies (3). Let D be such a disk and D' a closed disk (around p) lying in the non-empty interior of $\cap_{h \in H_p} h \cdot D \subset D$. For $h \in H_p$, $h \cdot D'$ lies in $\mathring{D}$. Since all the disks $h \cdot D'$ lie in $\mathring{D}$, it follows form a general argument on intersection of disks in $\mathbb{R}^2$ ([CK94], proposition 2.4) that the closure J_p of the connected component of $\cap_{h \in H_p} h \cdot \mathring{D}'$ containing p is a closed disk. Clearly J_p satisfies (1) and (3). We still have to prove that (2) is satisfied for J_p and H_p. An orientation preserving homeomorphism f of finite order k of the unit closed disc $D" \subset \mathbb{R}^2$ is conjugate to a rotation of $D"$ ([CK94], theorem 3.1). Hence, it suffices to prove that H_p is cylic. By the same conjugacy result H_p acts freely on the boundary $\partial J_p \simeq S^1$ of J_p. In particular, $\partial J_p \to (\partial J_p)/H_p$ is a normal cover and H_p is a quotient of $\pi_1((\partial J_p)/H_p)$. The quotient $(\partial J_p)/H_p$ is a circle, since it is a compact boundaryless 1-dimensional manifold. Hence, H_p is a quotient of $\mathbb{Z}$ and H_p is cyclic. We have proved the proposition.

Corollary 6.2. *The set of points in S having a non-trivial stabilizer is finite if S is closed.*

PROOF. Denote by X the set in the corollary. Point (4) of the last proposition implies that X is discrete. The corollary follows, since X is closed. We prove that X is closed. For $h \in H$ denote by f_h the map $S \to S \times S, p \mapsto (p, h \cdot p)$. The diagonal $\Delta_S \subset S \times S$, is closed and $X = \cup_{h \in H} f_h^{-1}(\Delta_S)$. Hence, X is closed.

Proposition 6.3. *Let H be a finite group acting by orientation preserving homeomorphisms on a connected closed oriented surface. Denote by Y the set of points having a non-trivial stabilizer. The quotient space S/H is a surface and the quotient map $(S \setminus Y) \to (S \setminus Y)/H$ is a normal cover.*

PROOF. We will use the notations of the previous proposition. The quotient map $\pi : S \to S/H$ is open. The space S/H is Hausdorff, since S is Hausdorff and H is finite. Point (2) of the previous proposition implies that $(S \setminus Y) \to (S \setminus Y)/H$ is a (normal) cover and that $\pi(D_p) \simeq D_p/H_p$. By (3) of the previous proposition D_p/H_p is a disk this proves that the neighborhood $\pi(D_p)$ (π is open) of $\pi(p)$ is Euclidean. We have proved the proposition.

Lemme 6.4. *Let H, S and Y be as in the previous proposition. Denote by π the quotient map $S \to S/H$. The surfaces S and S/H admit complex structures (Riemann surface structures) such that π is holomorphic.*

PROOF. The quotient space S/H is a closed surface and hence admits a complex structure. Such a structure can be lifted to S via π (Cf. [Ful95] §19.B) and π becomes holomorphic.

Proposition 6.5. *If H is a finite group acting by orientation preserving homeomorphisms on a closed surface S then there is a complex structure on S in which H acts holomorphically.*

PROOF. By the previous lemma, we have complex structures over S and S/H such that $\pi : S \to S/H$ is holomorphic. Take $p \in S \setminus Y$ and U_p a neighborhood of p such that $\pi_{|U_p} : U_p \to \pi(U_p)$ is an homeomorphism ($(S \setminus Y) \to (S \setminus Y)/H$ is a cover by proposition 6.3). The map $\pi_{|h \cdot U_p} : h \cdot U_p \to \pi(h \cdot U_p) = \pi(U_p)$ is also a homeomorphism and $U_p \to h \cdot U_p$, induced by $h \in H$ is equal to $\pi^{-1}_{|h \cdot U_p} \pi_{|U_p}$. Hence, h acts holomorphically on $S \setminus Y$, since π is holomorphic. Using this and the fact that $x \mapsto h \cdot x$ is continuous at $p \in Y$ we deduce from Riemann's theorem on removable singularities that the action on S is holomorphic. We have proved the proposition.

Proposition 6.6. *Let S be a closed surface, $Y \subset S$ a finite set and f a self homeomorphism of $S \setminus Y$. There exists a unique self homeomorphism $\tilde{f}$ of S extending f. Moreover, if S is oriented and f preserves the orientation, then $\tilde{f}$ is orientation preserving.*

PROOF. Denote by $y_1, \dots y_n$ the elements of Y and take pairwise disjoint closed topological disks $D_i \subset S$ (for $n \in [1, n]$) such that y_i lies in the interior of D_i. Denote by ∂D_i the boundary of D_i and set $D'_i = D_i \setminus \{y_i\}$. The punctured disc $f(D'_i)$ is closed in $S \setminus Y$ but not compact. Hence, its closure in S is obtained by adding points of Y. Take $y_{j_i} \in Y$ adherent to $f(D'_i)$. Since $f(\partial D_i)$ is closed in S, small neighborhoods of y_{j_i} do not intersect $f(\partial D_i)$ and such neighborhoods are included in $\{y_i\} \cup (f(D'_i) \setminus f(\partial D_i))$ ($f(\partial D_i)$ disconnects the surface). In particular y_{j_i}, is not adherent to $f(D'_k)$ for $k \neq i$. Hence, we have a bijection $y_i \mapsto y_{j_i}$ of Y and the closure of $f(D'_i)$ in S is $f(D'_i) \cup y_{j_i}$. Since D_i and $f(D'_i) \cup y_{j_i}$ are one point compactifications of D'_i and $f(D'_i)$ respectively, the homeomorphism $f_i : D'_i \to f(D'_i)$ induced by f extends to a homeomorphism $\tilde{f}_i : D_i \to f(D'_i) \cup y_{j_i}$ mapping y_i to y_{j_i}. This gives an open and bijective extension $\tilde{f} : S \to S$ of f. Moreover, $\tilde{f}$ is unique, since $S \setminus Y$ is dense in S. The orientation part can be proved by manipulating the continuous local homological orientations on S and $S \setminus Y$.

Corollary 6.7. *Take S, Y as in the proposition. Let H be a group acting by homeomorphisms on $S \setminus Y$. The action of H on $S \setminus Y$ extends to a unique action of H on S. Moreover, if S is oriented and the action on $S \setminus Y$ preserves the orientation, then the extended action preserves the orientation.*

Proposition 6.8. *Let S be a closed, oriented surface, Y a finite (eventually empty) subset of S and H be a finite group acting by orientation preserving homeomorphisms on $S \setminus Y$. The action of H extends to a unique action of H on S stabilizing Y. Moreover, S admits a complex structure in which the extended action of H is holomorphic.*

PROOF. The proposition follows from proposition 6.5 and corollary 6.7.

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