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Hausdorff dimension of multiple expansions

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Abstract

Let M be a positive integer and $q \in (1, M + 1]$. A q -expansion of a real number x is a sequence $(c_i) = c_1c_2 \cdots$ with $c_i \in \{0, 1, \dots, M\}$ such that $x = \sum_{i=1}^{\infty} c_i q^{-i}$. Let \mathcal{U}_q^j denote the set of numbers having exactly j expansions. Contrary to the unique expansions, we prove for each $j \geq 2$ that the set \mathcal{U}_q^j is closed only if it is empty. Then we generalize an important example of Sidorov by exhibiting many non-trivial bases q for which the Hausdorff dimension of \mathcal{U}_q^j is independent of j .

Keywords: Non-integer bases, β -expansion, greedy expansion, multiple expansions, unique expansion, Hausdorff dimension, topological entropy

2010 MSC: 11A63, 11K55, 28A80, 37B10

1. Introduction

Let $M \in \mathbb{N} := \{1, 2, \dots\}$. Given a sequence $c = (c_i) \in \{0, 1, \dots, M\}^{\mathbb{N}}$, we define its *evaluation* in a base $q > 1$ by the formula

$$\pi_q(c) = \pi_q((c_i)) := \sum_{i=1}^{\infty} \frac{c_i}{q^i},$$

and we say that c is an *expansion* of $x \in \mathbb{R}$ in a base q if $\pi_q(c) = x$. The study of such expansions started with a seminal paper of Rényi [35] in 1957. A new

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impetus was given by the discovery of strange uniqueness phenomena by Erdős et al. around 1990 [14, 15, 16].

Note that we have always

$$\pi_q(c) \in J := \left[0, \frac{M}{q-1}\right].$$

If $q \in (1, M+1]$, then a number x has an expansion if and only if $x \in J$. Moreover, a classical theorem of Rényi [35] asserts that every $x \in J$ has a lexicographically largest expansion. Such expansion is called the *greedy expansion* of x in base q , and it is denoted by $b(x, q)$. Here and in the sequel we follow the notation of the papers [30, 11, 24, 13, 26], whose results will be frequently used.

In the familiar case $q = M+1$ we have $J = [0, 1]$, each $x \in [0, 1]$ has one or two expansions, and only countably many special rational numbers have two expansions. On the other hand, if $q > M+1$, then no number $x \in J$ has more than one expansion, and in fact not every $x \in J$ has an expansion. For example, if $\frac{M}{q(q-1)} < x < \frac{1}{q}$, then x has no expansion. Finally, if $q < M+1$, then almost every $x \in J$ has a continuum of expansions [36, 9].

In this paper we are interested in the existence of multiple expansions of a real number $x \in J$. Fix $M \geq 1$, and let \mathcal{U}_q^j denote the set of real numbers having exactly j expansions. Since M is fixed, we do not indicate the dependence of \mathcal{U}_q^j and of other related quantities on M in the sequel. The superscript 1 will be omitted, so that we write \mathcal{U}_q instead of \mathcal{U}_q^1 . Furthermore, since a number having uncountably many expansions necessarily has a continuum of expansions by [6, Theorem 4.1] and [12, Theorem 2.3.1], it suffices to consider $j \in \mathbb{N} \cup \{\aleph_0\} \cup \{2^{\aleph_0}\}$. The combinatorial, topological and fractal structure of the sets \mathcal{U}_q is well understood today [10, 20, 21, 18, 11, 32, 31, 26, 2]. For $j > 1$, many important theorems have been obtained in [15, 37, 7, 5, 6, 33, 38, 39, 25]. Much research has also been devoted to the properties of the sets of expansions of $x \in J$; see for example [36, 8, 6, 22] and the references therein. However, the theory is far from complete. In order to state our results we let \mathcal{U} denote the set of bases $q > 1$ in which 1 has a unique expansion. We recall from [16, 10, 30, 13] that its topological closure $\overline{\mathcal{U}}$ is a Cantor set of zero Lebesgue measure and Hausdorff dimension one, and that $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable.

The set \mathcal{U}_q is closed for Lebesgue almost every $q > 1$, but not for all of them: \mathcal{U}_q is closed if and only if $q \notin \overline{\mathcal{U}}$ [11]. We will show that the situation is much simpler for \mathcal{U}_q^j if $j > 1$:

Theorem 1.1. *If \mathcal{U}_q^j is closed for some $q > 1$ and $j > 1$, then it is empty.*

Most of the present work is motivated by an intriguing example of Sidorov [37]. He proved that if $M = 1$ and $T \approx 1.83929$ is the *Tribonacci number*, i.e., the positive root of the equation $q^3 = q^2 + q + 1$, then the following *equal dimension property* holds:

$$\dim_H \mathcal{U}_T^j = \dim_H \mathcal{U}_T \quad \text{for all finite } j. \quad (1.1)$$

Here $\dim_H F$ denotes the Hausdorff dimension of the set F . The equal dimension property trivially holds also whenever $\dim_H \mathcal{U}_q = 0$, because

$$\dim_H \mathcal{U}_q^j \leq \dim_H \mathcal{U}_q \quad \text{for all } q > 1 \quad \text{and } j \leq \aleph_0 \quad (1.2)$$

by a bifurcation argument of Sidorov [37] and Baker [6]; see Lemma 5.5 and Proposition 5.6 below. If we let $q_{KL} \in (1, M+1)$ denote the Komornik–Loreti constant, i.e., the smallest element of \mathcal{U} [28, 29], then the equality $\dim_H \mathcal{U}_q = 0$ holds if and only if $q \leq q_{KL}$ [18, 32, 31]. We stress that q_{KL} depends on M . On the other hand, the equal dimension property fails in the classical integer base case $q = M+1$ because then \mathcal{U}_q^2 is countable and $\mathcal{U}_q = [0, 1] \setminus \mathcal{U}_q^2$, so that $\dim_H \mathcal{U}_q^2 = 0 < 1 = \dim_H \mathcal{U}_q$. The equal dimension property also fails if $q > M+1$, because then every expansion is unique, and $\dim_H \mathcal{U}_q > 0$ by the relation $M+1 > q_{KL}$.

In order to state the main theorem of this paper, it is convenient to use some notations of symbolic dynamics [34]. Given a finite *word* or *block*

$$w = a_1 \dots a_{m-1} a_m \in \{0, \dots, M\}^m,$$

we introduce the following notation:

$$\begin{aligned} w^+ &:= a_1 \dots a_{m-1} (a_m + 1) \quad \text{if } a_m < M, \\ w^- &:= a_1 \dots a_{m-1} (a_m - 1) \quad \text{if } a_m > 0, \\ \bar{w} &:= (M - a_1) \dots (M - a_{m-1}) (M - a_m). \end{aligned}$$

In particular, in case $m = 1$ we define $w^+ = a_1 + 1$, $w^- = a_1 - 1$ and $\bar{w} := M - a_1$. The word \bar{w} is called the *reflection* of w . For example, the greedy expansion of 1 in a base $q \in \bar{\mathcal{U}} \setminus \mathcal{U}$ is *finite*, i.e., it is of the form $w^+ 0^\infty$ [30, 13].

We recall from [26] that the Hausdorff dimension of \mathcal{U}_q is given by the formula

$$\dim_H \mathcal{U}_q = \frac{h(\mathcal{U}'_q)}{\log q}, \quad q \in (1, +\infty),$$

where \mathcal{U}'_q denotes the set of unique expansions (i.e., sequences) of the numbers $x \in \mathcal{U}_q$, and $h(\mathcal{U}'_q)$ denotes the topological entropy of \mathcal{U}'_q . Furthermore, the topological entropy is constant on each connected component of $(1, +\infty) \setminus \bar{\mathcal{U}}$, so that the dimension function $q \mapsto \dim_H \mathcal{U}_q$ has a negative derivative almost everywhere in $(q_{KL}, +\infty)$. Finally, the dimension and entropy functions are continuous and non-decreasing [26, 2], and hence they have a *Devil's staircase* behavior.

Although the entropy is constant on the connected components of $(1, +\infty) \setminus \bar{\mathcal{U}}$, its maximal constancy or *stability intervals* are larger: they were determined in [3]. They are $(1, q_{KL}]$, $[M+1, +\infty)$, and infinitely many compact intervals $[p_L, p_R]$, where p_L runs over some proper subset of $\bar{\mathcal{U}} \setminus \mathcal{U}$. Furthermore,

$$\text{if } b(1, p_L) = w^+ 0^\infty, \quad \text{then } b(1, p_R) = w^+ \bar{w}^\infty. \quad (1.3)$$

Now we are ready to state our next result:

Theorem 1.2. *Let $p_L \in \overline{\mathcal{U}} \setminus \mathcal{U}$ be the left endpoint of some maximal stability interval $[p_L, p_R]$, and write $b(1, p_L) = w^+0^\infty$. Then the equal dimension property*

$$\dim_H \mathcal{U}_{r_k}^j = \dim_H \mathcal{U}_{r_k}^1 \quad \text{for all finite } j$$

holds for all bases r_k , defined by the implicit equations

$$\pi_{r_k}(w^+(\overline{w})^k 0^\infty) = 1, \quad k = 0, 1, \dots \quad (1.4)$$

Remark 1.3.

- (i) *It follows from (1.3) and (1.4) that (r_k) is an increasing sequence with $r_0 = p_L$ and $r_k \nearrow p_R$.*
- (ii) *If $M = 1$, then the Tribonacci number is the left endpoint of a maximal stability interval $[p_L, p_R]$, so that Sidorov's result [37] is a special case of Theorems 1.2. Moreover, Theorem 1.2 provides infinitely many further bases r_k in this interval for which the equal dimension property holds.*
- (iii) *More generally, if $M = 1$, then the multinacci numbers $q > 1$, defined by the implicit equations*

$$q^n = q^{n-1} + \dots + 1, \quad n = 3, 4, \dots$$

are also left endpoints of maximal stability intervals $[p_L, p_R]$, so that each of them determines infinitely many further bases $r_k \in [p_L, p_R]$ for which the equal dimension property holds.

- (iv) *We recall that q_{KL} denotes the smallest element of \mathcal{U} . Writing $b(1, q_{KL}) = (\lambda_i)$, for each choice of positive integers n and j there exists a base $q_{n,j}$ such that*

$$b(1, q_{n,j}) = \lambda_1 \cdots \lambda_{2^n} \overline{(\lambda_1 \cdots \lambda_{2^n})^{j+1} 0^\infty}.$$

It follows from [3, Lemma 4.5, Theorem 2] that the numbers $q_{n,j}$ are left endpoints of suitable maximal stability intervals $[p_L, p_R]$, so that each of them determines infinitely many further bases $r_k \in [p_L, p_R]$ for which the equal dimension property holds.

The plan of the paper is the following. For the convenience of the reader we give in Sections 2 and 4 a short review on expansions, symbolic dynamics and Hausdorff dimension. Theorem 1.1 is proved in Section 3, and Theorem 1.2 is proved in Sections 5 and 6. We end the paper by raising some open problems.

2. A review of expansions in non-integer bases

In this section we recall from [4, 23, 12, 13] some notions and results that we will need in the sequel. We fix a positive integer M , and by a *sequence* denoted by (c_i) or $c_1 c_2 \cdots$ we mean an element of $\{0, \dots, M\}^{\mathbb{N}}$. By a *word* w we mean a finite string of digits $w = c_1 \cdots c_n$ with $c_i \in \{0, 1, \dots, M\}$. Given two finite

words $w = c_1 \cdots c_n$ and $v = d_1 \cdots d_m$, we denote by $wv = c_1 \cdots c_n d_1 \cdots d_m$ their *concatenation*. Accordingly, for $k \in \mathbb{N}$ we denote by w^k or $(w)^k$ the concatenation of w with itself k times, and by $w^\infty = ww \cdots$ or $(w)^\infty$ the concatenation $ww \cdots$ of w with itself infinitely many times.

We recall from the introduction that a sequence or expansion (c_i) is called *finite* if it has a last non-zero digit; otherwise, it is called *infinite*. Furthermore, a sequence or expansion (c_i) is called *doubly infinite* if both (c_i) and its *reflection* $(\overline{c_i}) := (M - c_i)$ are infinite. A sequence (c_i) is called *periodic* if there exists an $n \geq 1$ such that $(c_i) = (c_1 \cdots c_n)^\infty$. Then the smallest such n is called the *period* of (c_i) , and the word $c_1 \cdots c_n$ is called the *period block* of (c_i) . For example, 0^∞ and M^∞ are both periodic and doubly infinite.

We will use the lexicographic order between words and sequences. If $c_1 \cdots c_n$ and $d_1 \cdots d_n$ are two words having the same length $n \geq 1$, then we write $c_1 \cdots c_n \prec d_1 \cdots d_n$ if there exists a $k \in \{1, \dots, n\}$ such that $c_i = d_i$ for $i = 1, \dots, k-1$, and $c_k < d_k$, and $c_1 \cdots c_n \preceq d_1 \cdots d_n$ if $c_1 \cdots c_n \prec d_1 \cdots d_n$ or $c_1 \cdots c_n = d_1 \cdots d_n$. Symmetrically, we also write $d_1 \cdots d_n \succ c_1 \cdots c_n$ if $c_1 \cdots c_n \prec d_1 \cdots d_n$, and $d_1 \cdots d_n \succeq c_1 \cdots c_n$ if $c_1 \cdots c_n \preceq d_1 \cdots d_n$. Similarly, if (c_i) and (d_i) are two sequences, then we write $(c_i) \prec (d_i)$ if there exists a $k \in \mathbb{N}$ such that $c_i = d_i$ for $i = 1, \dots, k-1$ and $c_k < d_k$, and $(c_i) \preceq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$. We also write $(d_i) \succ (c_i)$ if $(c_i) \prec (d_i)$, and $(d_i) \succeq (c_i)$ if $(c_i) \preceq (d_i)$.

By the *subwords* of a sequence (c_i) we mean the words of the form $c_m \cdots c_n$ with $n \geq m \geq 1$. The subword $c_1 \cdots c_n$ is called the *prefix of length n* of (c_i) . If w is a prefix of (c_i) , then we also say the sequence (c_i) *starts* with w .

For any fixed base $q \in (1, M+1]$, every $x \in J$ has a lexicographically largest expansion $b(x, q) = (b_i)$, obtained by the greedy algorithm, and a lexicographically largest infinite expansion $a(x, q) = (a_i)$; see [4, 13]. Such expansions are called the *greedy* and *quasi-greedy expansions* of x in base q , respectively. If the greedy expansion of a number of x is infinite, then they coincide. Otherwise, $b(x, q)$ has a last nonzero digit b_n , and then $a(x, q) = b_1 \cdots b_{n-1} b_n^- a(1, q)$; see [11] and [4, Proposition 3.4]. If $q \in (1, M+1)$, then all quasi-greedy expansions are doubly infinite [24]. The case $x = 1$ being particularly important, we introduce the shorter notations $\beta(q) := b(1, q)$ and $\alpha(q) := a(1, q)$. If $\beta(q) = (\beta_i)$ has a last nonzero digit β_n , then $\alpha(q) = (\beta_1 \cdots \beta_{n-1} \beta_n^-)^\infty$.

The following lexicographic characterization of greedy and quasi-greedy expansions was given in [4].

Proposition 2.1. *Fix $q \in (1, M+1]$. Then:*

- (i) *The map $x \mapsto b(x, q)$ is a strictly increasing bijection between the interval J and the sequences (b_i) satisfying the lexicographic inequalities*

$$(b_{n+i}) \prec \alpha(q) \quad \text{whenever} \quad b_1 \cdots b_n \neq M^n.$$

- (ii) *The map $x \mapsto a(x, q)$ is a strictly increasing bijection between the interval J and the infinite sequences (a_i) satisfying the lexicographic inequalities*

$$(a_{n+i}) \preceq \alpha(q) \quad \text{whenever} \quad a_1 \cdots a_n \neq M^n.$$

There is an elementary relation between the expansions and their reflections:

Lemma 2.2. *For any fixed $x \in J$, the map $(c_i) \mapsto (M - c_i)$ is a strictly decreasing bijection between the expansions of x and $\frac{M}{q-1} - x$.*

PROOF. The decreasing property is obvious. Furthermore, the identity

$$\pi_q((M - c_i)) = \sum_{i=1}^{\infty} \frac{M - c_i}{q^i} = \frac{M}{q-1} - \sum_{i=1}^{\infty} \frac{c_i}{q^i} = \frac{M}{q-1} - x$$

shows that (c_i) is an expansion of x if and only if $(M - c_i)$ is an expansion of $\frac{M}{q-1} - x$.

An expansion of x is unique if and only if it is at the same time the lexicographically largest and smallest (called *lazy*) expansion of x . Therefore we deduce from Lemma 2.2 and Proposition 2.1 the following result:

Corollary 2.3. *An expansion (c_i) of a number x in a base q is unique if and only if the following two conditions are satisfied:*

$$(c_{n+i}) \prec \alpha(q) \quad \text{whenever} \quad c_1 \cdots c_n \neq M^n.$$

and

$$\overline{(c_{n+i})} \prec \alpha(q) \quad \text{whenever} \quad c_1 \cdots c_n \neq 0^n.$$

There is a similar characterization of the sequences $\beta(q)$ and $\alpha(q)$ in [4]. Setting $\beta(1) = 10^\infty$ and $\alpha(1) = 0^\infty$ for commodity, we have the following statement:

Proposition 2.4.

- (i) *The map $q \mapsto \beta(q)$ is an increasing bijection between the interval $[1, M+1)$ and the set of sequences (β_i) satisfying the lexicographic inequalities*

$$(\beta_{n+i}) \prec (\beta_i) \quad \text{for all} \quad n \geq 1.$$

- (ii) *The map $q \mapsto \alpha(q)$ is an increasing bijection between the interval $[1, M+1]$ and the set of infinite sequences (α_i) satisfying the lexicographic inequalities*

$$(\alpha_{n+i}) \preceq (\alpha_i) \quad \text{for all} \quad n \geq 0.$$

Following [30, 24, 13], let \mathcal{V} denote the set of bases $q > 1$ in which 1 has a unique doubly infinite expansion. Then \mathcal{V} is a closed set, and

$$\mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V} \subset [0, M+1].$$

Furthermore, both $\mathcal{V} \setminus \overline{\mathcal{U}}$ and $\overline{\mathcal{U}} \setminus \mathcal{U}$ are countably infinite sets, and therefore the three sets \mathcal{U} , $\overline{\mathcal{U}}$ and \mathcal{V} have the same Lebesgue measure and Hausdorff dimension; see [13] for proofs. Finally we recall that 1 has a finite greedy expansion $\beta(q)$ in each base $q \in \mathcal{V} \setminus \mathcal{U}$ [30, 11, 12].

The sets \mathcal{U} , $\overline{\mathcal{U}}$ and \mathcal{V} were characterized in [16, 30, 13] by using the sequences $\alpha(q)$ and $\beta(q)$. Proposition 2.4 is used in the proof of the following proposition:

Proposition 2.5. *Let $q \in (1, M + 1]$, and write $\alpha(q) = (\alpha_i)$, $\beta(q) = (\beta_i)$. We have*

$$\begin{aligned} q \in \mathcal{U} &\iff \overline{(\beta_{n+i})} \prec (\beta_i) \quad \text{for all } n \geq 0; \\ q \in \overline{\mathcal{U}} &\iff \overline{(\alpha_{n+i})} \prec (\alpha_i) \quad \text{for all } n \geq 0; \\ q \in \mathcal{V} &\iff \overline{(\alpha_{n+i})} \preceq (\alpha_i) \quad \text{for all } n \geq 0. \end{aligned}$$

We will need from [13] the following properties of $\alpha(q)$:

Proposition 2.6. *Let $q \in \mathcal{V} \setminus \mathcal{U}$, and write $\alpha(q) = \alpha_1 \alpha_2 \cdots$.*

- (i) *The sequence $\alpha(q)$ is periodic, say $\alpha_1 \alpha_2 \cdots = (\alpha_1 \cdots \alpha_N)^\infty$, and $\alpha_N < M$.*
- (ii) *If $\alpha(q) = (\alpha_1 \cdots \alpha_N)^\infty$ with the period N , then $q^+ := \min \{p \in \mathcal{V} : p > q\}$ is well defined, and*

$$\alpha(q^+) = \left(\alpha_1 \cdots \alpha_N^+ \overline{\alpha_1 \cdots \alpha_N^+} \right)^\infty.$$

Hence $q^+ \in \mathcal{V} \setminus \overline{\mathcal{U}}$.

- (iii) *If M is even and $q = \min \mathcal{V}$, then $\overline{\alpha_1} = \alpha_1$. Otherwise, we have $\overline{\alpha_1} < \alpha_1$.*
- (iv) *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, $q > \min \mathcal{V}$ and*

$$\alpha(q) = (\alpha_1 \cdots \alpha_N \overline{\alpha_1 \cdots \alpha_N})^\infty,$$

then

$$\overline{\alpha_{i+1} \cdots \alpha_N} \prec \alpha_1 \cdots \alpha_{N-i} \quad \text{for } i = 0, \dots, N-1.$$

By analogy with \mathcal{V} , we introduce for each $q > 1$ the set \mathcal{V}_q of real numbers having at most one doubly infinite expansion in base q . For example, $\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q = J_q$ if $q > M + 1$, and $\mathcal{U}_q \subsetneq \overline{\mathcal{U}_q} = \mathcal{V}_q = J_q = [0, 1]$ if $q = M + 1$. The next statement was proved in [11, 24]:

Lemma 2.7.

- (i) $\mathcal{U}_q \subseteq \mathcal{V}_q$, \mathcal{V}_q is closed, and $\mathcal{V}_q \setminus \mathcal{U}_q$ is countable.
- (ii) If $q \in (1, M + 1)$ and $x \in J$, then

$$x \in \mathcal{V}_q \iff x \text{ has a unique doubly infinite expansion.}$$

In Corollary 2.3 we gave a lexicographic characterization of the set

$$\mathcal{U}'_q := \{a(x, q) : x \in \mathcal{U}_q\}.$$

In the following Lemma 2.8 we recall from [11] a similar lexicographic characterization of the set

$$\mathcal{V}'_q := \{a(x, q) : x \in \mathcal{V}_q\}.$$

Lemma 2.8. *Let $q \in (1, M + 1]$. A sequence (c_i) belongs to \mathcal{V}'_q if and only if*

$$(c_{n+i}) \preceq \alpha(q) \quad \text{whenever} \quad c_1 \cdots c_n \neq M^n,$$

and

$$(c_{n+i}) \succeq \overline{\alpha(q)} \quad \text{whenever} \quad c_1 \cdots c_n \neq 0^n.$$

3. Proof of Theorem 1.1

First we establish two elementary properties of the sets \mathcal{U}_q^j . The symmetry of \mathcal{U}_q was stated and proved in [11].

Lemma 3.1. *Fix j arbitrarily.*

(i) *The set \mathcal{U}_q^j is symmetric:*

$$x \in \mathcal{U}_q^j \iff \frac{M}{q-1} - x \in \mathcal{U}_q^j.$$

(ii) *If $x \in \mathcal{U}_q^j \cap (0, 1)$, then $x/q \in \mathcal{U}_q^j \cap (0, 1)$.*

PROOF. (i) For any fixed $x \in J$, by Lemma 2.2 the map $(c_i) \mapsto (M - c_i)$ is a bijection between the expansions of x and $\frac{M}{q-1} - x$. Therefore x and $\frac{M}{q-1} - x$ have the same number of expansions.

(ii) Since $x/q \in (0, 1)$, it suffices to show that the map $(c_i) \mapsto 0(c_i)$ is a bijection between the expansions of x and x/q . It is clear that if (c_i) is an expansion of x , then $0(c_i)$ is an expansion of x/q . Conversely, if (d_i) is an expansion of x/q , then it starts with 0 because $x/q < 1/q$. Therefore $(c_i) := (d_{1+i})$ is an expansion of x , and $(d_i) = 0(c_i)$.

Next we recall a theorem of Baker [5] (he gave a different but equivalent formula for odd values of M):

Lemma 3.2. *The smallest element q_{GR} of the set \mathcal{V} is given by the formula*

$$q_{GR} = \begin{cases} (m + \sqrt{m^2 + 4m})/2 & \text{with } \beta(q_{GR}) = mm0^\infty & \text{if } M = 2m - 1, \\ m + 1 & \text{with } \beta(q_{GR}) = (m + 1)0^\infty & \text{if } M = 2m \end{cases}$$

for $m \in \mathbb{N}^*$.

Concerning the bases q_{GR} we need a third property:

Lemma 3.3. *If $M = 2m$, then $(m + 1)^{-k} \in \mathcal{U}_{m+1}^{\mathbb{N}^0}$ for all nonnegative integers k .*

PROOF. First we show that $1 \in \mathcal{U}_{m+1}^{\aleph_0}$ on the alphabet $\{0, \dots, 2m\}$. By a simple computation we have infinitely many expansions of 1 in base $m+1$:

$$\sum_{i=1}^{\infty} \frac{m}{(m+1)^i} = 1, \quad (3.1)$$

$$\sum_{i=1}^k \frac{m}{(m+1)^i} + \frac{m+1}{(m+1)^{k+1}} = 1, \quad k = 0, 1, \dots, \quad (3.2)$$

$$\sum_{i=1}^k \frac{m}{(m+1)^i} + \frac{m-1}{(m+1)^{k+1}} + \sum_{i=k+2}^{\infty} \frac{2m}{(m+1)^i} = 1, \quad k = 0, 1, \dots \quad (3.3)$$

By the usual convention $\sum_{i=1}^0 a_i = 0$ for empty sums, for $k = 0$ the relations (3.2) and (3.3) reduce to

$$\frac{m+1}{m+1} = 1 \quad \text{and} \quad \frac{m-1}{m+1} + \sum_{i=2}^{\infty} \frac{2m}{(m+1)^i} = 1,$$

respectively. The relation $1 \in \mathcal{U}_{m+1}^{\aleph_0}$ will follow if we show that 1 has no other expansions in base $m+1$. To show this, let (c_i) be an arbitrary expansion of 1 in base $m+1$. If $(c_i) \neq m^\infty$, then there is an integer $k \geq 0$ such that (c_i) starts with exactly k consecutive digits m . If $c_{k+1} > m$, then we infer from (3.2) that $c_{k+1} = m+1$, and $c_i = 0$ for all $i > k+1$; for otherwise we would have $\pi_{m+1}((c_i)) > 1$. Similarly, if $c_{k+1} < m$, then we infer from (3.3) that $c_{k+1} = m-1$, and $c_i = 2m$ for all $i > k+1$. We have thus shown that every expansion of 1 appears in (3.1)–(3.3).

It remains to show that $(m+1)^{-k}$ has exactly \aleph_0 expansions for each fixed integer $k \geq 1$. If (c_i) is an expansion of 1, then $0^k(c_i)$ is an expansion of $(m+1)^{-k}$, so that $(m+1)^{-k}$ has at least \aleph_0 expansions. On the other hand, if (d_i) is an expansion of $(m+1)^{-k}$, then either $d_k = 1$ and $d_i = 0$ for all $i \neq k$, or $d_1 = \dots = d_k = 0$, and then $(c_i) := (d_{k+i})$ is an expansion of 1. Hence $(m+1)^{-k}$ has at most $1 + \aleph_0 = \aleph_0$ expansions.

Now we are ready to prove Theorem 1.1.

PROOF (PROOF OF THEOREM 1.1). Since $j > 1$, and 0 has a unique expansion, $0 \notin \mathcal{U}_q^j$. We will show that if $\mathcal{U}_q^j \neq \emptyset$, then $0 \in \overline{\mathcal{U}_q^j}$, and hence \mathcal{U}_q^j is not closed. We distinguish several cases:

(i) If $q < q_{GR}$, then every $x \in \left(0, \frac{M}{q-1}\right)$ has a continuum of expansions by [16, 5]. Since $j > 1$, the assumption $\mathcal{U}_q^j \neq \emptyset$ implies that $j = 2^{\aleph_0}$. Then $\mathcal{U}_q^j = \left(0, \frac{M}{q-1}\right)$ and therefore $0 \in \overline{\mathcal{U}_q^j}$ as required.

(ii) If $q > q_{GR}$, then $\frac{M}{q-1} < 2$. Indeed, if M is odd, say $M = 2m - 1$, then

$$\frac{M}{q-1} < \frac{M}{q_{GR}-1} = \frac{4m-2}{m + \sqrt{m^2 + 4m} - 2} < \frac{4m-2}{m + (m+1) - 2} = 2, \quad (3.4)$$

while in case $M = 2m$ we have

$$\frac{M}{q-1} < \frac{M}{q_{GR}-1} = \frac{2m}{(m+1)-1} = 2.$$

Since $\frac{M}{q-1} < 2$, we have

$$\left(0, \frac{M}{q-1}\right) = (0, 1) \cup \left(\frac{M}{q-1} - 1, \frac{M}{q-1}\right).$$

Since \mathcal{U}_q^j is non-empty, and symmetric by Lemma 3.1 (i), there exists a point $x \in \mathcal{U}_q^j \cap (0, 1)$, and then $q^{-k}x \in \mathcal{U}_q^j \cap (0, 1)$ for all $k \in \mathbb{N}$ by Lemma 3.1 (ii). Since $q^{-k}x \rightarrow 0$, we conclude again that $0 \in \overline{\mathcal{U}_q^j} \setminus \mathcal{U}_q^j$.

(iii) It remains to consider the case $q = q_{GR}$. If M is odd, then we may repeat the reasoning of the preceding step, because we still have $\frac{M}{q-1} < 2$: the second one of the two inequalities in (3.4) remains strict. If $M = 2m$ is even, then we have $\frac{M}{q-1} = 2$, so there exists again a point $x \in \mathcal{U}_q^j \cap (0, 1]$. If $x \in \mathcal{U}_q^j \cap (0, 1)$, then we conclude as before. If $x = 1 \in \mathcal{U}_q^j$, then $j = \aleph_0$ because $1 \in \mathcal{U}_{m+1}^{\aleph_0}$ by Lemma 3.3. Again by Lemma 3.3 we have $(m+1)^{-k} \in \mathcal{U}_{m+1}^{\aleph_0}$ for all $k \geq 1$, implying that $0 \in \overline{\mathcal{U}_{m+1}^{\aleph_0}} \setminus \mathcal{U}_{m+1}^{\aleph_0}$.

4. A review of topological entropy and Hausdorff dimension

In this section we recall from [17, 34, 31, 3] some definitions and results on Hausdorff dimension and on symbolic dynamics that will be used in the rest of the paper.

Given a set $F \subset \mathbb{R}$ and a nonnegative real number s , the *s-dimensional Hausdorff measure* is defined by the formula

$$\mathcal{H}^s(F) := \lim_{\delta \searrow 0} \left(\inf \left\{ \sum_{i=1}^{\infty} |I_i|^s \right\} \right),$$

where the infimum is taken for all countable covers of F by intervals of length less than δ . It may be shown [17] that there exists a critical value $\dim_H F \in [0, 1]$, called the *Hausdorff dimension* of F , such that

$$\mathcal{H}^s(F) = 0 \quad \text{if } s > \dim_H F, \quad \text{and} \quad \mathcal{H}^s(F) = \infty \quad \text{if } s < \dim_H F.$$

The following results are elementary and well-known:

Lemma 4.1. *The Hausdorff dimension has the following properties:*

- (i) $\dim_H F = 0$ for every finite set.
- (ii) $\dim_H(a + rF) = \dim_H F$ for every $a \in \mathbb{R}$ and $r > 0$.

- (iii) If $F \subseteq G$, then $\dim_H F \leq \dim_H G$.
- (iv) $\dim_H \cup_i F_i = \sup_i \dim_H F_i$ for every finite or countable family of sets F_i .
- (v) Two sets differing by a countable set have the same Hausdorff dimension.

A set $\mathcal{S}' \subseteq \{0, 1, \dots, M\}^\infty$ is called a *subshift* if there exists a set of words \mathcal{W} such that

$$\mathcal{S}' = \left\{ (c_i) \in \{0, 1, \dots, M\}^\mathbb{N} : (c_i) \text{ does not contain any word from } \mathcal{W} \right\}.$$

The elements of \mathcal{W} are called *forbidden words*. If \mathcal{W} can be chosen to be a finite set then \mathcal{S}' is called a *subshift of finite type*.

Given a subshift \mathcal{S}' and a positive integer n , we denote by $B_n(\mathcal{S}')$ the set of different prefixes of length n appearing in the sequences of \mathcal{S}' . Since prefixes of sequences and subwords of sequences of a subshift are in a one-to-one correspondence, $B_n(\mathcal{S}')$ is also the set of different subwords of length n appearing in the sequences of \mathcal{S}' . Furthermore, we denote by $B_*(\mathcal{S}')$ the set of finite subwords (of arbitrary length) of sequences in \mathcal{S}' together the empty word ϵ . A subshift \mathcal{S}' is called *topologically transitive* (or simply *transitive*) if for every $\omega, \nu \in B_*(\mathcal{S}')$ there exists a $\delta \in B_*(\mathcal{S}')$ such that $\omega\delta\nu \in B_*(\mathcal{S}')$.

We will need the following stronger property.

Lemma 4.2. *Let \mathcal{S}' be a transitive subshift of finite type. Then for every $\omega \in B_*(\mathcal{S}')$ and $(c_i) \in \mathcal{S}'$ there exists a $\delta \in B_*(\mathcal{S}')$ such that $\omega\delta(c_i) \in \mathcal{S}'$.*

Although this property is known for the specialists, for the reader's convenience we give a short proof:

PROOF. We recall from [34, Theorem 2.3.2] that \mathcal{S}' is generated by a suitable strongly connected labeled graph Γ . Here the strong connectedness of the labeled graph Γ [34, Definition 2.2.13] means that for any two vertices I, J of Γ there exists a path from I to J . Now consider a path P_1 generating ω and a path P_2 generating (c_i) . By the strong connectedness there exists a path P_3 from the last vertex of P_1 to the first vertex of P_2 . If we denote by δ the word generated by P_3 , then the combined path $P_1P_3P_2$ generates the sequence $\omega\delta(c_i)$.

Given a set $\mathcal{F}' \subseteq \{0, 1, \dots, M\}^\mathbb{N}$ and a base $q > 1$, we introduce the notation

$$\mathcal{F}_q := \pi_q(\mathcal{F}') = \{ \pi_q((c_i)) : (c_i) \in \mathcal{F}' \}.$$

We will be interested in the Hausdorff dimension of sets of the form \mathcal{F}_q .

Lemma 4.3. *Let $\mathcal{S}' \subseteq \{0, 1, \dots, M\}^\mathbb{N}$ be a transitive subshift of finite type, $w \in B_*(\mathcal{S}')$ a given word, and set*

$$\mathcal{F}' := \{ (c_i) \in \mathcal{S}' : (c_i) \text{ starts with } w \}.$$

Then

$$\dim_H \mathcal{F}_q = \dim_H \mathcal{S}'_q$$

for all $q > 1$.

PROOF. Setting

$$\mathcal{S}^{v'} := \{(c_i) \in \mathcal{S}' : wv(c_i) \in \mathcal{F}'\}$$

for each finite word v , we have

$$\pi_q(wv0^\infty) + \frac{1}{q^n} \mathcal{S}_q^v \subseteq \mathcal{F}_q$$

where n is the length of the word wv , and therefore $\dim_H \mathcal{S}_q^v \leq \dim_H \mathcal{F}_q$. Since \mathcal{S}' is a transitive subshift of finite type, we have

$$\mathcal{S}_q = \cup_{v \in B_*(\mathcal{S}')} \mathcal{S}_q^v$$

by Lemma 4.2. Since $B_*(\mathcal{S}')$ is a countable set, it follows that

$$\dim_H \mathcal{S}_q = \sup_{v \in B_*(\mathcal{S}')} \dim_H \mathcal{S}_q^v \leq \dim_H \mathcal{F}_q.$$

The inverse inequality $\dim_H \mathcal{F}_q \leq \dim_H \mathcal{S}_q$ is clear because $\mathcal{F}_q \subseteq \mathcal{S}_q$.

In many important cases, the Hausdorff dimension of \mathcal{S}_q may be expressed using the *topological entropy* of \mathcal{S}' , denoted by $h(\mathcal{S}')$. We recall from [34, Definition 4.1.1] the definition

$$h(\mathcal{S}') := \lim_{n \rightarrow \infty} \frac{\log |B_n(\mathcal{S}')|}{n},$$

where $|B_n(\mathcal{S}')|$ denotes the number of elements of $B_n(\mathcal{S}')$. By [34, Proposition 4.1.8] this limit always exists, and it is equal to

$$\inf_{n \geq 1} \frac{\log |B_n(\mathcal{S}')|}{n}.$$

Let us introduce the following sets:

$$\tilde{\mathcal{V}}'_q := \left\{ (c_i) : \overline{\alpha(q)} \preceq (c_{n+i}) \preceq \alpha(q) \text{ for all } n \geq 0 \right\}, \quad (4.1)$$

$$\tilde{\mathcal{V}}_q := \left\{ \pi_q((c_i)) : (c_i) \in \tilde{\mathcal{V}}'_q \right\}. \quad (4.2)$$

By Lemma 2.8 we have

$$\tilde{\mathcal{V}}'_q \subset \mathcal{V}'_q, \quad \tilde{\mathcal{V}}_q \subset \mathcal{V}_q \quad \text{and} \quad \tilde{\mathcal{V}}'_q = \left\{ a(x, q) : x \in \tilde{\mathcal{V}}_q \right\}.$$

Remark 4.4. *If a sequence (c_i) satisfies the seemingly weaker conditions*

$$(c_{n+i}) \preceq \alpha(q) \text{ for } n = 0, \quad \text{and whenever } c_n < M \quad (4.3)$$

and

$$\overline{(c_{n+i})} \preceq \alpha(q) \text{ for } n = 0, \quad \text{and whenever } c_n > 0, \quad (4.4)$$

then in fact $(c_i) \in \tilde{\mathcal{V}}'_q$. Indeed, since $(c_i) \preceq \alpha(q)$ by (4.3), for any index k with $c_k = M$, there exists a nonnegative integer $n < k$ such that

$$(c_{n+i}) \preceq \alpha(q), \quad \text{and} \quad c_j = M \quad \text{for all} \quad n < j \leq k. \quad (4.5)$$

Then the definition of the lexicographic inequality and (4.5) imply that

$$(c_{k+i}) \preceq (c_{n+i}) \preceq \alpha(q).$$

The first inequality is even strict unless $(c_i) = M^\infty$. We obtain similarly that $\frac{(c_{k+i})}{(c_{k+i})} \preceq \alpha(q)$ for all $k \geq 0$, so that $(c_i) \in \mathcal{V}'_q$ by (4.1).

Lemma 4.5. *Let $q \in (1, M + 1] \setminus \mathcal{U}$.*

- (i) \mathcal{V}'_q is a subshift of finite type.
- (ii) $\tilde{\mathcal{V}}'_q$ is a subshift of finite type.
- (iii) The topological entropies $h(\mathcal{V}'_q)$ and $h(\tilde{\mathcal{V}}'_q)$ exist, and are equal.
- (iv) If $\mathcal{S}' \subseteq \tilde{\mathcal{V}}'_q$ is a subshift of finite type, then

$$\dim_H(\mathcal{S}'_p) = \frac{h(\mathcal{S}')}{\log p}$$

for all $p \geq q$.

The assertions of Lemma 4.5 were proved in [11] and [31]; in [31] the case $\mathcal{S}' = \tilde{\mathcal{V}}'_q$ was only considered, but the proof remains valid for $\mathcal{S}' \subseteq \tilde{\mathcal{V}}'_q$.

Lemma 4.6. *The sets $\mathcal{U}_q, \overline{\mathcal{U}}_q, \mathcal{V}_q$ and $\tilde{\mathcal{V}}_q$ have the same Hausdorff dimension for every $q \in (1, M + 1)$.*

Lemma 4.6 follows from Lemma 2.7 (i) and the remark in [26, p. 171].

Now we consider the entropy function $q \mapsto h(\mathcal{V}'_q)$. We recall from [26] that the topological entropy of \mathcal{V}'_q is well defined for all $q \in (1, \infty)$, and that it is constant on each connected component of $(1, +\infty) \setminus \overline{\mathcal{U}}$. The maximal stability intervals of this function, i.e., the maximal intervals on which h is constant, were determined by Alcaraz Barrera et al. [3].

We will need two theorems from [3]. Theorem 4.7 is a simplified form of [3, Theorem 2 and Lemmas 4.8 and 4.9], while Theorem 4.8 follows from the proof of [3, Lemma 5.1]; see Remark 4.9 below.

Theorem 4.7. *Consider the entropy function $q \mapsto h(\mathcal{V}'_q)$ on $(1, \infty)$.*

- (i) *The maximal stability intervals of h are formed by $(1, q_{KL}]$, $[M + 1, \infty)$, and an infinite family of disjoint closed intervals $[p_L, p_R] \subset (q_{KL}, M + 1)$, where p_L runs over some proper subset of $\overline{\mathcal{U}} \setminus \mathcal{U}$, and p_R runs over some proper subset of \mathcal{U} . Furthermore,*

$$\text{if } \beta(p_L) = w^+ 0^\infty, \quad \text{then } \beta(p_R) = w^+ \overline{w}^\infty. \quad (4.6)$$

- (ii) Let $q_L \in \mathcal{V} \setminus \mathcal{U}$, and write $\beta(q_L) = w^+0^\infty$. Then there exists a base $q_R \in \mathcal{U}$ such that $\beta(q_R) = w^+\bar{w}^\infty$, and $[q_L, q_R]$ is contained by some maximal stability interval of h .

We refer to [3] for the precise technical characterization of the words w appearing in (4.6).

Theorem 4.8. *Let $[p_L, p_R] \subset (q_{KL}, M+1)$ be a maximal stability interval of the entropy function, and $q \in [p_L, p_R]$. Then $\tilde{\mathcal{V}}'_q$ contains a unique transitive subshift $\mathcal{S}'(q)$ of finite type such that $h(\mathcal{S}'(q)) = h(\tilde{\mathcal{V}}'_q)$. Furthermore, $\mathcal{S}'(q) \subseteq \tilde{\mathcal{V}}'_{p_L}$ and $\alpha(p_L) \in \mathcal{S}'(q)$.*

Remark 4.9. *For the reader's convenience we briefly explain how Theorem 4.8 follows from the results of [3]. First assume that if $q = p_L$ and $[p_L, p_R]$ is a *-irreducible interval. Then by [3, Lemma 5.9] there exists a unique transitive subshift of finite type $X'_{p_L} \subset \tilde{\mathcal{V}}'_{p_L}$ such that $h(X'_{p_L}) = h(\tilde{\mathcal{V}}'_{p_L})$. Furthermore, $\alpha(p_L) \in X'_{p_L}$.*

If $q \in (p_L, p_R)$, then the proof of [3, Proposition 5.10] shows that X'_{p_L} is still the unique transitive subshift of finite type in $\tilde{\mathcal{V}}'_{p_L}$ such that $h(X'_{p_L}) = h(\tilde{\mathcal{V}}'_{p_L})$.

Next assume that $[p_L, p_R]$ is an irreducible interval. Then $\tilde{\mathcal{V}}'_{p_L}$ is a transitive subshift of finite type by [3, Theorem 1], and $\alpha(p_L) \in \tilde{\mathcal{V}}'_{p_L}$. Since $\tilde{\mathcal{V}}'_{p_L} \subset \tilde{\mathcal{V}}'_q$ for every $q > p_L$, it remains to show that in case $q \in [p_L, p_R)$ every other transitive subshift of finite type $\mathcal{S}' \subset \tilde{\mathcal{V}}'_q$ has a strictly smaller topological entropy than $\tilde{\mathcal{V}}'_{p_L}$. The proof of [3, Lemma 5.1] shows that a transitive subshift of finite type $\mathcal{S}' \subset \mathcal{V}'_q$ cannot intersect both $\tilde{\mathcal{V}}'_{p_L}$ and $\tilde{\mathcal{V}}'_q \setminus \tilde{\mathcal{V}}'_{p_L}$. If $\mathcal{S}' \subsetneq \tilde{\mathcal{V}}'_{p_L}$, then $h(\mathcal{S}') < h(\tilde{\mathcal{V}}'_{p_L})$ by [34, Corollary 4.4.9]. If $\mathcal{S}' \subset \tilde{\mathcal{V}}'_q \setminus \tilde{\mathcal{V}}'_{p_L}$, then

$$h(\mathcal{S}') \leq h(\tilde{\mathcal{V}}'_q \setminus \tilde{\mathcal{V}}'_{p_L}) < h(\tilde{\mathcal{V}}'_{p_L}),$$

where the last inequality also follows from the proof of [3, Lemma 5.1]; see also [1, Corollary 3.10] for an explicit statement.

5. Preparation for the proof of Theorem 1.2

In this section we consider only bases q belonging to $\mathcal{V} \setminus \mathcal{U}$. Since $q \in \mathcal{V}$, by Proposition 2.5, the quasi-greedy expansion $\alpha(q) = \alpha_1\alpha_2\cdots$ of 1 in base q satisfies the lexicographic inequalities

$$\alpha_{k+1}\alpha_{k+2}\cdots \preceq \alpha(q) \quad \text{and} \quad \overline{\alpha_{k+1}\alpha_{k+2}\cdots} \preceq \alpha(q) \quad (5.1)$$

for all $k \geq 0$. Moreover, since $q \in \mathcal{V} \setminus \mathcal{U}$, the sequence $\alpha(q)$ is periodic, the last digit of its period block $\alpha_1 \cdots \alpha_N$ is smaller than M , and the greedy expansion of 1 in base q is given by the formula $\beta(q) = \alpha_1 \cdots \alpha_{N-1} \alpha_N^+ 0^\infty$.

Let \mathcal{F}' be a family of sequences (not necessarily a subshift), and set

$$\mathcal{F}_q := \{\pi_q((c_i)) : (c_i) \in \mathcal{F}'\}.$$

The following lemma is crucial for the proof of Theorem 1.2. Its statement uses the assumption that $\beta(q)$ is a finite expansion, and at the end of the proof we will use the relation $q \in \mathcal{V}$. These two conditions are equivalent to our assumption $q \in \mathcal{V} \setminus \mathcal{U}$.

Lemma 5.1. *Let $q \in \mathcal{V} \setminus \mathcal{U}$, and write $\beta(q) = \alpha_1 \cdots \alpha_{N-1} \alpha_N^+ 0^\infty$. Assume that each $(c_i) \in \mathcal{F}'$ satisfies the following lexicographic conditions:*

$$(c_{n+i}) \preceq \alpha(q) \quad \text{for } n = 0, \quad \text{and whenever } c_n < M; \quad (5.2)$$

$$\overline{(c_{n+i})} \preceq \alpha(q) \quad \text{for } n = 0, \quad \text{and whenever } c_n > 0; \quad (5.3)$$

$$\alpha_{k+1} \cdots \alpha_N^+ c_1 c_2 \cdots \preceq \alpha(q) \quad \text{whenever } 1 \leq k < N \quad \text{and } \alpha_k < M. \quad (5.4)$$

Then

$$\dim_H \mathcal{U}_q^j \geq \dim_H \mathcal{F}_q \quad \text{for all } j < \aleph_0. \quad (5.5)$$

PROOF. It suffices to prove the inequalities (5.5) under the following stronger conditions:

$$(c_{n+i}) \prec \alpha(q) \quad \text{for } n = 0, \quad \text{and whenever } c_n < M; \quad (5.6)$$

$$\overline{(c_{n+i})} \prec \alpha(q) \quad \text{for } n = 0, \quad \text{and whenever } c_n > 0; \quad (5.7)$$

$$\alpha_{k+1} \cdots \alpha_N^+ c_1 c_2 \cdots \prec \alpha(q) \quad \text{whenever } 1 \leq k < N \quad \text{and } \alpha_k < M. \quad (5.8)$$

Indeed, by weakening the conditions (5.6)–(5.8) to (5.2)–(5.4) the family \mathcal{F}' may be increased only by countably many new sequences, so that $\dim_H \mathcal{F}_q$ remains the same.

It suffices to show for each $(c_i) \in \mathcal{F}'$ the relations

$$x_j := \pi_q(10^{(j-1)N} c_1 c_2 \cdots) \in \mathcal{U}_q^j, \quad j = 1, 2, \dots$$

Indeed, then

$$\frac{1}{q} + \frac{1}{q^{1+(j-1)N}} \mathcal{F}_q \subseteq \mathcal{U}_q^j,$$

i.e., \mathcal{U}_q^j contains a set similar to \mathcal{F}_q .

The case $j = 1$ follows by observing that $1c_1c_2 \cdots$ is a unique expansion by (5.6) and (5.7). Henceforth we assume that $j \geq 2$, and we write $w := \alpha_1 \cdots \alpha_N$ for brevity.

Since $0^{(j-1)N} c_1 c_2 \cdots$ is greedy by (5.6) and $0^{(j-1)N} c_1 c_2 \cdots \prec w^+ 0^\infty$, by Proposition 2.1 (i) we have

$$\pi_q(0^{(j-1)N} c_1 c_2 \cdots) < \pi_q(w^+ 0^\infty) = 1.$$

This implies that $x_j = \pi_q(10^{(j-1)N} c_1 c_2 \cdots)$ has no expansion starting with a digit greater than 1. Furthermore, since $0^{(j-1)N} c_1 c_2 \cdots$ is a unique expansion by (5.6) and (5.7), $10^{(j-1)N} c_1 c_2 \cdots$ is the only expansion of x_j that starts with 1.

Next we show that x_j has exactly $j-1$ expansions starting with 0. Applying repeatedly the equality $\pi_q(w^+0^\infty) = 1$, we may construct from $10^{(j-1)N}c_1c_2\cdots$ $(j-1)$ further expansions of x_j , starting with 0:

$$x_j = \pi_q(0w^k w^+ 0^{(j-2-k)N} c_1 c_2 \cdots), \quad k = 0, 1, \dots, j-2. \quad (5.9)$$

It remains to show that (5.9) contains every expansion of x_j that starts with 0.

First we show that $(d_i) := 0w^{j-2}w^+c_1c_2\cdots$ is the lazy expansion of x_j . By Proposition 2.1 (i) it suffices to show that

$$\overline{(d_{n+i})} \prec \alpha(q) \quad \text{whenever} \quad \overline{d_n} < M.$$

For $n \geq (j-1)N+1$ this condition is satisfied by (5.7). For $1 \leq n < (j-1)N+1$ the condition follows from (5.1) and the equality $\alpha(q) = w^\infty$, because $w \prec w^+$, and therefore

$$\overline{(d_{n+i})} \prec \overline{(\alpha_{n+i})} \preceq \alpha(q).$$

Now let (e_i) be an arbitrary expansion of x_j that starts with 0. Then

$$e_1 = 0 \quad \text{and} \quad e_1 e_2 \cdots \succeq 0w^{j-2}w^+c_1c_2\cdots,$$

whence there exists a $k \in \{0, \dots, j-2\}$ and a word $v \succeq w^+$ of length N , such that (e_i) starts with $0w^k v$. We have

$$\pi_q(0w^k v e_{(k+1)N+2} e_{(k+1)N+3} \cdots) = x_j = \pi_q(0w^k w^+ 0^{(j-2-k)N} c_1 c_2 \cdots),$$

and hence

$$\pi_q(v e_{(k+1)N+2} e_{(k+1)N+3} \cdots) = \pi_q(w^+ 0^{(j-2-k)N} c_1 c_2 \cdots).$$

Since $v \succeq w^+$, and the sequence $w^+ 0^{(j-2-k)N} c_1 c_2 \cdots$ is greedy by (5.6) and (5.8), we conclude that $v = w^+$, and therefore

$$\pi_q(e_{(k+1)N+2} e_{(k+1)N+3} \cdots) = \pi_q(0^{(j-2-k)N} c_1 c_2 \cdots).$$

Since $0^{(j-2-k)N} c_1 c_2 \cdots$ is a unique expansion by (5.6) and (5.7), we conclude that

$$e_{(k+1)N+2} e_{(k+1)N+3} \cdots = 0^{(j-2-k)N} c_1 c_2 \cdots,$$

and therefore

$$(e_i) = 0w^k w^+ 0^{(j-2-k)N} c_1 c_2 \cdots,$$

i.e., (e_i) is one of the expansion listed in (5.9). This completes the proof of the lemma.

As we explain in the following statement, in some cases the condition (5.4) of Lemma 5.1 may be replaced by a simpler one:

Lemma 5.2. *Let $q \in \mathcal{V} \setminus \mathcal{U}$ with $q > q_{GR}$ and $\beta(q) = \alpha_1 \cdots \alpha_{N-1} \alpha_N^+ 0^\infty$. If a sequence $(c_i) \in \{0, \dots, M\}^{\mathbb{N}}$ satisfies the inequalities*

$$c_1 \cdots c_k \preceq \overline{\alpha_1 \cdots \alpha_k} \quad (5.10)$$

and

$$c_{k+1} c_{k+2} \cdots \preceq \alpha(q) \quad (5.11)$$

whenever $1 \leq k < N$ and $\alpha_k < M$, then (c_i) also satisfies the inequalities (5.4) of Lemma 5.1:

$$\alpha_{k+1} \cdots \alpha_N^+ c_1 c_2 \cdots \preceq \alpha(q) \quad \text{whenever } 1 \leq k < N \quad \text{and } \alpha_k < M.$$

PROOF. Since $\alpha(q) = (\alpha_1 \cdots \alpha_N)^\infty$, it suffices to show that if $1 \leq k < N$ and $\alpha_k < M$, then

$$\alpha_{k+1} \cdots \alpha_N^+ c_1 \cdots c_k \preceq \alpha_1 \cdots \alpha_N.$$

Indeed, using (5.11) we will then obtain that

$$\alpha_{k+1} \cdots \alpha_N^+ c_1 c_2 \cdots \preceq (\alpha_1 \cdots \alpha_N) \alpha(q) = \alpha(q).$$

In view of (5.10) it suffices to prove that

$$\alpha_{k+1} \cdots \alpha_N^+ \overline{\alpha_1 \cdots \alpha_k} \preceq \alpha_1 \cdots \alpha_N,$$

or equivalently that

$$\alpha_{k+1} \cdots \alpha_N^+ \overline{\alpha_1 \cdots \alpha_k} \prec \alpha_1 \cdots \alpha_N^+. \quad (5.12)$$

Applying Proposition 2.4 (i) to $\beta(q) = \alpha_1 \cdots \alpha_N^+ 0^\infty$ we see that

$$\alpha_{k+1} \cdots \alpha_N^+ \preceq \alpha_1 \cdots \alpha_{N-k}. \quad (5.13)$$

If the inequality (5.13) is strict, then (5.12) follows. Otherwise, we have to show that $\overline{\alpha_1} < \alpha_N^+$ if $k = 1$, and

$$\overline{\alpha_1 \cdots \alpha_k} \prec \alpha_{N-k+1} \cdots \alpha_N^+ \quad \text{if } 1 < k < N.$$

or equivalently, by taking reflections, that $\overline{\alpha_N^+} < \alpha_1$ if $k = 1$, and

$$\overline{\alpha_{N-k+1} \cdots \alpha_N^+} \prec \alpha_1 \cdots \alpha_k \quad \text{if } 1 < k < N.$$

This follows by applying Proposition 2.6 (iv) for $q^+ := \min\{p \in \mathcal{V} : p > q\}$ because

$$\alpha(q^+) = \left(\alpha_1 \cdots \alpha_N^+ \overline{\alpha_1 \cdots \alpha_N^+} \right)^\infty$$

by Proposition 2.6 (ii).

In the rest of this section we adopt a notation from [3], but we do not assume any irreducibility or *-irreducibility condition on the interval $[q_L, q_R]$ defined below. We refer to [3] for the definitions of irreducible and *-irreducible sequences. The maximal stability intervals $[p_L, p_R]$ of Theorems 4.7 and 4.8 are special cases of these intervals $[q_L, q_R]$. As we will see, some results of [3] remain valid without such assumptions.

For the following two lemmas we fix an *arbitrary* base $q_L \in \mathcal{V} \setminus \mathcal{U}$ and we write

$$\beta(q_L) = w^+ 0^\infty \quad \text{with} \quad w = a_1 \cdots a_m,$$

so that $\alpha(q_L) = w^\infty$. Furthermore, we define the bases r_0, r_1, \dots and q_R by the implicit equations

$$\pi_{r_j} (w^+ (\bar{w})^j 0^\infty) = 1 \quad \text{for} \quad j = 0, 1, \dots,$$

and

$$\pi_{q_R} (w^+ (\bar{w})^\infty) = 1.$$

Part (i) of the following lemma was proved in [1, 3] when $q_L = p_L$ is the left endpoint of a maximal stability interval.

Lemma 5.3. *We have*

- (i) $q_R \in \mathcal{U}$ and $\beta(q_R) = w^+ (\bar{w})^\infty$.
- (ii) $r_1 \in \mathcal{V} \setminus \bar{\mathcal{U}}$, $r_j \in \bar{\mathcal{U}} \setminus \mathcal{U}$ for all $j \geq 2$, and $\beta(r_j) = w^+ (\bar{w})^j 0^\infty$ for all $j \geq 1$.
- (iii) $q_L = r_0 < r_1 < \cdots < q_R$.

PROOF. First we establish the following equalities:

$$\overline{a_1 \cdots a_{m-i}} \preceq a_{i+1} \cdots a_m \prec a_{i+1} \cdots a_m^+ \preceq a_1 \cdots a_{m-i} \quad \text{for all} \quad 0 < i < m. \quad (5.14)$$

Since $\beta(q_L) = a_1 \cdots a_m^+ 0^\infty$, applying Proposition 2.4 (i) we have

$$a_{i+1} \cdots a_m \prec a_{i+1} \cdots a_m^+ \preceq a_1 \cdots a_{m-i} \quad \text{for all} \quad 0 < i < m,$$

proving the second and the third inequalities in (5.14). Since $q_L \in \mathcal{V}$ and $\alpha(q_L) = (a_1 \cdots a_m)^\infty$, the first inequality in (5.14) is obtained by applying Proposition 2.5.

(i) Setting $(\beta_i) := w^+ (\bar{w})^\infty$ for brevity, in view of Propositions 2.4 and 2.5 it suffices to prove that

$$(\beta_{n+i}) \prec (\beta_i) \quad \text{and} \quad \overline{(\beta_{n+i})} \prec (\beta_i)$$

for all $n > 0$. By periodicity, it suffices to consider $0 < n < 2m$. Using (5.14) and the reflected inequalities, we will prove the following stronger relations:

$$\begin{aligned} \beta_{n+1} \cdots \beta_{n+m} \prec \beta_1 \cdots \beta_m & \quad \text{if} \quad 0 < n < 2m, \\ \overline{\beta_{n+1} \cdots \beta_m} \prec \beta_1 \cdots \beta_{m-n} & \quad \text{if} \quad 0 < n < m, \\ \overline{\beta_{n+1} \cdots \beta_{2m}} \prec \beta_1 \cdots \beta_{2m-n} & \quad \text{if} \quad m \leq n < 2m. \end{aligned}$$

If $0 < n < m$, then

$$\begin{aligned}\beta_{n+1} \cdots \beta_{n+m} &= a_{n+1} \cdots a_m^+ \overline{a_1 \cdots a_n} \preceq a_1 \cdots a_{m-n} \overline{a_1 \cdots a_n} \\ &\prec a_1 \cdots a_{m-n} a_{m-n+1} \cdots a_m^+ = \beta_1 \cdots \beta_m\end{aligned}$$

and

$$\overline{\beta_{n+1} \cdots \beta_m} = \overline{a_{n+1} \cdots a_m^+} \prec a_1 \cdots a_{m-n} = \beta_1 \cdots \beta_{m-n}.$$

If $n = m$, then

$$\beta_{n+1} \cdots \beta_{n+m} = \overline{a_1 \cdots a_m} \prec a_1 \cdots a_m \prec \beta_1 \cdots \beta_m$$

and

$$\overline{\beta_{n+1} \cdots \beta_{2m}} = a_1 \cdots a_m \prec a_1 \cdots a_m^+ = \beta_1 \cdots \beta_m.$$

Finally, if $m < n = m + \ell < 2m$, then

$$\begin{aligned}\beta_{n+1} \cdots \beta_{n+m} &= \overline{a_{\ell+1} \cdots a_m a_1 \cdots a_\ell} \preceq a_1 \cdots a_{m-\ell} \overline{a_1 \cdots a_\ell} \\ &\prec a_1 \cdots a_{m-\ell} a_{m-\ell+1} \cdots a_m^+ = \beta_1 \cdots \beta_m\end{aligned}$$

and

$$\overline{\beta_{n+1} \cdots \beta_{2m}} = a_{\ell+1} \cdots a_m \prec a_1 \cdots a_{m-\ell} = \beta_1 \cdots \beta_{m-\ell}.$$

(ii) First we note that $r_j \notin \mathcal{U}$ for all $j \geq 1$ because the unique expansion of 1 is infinite in every base, and r_j is defined by a finite expansion of 1. Now fix $j \geq 1$, and set

$$(c_i) := \left(w^+ (\overline{w})^{j-1} w^+ \right)^\infty = (a_1 \cdots a_m^+ (\overline{a_1 \cdots a_m})^{j-1} \overline{a_1 \cdots a_m^+})^\infty.$$

It follows from the definition of r_j that (c_i) is an infinite expansion of 1 in base r_j . We will prove the following inequalities:

$$\overline{(c_i)} \prec c_{n+1} c_{n+2} \cdots \preceq (c_i) \quad \text{for all } n \geq 0 \quad \text{if } j \geq 2, \quad (5.15)$$

$$\overline{(c_i)} \preceq c_{n+1} c_{n+2} \cdots \preceq (c_i) \quad \text{for all } n \geq 0 \quad \text{if } j = 1. \quad (5.16)$$

The second inequalities in (5.15) and (5.16) will imply that $(c_i) = \alpha(r_j)$ for all $j \geq 1$, and hence $\beta(r_j) = w^+ (\overline{w})^{j-1} w^+$ for all $j \geq 1$ by the general relation between $\alpha(q)$ and $\beta(q)$ for any base $q \in (1, M+1]$. Since $\alpha(r_j) = (c_i)$, it will follow from (5.16) that $r_1 \in \mathcal{V}$, and it will follow from (5.15) that $r_1 \in \overline{\mathcal{U}}$ for all $j \geq 2$. Finally, $r_1 \notin \overline{\mathcal{U}}$ because $\alpha(r_1) = (w^+ \overline{w^+})^\infty$, so that $\overline{(c_{m+i})} = (c_i)$ instead of $\overline{(c_{m+i})} \prec (c_i)$.

It remains to establish (5.15) and (5.16). By the periodicity of (c_i) it suffices to prove them for $0 \leq n < (j+1)m$. First we prove the right inequalities in (5.15) and (5.16). More explicitly, we prove for all $j \geq 1$ and $0 \leq n < (j+1)m$ the inequalities

$$c_{n+1} c_{n+2} \cdots \preceq \left(a_1 \cdots a_m^+ (\overline{a_1 \cdots a_m})^{j-1} \overline{a_1 \cdots a_m^+} \right)^\infty.$$

We distinguish five cases. If $n = 0$, then we have an equality. The case $0 < n < m$ follows from the relations

$$a_{n+1} \cdots a_m^+ \preceq a_1 \cdots a_{m-n} \quad \text{and} \quad \overline{a_1 \cdots a_n} \prec a_{m-n+1} \cdots a_m^+$$

in (5.14). The case $n = km$ for $k \in \{1, \dots, j\}$ follows from the inequality $\overline{a_1} < a_1$: see Proposition 2.6 (iii). The case $km < n = km + \ell < (k+1)m$ for $k \in \{1, \dots, j-1\}$ follows from the relations

$$\overline{a_{\ell+1} \cdots a_m} \preceq a_1 \cdots a_{m-\ell} \quad \text{and} \quad \overline{\alpha_1 \cdots \alpha_\ell} \prec \alpha_{m-\ell+1} \cdots \alpha_m^+,$$

both coming from (5.14) by reflection. Finally, the case $jm < n = jm + \ell < (j+1)m$ follows from the relation

$$\overline{a_{\ell+1} \cdots a_m^+} \prec a_1 \cdots a_{m-\ell}$$

in (5.14).

Now we turn to the proof of the left inequalities in (5.15) and (5.16). More explicitly, we establish the inequalities

$$\left(\overline{a_1 \cdots a_m^+} (a_1 \cdots a_m)^{j-1} a_1 \cdots a_m^+ \right)^\infty \prec c_{n+1} c_{n+2} \cdots \quad (5.17)$$

for all $j \geq 1$ and $0 \leq n < (j+1)m$, *except if $j = 1$ and $n = m$* . We distinguish again five cases. The case $n = 0$ follows from the inequality $\overline{a_1} < a_1$. The case $0 < n < m$ follows from the relation $\overline{a_1 \cdots a_{m-n}} \prec a_{n+1} \cdots a_m^+$ in (5.14) by reflection. The case $n = km$ for $k \in \{1, \dots, j-1\}$ is clear because $a_1 \cdots a_m^+ \prec \overline{a_1 \cdots a_m}$. The case $km < n = km + \ell < (k+1)m$ for $k \in \{1, \dots, j-1\}$ follows from the relation $a_{\ell+1} \cdots a_m \prec a_1 \cdots a_{m-\ell}$ in (5.14). The case $jm < n = jm + \ell < (j+1)m$ follows from the relations

$$\overline{a_1 \cdots a_{m-\ell}} \preceq \overline{a_{\ell+1} \cdots a_m^+} \quad \text{and} \quad \overline{a_{m-\ell+1} \cdots a_m^+} \prec a_1 \cdots a_\ell,$$

coming from (5.14) by reflection.

If $j \geq 2$, then (5.17) holds for $n = jm$, too, because

$$\overline{a_1 \cdots a_m^+ a_1 \cdots a_m} \prec \overline{a_1 \cdots a_m^+ a_1 \cdots a_m^+}.$$

This completes the proof of (5.15).

On the other hand, if $j = 1$ and $n = jm = m$, then we have $\overline{(c_i)} = c_{n+1} c_{n+2} \cdots$. This completes the proof of (5.16).

(iii) Since $\beta(q_L) = \beta(r_0) \prec \beta(r_1) \prec \cdots \prec \beta(q_R)$, the relations $q_L = r_0 < r_1 < \cdots < q_R$ follow by applying Proposition 2.4 (i).

Lemma 5.4. *Assume that $q_L \in \mathcal{V} \setminus \mathcal{U}$ with $q_L > q_{GR}$. Let us denote by \mathcal{F}' the set of sequences (c_i) starting with \overline{w} and satisfying the following conditions:*

$$c_{n+1} c_{n+2} \cdots \preceq \alpha(q_L) \quad \text{for } n = 0, \quad \text{and whenever } c_n < M; \quad (5.18)$$

$$\overline{c_{n+1} c_{n+2} \cdots} \preceq \alpha(q_L) \quad \text{for } n = 0, \quad \text{and whenever } c_n > 0. \quad (5.19)$$

Fix $j \geq 0$ arbitrarily, and write $\alpha_1 \cdots \alpha_N^+ = w^+(\bar{w})^j$, so that $\beta(r_j) = \alpha_1 \cdots \alpha_N^+ 0^\infty$. Then we have

$$\alpha_{k+1} \cdots \alpha_N^+ c_1 c_2 \cdots \preceq \alpha(q_L) \quad \text{whenever } 1 \leq k < N \quad \text{and} \quad \alpha_k < M. \quad (5.20)$$

PROOF. Since $\beta(q_L) = w^+ 0^\infty$, we have

$$\alpha(q_L) = w^\infty, \quad \alpha(q_R) = w^+ \bar{w}^\infty, \quad \text{and} \quad \beta(r_j) = \alpha_1 \cdots \alpha_N^+ 0^\infty. \quad (5.21)$$

The assumptions (5.18) and (5.19) imply by Remark 4.4 the stronger relations

$$\overline{\alpha(q_L)} \preceq c_{n+1} c_{n+2} \cdots \preceq \alpha(q_L) \quad \text{for all } n \geq 0. \quad (5.22)$$

Now we prove that

$$(\bar{w})^\ell c_{n+1} c_{n+2} \cdots \preceq c_{n+1} c_{n+2} \cdots \quad (5.23)$$

for all integers $\ell, n \geq 0$. If $\ell = 0$ or if $c_{n+1} c_{n+2} \cdots = (\bar{w})^\infty$, then we have equality. Otherwise, we have $\ell \geq 1$, and $c_{n+1} c_{n+2} \cdots \succ \alpha(q_L) = (\bar{w})^\infty$ by (5.21) and (5.22). Therefore there exist a nonnegative integer k and a word $u \succ \bar{w}$ of length m such that $c_{n+1} c_{n+2} \cdots$ starts with $(\bar{w})^k u$. Then $(\bar{w})^\ell c_{n+1} c_{n+2} \cdots$ starts with $(\bar{w})^{k+\ell} u$, and this implies (5.23).

Now we are ready to prove (5.20). We recall the notation $w = a_1 \cdots a_m$. If $j = 0$, then $r_0 = q_L$, and (5.20) follows by applying Lemma 5.2 with $q = q_L$, $N = m$ and $\alpha_1 \cdots \alpha_N = a_1 \cdots a_m$. The assumption (5.10) of Lemma 5.2 is satisfied because $c_1 \cdots c_{N-1} = \overline{a_1 \cdots a_{N-1}}$, and the assumption (5.11) is satisfied by (5.22).

If $j \geq 1$, then $\alpha_1 \cdots \alpha_N^+ = a_1 \cdots a_m^+ (\overline{a_1 \cdots a_m})^j$ and $N = (j+1)m$. Let $1 \leq k < N$ be such that $\alpha_k < M$.

If $1 \leq k \leq m$, then using (5.23) we get

$$\alpha_{k+1} \cdots \alpha_N^+ c_1 c_2 \cdots = a_{k+1} \cdots a_m^+ (\bar{w})^j c_1 c_2 \cdots \preceq a_{k+1} \cdots a_m^+ c_1 c_2 \cdots \preceq \alpha(q_L).$$

The last inequality follows from the already settled case $j = 0$.

If $im + 1 \leq k \leq (i+1)m$ for some $i = 1, \dots, j$, then writing $\ell := k - im$ and using the equality $c_1 \cdots c_m = \bar{w}$ and the relation (5.23), we get

$$\begin{aligned} \alpha_{k+1} \cdots \alpha_N^+ c_1 c_2 \cdots &= \overline{a_{\ell+1} \cdots a_m} (\bar{w})^{j-i} c_1 c_2 \cdots \\ &= \overline{a_{\ell+1} \cdots a_m} (\bar{w})^{j-i+1} c_{m+1} c_{m+2} \cdots \\ &\preceq \overline{a_{\ell+1} \cdots a_m} c_{m+1} c_{m+2} \cdots \\ &= c_{\ell+1} c_{\ell+2} \cdots \preceq \alpha(q_L); \end{aligned}$$

the last inequality holds by (5.22).

The following lemma is an application of Sidorov's bifurcation argument [37]; see also [6].

Lemma 5.5. *Let $q > 1$, and $x \in \mathcal{U}_q^j$ for some $j \leq \aleph_0$. Then there exists an expansion (c_i) of x and a positive integer k such that (c_{k+i}) is a unique expansion.*

PROOF. If $j < \aleph_0$, then there exists a positive integer k such that all expansions of x have different prefixes of length k , and hence (c_{k+i}) is a unique expansion for each expansion (c_i) of x .

Henceforth let $j = \aleph_0$, and assume on the contrary that for every expansion (c_i) of x and for every positive integer n , there exists another expansion (d_i) such that $c_1 \cdots c_n = d_1 \cdots d_n$. We will construct for each sequence $\gamma = (\gamma_i) \in \{0, 1\}^{\mathbb{N}}$ a unique expansion (c_i^γ) of x such that if $\gamma, \delta \in \{0, 1\}^{\mathbb{N}}$ are different sequences, then $(c_i^\gamma) \neq (c_i^\delta)$. This will contradict our assumption $x \in \mathcal{U}_q^{\aleph_0}$. First we construct an expansion (c_i^w) of x for every non-empty word $w \in \{0, 1\}^*$, and an increasing sequence (i_n) of positive integers, satisfying for every $n \geq 1$ the following conditions:

$$\text{if } w, z \in \{0, 1\}^n \text{ and } w < z, \text{ then } c_1^w \cdots c_{i_n}^w < c_1^z \cdots c_{i_n}^z; \quad (5.24)$$

$$\text{if } w \in \{0, 1\}^n, \text{ then } c_1^w \cdots c_{i_n}^w \text{ is a prefix of } (c_i^{w^0}) \text{ and } (c_i^{w^1}). \quad (5.25)$$

We proceed by induction on n . For $n = 1$ we choose two different expansions (c_i^0) and (c_i^1) of x such that $(c_i^0) < (c_i^1)$, and then we choose a positive integer i_1 such that $c_1^0 \cdots c_{i_1}^0 < c_1^1 \cdots c_{i_1}^1$. Then (5.24) is satisfied for $n = 1$. Now let $n \geq 1$, and assume that (5.24) is satisfied. For each $w \in \{0, 1\}^n$, by our assumption there exists an expansion (d_i) of x such that $(c_i^w) \neq (d_i)$ and $c_1^w \cdots c_{i_n}^w$ is a prefix of (d_i) . Define

$$(c_i^{w^0}) := \min \{(c_i^w), (d_i)\} \quad \text{and} \quad (c_i^{w^1}) := \max \{(c_i^w), (d_i)\},$$

then (5.25) is satisfied. Furthermore, since there are only finitely many words $w \in \{0, 1\}^n$, we may choose a positive integer $i_{n+1} > i_n$ such that

$$c_1^{w^0} \cdots c_{i_{n+1}}^{w^0} < c_1^{w^1} \cdots c_{i_{n+1}}^{w^1}$$

for every $w \in \{0, 1\}^n$. Combining this with (5.24), the condition (5.24) for $n+1$ follows:

$$\text{if } w, z \in \{0, 1\}^{n+1} \text{ and } w < z, \text{ then } c_1^w \cdots c_{i_{n+1}}^w < c_1^z \cdots c_{i_{n+1}}^z.$$

Using the expansions (c_i^w) may obtain a continuum of expansions of x as follows. For each sequence $\gamma = (\gamma_i) \in \{0, 1\}^{\mathbb{N}}$, by (5.25) there exists a unique sequence (c_i^γ) such that $c_1^{\gamma^1 \cdots \gamma^n} \cdots c_{i_n}^{\gamma^1 \cdots \gamma^n}$ is a prefix of (c_i^γ) for every n . They are also expansions of x by continuity, because $(c_i^{\gamma^1 \cdots \gamma^n}) \rightarrow (c_i^\gamma)$ component-wise. Finally, if $\gamma, \delta \in \{0, 1\}^{\mathbb{N}}$ are different sequences, then $(c_i^\gamma) \neq (c_i^\delta)$ because by (5.24) there exists an integer $n \geq 1$ such that $c_1^\gamma \cdots c_{i_n}^\gamma \neq c_1^\delta \cdots c_{i_n}^\delta$.

We end this section by proving the inequalities (1.2) of the introduction:

Proposition 5.6. *The inequalities*

$$\dim_H \mathcal{U}_q^j \leq \dim_H \mathcal{U}_q$$

hold for every base $q > 1$ and for every $j \leq \aleph_0$.

PROOF. Fix $j \leq \aleph_0$ arbitrarily. If $x \in \mathcal{U}_q^j$, then by Lemma 5.5 there exists an expansion (c_i) of x and a positive integer n , such that (c_{n+i}) is a unique expansion. Therefore

$$x \in \sum_{i=1}^n \frac{c_i}{q^i} + \frac{1}{q^n} \mathcal{U}_q,$$

and then

$$\mathcal{U}_q^j \subseteq \bigcup_{n=1}^{\infty} \bigcup_{w \in \{0, \dots, M\}^n} \left(\pi_q(w 0^\infty) + \frac{1}{q^n} \mathcal{U}_q \right).$$

Since \mathcal{U}_q^j is covered by countably many sets, each of which is similar to \mathcal{U}_q , using Lemma 4.1 (iv) we conclude that $\dim_H \mathcal{U}_q^j \leq \dim_H \mathcal{U}_q$.

6. Proof of Theorem 1.2

As in Lemmas 5.3 and 5.4 of Section 5, we fix an arbitrary base $q_L \in \mathcal{V} \setminus \mathcal{U}$, and we write

$$\beta(q_L) = w^+ 0^\infty \quad \text{and} \quad \alpha(q_L) = w^\infty \quad \text{with} \quad w = a_1 \cdots a_m.$$

First we prove the following result:

Proposition 6.1. *Let $q_L \in \overline{\mathcal{U}} \setminus \mathcal{U}$, and write $\beta(q_L) = w^+ 0^\infty$. Assume that \mathcal{V}'_{q_L} has a transitive subshift \mathcal{S}' of finite type, containing the word \overline{w} , and having the same topological entropy as \mathcal{V}'_{q_L} . Then the equal dimension property*

$$\dim_H \mathcal{U}_{r_k}^j = \dim_H \mathcal{U}_{r_k}^1 \quad \text{for all finite } j$$

holds for all bases r_k , defined by the implicit equations

$$\pi_{r_k}(w^+(\overline{w})^k 0^\infty) = 1, \quad k = 0, 1, \dots \quad (6.1)$$

PROOF. We write $w = a_1 \cdots a_m$ as usual, so that m denotes the length of the word w . First we show that $\overline{w} \preceq v \preceq w$ for every word $v \in B_m(\mathcal{S}')$. Since \mathcal{S}' is transitive and $\overline{w}, v \in B_m(\mathcal{S}')$, there exist a sequence $(c_i) \in \mathcal{S}'$ and two integers $k \geq 0$ and n such that

$$c_{k+1} \cdots c_{k+m} = \overline{w}, \quad k+m \leq n, \quad \text{and} \quad c_{n+1} \cdots c_{n+m} = v.$$

Since w is different from 0^m and M^m , $c_1 \cdots c_n$ is different from 0^n and M^n , and therefore

$$(\overline{w})^\infty \preceq (c_{n+i}) \preceq (w)^\infty$$

by the last part of Lemma 2.8. In particular, $\overline{w} \preceq c_{n+1} \cdots c_{n+m} \preceq w$, i.e., $\overline{w} \preceq v \preceq w$.

Since $\overline{w} \preceq v \preceq w$ for every word $v \in B_m(\mathcal{S}')$, we have

$$(\overline{w})^\infty \preceq (c_{k+i}) \preceq (w)^\infty$$

for every sequence $(c_i) \in \mathcal{S}'$ and for all $k \geq 0$, so that $\mathcal{S}' \subseteq \mathcal{V}'_{q_L}$ by the definition of \mathcal{V}'_{q_L} .

Now let \mathcal{F}' be the set of sequences $(c_i) \in \mathcal{S}' \subseteq \tilde{\mathcal{V}}'_{q_L}$ starting with \bar{w} , and fix an arbitrary integer $k \geq 0$. By Lemma 5.3 (ii), we have $\beta(r_k) = w^+(\bar{w})^k 0^\infty$. Writing $\beta(r_k) = \alpha_1 \cdots \alpha_N^+ 0^\infty$, and applying Lemma 5.4, we obtain that

$$\alpha_{\ell+1} \cdots \alpha_N^+ c_1 c_2 \cdots \preceq \alpha(q_L) \quad \text{whenever} \quad 1 \leq \ell < N \quad \text{and} \quad \alpha_\ell < M. \quad (6.2)$$

Since $\mathcal{F}' \subseteq \mathcal{S}' \subseteq \tilde{\mathcal{V}}'_{q_L}$, the conditions (5.18) and (5.19) of Lemma 5.4 are satisfied. Since $\alpha(q_L) \preceq \alpha(r_k)$ and therefore $\mathcal{F}' \subseteq \mathcal{S}' \subseteq \tilde{\mathcal{V}}'_{q_L} \subseteq \tilde{\mathcal{V}}'_{r_k}$, the conditions (5.2) and (5.3) of Lemma 5.1 are satisfied. By (6.2) the condition (5.4) of Lemma 5.1 is satisfied, too. Applying Lemma 5.1 we obtain that

$$\dim_H \mathcal{U}_{r_k}^j \geq \dim_H \mathcal{F}_{r_k}. \quad (6.3)$$

Furthermore, we have the following equalities:

$$\dim_H \mathcal{F}_{r_k} = \dim_H \mathcal{S}_{r_k} = \frac{h(\mathcal{S}')}{\log r_k} = \frac{h(\tilde{\mathcal{V}}'_{q_L})}{\log r_k}. \quad (6.4)$$

Indeed, the first and second equalities in (6.4) follow from Lemmas 4.3 and 4.5 (iv), respectively. The third equality in (6.4) follows from the assumptions of the theorem.

On the other hand, we have

$$\dim_H \mathcal{U}_{r_k}^j \leq \dim_H \mathcal{U}_{r_k} = \dim_H \tilde{\mathcal{V}}_{r_k} = \frac{h(\tilde{\mathcal{V}}'_{r_k})}{\log r_k} = \frac{h(\tilde{\mathcal{V}}'_{q_L})}{\log r_k}. \quad (6.5)$$

Indeed, in (6.5) the first inequality follows from Proposition 5.6, and the next two equalities follow from Lemmas 4.6 and 4.5 (iv), respectively. To prove the final equality in (6.5), we observe that r_k and q_L belong to the same maximal stability interval of the entropy function. Indeed, we have $r_k \in [q_L, q_R]$ by Lemma 5.3, and by Theorem 4.7 there exists a maximal stability interval $[p_L, p_R]$ such that $[q_L, q_R] \subseteq [p_L, p_R]$. Now the equal dimension property for the base r_k follows from (6.3)–(6.5).

In Proposition 6.2 we describe the quasi-greedy expansion $\alpha(q)$ for every $q \in \mathcal{V} \cap (q_L, q_R)$. The result follows from the proof of [1, Proposition 3.9] under the further assumption that q_L is the left endpoint of a maximal stability interval. Since the proposition has an independent interest, here we give a direct proof.

Proposition 6.2. *If $q \in \mathcal{V} \cap (q_L, q_R)$, then there exists a sequence (k_j) of non-negative integers such that*

$$\alpha(q) = w^+ \left(\overline{w^{k_1} w^+} \right) \left(w^{k_2} w^+ \right) \left(\overline{w^{k_3} w^+} \right) \left(w^{k_4} w^+ \right) \cdots,$$

and $k_j \leq k_1$ for all $j \geq 2$.

Moreover, if $q \in (\mathcal{V} \setminus \mathcal{U}) \cap (q_L, q_R)$, then the sequence (k_j) is periodic.

PROOF. Since $\beta(q_L) = w^+0^\infty$ and $q_L < q < q_R$, $\alpha(q)$ starts with w^+ , and $\alpha(q) \prec \alpha(q_R) = w^+(\bar{w})^\infty$. Therefore there exists a maximal integer $k_1 \geq 0$ and a word $u \prec \bar{w}$ of length m such that $\alpha(q)$ starts with $w^+\overline{w^{k_1}u}$.

Applying (5.1) it follows that $u \succeq \overline{w^+}$. Therefore we have $u = \overline{w^+}$, and $\alpha(q)$ starts with $w^+(\overline{w^{k_1}w^+})$.

Now there exists an integer $k'_2 \geq 0$ and a word u of length m such that $\alpha(q)$ starts with $w^+(\overline{w^{k_1}w^+})w^{k'_2}u$. Applying (5.1) we see that $u \preceq w^+$ and $\overline{w^+}w^{k'_2}u \succeq \overline{w^+}w^{k_1}w^+$. Hence $k'_2 \leq k_1$, and

$$u = \begin{cases} w^+ & \text{if } k_1 = k'_2, \\ w^+ \text{ or } w & \text{if } k_1 > k'_2. \end{cases}$$

There exists therefore an integer k_2 satisfying $0 \leq k_2 \leq k_1$, and such that $\alpha(q)$ starts with $w^+(\overline{w^{k_1}w^+})(w^{k_2}w^+)$.

Continuing by induction, assume for some positive integer j that $\alpha(q)$ starts with

$$w^+(\overline{w^{k_1}w^+})(w^{k_2}w^+) \dots (\overline{w^{k_{2j-1}}w^+})(w^{k_{2j}}w^+).$$

Then there exists an integer $k'_{2j+1} \geq 0$ and a word u of length m such that $\alpha(q)$ starts with

$$w^+(\overline{w^{k_1}w^+})(w^{k_2}w^+) \dots (\overline{w^{k_{2j-1}}w^+})(w^{k_{2j}}w^+)(\overline{w^{k'_{2j+1}}})u.$$

Applying (5.1) we obtain the lexicographic inequalities

$$u \succeq \overline{w^+} \quad \text{and} \quad w^+(\overline{w^{k'_{2j+1}}})u \preceq w^+(\overline{w^{k_1}})\overline{w^+}.$$

Hence $k'_{2j+1} \leq k_1$, and

$$u = \begin{cases} \overline{w^+} & \text{if } k'_{2j+1} = k_1, \\ w^+ \text{ or } \bar{w} & \text{if } k'_{2j+1} < k_2. \end{cases}$$

There exists therefore an integer k_{2j+1} satisfying $0 \leq k_{2j+1} \leq k_1$ and such that $\alpha(q)$ starts with

$$w^+(\overline{w^{k_1}w^+})(w^{k_2}w^+) \dots (\overline{w^{k_{2j-1}}w^+})(w^{k_{2j}}w^+)(\overline{w^{k_{2j+1}}w^+}).$$

Next, there exists an integer $k'_{2j+2} \geq 0$ and a word u of length m such that $\alpha(q)$ starts with

$$w^+(\overline{w^{k_1}w^+})(w^{k_2}w^+) \dots (\overline{w^{k_{2j-1}}w^+})(w^{k_{2j}}w^+)(\overline{w^{k_{2j+1}}w^+})(w^{k'_{2j+2}})u.$$

Applying (5.1) we obtain the lexicographic inequalities

$$u \preceq w^+ \quad \text{and} \quad \overline{w^+}(w^{k'_{2j+2}}u) \succeq \overline{w^+}(w^{k_1}w^+).$$

Hence $k'_{2j+2} \leq k_1$, and

$$u = \begin{cases} w^+ & \text{if } k'_{2j+2} = k_1, \\ w^+ \text{ or } w & \text{if } k'_{2j+2} < k_2. \end{cases}$$

There exists therefore an integer k_{2j+2} satisfying $0 \leq k_{2j+2} \leq k_1$ and such that $\alpha(q)$ starts with

$$w^+ \left(\overline{w^{k_1} w^+} \right) (w^{k_2} w^+) \dots \left(\overline{w^{k_{2j-1}} w^+} \right) (w^{k_{2j}} w^+) \left(\overline{w^{k_{2j+1}} w^+} \right) (w^{k_{2j+2}} w^+).$$

This completes the proof of the proposition for $q \in \mathcal{V} \cap (q_L, q_R)$.

If $q \in (\mathcal{V} \setminus \mathcal{U}) \cap (q_L, q_R)$, then $\alpha(q)$ is periodic, and therefore the sequence (k_j) is also periodic.

Theorem 1.2 follows from the following theorem:

Theorem 6.3. *A base $q_L \in \overline{\mathcal{U}} \setminus \mathcal{U}$ satisfies the assumptions of Proposition 6.1 if and only if $q_L = p_L$, where $[p_L, p_R]$ is a maximal stability interval.*

PROOF (PROOF OF THEOREM 6.3.). First let $[q_L, q_R] \subset (q_{KL}, M+1)$ be a maximal stability interval of the entropy function. Applying Theorem 4.8 we see that $\tilde{\mathcal{V}}'_{q_L}$ has a unique transitive subshift \mathcal{S}' of finite type, having the same topological entropy as \mathcal{V}'_{q_L} , and containing $\alpha(q_L)$. Therefore the conditions of Proposition 6.1 are satisfied.

Now assume that $[q_L, q_R] \subset (q_{KL}, M+1)$ is not a maximal stability interval. Then applying Theorem 4.7 (ii) we see that there exists a maximal stability interval $[p_L, p_R]$ such that $q_L \in (p_L, p_R)$. Let $\mathcal{S}' \subseteq \tilde{\mathcal{V}}'_{q_L}$ be a transitive subshift of finite type, having the same topological entropy as $\tilde{\mathcal{V}}'_{q_L}$. Then $\mathcal{S}' \subseteq \tilde{\mathcal{V}}'_{p_L}$ by Theorem 4.8.

Writing $\beta(p_L) = v^+ 0^\infty$ and $\beta(q_L) = w^+ 0^\infty$, we complete the proof of the theorem by showing that \mathcal{S}' cannot contain the word \overline{w} . Assume on the contrary that \mathcal{S}' contains \overline{w} . Then it follows from Proposition 6.2 that

$$(\overline{w})^\infty = \overline{v^+} (v^{k_1} v^+) \left(\overline{v^{k_2} v^+} \right) (v^{k_3} v^+) \left(\overline{v^{k_4} v^+} \right) \dots,$$

so that \overline{w} contains $\overline{v^+}$. This, however, contradicts the relation $\mathcal{S}' \subseteq \tilde{\mathcal{V}}'_{p_L}$, because

$$\tilde{\mathcal{V}}'_{p_L} = \{(c_i) : (\overline{v})^\infty \preceq (c_{n+i}) \preceq v^\infty \text{ for all } n \geq 0\}$$

by (4.1).

We end this paper by formulating two open problems:

- (i) Does the equal dimension property hold for all $q \in \mathcal{V} \setminus \mathcal{U}$?
- (ii) Does the equal dimension property hold for any $q \in \mathcal{U}$, different from q_{KL} ?

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