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# Resolving the singularity by looking at the dot and demonstrating the undecidability of the continuum hypothesis

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## Abstract

Einstein's theory of general relativity, which Newton's theory of gravity is a part of, is fraught with the problem of singularity that has been established as a theorem by Hawking and Penrose, the later being awarded the Nobel Prize in recent years. The crucial *hypothesis* that founds the basis of both Einstein's and Newton's theories of gravity is that bodies with unequal magnitudes of masses fall with the same acceleration under the gravity of a source object. Since, the validity of the Einstein's equations is one of the assumptions based on which Hawking and Penrose have proved the theorem, therefore, the above hypothesis is implicitly one of the founding pillars of the singularity theorem.

In this work, I demonstrate how one can possibly write a non-singular theory of gravity which manifests that the above mentioned hypothesis is only valid in an approximate sense in the "large distance" scenario. To mention a specific instance, under the gravity of the earth, a 5 kg and a 500 kg fall with accelerations which differ by approximately  $113.148 \times 10^{-32}$  meter/sec<sup>2</sup> and the more massive object falls with less acceleration. Further, I demonstrate why the concept of gravitational field is not definable in the "small distance" regime which automatically justifies why the Einstein's and Newton's theories fail to provide any "small distance" analysis. In course of writing down this theory, I demonstrate why the continuum hypothesis as spelled out by Goedel, is undecidable. The theory has several aspects which provide the following realizations: (i) Descartes' self-skepticism concerning exact representation of numbers by drawing lines (ii) Born's wish of taking into account "natural uncertainty in all observations" while describing "a physical situation" by means of "real numbers" (iii) Klein's vision of having "a fusion of arithmetic and geometry" where "a point is replaced by a small spot" (iv) Goedel's assertion about "non-standard analysis, in some version" being "the analysis of the future".

A major drawback of this work is that it can easily appear to the authorities of modern science as too simple to believe in. This is, firstly due to the origin of the motivations being rooted to the truthfulness of the language in which physics is written and secondly due to the lucidity of the calculations involved. However, at the same time, this work can also appear as a fresh and non-standard approach to do physics from its roots, where the problem of singularity is not even there to begin with. The credibility of this work depends largely on whether the reader is willing to adopt the second mindset.

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# 1 Introduction: a “philosophical nonsense” and a legitimate dilemma

One of the most critical open problems in modern theoretical physics is to present a resolution of the singularity problem in the theories of gravity[1, 2]. The problem has given birth to several directions of research under the name “quantum gravity” e.g. see ref.[3] and the references therein. However, none of them has been able to present a solution that can be deemed as acceptable. I discuss here a possible solution to this problem based on direct demonstrative reasoning rather than on some axiomatic framework.

To understand the solution it is necessary to understand the origin of the problem at first. The root of the singularity problem goes back to the antiquity when Newton used the axioms of geometry to write Principia[4, 5]. The axiom which founds the problem of singularity is the first axiom of geometry and it is common to both Euclidean[6] and non-Euclidean geometries[8, 9]:

*“A point is that which has no part.”* (1)

– see page 153 of ref.[6]. Such notion of “a point” is taken for granted with so much belief that Hilbert did not even bother to state it as an axiom while formalizing the foundations of geometry[7]. It is no surprise that the singularity problem plagues both Newton’s and Einstein’s theories of gravity[2]. I may explain this as follows.

Since physics is intended to explain our experiences (experimentally observed phenomena), the expression of the physicist must be truthful to his perception. However, this has not necessarily been the case during the development of the subject[34]. For example, Einstein used the word “experience” on several occasions in the introductory sections of his famous paper on special relativity to explain simultaneous events, clock synchronization and especially to put forward his second postulate[10]. Such explication of concepts based on practical experience, is what Bridgman called being “operational” [11]<sup>1</sup>. Now, if I take this lesson from Einstein to become operational and judge myself on how I express my experience of seeing “a point”, then the first axiom of geometry appears to be a false statement due to the following reason. Whenever I demonstrate to someone what “a point” is, I put a dot on the paper with my pencil and the dot must be visible for the demonstration process to proceed<sup>2</sup>. If I can see the dot, it definitely has some extension. Then, it is a straightforward lie to say that “a dot has no extension”. And, whatever has extension, certainly has parts. So, it is also a straightforward lie to say that “a dot has no part”. However, it is such “a dot” that I call “a point”. A direct switch of words shows that, it is a straightforward lie to say that “a point is that which has no part”. In order to make it true, I must stop the demonstration process because I can not put “a dot” on the paper to demonstrate what “a point” is. So, once I start the demonstration process, the situation becomes such that, although “I can see the dot” I must believe that “I can not see the dot” so that the first axiom of geometry holds true. For me, it is a direct denial of an empirically demonstrated truth i.e. I am denying my experience of vision of the dot. In other words, I am not being operational while expressing the perception of the dot. Bridgman might have written that the first axiom of geometry is “anti-operational” (see Appendix (A)). However, I need to emphasize that this operational viewpoint is directed towards maintaining a truthful connection between what I can speak/write and what I can draw i.e. a dot on the paper is a method of demonstration by which I express what I mean by “a point”<sup>3</sup>. This is different from writing about my experience regarding the measurement process while doing an experiment in the laboratory. It was such experimental experience that the great men like Mach[12], Einstein[10] and Bridgman[11] focused on, but not on the experience of their own expressions, while being operational. The truthfulness of our own expressions is also judged by ourselves through our own perceptions. For example – when I write, I need to keep my eyes open to be able see whether I am writing correctly; when I speak, I need to keep my ears open so that I can listen whether I am speaking correctly. One generally takes such processes as granted, and become unaware of his reliance on such perceptions, unless he loses one or more of his mode of perceptions.

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<sup>1</sup>“Operational” way of explaining concepts was advocated by Mach in ref.[12] which especially becomes clear from his use of words “experimental propositions” among other writings and it is not unknown today that he was one of those who had a profound influence on Einstein e.g. see pp. 141-145 of ref.[13].

<sup>2</sup>Such explanations are necessary alongside the equations of theoretical physics.

<sup>3</sup>Poincare pointed towards a “star” to demonstrate what a “point” means – see page no. 38 of ref.[26]. As Born noted, Klein indeed envisioned to replace “a point” with “a small spot”; in Klein’s words, there needs to be something “concrete” to explicate the “abstract” – see Appendix (C).

In spite of the simplicity and direct realizability of the above example concerning the demonstration of “a point”, the modern physicist may just dismiss the above arguments as “philosophical nonsense” or just “simply nonsense”<sup>4</sup>. So, I provide now a different argument to explain the problem at hand, that concerns the most cherished tool of the physicist to check the consistency of any expression involving quantities, namely, dimensional analysis[14]. With a dot, I generally represent the statement “ $r = 0$ ”, rooted to which is the singularity problem, as I draw the following conclusion from Newton’s law of gravitation:

$$\lim_{r \rightarrow 0} F = \lim_{r \rightarrow 0} G \frac{m_1 m_2}{r^2} = \infty \quad (2)$$

where  $m_1$  and  $m_2$  are the two masses involved and “ $r$ ” is said to be the distance between the two “point” masses[4, 5]. It is easy to see that on dimensional grounds[14], “ $r \rightarrow 0$ ” is an incorrect statement because “ $r$ ” has length dimension, but “0” is a number i.e. without any physical dimension! An immediate counter argument from the modern physicist may be that one can easily resolve this problem by simply writing the chosen unit of measurement in the expression e.g. “ $r = 0\lambda_0$  where  $\lambda_0$  stands for the chosen unit of measurement like metre, kilometre, etc.” However, this counter argument is anti-operational because no human being has ever experienced or experimentally measured “length of zero unit” e.g. “0 metre”, “0 kilometre”, etc. So, any operational notion of length, or experimentally measured (experienced) length, can at most be negligible with respect to the chosen unit of length, but not exactly zero. This is why any length must always be written in terms of the chosen length unit if I stick to operational perspective and my expression is truthful to my experience. Thus, any experienced length is relational length i.e. the corresponding expression should be a relation between two lengths – the experienced length ( $L$ ) and the chosen length unit ( $\lambda_0$ ). Such a relational expression must look like

$$L = n_L^{\lambda_0} \lambda_0, \quad \text{where } n_L^{\lambda_0} \text{ is some real positive number.} \quad (3)$$

The indices of “ $n$ ” signify the relation between the two lengths  $L$  and  $\lambda_0$ . Now, if  $n_L^{\lambda_0} \lll 1$ , or equivalently  $L \lll \lambda_0$ , then the experimenter can say “the experienced length is negligibly small compared to chosen length unit”. But, if one allows  $n_L^{\lambda_0} = 0$  on theoretical grounds (which one can always do by choice), can the experimenter verbally express the situation? I believe he can not because this “absolute zero” can not be experienced so that a corresponding expression can be given verbally<sup>5</sup> and allowing this in theory renders it anti-operational (nevertheless, not incorrect).

Therefore, the modern physicist arrives at a dilemma – either he has to let go of the operational viewpoint on which Einstein founded special relativity<sup>6</sup> (as Bridgman explained in ref.[11]), or he has to admit of the above arguments of mine to be reasonable. This, of course, brings in the question that even if my arguments are considered to be reasonable and such objections are considered as valid, is there a better alternative that can overcome these obstacles of reason? According to me, the answer is in the affirmative and my motto is to explore this alternative in what follows – it is technically quite simple, especially when compared to any of the modern existing theories in the literature that intend to resolve the problem of singularity.

## 2 “Truth” of the first axiom of geometry, experience of the dot and representation of an object as a whole

In view of what I have discussed in the previous section, the singularity problem can be resolved if I can overcome two obstacles of reason – firstly, there is a conflict between the logical truth of the first axiom of geometry regarding “a point” and the empirical truth of the vision of a dot that is indispensable for demonstrating what “a point” is; secondly, the statement “ $r = 0$ ” is incorrect on dimensional ground and the statement “ $r = 0\lambda_0$  (length of zero unit)” is not a truthful expression of experience. Interestingly, none of the above objections arise if I write – *a dot is a negligible extension compared to any other extension that*

<sup>4</sup>The skeptic reader should necessarily consult refs.[34, 35]

<sup>5</sup>The experimenter may say “I have measured zero unit of length” but he can not point out “what” he has measured because there is nothing to be measured in the first place. Thus the experience is empty i.e. it is not an experience at all. So the expression becomes meaningless. Consult refs.[34, 35] for further reading.

<sup>6</sup>Einstein relied on logical truths of the axioms of geometry to formulate his theories of relativity. Bridgman analyzed only the operational aspect of Einstein’s formulation and did not question the truth of the axioms of geometry from an operational viewpoint. See Appendix (A) for a discussion.

is called a line, where “a dot” can be replaced with “a point” i.e. instead of the first axiom of geometry, I consider the following statement:

*A point is that which has negligible extension compared to any other extension which I call “a line”.*

Now, it seems consistent with the immediate experience of seeing a dot and the corresponding realization of the smallness of its extension with respect to any extension that I call “a line”. With such a line I may represent the chosen length unit, on paper. Also, I may point out a crucial difference between the first axiom of geometry and the statement that I consider instead. The “truth” of the first axiom of geometry is a logical truth, as Einstein had stated with emphasis in Chapter 1 of ref.[17], and hence, such a truth is universal or absolute. In contrast to that, my statement is only a relational truth because of the comparison of the extension of “a dot” or “a point” with “a line”<sup>7</sup>. With such clarifications now I proceed to explicate further in what sense such ideas can be put to use so as to resolve the singularity problem. However, for further discussions one may consult Appendix (B).

Newton represented an object, considered as a whole, by putting a dot on the paper (see relevant diagrams in ref.[4, 5]), and called it “body or mass” right at the very beginning of Principia; see page no. 1 of ref.[4]. This is what we refer today as “point mass”[15]. The depiction of any geometric curve with that dot represents the motion of the object from the observer’s perspective. The subject concerning such study is what we call “point (or classical) mechanics”[15]. Using the first axiom of geometry Newton denied the visibility of the dot that he put on paper. Due to such denial, he could not represent the actual object of experiment (“mass” in his own words), with the extension of the dot itself. So, he had to consider “mass” as a physical dimension other than length, in order to represent in theory the actual object of experiment. Since he considered the object as a whole (which has no part), the words “point mass” stand justified. Einstein called it “material point” e.g. see ref.[10, 17]. I may emphasize that by “body or mass” Newton did not refer to the type, the shape or the external appearance of the particular object of investigation, rather he expressed the thought of the object irrespective of its type or shape or external features.

In the present scenario, I represent an object as a whole, by a dot which has an extension that is negligible compared to any other extension that I can call a line. With such a line I represent the chosen unit of length in terms of which experimental measurement is performed. Generally, the external shape of the object has the characteristic extension comparable to the length unit. Thus, what Newton called “mass”, in the present scenario, is not an independent physical dimension other than length. Rather it is understood as an extension in relation to the chosen length unit, as follows:

$$s_i \lll \lambda_0 \quad \equiv \quad s_i = \epsilon_{s_i}^{\lambda_0} \lambda_0 \quad \ni \quad 0 < \epsilon_{s_i}^{\lambda_0} \lll 1, \quad (\text{“}\ni\text{” means “such that”}) \quad (4)$$

where  $s_i$  is the characteristic extension of the  $i$ -th dot representing the  $i$ -th object in case there are more than one object in consideration and  $\lambda_0$  stands for length unit (like meter, kilometer, etc.) which I choose according to requirement, but within my restricted ability. I provide the visual demonstration of the relation between  $s_i$  and  $\lambda_0$  one after the other i.e. at first I demonstrate the visual realization of the dot with characteristic extension  $s_i$  and then I bring into the scenario the role of the length unit  $\lambda_0$ .

### 3 The dot, the void and distance: How many points are there on a line?

When I see the dot ( $s_i$ ), I also have the realization of the surrounding void which is *not* the dot. The realization of the one depends on the other. Without the realization of the dot, the realization of the void is not possible. Also, without the realization of the void, the realization of the dot is not possible. The realizations of the two opposite categories occur in a dependent way and not in isolation i.e. the dot can not be realized without the realization of the void and vice versa. This is what I have demonstrated in fig.(1).

Further, the totality of the perception of the dot and the void provides me with a realization of distance ( $d$ ), in any particular direction through the void but founded on the dot. That is, the thought of distance in any direction along the surface of the paper, originates from the realization of the dot. Thus, I can realize

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<sup>7</sup>My statement guarantees that I need to put the dot in such a way that I can slide my pencil over the paper to demonstrate what “a line” is. This is not possible if, instead of a pencil, if I use a very thick paint-brush and slide it over the same paper. This is because on touching the paper with the brush, I get “a patch” rather than “a dot” and on sliding the brush I get “an extended patch” rather than “a line”.

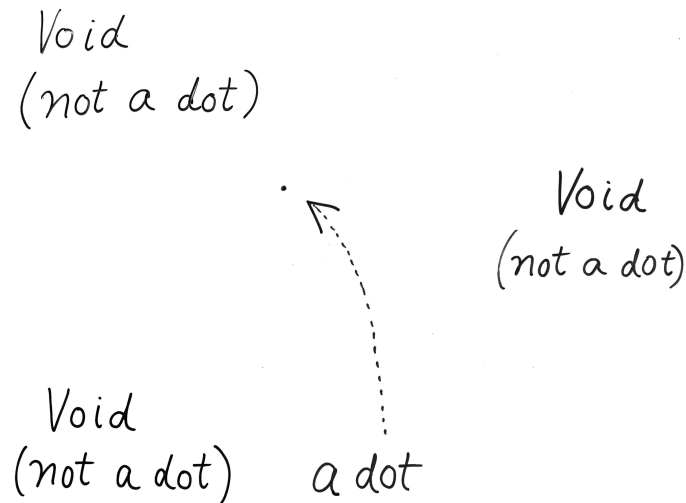


Figure 1: I see a dot and also the void which is not a dot. Realizations of the dot and the void, which are seemingly of opposite categories, happen in relation to each other. Without the realization of the dot it is not possible for me to realize what is not the dot, i.e. the void, and without the realization of the void it is not possible for me to realize what a dot is.

neither the dot nor the distance in isolation. Rather, I can realize both, in association with each other – an association that I can not dissociate – as if the two seemingly opposite categorical realizations originate in an interdependent way. So, the notion of distance must include the dot. To demonstrate this, I have provided a slightly enlarged view of the dot in fig.(2) so that it becomes clearly visible that the brackets, which are meant for the expressions of the thought of “distance” in different directions, include the dot. In order to provide reference for the reader’s trust on an authority, I may mention that the dot represents what Einstein may have called “body of reference” based on which the thought of “distance” is perceived of [17]. Nevertheless, this body of reference did not play a role in the associated calculations of Einstein because he denied the truth of the dot owing to his reliance on the “truth” of the first axiom of geometry. However, in the present scenario, the dot is the indispensable foundation for the thought of distance and therefore, of any further demonstration.

Since the thought of distance originates in association with, and hence founded on, the experience of the dot owing to its characteristic extension, then certainly, now I can *imagine* several dots are needed to fill up the distance i.e. a comparison between the dot and the distance arises. The demonstration of such an imagination, with a zoomed dot is *expressed* through a drawing in the fig.(3). I continue to repeatedly iterate with the dot along the direction in which the distance is thought, until I exhaust the required distance. It is this direction that I call “straight” and, the line that I draw by filling the distance with such dots, a straight line. However, after each iteration I put a cut in order to memorize the mark where the last iteration finished. These “cuts” are like the cross-wire of a microscope or the marks on a scale, which introduce an irremovable error in the measurement process. Undeniably, the removal of the cross-wire or the marks on a scale, makes measurement impossible. Likewise, removal of the “cuts” renders my *expression* through drawing, and hence any demonstration, impossible. Thus, the cuts are indispensable for the demonstration procedure. And, since the cuts are visible, then each cut has certain amount of thickness. Consequently there is some error involved due to these cuts that needs to be taken into account if I want to express my experience of the cuts in a truthful manner, or in Bridgman’s words, in an operational manner.

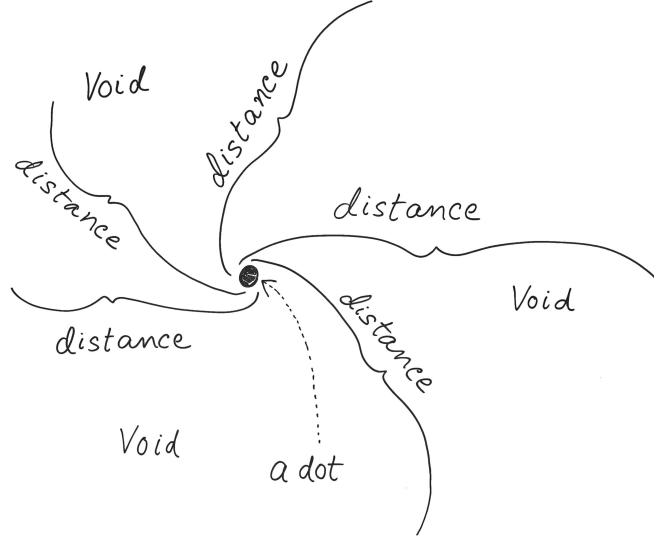


Figure 2: “Distance” is a thought that arises in association with the dot. So, the thought of distance, in any direction, includes the dot itself. This is why the brackets include the dot irrespective of the direction in which the distance is thought. So, the dot is the foundation of any following demonstration like drawing a line with more dots. I may emphasize that, in order to demonstrate this inclusion of the dot, I have shown the dot of fig.(1) with an enlarged view in this present diagram.

So, I write

$$d > s_i \equiv d = (N_d^{s_i} + 1 + \delta_d^{s_i})s_i = r_i + s_i \ni r_i := (N_d^{s_i} + \delta_d^{s_i})s_i, 0 < \delta_d^{s_i} < 1, N_d^{s_i} = 0, 1, 2, \dots \quad (5)$$

The symbol “:=” stands for “defined as”<sup>8</sup>. Here,  $N_d^{s_i}$  is the number of iterations that I need to make in order to fill the associated distance, “1” signifies that the dot itself can not be removed as it is the foundation of this whole demonstration process,  $\delta_d^{s_i}s_i$  is the collective thickness of all the cuts. Since the cuts can not be removed for the counting process to be demonstrated, therefore the number of dots can not be exactly counted i.e. an element of doubt always accompanies in the demonstration process. This is the essence of having  $\delta_d^{s_i} \neq 0$ . Considering the scenario altogether, the dot  $s_i$  and the associated distance  $d$  are interdependent realizations and the cuts are the premise based on which the measurement of  $d$  in terms of  $s_i$  can be demonstrated. Hence, the thickness of the cuts (i.e.  $\delta_d^{s_i}s_i$ ), itself, can not be measured within this process of demonstration. Thus,  $\delta_d^{s_i}s_i$  represents the incompleteness of this measurement process. Consequently, the number of dots required to fill the distance is neither completely countable nor it is completely not countable. Rather the number of dots required to fill the distance can be known approximately for serving a practical purpose, while the whole truth of the demonstration process always contains an element of doubt owing to the mode of demonstration itself. In other words, the thought of exact numbers, which are expressed through the numerals like 0, 1, 2, .... etc., can not be demonstrated with complete perfection because the mode of demonstration itself is the imperfection<sup>9</sup>.

<sup>8</sup>Here, the words “defined as” carry the meaning “abbreviation for” in symbolic terms i.e. a short hand for typographical purpose.

<sup>9</sup> Possibly due to such reasons Descartes wrote “as closely as possible” and not “exactly” while representing the concept of “number” through “geometry” on page no. 2 of ref.[16]: “..... taking one line which I shall call unity to relate it as closely as possible to numbers.”. If I may use an everyday example to demonstrate this fact, then I must write that any reasonably honest person should admit that when he says that he has five fingers in one hand, then certainly he makes an approximation because every finger is different from the other. So, it is just a working convention to write “exactly five fingers”, which is nevertheless extremely useful for daily purpose and considered to be logical and therefore, assumed to be exact. In this regard, the reader may consult Appendix (C) for some relevant statements by Born and Klein.



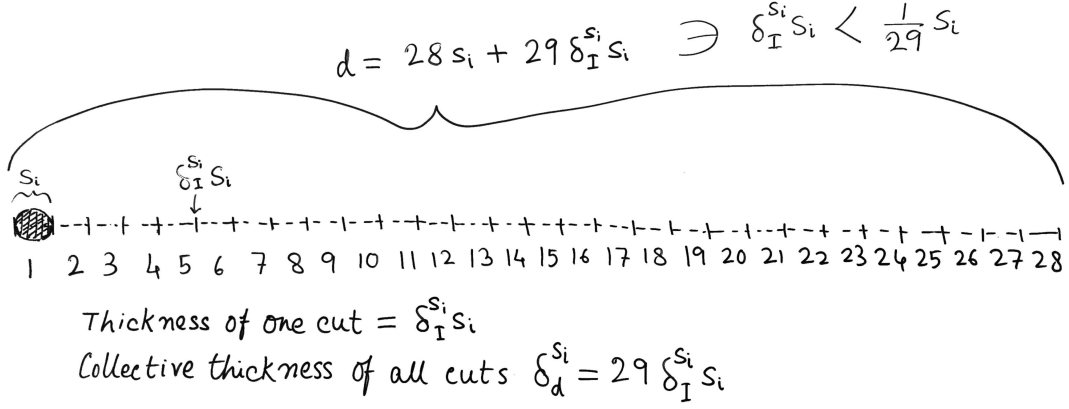


Figure 3: This figure is a demonstration of the notion of a distance  $d$  measured in terms of the dot  $s_i$ . The cuts are necessary as the mode of demonstration. Here,  $d = 28s_i + 29\delta_I^{s_i}s_i \ni \delta_I^{s_i}s_i < \frac{1}{29}s_i$ . Thickness of one cut is  $\delta_I^{s_i}s_i$ . Collective thickness of all the cuts is  $\delta_d^{s_i}s_i = 29\delta_I^{s_i}s_i$ .

Now, the collective thickness of the cuts is demonstrable in terms of the individual ones in a straightforward manner as I can write the following:

$$\delta_d^{s_i}s_i = (N_d^{s_i} + 2)\delta_I^{s_i}s_i < s_i \quad \ni \delta_I^{s_i}s_i \text{ is the thickness of an individual cut.} \quad (6)$$

The subscript “I” in  $\delta_I^{s_i}$  stands for “individual”. In view of this, (5) can be rewritten as follows:

$$\begin{aligned} d &= (N_d^{s_i} + 1 + \delta_d^{s_i})s_i \quad \ni \delta_d^{s_i}s_i < s_i \\ &= [N_d^{s_i} + 1 + (N_d^{s_i} + 2)\delta_I^{s_i}]s_i \quad \ni (N_d^{s_i} + 2)\delta_I^{s_i}s_i < s_i \\ &= [N_d^{s_i} + (N_d^{s_i} + 2)\delta_I^{s_i}]s_i + s_i \quad \ni \delta_I^{s_i}s_i < \frac{s_i}{N_d^{s_i} + 2} \end{aligned} \quad (7)$$

$$\therefore r_i = [N_d^{s_i} + (N_d^{s_i} + 2)\delta_I^{s_i}]s_i \quad \ni \delta_I^{s_i}s_i < \frac{s_i}{N_d^{s_i} + 2}, \quad N_d^{s_i} = 0, 1, 2, \dots$$

Now, I write down the following cases explicitly.

- For  $N_d^{s_i} = 0$ ,  $r_i = (0 + 2\delta_I^{s_i})s_i \quad \ni \delta_I^{s_i}s_i < \frac{s_i}{0+2}. \quad \therefore r_i < s_i, \quad d < 2s_i.$
- For  $N_d^{s_i} = 1$ ,  $r_i = (1 + 3\delta_I^{s_i})s_i \quad \ni \delta_I^{s_i}s_i < \frac{s_i}{1+2}. \quad \therefore r_i > s_i, \quad d > 2s_i.$
- For  $N_d^{s_i} = 2$ ,  $r_i = (2 + 4\delta_I^{s_i})s_i \quad \ni \delta_I^{s_i}s_i < \frac{s_i}{2+2}. \quad \therefore r_i > s_i, \quad d > 2s_i.$
- For  $N_d^{s_i} = 3$ ,  $r_i = (3 + 5\delta_I^{s_i})s_i \quad \ni \delta_I^{s_i}s_i < \frac{s_i}{3+2}. \quad \therefore r_i > s_i, \quad d > 2s_i$  and so on.

So, I can write that for  $N_d^{s_i} > 2$ ,  $r_i = [N_d^{s_i} + (N_d^{s_i} + 2)\delta_I^{s_i}]s_i$  such that  $\delta_I^{s_i}s_i < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1} \simeq \frac{s_i}{N_d^{s_i}}$ .

### 3.1 Explication of Klein’s envisioned distinction between “naive intuition” and “refined intuition (logic)”

I may note that the condition  $\delta_I^{s_i}s_i < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$  has been imposed in order to refine the demonstration process of counting dots by minimizing the errors incorporated due to the cuts. It does not look illogical or like a mistake if I omit  $s_i$  from both sides and write  $\delta_I^{s_i} < \frac{1}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$ , because I just “cancel” the same physical dimension (of length) from both sides as if  $s_i$  represents some non-zero number[34]. Then,

for  $N_d^{s_i} \rightarrow \infty$ , I obtain  $\delta_I^{s_i} < 0$ . While mathematically there is no problem with such a conclusion, this particular step of calculation is not demonstrable. That is, the relation “ $\delta_I^{s_i} s_i < 0s_i$ ” (written by choice) has no corresponding demonstration in terms of the cuts i.e. it is not a relation that is explicable, if a relation at all. Therefore, I may conclude that infinite measurement and arbitrary refinement of the demonstration process is not possible. This raises the doubt whether exact “zero” and exact “infinity” are demonstrable at all. In view of this and ref.[32] (see Appendix(C) for specific quotes), Klein could have explicated the present situation as follows. The expression “ $\delta_I^{s_i} s_i < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$ ” represents “*naive intuition*” because the “*concrete*” dot is taken into account as  $s_i$  and then complete refinement is not possible as “ $\delta_I^{s_i} s_i < 0s_i$ ” is not demonstrable. The expression “ $\delta_I^{s_i} < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$ ” allows complete refinement and therefore, it represents completely “*refined intuition*” and “*logical*”, but it is “*not properly intuition at all*” because of the loss of “*concrete*”-ness owing to the absence of  $s_i$ . Although the “*exact mathematician*” may not find a problem with the expression devoid of  $s_i$ , I must “*maintain*” my stance in using the expression with  $s_i$  that is demonstrable and useful for practical purpose for explaining phenomena observed in “*ordinary life*”, which is the aim of the present theoretical discussion. Henceforth, I shall call the expressions involving physical dimensions, like  $\delta_I^{s_i} s_i < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$ , as *physico-mathematical* expressions in order to make a distinction from mathematical expressions, like  $\delta_I^{s_i} < \frac{1}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$ , which are devoid of physical dimensions[34].

Now, I must provide the following clarification regarding fig.(3). It is important to note that I have drawn a zoomed version of the dot so as to make the situation understandable for the reader. If it were the original dot of fig.(1), then the cuts would not have been visible. Therefore, in this sense, I may write that I have demonstrated a “hypothetical” measurement of the distance in terms of the associated dot. It is just a visual demonstration of how I think of the distance on the basis of my realization of the dot and due to the association of the dot how its characteristic extension plays the role in realization of the distance in relation to the dot.

### 3.2 Meaning of “large distance” and “small distance” in relation to the dot

What I have already demonstrated is the fact that the realizations of the dot and the associated distance occur interdependently and then the estimate of the distance in terms of numbers is given by the characteristic extension of the dot, albeit always with the error due to the mode of demonstration – the cuts. Therefore, it is now easy to understand that the words “large distance” and “small distance” are simply meaningless or rather carry incomplete sense because there need to be a mention of “compared to what”. This completion is now automatic as the dot is taken as the foundation of any demonstration and it is with respect to the characteristic extension of the dot that the largeness of the associated distance (along any direction) can be specified.

So, from (5), I write down the two following cases which I call “small distance” and “large distance” respectively as follows:

- **Small Distance:**  $N_d^{s_i} = 0 \equiv r_i < s_i \equiv s_i < d < 2s_i$
- **Large Distance:**  $N_d^{s_i} = 1, 2, \dots \equiv r_i > s_i \equiv d > 2s_i$

where “ $\equiv$ ” stands for “equivalently”. So, the following two cases arise.

- **Small Distance Expansion (SDE):**

For  $r_i < s_i \equiv s_i < d < 2s_i$  I can write

$$\frac{1}{2} < \frac{s_i}{d} = \frac{s_i}{r_i + s_i} = \sum_{n=0}^{\infty} \left( -\frac{r_i}{s_i} \right)^n < 1. \quad (8)$$

- **Large Distance Expansion (LDE):** For  $r_i > s_i \equiv d > 2s_i$  I can write

$$0 < \frac{s_i}{d} = \frac{s_i}{r_i + s_i} = \frac{s_i}{r_i} \sum_{n=0}^{\infty} \left( -\frac{s_i}{r_i} \right)^n < \frac{1}{2}. \quad (9)$$

Here, the words “large” and “small” carry a clear and obvious comparative sense because “compared to what” is now declared<sup>10</sup>. So, I may emphasize at this point that if I need to draw a line to demonstrate some experimental phenomenon where an object is observed as a whole, then I may call it “large distance physics” and condition (13) needs to be satisfied. If it happens that the theory written down with such a condition fails to explain the observed phenomenon, then it means that the assumption of an object as a whole fails or not suitable for the description of that particular observed phenomenon. I believe that such assertions will appear to be more meaningful as I proceed.

### 3.3 Realization of Descartes’ doubt and Born’s wish: the “natural uncertainty” in the physical act of explicating numbers

Now, I introduce the role of  $\lambda_0$  in the context so as to explain the role of the length unit in terms of which actual measurement is to be done. As Einstein did specify in ref.[17] (specific quote to be given in the next subsection), the chosen standard of length is also a “distance”<sup>11</sup>. Thus, I can write

$$\lambda_0 = (N_{\lambda_0}^{s_i} + 1 + \delta_{\lambda_0}^{s_i})s_i \ni N_{\lambda_0}^{s_i} \ggg 1, \quad \delta_{\lambda_0}^{s_i} s_i = (N_{\lambda_0}^{s_i} + 2)\delta_I^{s_i} s_i < s_i, \quad (10)$$

which can be recast as

$$s_i = \epsilon_{s_i}^{\lambda_0} \lambda_0 \quad \text{where} \quad \epsilon_{s_i}^{\lambda_0} := (N_{\lambda_0}^{s_i} + 1 + \delta_{\lambda_0}^{s_i})^{-1} \quad (11)$$

and it is easy to see that

$$0 < \epsilon_{s_i}^{\lambda_0} \lll \frac{1}{2} \quad \text{as} \quad N_{\lambda_0}^{s_i} \ggg 1. \quad (12)$$

Any distance  $d$  that is to be measured in terms of  $\lambda_0$  must be greater than  $\lambda_0$  so that at least one iteration is possible. So, to represent the length unit  $\lambda_0$  by a line on the paper and to demonstrate the measurement of any distance in terms of this line, I must have the following condition to hold:

$$d > \lambda_0 \ggg 2s_i. \quad (13)$$

This is what I have tried to demonstrate in fig.(4) in relation to fig.(3). Therefore, considering such

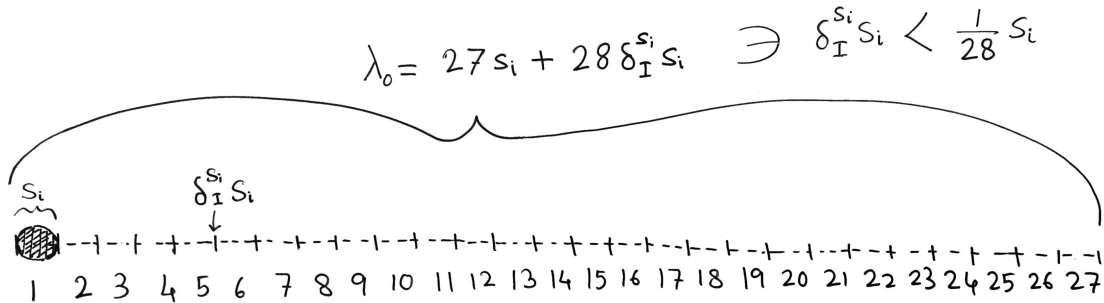


Figure 4: Compared to fig.(3), here there is one more dot. This is to demonstrate the condition  $d > \lambda_0$ .

explications, the distance  $d$  founded on the dot  $s_i$ , measured in terms of the length unit  $\lambda_0$ , can be expressed

<sup>10</sup>This is not a mere useless nitpicking of words. In case such a thought arises in the reader’s mind, I suggest a consultation of Appendix(D) and of refs.[34, 35].

<sup>11</sup>Cantor might call  $\lambda_0$  “unit distance”[38].

from (5) as follows:

$$\begin{aligned}
d &= (N_d^{s_i} + 1 + \delta_d^{s_i})s_i \\
&= (N_d^{s_i} + 1 + \delta_d^{s_i})\epsilon_{s_i}^{\lambda_0}\lambda_0 \\
&= [(N_d^{s_i} + \delta_d^{s_i})\epsilon_{s_i}^{\lambda_0} + \epsilon_{s_i}^{\lambda_0}]\lambda_0.
\end{aligned} \tag{14}$$

Now, (14) represents the act of putting a dot, the subsequent operations of drawing a line to represent  $d$  and the iterations by  $\lambda_0$  along with the errors introduced by the cuts, each of which has the same thickness as that of the dot by assumption. This can be represented as

$$d = [(N_d^{\lambda_0} + 1) + (N_d^{\lambda_0} + 2)\epsilon_{s_i}^{\lambda_0}]\lambda_0 \tag{15}$$

and then the following relation needs to hold:

$$(N_d^{s_i} + \delta_d^{s_i})\epsilon_{s_i}^{\lambda_0}\lambda_0 = (N_d^{\lambda_0} + 1)(1 + \epsilon_{s_i}^{\lambda_0})\lambda_0 \quad \ni N_d^{\lambda_0} = 0, 1, 2, 3 \dots \tag{16}$$

Then, I may recast (15) as follows:

$$d = [(N_d^{\lambda_0} + 1) + (N_d^{\lambda_0} + 2)\epsilon_{s_i}^{\lambda_0}]\lambda_0 \tag{17}$$

Considering such a construction of the iteration process, it provides a good estimate of the distance ( $d$ ) measurement with the length unit ( $\lambda_0$ ) until the following condition is satisfied:  $(N_d^{\lambda_0} + 2)\epsilon_{s_i}^{\lambda_0}\lambda_0 < \lambda_0 \equiv (N_d^{\lambda_0} + 2)s_i < \lambda_0$ ; otherwise this will lead to mistake. I shall designate “ $(N_d^{\lambda_0} + 1)(1 + \epsilon_{s_i}^{\lambda_0})$ ” as “ $x_{s_i}^{\lambda_0}$ ” which is an integer but always associated with an error of at least  $\epsilon_{s_i}^{\lambda_0}$ . With such clarifications of the symbols, I may now write

$$d = r_i + s_i = (x_{s_i}^{\lambda_0} + \epsilon_{s_i}^{\lambda_0})\lambda_0. \tag{18}$$

This explains why Descartes doubted whether his own demonstration<sup>12</sup> of “numbers” by drawing line was exact in ref.[16] (see footnote (9)). Also, this is a possible explanation of how one can demonstrate, as Born would write[30] (see Appendix (C)), the “*natural uncertainty*” of physical observations by writing real numbers in such a way, where the physical aspect is expressed by the physical dimension of  $s_i, d, \lambda_0$ . This in turn further justifies my use of the word “physico-mathematical”.

### 3.4 Einstein’s “mark off” for simple length measurement and his ignorance of the experienced truth of the marks

In view of what I have discussed regarding the “natural uncertainty” (in Born’s words) in the physical act of distance measurement to explicate the concept of numbers, it should be now of importance to have a comparison with Einstein’s views regarding the same so as to understand more clearly in what sense the present scenario is different from that of Einstein which is however the prevalent and accepted viewpoint in the standard literature of physics. Although Einstein did not consider the extension of the dot and never tried to answer the question regarding the number of dots on a line, however he explained distance measurement in the same way i.e. by repeated iterations with a chosen “distance” (length) which is considered as standard. On p 4 of ref.[17], Einstein wrote:

*“On the basis of the physical interpretation of distance which has been indicated, we are also in a position to establish the distance between two points on a rigid body by means of measurements. For this purpose we require a “distance” (rod  $S$ ) which is to be used once and for all, and which we employ as a standard measure. If, now,  $A$  and  $B$  are two points on a rigid body, we can construct the line joining them according to the rules of geometry; then, starting from  $A$ , we can mark off the distance  $S$  time after time until we reach  $B$ . The number of these operations required is the numerical measure of the distance  $AB$ . This is the basis of all measurement of length. Here we have assumed that there is nothing left over, i.e. that the measurement gives a whole number. This difficulty is got over by the use of divided measuring-rods, the introduction of which does not demand any fundamentally new method.”*

<sup>12</sup>I consider Descartes’ act of casting and analyzing doubts of his own reasoning as an example of self-inquiry.

Certainly here Einstein, as I have emphasized earlier also, did base his explanations on “the rules of geometry”. Therefore, not only he did not take into consideration the dots which he needed to put in order to demonstrate the points A and B, but he also did not consider the thickness of the marks so as to keep track of how many times he had to “mark off” the standard length. Due to such ignorance of direct experiences, Einstein concluded with confidence that “measurement gives a whole number” without mentioning the irremovable error due to the marks that he needed to put so as to make the measurement process possible. Even though there is “nothing left over”, there is the collective thickness of the marks which, itself, can not be measured within this measurement process and appear as, as Born might say (see Appendix(C)), “*natural uncertainty*” in the process. In view of this I may write that in order to take into account such errors it “does not demand any fundamentally new method” but it demands a fundamentally new attitude – an attitude of truthfulness of expressions of experience along with the admittance of the incompleteness of the measurement process. Ignorance of the experience of such errors is a mistake on the scientific grounds in pursuit of truth.

### 3.5 Goedel’s translation of Cantor’s “continuum problem” without any demonstration of “a point” and “a line”

In relation to what I have discussed in the introduction of Section(3), I may bring to the attention of the reader the following. It was Goedel who posed the following question on page no. 58 of ref.[27]:

“*How many points are there on a straight line on a Euclidean space?*”

This was his translation of Cantor’s continuum problem[28]. Now, first and foremost, the reader may wonder why I should indulge in such a question at all while the physicist can just comfortably ignore it and remain content with writing theories which satisfactorily explain experimental observations. For example, Einstein explained, in the *Abstract* of ref.[25], his theory of relativity as “*evolution of the notion of space and time into that of the continuum with metric structure.*”, without worrying about the word “continuum”. Such ignorance of the physicist might have been justified by Einstein in the following manner (in the same *Abstract*):

“*Physics constitutes a logical system of thought....The justification (truth content) of the system rests in the proof of usefulness of the resulting theorems on the basis of sense experiences, where the relations of the latter to the former can only be comprehended intuitively.*”

While I strongly believe that the essence of ideas lie in their practical use, however, an inquiry regarding the “truth content” of the parts of the “logical system”, i.e. the axioms, becomes necessary when “a logical system of thought” is pushed to its limits leading to a logical catastrophe like the singularity theorem[1, 2]. If I call Einstein’s ‘intuitive comprehension of the relations among the resulting theorems with sense experiences’ as “outer intuition”, then I may call ‘intuitive comprehension of the relations among the starting axioms with sense experiences’ as “inner intuition”. Brouwer might have written in this present context that such inner intuition “subtilizes logic” and possibly he could have called such an act of inward directed inquiry as “inner inquiry”[33], which I may call “self-inquiry”[34]. Such self-inquiry was not a matter of concern for Goedel, like Einstein, which I may elucidate as follows. On p 1 of ref.[29], Goedel adopted “*the system  $\Sigma$  of axioms for set theory*” and wrote about “*theorems demonstrable in  $\Sigma$* ” to deal with the continuum problem. However, he did not bother to demonstrate the axioms themselves i.e. the intent was to make outward inquiry and not inward inquiry. Such intent becomes even more vivid from the fact that Goedel did not demonstrate his translation of the continuum problem through a drawing, neither did Cantor do himself [38]. Such a demonstration process is necessary because, as far as I understand, without putting a dot on a paper, one can not possibly convey the meaning of the word “point” to somebody else. Consequently, the words “point” and “line” become empty of any essence without such demonstration. That is, one could have simply asked Goedel (or Cantor) the following question: *Can you explain what you mean by the words “point” and “line”?* If this were the case, I believe, Goedel would have picked up a pencil and put a dot on a paper to start his demonstration in order to intuitively comprehend the relations among the starting axioms with his own sense experiences. Such an act of self-inquiry is the subject matter of the present discussion so as to elucidate why the singularity problem is a logical catastrophe.

Now, it does not take a super-intelligence to sense an association of the present discussion with the foundations of set theory[22]. However, I must refrain myself from bringing into this discussion any such

abstractions which are nevertheless plagued with several antinomies[22]. Due to such reasons any discussion regarding set theory will lead to further discussions concerning philosophical, logical and linguistic issues which will add unnecessary complications to the present context. Most importantly, such abstractions do not have any immediate connection with the experimental works which support the theories of gravity because the experimental physicists are only worried about the equations provided by the theoretical physicist and the corresponding interpretations[24].

## 4 The two body interaction with each of the bodies considered as a whole

In terms of measurement with  $\lambda_0$  i.e. from the observer's perspective, taking into account what I have explained regarding (18), I write the following:

$$\mathbf{O}_{\lambda_0} \left[ \frac{s_i}{d} \right] := \frac{s_i}{r_i + s_i} \ni s_i = \epsilon_{s_i}^{\lambda_0} \lambda_0, \quad r_i = x_{s_i}^{\lambda_0} \lambda_0, \quad 0 < \epsilon_{s_i}^{\lambda_0} \lll 1/2, \quad (19)$$

where “ $\mathbf{O}$ ” stands for “observed”. Here, I have used the term “observed” in the sense of “measured in terms of  $\lambda_0$ ”, but without explicating who/what an observer is[34]. I may note that

$$\begin{aligned} 0 < \mathbf{O}_{\lambda_0} \left[ \frac{s_i}{d} \right] < \frac{1}{2} & \quad \text{for} \quad d > 2s_i \equiv r_i > s_i \\ \frac{1}{2} < \mathbf{O}_{\lambda_0} \left[ \frac{s_i}{d} \right] < 1 & \quad \text{for} \quad s_i < d < 2s_i \equiv r_i < s_i \end{aligned}$$

Now, I consider two dots  $s_1$  and  $s_2$  to represent two objects (each considered as a whole) in order to analyze the gravitational two body interaction and write a ratio of two forces  $F$  and  $F_0$  as follows:

$$\begin{aligned} \frac{F}{F_0} & := \mathbf{O}_{\lambda_0} \left[ \frac{s_1}{d} \right] \cdot \mathbf{O}_{\lambda_0} \left[ \frac{s_2}{d} \right] = \frac{s_1}{r_1 + s_1} \cdot \frac{s_2}{r_2 + s_2} \\ & \ni s_i = \epsilon_{s_i}^{\lambda_0} \lambda_0, \quad r_i = x_{s_i}^{\lambda_0} \lambda_0, \quad 0 < \epsilon_{s_i}^{\lambda_0} \lll 1/2, \quad \forall i \in [1, 2]. \end{aligned} \quad (20)$$

The symbols “ $\forall$ ”, “ $\in$ ” stand for “for all”, “belongs to” respectively. Since  $d > s_i$ , each of the series involved converges to some value between 0 and 1. Consequently,  $0 < F/F_0 < 1$  i.e.  $F_0$  is some upper bound on  $F$ . Eq.(20) can be analyzed for various cases depending on the relation among  $d, s_1, s_2$ . Here, I discuss the two most relevant cases in what follows.

### 4.1 “Geometric results” in the large distance approximation and for measurable distance

Considering what I have already discussed, the condition that I call “large distance” and “measurable distance” is written as

$$d > \lambda_0 \ggg 2s_i \quad \forall i \in [1, 2]. \quad (21)$$

Here, I intend to explicate the situation where one of the two objects is considered as a test object and the other one is considered as the source object e.g. the scenario where the experimenter drops objects on the earth surface to analyze gravitational phenomena. To do this, I *assume*:  $s_2 = n_0 s_1 \ni n_0 > 1$ . Hence,  $d > \lambda_0 \ggg 2s_2 > 2s_1$  because  $n_0 > 1$  by the above assumption. Then, some elementary calculations, involving LDEs with  $s_1, s_2, d$  from the observer's perspective, yield the following:

$$\frac{F}{F_0} = \frac{s_2 s_1}{r_1^2} \left[ 1 - \frac{2s_1}{r_1} + \frac{3s_1^2}{r_1^2} - \dots \right] \quad (22)$$

Certainly it is possible to reconstruct theoretical physics, from scratch, by attaching the words “point mass” with “ $2s \lll \lambda_0$ ”. In that case “mass” is not an independent physical dimension. However, I do not wish to explore such possibility in this article. Rather I shall take the opportunity to showcase how this singularity-free analysis consistently demonstrates the presently accepted scenario where “mass” is an independent

physical dimension by writing  $s_i := Gm_i/c^2$  and  $F_0 := c^4/G$ , where  $G, c$  have the usual interpretations of “fundamental constants” with appropriate physical dimensions. Then, eq.(22) takes the following form:

$$F = \frac{Gm_2m_1}{r_1^2} \left[ 1 - \frac{G}{c^2} \frac{2m_1}{r_1} + \frac{G^2}{c^4} \frac{3m_1^2}{r_1^2} - \dots \right] \quad (23)$$

where obviously  $m_2 = n_0m_1$ .

Now, let me consider the quantity  $F/m_1$  and call it the gravitational field due to  $m_2$  as felt by the test object with mass  $m_1$ , which I designate as  $g_{2 \rightarrow 1}$ . Then, from eq.(23) I have

$$g_{2 \rightarrow 1} := \frac{F}{m_1} = \frac{Gm_2}{r_1^2} \left[ 1 - \frac{G}{c^2} \frac{2m_1}{r_1} + \frac{G^2}{c^4} \frac{3m_1^2}{r_1^2} - \dots \right] \quad (24)$$

The sub-leading  $m_1$ -dependent terms signify the back-reaction of the test object. The leading back-reaction term, namely  $-2Gm_1m_2/c^2r_1^3$ , contributes as a negative acceleration, but suppressed by  $G/c^2$  factor.

Now, at this point, applying a bit of intuition, if I consider all such test masses for which  $d \gg 2s_1 = 2Gm_1/c^2$ , then I can write  $d = r_1 + s_1 \simeq r_1$  for all such test masses i.e. the subscript “1” of “ $r_1$ ” can now be erased and I can write simply “ $r$ ” in place of “ $r_1$ ”. This can be visualized as follows – two dots of different sizes look approximately the same if the lines drawn with them are very large compared to their characteristic extensions. In such a scenario, I can rewrite eq.(23) as follows:

$$F = \frac{Gm_2m_1}{r^2} \left[ 1 - \frac{G}{c^2} \frac{2m_1}{r} + \frac{G^2}{c^4} \frac{3m_1^2}{r^2} - \dots \right] \quad (25)$$

and eq.(24) as follows:

$$g_{2 \rightarrow 1} := \frac{F}{m_1} = \frac{Gm_2}{r^2} \left[ 1 - \frac{G}{c^2} \frac{2m_1}{r} + \frac{G^2}{c^4} \frac{3m_1^2}{r^2} - \dots \right] \quad (26)$$

If I consider only the first term on the right hand side of eq.(25), then I obtain what we know today as Newton’s law of gravity. If I consider the first term of eq.(26), then I obtain what we know today as the gravitational field due to a source of mass  $m_2$ , which appears to be independent of the test mass  $m_1$  from and this apparently is supported by the experimentally verified fact that all masses fall with same acceleration. Certainly, the full expression on the right hand side of eq.(26) is dependent on the test mass  $m_1$ . Consequently the question arises whether all masses fall with same acceleration.

## 4.2 Do all masses fall with same acceleration?

Certainly the answer to the above question is negative according to all the currently existing theories of gravity. Indeed, the fact that all masses fall with same acceleration, is the founding premises of Einstein’s theory of general relativity in a limit of which falls Newton’s theory of gravity[17]. However, the present analysis provides a ground to realize how such a hypothesis is only valid in an approximate limit of large distance physics. Now, as far as the experimental verification of such a hypothesis is concerned, the results of experimental measurements are always subject to refinement due to availability of means to make more precise measurements. Therefore, as it stands, if the above discussion of mine can be considered as valid, then the difference in accelerations of two different masses must be beyond current experimental precision. To verify whether this is the case, I consider only up to the first sub-leading order term of eq.(26) to do calculations, that is,

$$g_{2 \rightarrow 1} := \frac{F}{m_1} \simeq \frac{Gm_2}{r^2} \left[ 1 - \frac{G}{c^2} \frac{2m_1}{r} \right]. \quad (27)$$

From eq.(27) it is easy to obtain the following:

$$g_{2 \rightarrow 1'} - g_{2 \rightarrow 1''} \simeq -\frac{2G^2}{c^2} \frac{m_2(m_{1'} - m_{1''})}{r^3}. \quad (28)$$

where  $m_{1'}$  and  $m_{1''}$  are two different test masses when measured in terms of a standard unit of mass.

Now, using such an approximation I may now provide an estimate of the difference in acceleration of two objects with different masses due to gravity at the surface of the earth. I consider  $m_{1'} = 5$  kg and  $m_{1''} = 500$  kg, radius of the earth  $r = 6.371 \times 10^6$  meter, mass of the earth  $m_2 = 5.972 \times 10^{24}$  kg,  $G = 6.674 \times 10^{-11}$  meter<sup>3</sup> kg<sup>-1</sup> sec<sup>-2</sup>,  $c = 3 \times 10^8$  meter/sec. It is straightforward to check that  $r \gg s_1$  in case of both the masses because  $s_1$  takes values  $3.708 \times 10^{-27}$  meter and  $3.708 \times 10^{-25}$  meter for 5 kg and 500 kg respectively. Using these data, one can calculate from eq.(28) that

$$g_{2 \rightarrow 1'} - g_{2 \rightarrow 1''} \simeq 113.148 \times 10^{-32} \text{meter/sec}^2. \quad (29)$$

It is now understandable that in the conventional units of measurement such a difference in acceleration is undetectable. Nevertheless, the interesting fact is that the more massive an object is, the slower it falls i.e. the acceleration is less. It will be interesting to see whether such an analysis can shed some light on the observations of the galaxies where the currently known and accepted theories of gravity (like that of Newton and Einstein) fail to provide satisfactory explanations leading to problems like missing mass or dark matter[18].

### 4.3 Non-singular small distance approximation

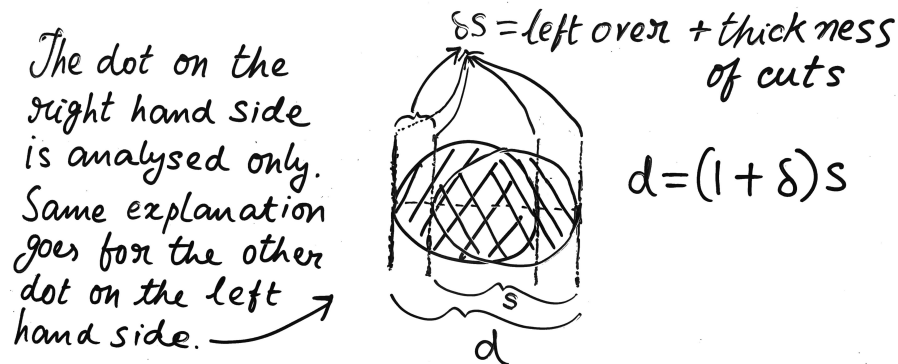
In the second case, I assume  $s_2 = n_0 s_1 \ni n_0 > 0$  and consider  $s_1 < d < 2s_1$ ,  $s_2 < d < 2s_2$  i.e. the two dots overlap. Alongside  $0 < \delta_1, \delta_2 < 1$ , the relation  $s_2 = n_0 s_1$  results in the following inequalities:  $(n_0 - 1) < \delta_1 < (2n_0 - 1)$ ,  $(n_0^{-1} - 1) < \delta_2 < (2n_0^{-1} - 1)$ . If  $n_0 > 1$ , then  $\delta_2 < 0$  and if  $n_0 < 1$ , then  $\delta_1 < 0$ , both of which violate the condition  $0 < \delta_1, \delta_2 < 1$  that is necessary for demonstrative purpose i.e. thickness of the cuts. Therefore, the only possibility is  $n_0 = 1$ . So, I consider  $s_1 = s_2 = s$  (say), consequently  $\delta_1 = \delta_2 = \delta$  (say) and write the series expansion involving two SDEs with  $s, s, d$  from the observer's perspective, as follows:

$$\frac{F}{F_0} = 1 - \frac{2r}{s} + \frac{3r^2}{s^2} \dots \ni s = \epsilon_s^{\lambda_0} \lambda_0, r = \delta \epsilon_s^{\lambda_0} \lambda_0, 0 < \epsilon_s^{\lambda_0} \lll 1/2, 0 < \delta < 1. \quad (30)$$

Writing  $s = Gm/c^2$  and  $F_0 = c^4/G$ , I recast eq.(30) as follows:

$$F = \frac{c^4}{G} \left[ 1 - \frac{c^2}{G} \frac{2r}{m} + \frac{c^4}{G^2} \frac{3r^2}{m^2} - \dots \right]. \quad (31)$$

I have tried to give below a visual demonstration of two overlapping dots, but only the zoomed version. From the observer's perspective it is impossible to demonstrate visually because it is undecidable whether there is only one dot or two overlapping dots as  $\epsilon_s^{\lambda_0} \lll 1/2$ . It is important to note that in this case the field



description is not possible because in eq.(31) the zeroth order term is devoid of any mass in the numerator and then the mass appears in the denominators of the sub-leading terms. So, a construction like " $F/m$ " is not possible. Since all the known theories of gravitational physics use the notion of gravitational field,



this particular situation where the “gravitational field” can not be defined as “ $F/m$ ”, is not captured by such theories and the scenario of overlapping dots is termed as singularity. In this current methodology, the infinite series in eq.(31) only suggests that there is nothing such as singularity, rather it is the foundation of physics on the first axiom of geometry (regarding “point”) that has led to the theoretical conclusion of singularity. Ignorance of the perception of a dot by saying “a point has zero extension”, to state the axiom, has created the problem of singularity. Therefore, the problem of singularity is a logical catastrophe.

#### 4.4 The approximate truth of a founding assumption of the singularity theorem by Hawking and Penrose

Now, in light of the present discussion, one may wonder about the status of the Hawking-Penrose singularity theorem, which is founded on a set theoretic basis[1, 2] and has also been credited with The Nobel Prize (to Penrose)[37] on the basis of experimental verification[23]. In view of this I venture to offer some critical comments in light of what I have discussed in subsection(4.2) and subsection(4.3).

Hawking and Penrose certainly legitimized and epitomized the singularity problem, based on analysis founded on set theoretic abstractions[1, 2]. Whereas, this present discussion paves a way to solve the problem by demonstrating that the singularity theorem is not an exact result. Rather, it holds only in a large distance approximation. I admit that a lot of work needs to be done along the lines of investigation that I have discussed here, but I believe there is a convincing enough argument that I can give in favour of my claim. To explicate this, I may note that one of the assumptions, for which the singularity theorem remains valid is that “*Einstein’s equations hold*”, as the authors declared in the *Abstract* of ref.[2]. Now, Einstein’s formulation of general relativity, and hence his equations, are based on the hypothesis that all masses fall with same acceleration under the gravity of a source object and it is due to such hypothesis that the concept of “gravitational field due to a source object” is independent of the mass of the test object[17]. But, I have demonstrated through my analysis that this hypothesis is valid only in the large distance approximation in Section(4.1). Thus, Einstein’s equations are only valid in the large distance approximation. Consequently, the singularity theorem is also valid only in the large distance approximation (even if I disregard any possible consequences of the present discussion concerning the other assumptions of the theorem).

## 5 Concluding Remarks

I believe that I have demonstrated, at least with partial satisfaction of the reader, how a singularity-free theory of gravity can be written. I conclude by bringing to the attention of the reader a few things which can be immediately noticed from this simple analysis. The notion of distance is only relative and not absolute. This is relativity of quantities i.e. largeness or smallness of some quantity only depends on the quantity with which it is being compared.

Further, I may note that the independence of the gravitational field (due to some source mass  $m_2$ ) from the mass of the test object ( $m_1$ ), is only restored when the conditions  $d > \lambda_0 \gg 2s_1, 2s_2$  are satisfied, where  $s_1 = Gm_1/c^2, s_2 = Gm_2/c^2$ . This manifests how the geometric foundation of physics goes hand in hand with the fact that all masses fall with same acceleration. This is because  $d \gg 2s_1$  means any notion of distance while expressed as a line on paper to denote the trajectory of the test object, contains many more than just two points where the extension of the point is negligible compared to the chosen length unit which itself is a line with very many points. Since both, Newton’s theory of gravity[5] and Einstein’s general relativity[17], are founded upon such a thesis (i.e. all masses fall with same acceleration), the mass of the test object does not play a role in such theories. Therefore, the methodology that I have demonstrated may be of help in certain situations which are not satisfactorily explained by any of the above two theories e.g. the missing mass problem of galaxies[18]. I hope to develop along such directions in near future. Further, there is another issue that I have not focused on in this work and that is the condition  $d > \lambda_0 \gg 2s_2$  where  $s_2$  is the dot that represents the source object as a whole. This condition does not have a direct implication in the calculation which I have shown, except the fact that it ensures the possibility of representation of the source object as a dot whose extension is negligible compared to the line that represents the chosen unit of length. An immediate question arises that what happens to the representation if  $d > \lambda_0 > 2s_2$ . I hope to elaborate on this issue in near future.

Last but not the least, I may conclude by mentioning that the case of two overlapping dots is a feature of the present analysis which is not possible to be demonstrated by any currently existing theories of gravity or “quantum gravity”[3]. This particular scenario, which manifests the singularity problem in standard literature of physics, is now explicated without any trouble at all, as I have explained in this non-standard scenario. Due to this single most reason, I believe, the present work is novel in its own right. What remains to be seen is whether it is possible to write theoretical physics in such singularity free scenario. Although I have provided a glimpse of, what I may call, Non-Standard Physics in Appendix(E), I plan to provide the details of such theoretical construction in near future.

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## A A critical commentary on Einstein’s views regarding “Physical Meaning of Geometrical Propositions”

In what follows, I shall offer some critical comments on a few relevant statements by Einstein from the first chapter of ref.[17] in order to show how Einstein was not operational and rather relied on logic, which by nature is abstract and detached from immediate truths of experience as Einstein himself pointed out. So, what Bridgman called as “operational” was only a contextual characteristic of Einstein’s reasoning that founded his relativity theories i.e. while Einstein explained the concepts verbally he was operational because his explanations were directly attached to experienced facts, but while he did mathematical analysis he relied on the axioms of geometry. I analyze Einstein’s statements part by part as follows.

*“Geometry sets out from certain conceptions such as “plane,” “point,” and “straight line,” with which we are able to associate more or less definite ideas, and from certain simple propositions (axioms) which, in virtue of these ideas, we are inclined to accept as “true.” ”*

It is manifest from Einstein’s words that geometric conceptions are associated with “more or less definite” ideas i.e. geometric conceptions are approximate and not exact or definite. Furthermore, such ideas are assumed to be universal truths and that is why we need to be “inclined to accept” such truths in order to proceed with the axiomatic framework of geometry. Such inclinations or biases are the necessities for working with such axiomatic framework irrespective of whether such assumed truths defy experience or not. Indeed Einstein pointed this out shortly.

*“The question of the “truth” of the individual geometrical propositions is thus reduced to one of the “truth” of the axioms. Now it has long been known that the last question is not only unanswerable by the methods of geometry, but that it is in itself entirely without meaning. ”*

Here, Einstein pointed out that the “truth” of any axiom of geometry can not be verified by remaining within the axiomatic framework of geometry. This certainly means that if any one axiom of geometry is considered as “false”, then the basis of such consideration can not be geometry itself. The experience of seeing a dot, which is an act of being operational, is thus necessary to be taken into consideration if the limited validity of the axioms of geometry is to be explored. And this is reflected from Einstein’s own admission that I point out next.

*“The concept “true” does not tally with the assertions of pure geometry, because by the word “true” we are eventually in the habit of designating always the correspondence with a “real” object; geometry, however, is not concerned with the relation of the ideas involved in it to objects of experience, but only with the logical connection of these ideas among themselves.”*

This statement by Einstein clearly justifies in what sense he was not operational and it was because of his reliance on the axioms of geometry to write down theories. As Einstein emphasized, the word “true” is only meaningful while we relate our ideas to some “real” objects of experience. So, considering Bridgman’s notion

of being “operational”, the axioms of geometry are not operational (or anti-operational) as those axioms are statements which do not explain our experience. Further, considering Einstein’s explication of the word “true”, a statement that expresses the visible dot of the pencil can be only considered to be “true” by virtue of our experience of vision of the dot. Therefore, I may conclude that although Einstein was operational while explaining concepts based on experience of observed phenomena, but he was not operational due to his use of the axioms of geometry to write down his theories.

## B More clarifications on the statement regarding “a point” and “a line”

Here I clarify certain issues with some expected objections in mind those may be raised by the logic minded reader. I enlist such expected objections and the possible refutations as follows:

- Is my statement a definition of “a point” or “a line”? I should say “it is neither” and “it is both” – the answer depends on the way the situation is analyzed. It is *neither* because I have not considered the term “definition” at all (except on a different occasion in the sense of symbolic abbreviation – see footnote(8)). Rather, I have adopted a practical method of demonstrating facts in simple language which depends of the mindset of the reader to be deemed as acceptable. If I consider the term “definition” as “explication”, then I have explicated *both* “a point” and “a line” in relation to each other. This is because I can not realize any of those concepts in isolation, but only in relation to each other. When I put a dot on the paper, it seems of negligible extension only if I have the experience of drawing a line with respect to which I write the dot’s extension to be “negligible”. Otherwise, the word “negligible” by itself is incomplete in the sense that one can always question “negligible with respect to what?”
- I can draw a line only after putting a dot. So, is my statement circular in reasoning? The answer is “yes” and “no” depending on the way one chooses to analyze the situation. The statement *is* circular if “a dot” (“a point”) and “a line” are considered as isolated and logical truths, as it is done in accord with the axioms of geometry. However, in that process, the truth of the experience is denied – an anti-operational process of reasoning. The statement *is not* circular if the experience of putting a dot and drawing a line is truthfully analyzed. Certainly the dot needs to be put first and then the line can be drawn while I draw with a pencil. However, the negligible extension of a dot can be realized only after drawing, and in relation to, a line. So, neither a dot nor a line has in itself the truth of experience in isolation. Rather, both can be realized in relation to the other. Therefore, my statement is just an expression of demonstration based on experience. I do not want to categorize only parts of the statement into logical and hence, isolated absolute truths that only leads to logical paradoxes like this objection of circular reasoning.

So, instead of putting logical analysis into the context, I consider my investigation as a series of reasonable expressions that explicate my experience. Nevertheless, I must admit that such operational process of reasoning brings in the danger of consideration of an infinite process of reasoning because one can raise a legitimate doubt regarding whether I can ever stop expressing my experience e.g. colour of the paper, colour of the pencil, the pressure I apply to put the dot and so on. So, I declare the incompleteness of my expression and my choice to cut off the reasoning where I find it suitable and sufficient for the present context.

Even after all these explanations, I expect the modern scientist, who puts his complete trust in logic and hence, his belief in complete definitions, to be unwilling to believe in the process of reasoning that I have adopted here. To germinate the seed of doubt in such a mind, I have analyzed Riemann’s definition of the concept of “line” in Appendix(D).

## C Born's wish of uncertain numbers, Klein's vision of "a small spot" and "fusion of arithmetic and geometry": an unfulfilled desire

The experience of the dot and the indispensability of the cuts, render the explication of numbers to be inexact. Such a demonstration process is always accompanied with errors due to the mode of demonstration itself and without such demonstration the ideas remain abstract and can not be applied for practical purpose, unless it is lied about. Apart from Descartes' doubt regarding the exactness of numbers being explicated in such a way (see footnote (9)), I find it interesting and worthy to note Born's and Klein's views regarding such issues so that the importance of the present work is manifested in a more convincing manner.

Born wrote the following on page no. 81 of ref.[30]:

*"Of course, I do not intend to banish from physics the idea of a real number. It is indispensable for the application of analysis. What I mean is that a physical situation must be described by means of real numbers in such a way that the natural uncertainty in all observations is taken into account."*

I believe, Born would have agreed that demonstration of the concept of number by drawing a line and making cuts, which nevertheless begins with putting a visible dot on the paper, is both a physical act and a physical observation as well. Thus, the uncertainty ( $\epsilon, \delta$  etc.), that is incorporated in my demonstration, is what Born tried to indicate (my guess). Assuming Born's agreement, I may consider that the demonstration procedure that I have adopted and the way I have analyzed the situation, can be considered as quite believable by the modern authorities of science. However, I may go further to bring to light some more convincing historical visions along such lines of thought, which Born himself provided in the statements which followed the above quoted ones. Born's comment concerning Klein's vision, that followed the above quoted statement only justifies the essence of the present discussion in an even more explicit and convincing way:

*"Felix Klein called for a similar step to be taken in geometry. Besides abstract, exact geometry, he desired to have a practical geometry, in which a point is replaced by a small spot, straight lines by narrow strips, etc. However, nothing much resulted from this."*

Although Born did not analyze further why Klein could not accomplish what he wanted to, it appears to me that once "a point" is replaced by "a small spot", then the question arises that "small with respect to what". An obvious, and the simplest (according to me), answer is that "with respect to any other extension that one may call a line" i.e. the relational aspect of the scenario (i.e. the visual cognition) needs to be necessarily taken into account and the involvement of units become a requirement because neither the dot nor the line, by itself, explicates the concept of number that is represented by numeral. Then the scenario can not be categorized as either "arithmetic" or "geometry", but somewhere in the middle – a practical method of demonstration and the associated analysis, which Klein himself might have called "*fusion of arithmetic and geometry*" and not any one in isolation – see page no. 2 of ref.[31]. However, any act of demonstration of this "fusion", which being a "practical method", must rely on a mode of demonstration. This mode of demonstration is certainly a physical act that leads to the "physical situation" (in Born's words) involving the extension of the dot that is negligible with respect to the extension of a line. Certainly Klein highlighted the issue of inexactness associated with such an explication of the scenario. However, what was missing in Klein's analysis was this understanding of the relational nature of the underlying situation that Born also could not get the grasp of.

Nevertheless, it is worth clarifying certain aspects of this work in tandem with some of the statements of Klein from pp. 42-43 of ref.[32]:

*"If we now ask how we can account for this distinction between the naive and refined intuition, I must say that, in my opinion, the root of the matter lies in the fact that the naive intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact. To explain the meaning of the first half of this statement it is my opinion that, in our naive intuition, when thinking of a point we do not picture to our mind an abstract mathematical point, but substitute something concrete for it. In imagining a line, we do not picture to ourselves "length without breadth", but a strip of a certain width."*

These are the statements, from which I have quoted the appropriate words while demonstrating the impossibility of a complete refinement of the act of counting dots in Section(3) and therefore, the importance

of the condition  $(N_d^{s_i} + 2)\delta_i^{s_i} s_i < s_i$  which I imposed. Added to these statements, it is important to note what Klein wrote next regarding the term “definition” while it comes to the words “point”, “line”, etc.

*“Now such a strip has of course always a tangent....; i.e. we can always imagine a straight strip having a small portion (element) in common with the curved strip; similarly with respect to the osculating circle. The definitions in this case are regarded as holding only approximately, or as far as may be necessary. The “exact” mathematicians will of course say that such definitions are not definitions at all. But I maintain that in ordinary life we actually operate with such inexact definitions. Thus we speak without hesitancy of the direction and curvature of a river or a road, although the “line” in this case has certainly considerable width.”*

Certainly the term “definition” is logical and has an inherent appeal of being complete or exact or perfect. Then, to be logical a complete refinement of intuition is necessary and then, in the process, directly experienced truth needs to be ignored in the case of the dot or the line. This is why, considering such a view in accord with Klein, I have used terms like “explication”, “demonstration”, etc. instead of the term “definition” in the context “point”, “line”, etc. In agreement with Klein’s view, Kant would prefer the word “*exposition*” instead of the word “definition”:

*“Instead of the term, definition, I prefer to use the term, exposition, as being a more guarded term, which the critic can accept as being up to a certain point valid, though still entertaining doubts as to the completeness of the analysis.”*

– see page no. 144, Vol. 1 of ref.[36]. For a very simple demonstration of such a logical dilemma while considering the term “definition”, one may consult Section (2.1.2) of ref.[34] and the relevant discussion regarding Frege’s demonstration of definition of a concept analyzed in Section (2.1) of ref.[35].

## D Analyzing Riemann’s expressions regarding “The hypotheses on which geometry is based” from the logician’s perspective

In what follows I analyze some of the statements of Riemann in ref.[9] so as to bring forth the subtleties of reasoning that gets associated if each of his verbal statements are carefully analyzed. The two issues that I intend to raise concern about are the following: (i) Riemann’s definition of “a line” is circular in reasoning (ii) Riemann’s statements regarding “infinitesimal” are incomplete.

### D.1 Circular reasoning in Riemann’s definition of the concept of “line”

Riemann wrote on pp 261-262 of ref.[9]:

*“Measurement requires that the measure of the entities being measured must be independent of their location, and this can be the case in more than one way.”*

If I suppose, a length  $L$  being measured in terms of some length unit  $\lambda_0$ , then the number  $n_L^{\lambda_0}$  that is yielded must be independent of the location where the measurement process takes place. I believe this is what Riemann meant in the above statement. Einstein assumed that such measurement process is exact and does not contain any error as he wrote on p 4 of ref.[17] “*that there is nothing left over, i.e. that the measurement gives a whole number.*”. Therefore,  $n_L^{\lambda_0}$ , which represents the number of times  $\lambda_0$  can be superposed on  $L$ , can only take values  $1, 2, 3, \dots$ . Now, Riemann went on to write,

*“The assumption which first suggests itself, and which I intend to pursue here, is that the length of lines is independent of their position, so that every line can be measured by comparing it with any other line.”*

The word “location” is now replaced by “position”. Then Riemann wrote the following.

*“If the determination of the position of a point in a given n-dimensional manifold is reduced to the determination of n variables  $x_1, x_2, x_3, \dots, x_n$ , then a line may be defined by the statement that the quantities  $x$  are given functions of a single variable.”*

Here, I raise the following question: *How is the position of a point determined?* Since Riemann did not explain further the word “determination”, I consider the following explanation.

- I choose some origin  $O$  and draw a line by joining  $O$  and the point  $P$  (say). The line  $OP$  is the distance  $d$  of the point  $P$  from  $O$ .

- I choose another line  $\lambda_0$  such that it can be superposed multiple times on  $d$  to generate a number  $n_d^{\lambda_0}$  (say). Then, I write  $d = n_d^{\lambda_0} \lambda_0$ .

The above steps explicate the meaning of “determination” of the position of a point. Then the quantities  $x$  (which, I believe, should be written as  $x_i \ni i = 1, 2, 3, \dots, n$ ) should be written in terms of  $\lambda_0$ . Therefore, this act of “determination” is itself a measurement process. Then, according to Riemann’s own statement, the yielded number  $n_d^{\lambda_0}$  should be independent of the “location” of the lines that represent the distance  $d$  and the length unit  $\lambda_0$ . The problem is now to give meaning to the word “location/position”. This is because the distance  $d$  itself is now a line that is getting measured for the determination process to be carried out. Then, one needs to choose another origin, say  $O'$ , with respect to which the “location/position” of  $d$  and  $\lambda_0$  needs to be “determined” so that the previous determination holds any meaning according to Riemann. However, the new “determination” is again a measurement process and to give meaning to it, another origin, say  $O''$ , needs to be chosen. The process goes on. Hence, the term “determination” can not be completely *defined* and can only be *explicated* with partial satisfaction as there is always a doubt retained in the process of reasoning owing to such self-inquiry[34].

Further, it must be clear from the above explanation that in order to carry out the “determination” process, the lines need to be drawn i.e. the concept of “line” needs to be used. However, instead of analyzing his own words in such a way, Riemann “defined” what a “line” is, without explication of the term “determination” for which he needed to use the concept of a “line” in the first place. Since geometry is a system of logical truths or axioms then, viewing from the logician’s perspective, such definition of “line” appears to be based on circular reasoning i.e. Riemann’s definition of “line” is not a logical definition. However, if one ignores the logical rigor of such a logical system of thoughts then Riemann’s analysis is definitely useful in an operational way because we do general relativity based on such concepts. Therefore, the foundations of Riemannian geometry, which forms the basis of general relativity, are both logical and not logical in the same process of reasoning – logical because such a definition of “line” is accepted to be true and considered as axiom; illogical because such a definition is circular in reasoning and hence, can not be considered as logical by the logician.

## D.2 Riemann’s incomplete statement: “infinitesimal” with respect to what?

Riemann continued to write the following:

*“The problem then is to find a mathematical expression for the length of a line, and for this purpose we need to consider the quantities  $x$  as expressible in terms of units.”*

As I have explained earlier while explicating the term “determination”, Riemann did acknowledge that the quantities  $x$  ( $x_i$ ) should be expressed in terms of units. However, Riemann’s verbal statements are not truthfully translated into his equations because the units are not written explicitly. This is important to note because of what Riemann wrote next.

*“I shall handle this problem only under certain restrictions, and confine myself in the first place to lines in which the relations between the quantities  $dx$  - the associated variations of the variables  $x$  - vary in continuous fashion. We can then visualize the line as being divided up into elements, within which the ratios of the increments  $dx$  can be regarded as constant, and the problem reduces to finding a general expression for line element starting from a given point, which will involve the variables  $x$  as well as the variables  $dx$ .”*

Here, there is no clarification regarding whether the elements of the line are bigger than or smaller than or equal to the length unit. This is important because of what Riemann wrote next.

*“Secondly, I shall assume that the length of the line element, disregarding quantities of the second order of magnitude, remains unchanged if all its points undergo the same infinitesimal displacement.”*

I believe that by the word “infinitesimal” Riemann meant “infinitesimally small”. If not, then the word “infinitesimal” needs to be clarified in a more elementary fashion. If yes, then an obvious question arises about the displacement and that is, infinitesimally small with respect to what? Is it with respect to the length unit or the quantity  $x$ ? If it is with respect to the length unit, then how does the theory look like when smaller length units are chosen because in that case the displacement does not remain “infinitesimal” anymore? If it is with respect to the quantity  $x$ , then the theory should be written in such a way that the quantity  $x$  must have, as Born would write (see Appendix(C)), a “natural uncertainty” much greater than

$dx$  irrespective of the role of the length unit. Neither do I find any answer to such basic questions nor do I find clarifications regarding such basic doubts anywhere in ref.[9]. Therefore, I find Riemann’s use of the word “infinitesimal” to have only an incomplete sense.

## E A glimpse of non-standard physics

In this work, I have used the concept of “mass” as an independent physical dimension so as to draw the connection with standard physics literature by writing  $s_i = Gm_i/c^2$ . However, the relevant equations can be written in terms of  $s_i$ , in relation to  $\lambda_0$ , only and this will result in the appearance extremely small numbers in the associated analysis which is quite akin to what one encounters in Non-Standard Analysis (NSA) [19]<sup>13</sup>. Goedel asserted that “*there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.*” (see the *Preface* of ref.[19]). Certainly, what I have discussed in this work has a priori nothing to do with NSA that is founded on mathematical logic and does not involve physical dimensions. However, it would not be a criminal offense to consider it as an “other version” of NSA. I may call this *Non-Standard Physics* (NSP) so as to distinguish it from standard physics literature and from NSA due to the distinctions that I mentioned before. While a detailed discussion regarding NSP is beyond the scope of this article, however, some immediate results from NSP can be showcased so as to convince the reader that it is a clear possibility. In NSP, instead of being considered as an independent physical dimension, the concept of “point mass” founded on which is classical mechanics of standard physics, is expressed as an intrinsic length of an object, considered as a whole, that is extremely small compared to the chosen conventional length unit. So, the unit of intrinsic length, if called “kilogram” and abbreviated as “kg”, then unlike standard physics, now we have “kg  $\lll$  meter”, where “meter” is the chosen unit of length. Any other such intrinsic lengths are some multiples of “kg”. I provide below a comparison between the standard physics and NSP, in light of the choice of units that we make in the beginning of classical mechanics while studying the laws of motion, followed by a simple problem to explicate the situation.

### Choice of units

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<sup>13</sup>For modern texts see, for example, refs.[20, 21]

Standard Physics	Non-standard Physics
<p>Force on an object, considered as a whole and represented as a “point mass”, is written as <math>F \propto ma</math>. Therefore,</p> $F = kma,$ <p>where <math>k</math> is a proportionality constant with appropriate physical dimension. For <math>m = 1\text{kg}</math>, <math>a = 1\text{meter}/\text{sec}^2</math>, we have</p> $F = k \text{ kg.meter}/\text{sec}^2.$ <p>We choose <math>k</math> such that 1 unit of mass, having 1 unit of acceleration is equivalent to 1 unit of force. So, we choose <math>k = 1</math> to write</p> $1\text{N} = 1\text{kg.meter}/\text{sec}^2,$ <p>where <math>N</math> stands for ‘Newton’, the unit of force. This is a convention to write the unit of force that is derived from the units of mass and acceleration.</p>	<p>Force on an object, considered as a whole and represented as “an intrinsic length that is extremely small compared to the chosen conventional unit of length”, is written as <math>F \propto sa</math>. Therefore,</p> $F = ksa,$ <p>where <math>k</math> is a proportionality constant with appropriate physical dimension. Chosen conventional unit of length is called “meter”. So, we write <math>s \ll\ll \text{meter}</math>. Unit of <math>s</math> (not <math>m</math>) is called “kilogram”, abbreviated as “kg” such that <math>\text{kg} \ll\ll \text{meter}</math>. Then, (instead of “<math>m = 1\text{kg}</math>”) we write “<math>s = 1\text{kg} = 1.\epsilon_{\text{kg}}^{\text{meter}} \text{meter} = \epsilon_{\text{kg}}^{\text{meter}} \text{meter} \ni 0 &lt; \epsilon_{\text{kg}}^{\text{meter}} \ll\ll 1</math>”. So, for <math>s = 1\text{kg} = \epsilon_{\text{kg}}^{\text{meter}} \text{meter}</math>, <math>a = 1 \text{meter}/\text{sec}^2</math>, we have</p> $\begin{aligned} F &= k(1\text{kg}).(1\text{meter}/\text{sec}^2) \ni 0 < \epsilon_{\text{kg}}^{\text{meter}} \ll\ll 1 \\ &= k\epsilon_{\text{kg}}^{\text{meter}} \text{meter}^2/\text{sec}^2. \end{aligned}$ <p>We choose <math>k</math> such that 1 unit of intrinsic length (extremely small compared to conventional length unit), having 1 unit of acceleration is equivalent to 1 unit of force. So, we choose <math>k = 1</math> to write</p> $1 \text{ N} = \epsilon_{\text{kg}}^{\text{meter}} \text{meter}^2/\text{sec}^2,$ <p>where <math>N</math> stands for ‘Newton’, the unit of force.</p>

### A simple problem

A force of 5 N gives a mass  $m_1$ , an acceleration of 10 meter/sec<sup>2</sup> and a mass  $m_2$  an acceleration of 20 meter/sec<sup>2</sup>. What acceleration would it give if both the masses were tied together?



Solution in Standard Physics	Solution in Non-standard Physics
<p>We note that <math>N = \text{kg}\cdot\text{meter}/\text{sec}^2</math>. Now, we solve the problem as follows.</p> $5 \text{ N} = m_1 \cdot 10 \text{ meter}/\text{sec}^2 \quad \equiv \quad m_1 = \frac{1}{2} \text{ kg}$ $5 \text{ N} = m_2 \cdot 20 \text{ meter}/\text{sec}^2 \quad \equiv \quad m_2 = \frac{1}{4} \text{ kg}$ <p>If <math>a</math> is the acceleration of the joint mass, then we can write</p> $5\text{N} = (m_1 + m_2)a = \left(\frac{1}{2} + \frac{1}{4}\right) \text{kg}\cdot a$ $\equiv a = \frac{20}{3} \text{meter}/\text{sec}^2.$	<p>First we replace <math>m_1, m_2</math> by <math>s_1, s_2</math> and also we note that <math>N = \epsilon_{kg}^{meter} \text{ meter}^2/\text{sec}^2 \ni 0 &lt; \epsilon \lll 1</math>. Now, we solve the problem as follows.</p> $5\text{N} = s_1 \cdot 10 \text{ meter}/\text{sec}^2 \quad \equiv \quad s_1 = \frac{1}{2} \epsilon_{kg}^{meter} \text{ meter},$ $5\text{N} = s_2 \cdot 20 \text{ meter}/\text{sec}^2 \quad \equiv \quad s_2 = \frac{1}{4} \epsilon_{kg}^{meter} \text{ meter}.$ <p>If <math>a</math> is the acceleration of the tied collection of bodies, then we can write</p> $5\text{N} = (s_1 + s_2)a = \left(\frac{1}{2} + \frac{1}{4}\right) \text{kg}\cdot a$ $\equiv a = \frac{20}{3} \text{meter}/\text{sec}^2.$

In view of this, I may write that it now becomes just a matter of further effort to understand how I can write down the known “laws” of standard physics as only approximate truths from NSP.

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