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► **To cite this version:**

Philippe Laurençot, Bogdan-Vasile Maticiuc. BOUNDED WEAK SOLUTIONS TO A CLASS OF DEGENERATE CROSS-DIFFUSION SYSTEMS. *Annales Henri Lebesgue*, 2023, 6, pp.847–874. 10.5802/ahl.179 . hal-03528167

**HAL Id: hal-03528167**

**<https://hal.science/hal-03528167v1>**

Submitted on 17 Jan 2022

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# BOUNDED WEAK SOLUTIONS TO A CLASS OF DEGENERATE CROSS-DIFFUSION SYSTEMS

PHILIPPE LAURENÇOT AND BOGDAN-VASILE MATIOC

ABSTRACT. Bounded weak solutions are constructed for a degenerate parabolic system with a full diffusion matrix, which is a generalized version of the thin film Muskat system. Boundedness is achieved with the help of a sequence  $(\mathcal{E}_n)_{n \geq 2}$  of Liapunov functionals such that  $\mathcal{E}_n$  is equivalent to the  $L_n$ -norm for each  $n \geq 2$  and  $\mathcal{E}_n^{1/n}$  controls the  $L_\infty$ -norm in the limit  $n \rightarrow \infty$ . Weak solutions are built by a compactness approach, special care being needed in the construction of the approximation in order to preserve the availability of the above-mentioned Liapunov functionals.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ ,  $N \geq 1$ , with smooth boundary  $\partial\Omega$  and let  $R$  and  $\mu$  be two positive real numbers. In a recent paper [11], we noticed that there is an infinite family  $(\mathcal{E}_n)_{n \geq 1}$  of Liapunov functionals associated with the thin film Muskat system

$$\begin{aligned} \partial_t f &= \operatorname{div} (f \nabla [(1 + R)f + Rg]) \quad \text{in } (0, \infty) \times \Omega, \\ \partial_t g &= \mu R \operatorname{div} (g \nabla [f + g]) \quad \text{in } (0, \infty) \times \Omega, \end{aligned}$$

supplemented with homogeneous Neumann boundary conditions and initial conditions, with the following properties: for all  $n \geq 2$ , there are  $0 < c_n < C_n$  such that

$$c_n \|f + g\|_n^n \leq \mathcal{E}_n(f, g) \leq C_n \|f + g\|_n^n, \quad (f, g) \in L_{n,+}(\Omega, \mathbb{R}^2),$$

and there are  $0 < c_\infty < C_\infty$  such that

$$c_\infty \|f + g\|_\infty \leq \liminf_{n \rightarrow \infty} \mathcal{E}_n(f, g)^{1/n} \leq \limsup_{n \rightarrow \infty} \mathcal{E}_n(f, g)^{1/n} \leq C_\infty \|f + g\|_\infty$$

for  $(f, g) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ , where  $L_{p,+}(\Omega, \mathbb{R}^m)$  denotes the positive cone of  $L_p(\Omega, \mathbb{R}^m)$  for  $m \geq 1$  and  $p \in [1, \infty]$ . On the one hand, the thin film Muskat system being of cross-diffusion type (i.e., featuring a diffusion matrix with no zero entry), the availability of such a family of Liapunov functionals is rather seldom within this class of systems and paves the way towards the construction of bounded weak solutions, a result that we were only able to show in one space dimension  $N = 1$  in [11]. On the other hand, it is tempting to figure out whether this property is peculiar to the thin film Muskat system or extends to the generalization thereof

$$\partial_t f = \operatorname{div} (f \nabla [af + bg]) \quad \text{in } (0, \infty) \times \Omega, \tag{1.1a}$$

$$\partial_t g = \operatorname{div} (g \nabla [cf + dg]) \quad \text{in } (0, \infty) \times \Omega, \tag{1.1b}$$

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*Date:* January 17, 2022.

*2020 Mathematics Subject Classification.* 35K65 - 35K51 - 37L45 - 35B65 .

*Key words and phrases.* degenerate parabolic system - cross-diffusion - boundedness - Liapunov functionals - global existence.

with  $(a, b, c, d) \in (0, \infty)^4$ , supplemented with homogeneous Neumann boundary conditions

$$\nabla f \cdot \mathbf{n} = \nabla g \cdot \mathbf{n} = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \quad (1.1c)$$

and non-negative initial conditions

$$(f, g)(0) = (f^{in}, g^{in}) \quad \text{in } \Omega. \quad (1.1d)$$

Obviously, the thin film Muskat system is a particular case of (1.1a)-(1.1b), corresponding to the choice  $(a, b, c, d) = (1 + R, R, \mu R, \mu R)$ .

The main result of this paper is to show that, for any quadruple  $(a, b, c, d)$  satisfying

$$(a, b, c, d) \in (0, \infty)^4 \quad \text{and} \quad ad > bc, \quad (1.2)$$

we can associate a similar family of Liapunov functionals with (1.1) and prove the global existence of bounded non-negative weak solutions to (1.1), whatever the dimension  $N \geq 1$ . More precisely, given a quadruple  $(a, b, c, d)$  satisfying (1.2), we define a sequence  $(\Phi_n)_{n \geq 1}$  of functions as follows. Setting  $L(r) := r \ln r - r + 1 \geq 0$ ,  $r \geq 0$ , we first define the function  $\Phi_1$  by the relation

$$\Phi_1(X) := L(X_1) + \frac{b^2}{ad}L(X_2), \quad X = (X_1, X_2) \in [0, \infty)^2. \quad (1.3)$$

Next, for each integer  $n \geq 2$ , let  $\Phi_n$  be the homogeneous polynomial of degree  $n$  defined by

$$\Phi_n(X) := \sum_{j=0}^n a_{j,n} X_1^j X_2^{n-j}, \quad X = (X_1, X_2) \in \mathbb{R}^2, \quad (1.4)$$

with  $a_{0,n} := 1$  and

$$a_{j,n} := \binom{n}{j} \prod_{k=0}^{j-1} \frac{ak + c(n-k-1)}{bk + d(n-k-1)} > 0, \quad 1 \leq j \leq n. \quad (1.5)$$

We then define, for  $n \geq 1$ , the functional

$$\mathcal{E}_n(u) := \int_{\Omega} \Phi_n(u(x)) dx, \quad u = (f, g) \in L_{\max\{2,n\},+}(\Omega, \mathbb{R}^2). \quad (1.6)$$

We finally observe that (1.2) guarantees that

$$\Theta_1 := \frac{b(ad+bc)}{2ad} > 0 \quad \text{and} \quad \Theta_2 := \frac{(ad-bc)(3ad+bc)}{4a^2d^2} > 0. \quad (1.7)$$

With this notation, the main result of this paper is the following:

**Theorem 1.1.** *Assume (1.2) and let  $u^{in} := (f^{in}, g^{in}) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$  be given. Then, there is a bounded weak solution  $u = (f, g)$  to (1.1) such that:*

(i) for each  $T > 0$ ,

$$(f, g) \in L_{\infty,+}((0, T) \times \Omega, \mathbb{R}^2) \cap L_2((0, T), H^1(\Omega, \mathbb{R}^2)) \cap W_2^1((0, T), H^1(\Omega, \mathbb{R}^2)'); \quad (1.8)$$

(ii) for all  $\varphi \in H^1(\Omega)$  and  $t \geq 0$ ,

$$\int_{\Omega} (f(t, x) - f^{in}(x))\varphi(x) dx + \int_0^t \int_{\Omega} f(s, x) \nabla[af + bg](s, x) \cdot \nabla\varphi(x) dx ds = 0 \quad (1.9a)$$

and

$$\int_{\Omega} (g(t, x) - g^{in}(x))\varphi(x) dx + \int_0^t \int_{\Omega} g(s, x) \nabla[cf + dg](s, x) \cdot \nabla\varphi(x) dx ds = 0; \quad (1.9b)$$

(iii) for all  $t \geq 0$ ,

$$\mathcal{E}_1(u(t)) + \frac{1}{a} \int_0^t \int_{\Omega} [|\nabla(af + \Theta_1 g)|^2 + \Theta_2 |\nabla g|^2](s, x) \, dx ds \leq \mathcal{E}_1(u^{in}), \quad (1.10)$$

where the positive constants  $\Theta_1$  and  $\Theta_2$  are defined in (1.7);

(iv) for all  $n \geq 2$  and all  $t \geq 0$ ,

$$\mathcal{E}_n(u(t)) \leq \mathcal{E}_n(u^{in}); \quad (1.11)$$

(v) for  $t \geq 0$ ,

$$\|f(t) + g(t)\|_{\infty} \leq \frac{d \max\{a, b\}}{b \min\{c, d\}} \|f^{in} + g^{in}\|_{\infty}. \quad (1.12)$$

Let us first mention that Theorem 1.1 improves [11] in two directions: on the one hand, it shows that the structural properties (1.10), (1.11), and (1.12), uncovered there for the thin film Muskat system, are also available for the whole class (1.1). On the other hand, it provides the existence of non-negative bounded weak solutions to (1.1) in all space dimensions, a result which was only established in one space dimension in [11]. Global weak solutions to the thin film Muskat system are also constructed in [1–3, 6, 9, 10], but they need not be bounded, except in [3]. The latter however requires some smallness condition on the initial data, in contrast to Theorem 1.1. Finally, the local well-posedness of the thin film Muskat system in the classical sense is investigated in [7].

We next outline the main steps of the proof of Theorem 1.1. As in [11], the starting point is to notice that, introducing the mobility matrix

$$M(X) = (m_{jk}(X))_{1 \leq j, k \leq 2} := \begin{pmatrix} aX_1 & bX_1 \\ cX_2 & dX_2 \end{pmatrix}, \quad X = (X_1, X_2) \in \mathbb{R}^2, \quad (1.13)$$

and  $u := (f, g)$ , an alternative formulation of the system (1.1a)-(1.1b) is

$$\partial_t u = \sum_{i=1}^N \partial_i (M(u) \partial_i u) \quad \text{in } (0, \infty) \times \Omega. \quad (1.14)$$

Then, given  $\Phi \in C^2(\mathbb{R}^2, \mathbb{R})$ , it readily follows from (1.14), the homogeneous Neumann boundary conditions (1.1c), and the symmetry of the Hessian matrix  $D^2(\Phi)$  that

$$\frac{d}{dt} \int_{\Omega} \Phi(u) \, dx + \sum_{i=1}^N \int_{\Omega} \langle D^2 \Phi(u) M(u) \partial_i u, \partial_i u \rangle \, dx = 0, \quad (1.15)$$

where  $\langle \cdot, \cdot \rangle$  stands for the scalar product on  $\mathbb{R}^2$ . As a straightforward consequence of (1.15) we note that  $\int_{\Omega} \Phi(u) \, dx$  is a Liapunov functional for (1.14) when the matrix  $D^2 \Phi(u) M(u)$  is positive semidefinite. We shall then show in Appendix A that, for all  $n \geq 2$ , it is possible to construct an homogeneous polynomial  $\Phi_n \in \mathbb{R}[X_1, X_2]$  of degree  $n$  which is convex on  $[0, \infty)^2$  and such that the matrix  $D^2 \Phi_n(X) M(X)$  is positive semidefinite for all  $X \in [0, \infty)^2$ . A closed form formula is actually available for the polynomial  $\Phi_n$ , see (1.4) and (1.5).

We next construct weak solutions to (1.14) by a compactness method. It is here of utmost importance to construct approximations which do not alter the inequalities (1.15) for  $\Phi = \Phi_n$  and  $n \geq 1$ . As a first step, it is well-known that implicit time discrete schemes are well-suited in

that direction. Thus, given  $\tau > 0$ , we shall first prove the existence of a sequence  $(u_l^\tau)_{l \geq 0}$  which satisfies  $u_0^\tau = u^{in} := (f^{in}, g^{in})$  and, for  $l \geq 0$ ,

$$u_{l+1}^\tau - \tau \sum_{i=1}^N \partial_i \left( M(u_{l+1}^\tau) \partial_i u_{l+1}^\tau \right) = u_l^\tau \quad \text{in } \Omega, \quad (1.16)$$

supplemented with homogeneous Neumann boundary conditions. Furthermore, the sequence  $(u_l^\tau)_{l \geq 0}$  has the property that, for  $n \geq 1$  and  $l \geq 0$ ,

$$\mathcal{E}_n(u_{l+1}^\tau) + \tau \sum_{i=1}^N \int_{\Omega} \langle D^2 \Phi_n(u_{l+1}^\tau) M(u_{l+1}^\tau) \partial_i u_{l+1}^\tau, \partial_i u_{l+1}^\tau \rangle \, dx \leq \mathcal{E}_n(u_l^\tau), \quad (1.17)$$

so that the structural property (1.15) is indeed preserved by the time discrete scheme. The existence of a solution to (1.16) is achieved by a compactness method relying on an approximation of the matrix  $M(\cdot)$  by bounded ones. This step is actually the more delicate one, as we have to construct matrices approximating  $M(\cdot)$  which do not alter (1.17). To this end, a two-parameter approximation procedure is required and it is detailed in Section 2.2. The existence of a weak solution to (1.16) satisfying (1.17) is shown in Section 2.4, building upon preliminary and intermediate results established in Section 2.1 and Section 2.3.

**Remark 1.2.** *A common feature of system (1.1) is that it has, at least formally, a gradient flow structure for the functional  $\mathcal{E}_2$  with respect to the 2-Wasserstein distance in the space  $\mathcal{P}_2(\Omega, \mathbb{R}^2)$  of probability measures with finite second moments, as pointed out in [1, 9] for the thin film Muskat system. In particular, there is a natural variational structure associated with (1.1) which is suitable to construct weak solutions. However, the connection between this variational structure and the whole family  $(\mathcal{E}_n)_{n \geq 2}$  of Liapunov functionals is yet unclear.*

**Notation.** For  $p \in [1, \infty]$ , we denote the  $L_p$ -norm in  $L_p(\Omega)$  by  $\|\cdot\|_p$  and set

$$L_p(\Omega, \mathbb{R}^2) := L_p(\Omega) \times L_p(\Omega), \quad H^1(\Omega, \mathbb{R}^2) := H^1(\Omega) \times H^1(\Omega).$$

The positive cone of a Banach lattice  $E$  is denoted by  $E_+$ . The space of  $2 \times 2$  real-valued matrices is denoted by  $\mathbf{M}_2(\mathbb{R})$ , while  $\mathbf{Sym}_2(\mathbb{R})$  is the subset of  $\mathbf{M}_2(\mathbb{R})$  consisting of symmetric matrices and  $\mathbf{SPD}_2(\mathbb{R})$  is the set of symmetric and positive definite matrices in  $\mathbf{M}_2(\mathbb{R})$ . Finally, we denote the positive part of a real number  $r \in \mathbb{R}$  by  $r_+ := \max\{r, 0\}$  and  $\langle \cdot, \cdot \rangle$  is the scalar product on  $\mathbb{R}^2$ .

## 2. A TIME DISCRETE SCHEME

In order to construct bounded non-negative global weak solutions to the evolution problem (1.1), we employ a compactness approach, paying special attention to preserve as much as possible the structural properties (1.10), (1.11), and (1.12) in the design of the approximation. It turns out that implicit time discrete schemes are well-suited for that purpose and we thus establish in this section the existence of solutions to the implicit time discrete scheme associated with (1.1), see (2.1a)-(2.1b).

**Proposition 2.1.** *Given  $\tau > 0$  and  $U = (F, G) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ , there is a solution*

$$u = (f, g) \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2)$$

to

$$\int_{\Omega} (f\varphi + \tau f \nabla [af + bg] \cdot \nabla \varphi) \, dx = \int_{\Omega} F\varphi \, dx, \quad \varphi \in H^1(\Omega), \quad (2.1a)$$

$$\int_{\Omega} (g\psi + \tau g \nabla [cf + dg] \cdot \nabla \psi) \, dx = \int_{\Omega} G\psi \, dx, \quad \psi \in H^1(\Omega), \quad (2.1b)$$

which also satisfies

$$\mathcal{E}_n(u) \leq \mathcal{E}_n(U) \quad \text{for } n \geq 2 \quad (2.2)$$

and

$$\mathcal{E}_1(u) + \frac{\tau}{a} \int_{\Omega} [|\nabla(af + \Theta_1 g)|^2 + \Theta_2 |\nabla g|^2] \, dx \leq \mathcal{E}_1(U), \quad (2.3)$$

recalling that, see (1.7),

$$\Theta_1 = \frac{b(ad + bc)}{2ad} > 0 \quad \text{and} \quad \Theta_2 = \frac{(ad - bc)(3ad + bc)}{4a^2 d^2} > 0.$$

As already mentioned, several steps are involved in the proof of Proposition 2.1. We begin with the existence of bounded weak solutions to an auxiliary elliptic system which shares the same structure with (2.1), but has bounded coefficients instead of linearly growing ones, see Section 2.1. As a next step, we introduce in Section 2.2 the approximation to (2.1) which is derived from (2.1) by replacing the matrix  $M(\cdot)$  defined in (1.13) by a suitable invertible and bounded matrix  $M_{\varepsilon}^{\rho}(\cdot)$  with  $(\varepsilon, \rho) \in (0, 1) \times (1, \infty)$ . We emphasize here once more that the matrix  $M_{\varepsilon}^{\rho}(\cdot)$  is designed in such a way that the inequalities (2.2) and (2.3) are not significantly altered. Passing to the limit, first as  $\rho \rightarrow \infty$ , and then as  $\varepsilon \rightarrow 0$ , is then performed in Section 2.3 and Section 2.4, respectively, this last step completing the proof of Proposition 2.1.

Throughout this section,  $C$  and  $(C_l)_{l \geq 0}$  denote various positive constants depending only on  $N, \Omega$ , and  $(a, b, c, d)$ . Dependence upon additional parameters will be indicated explicitly.

**2.1. An auxiliary elliptic system.** Let  $A = (a_{jk})_{1 \leq j, k \leq 2}$  and  $B = (b_{jk})_{1 \leq j, k \leq 2}$  be chosen such that  $A \in \mathbf{SPD}_2(\mathbb{R})$ ,  $B \in BC(\mathbb{R}^2, \mathbf{M}_2(\mathbb{R}))$ , and  $AB(X) \in \mathbf{SPD}_2(\mathbb{R})$  for all  $X \in \mathbb{R}^2$ . Moreover, we assume that there is  $\delta_1 > 0$  such that

$$\langle AB(X)\xi, \xi \rangle \geq \delta_1 |\xi|^2, \quad (X, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2. \quad (2.4)$$

Since  $A \in \mathbf{SPD}_2(\mathbb{R})$ , there is also  $\delta_2 > 0$  such that

$$\langle A\xi, \xi \rangle \geq \delta_2 |\xi|^2, \quad \xi \in \mathbb{R}^2. \quad (2.5)$$

**Lemma 2.2.** *Given  $\tau > 0$  and  $U = (U_1, U_2) \in L_2(\Omega, \mathbb{R}^2)$ , there is  $u = (u_1, u_2) \in H^1(\Omega, \mathbb{R}^2)$  which solves the nonlinear equation*

$$\int_{\Omega} \left[ \langle u, v \rangle + \tau \sum_{i=1}^N \langle B(u) \partial_i u, \partial_i v \rangle \right] \, dx = \int_{\Omega} \langle U, v \rangle \, dx, \quad v \in H^1(\Omega, \mathbb{R}^2). \quad (2.6)$$

Additionally:

(i) If

$$\begin{aligned} b_{11}(X) &\geq b_{12}(X) = 0, & X &\in (-\infty, 0) \times \mathbb{R}, \\ b_{22}(X) &\geq b_{21}(X) = 0, & X &\in \mathbb{R} \times (-\infty, 0), \end{aligned} \quad (2.7)$$

and if  $U(x) \in [0, \infty)^2$  for a.a.  $x \in \Omega$ , then  $u(x) \in [0, \infty)^2$  for a.a.  $x \in \Omega$ .

(ii) If there exists  $\rho > 0$  such that

$$\begin{aligned} b_{11}(X) &\geq b_{12}(X) = 0, & X &\in (\rho, \infty) \times \mathbb{R}, \\ b_{22}(X) &\geq b_{21}(X) = 0, & X &\in \mathbb{R} \times (\rho, \infty), \end{aligned} \quad (2.8)$$

and if  $\max\{U_1, U_2\} \leq \rho$  a.e. in  $\Omega$ , then  $\max\{u_1, u_2\} \leq \rho$  a.e. in  $\Omega$ .

*Proof.* The proof of Lemma 2.2 is rather classical and it is actually similar to that of [11, Lemma B.1]. We nevertheless sketch it below for the sake of completeness.

**Step 1.** To set up a fixed point scheme, we consider  $u \in L_2(\Omega, \mathbb{R}^2)$  and define a bilinear form  $b_u$  on  $H^1(\Omega, \mathbb{R}^2)$  by

$$b_u(v, w) := \int_{\Omega} \left[ \langle Av, w \rangle + \tau \sum_{i=1}^N \langle AB(u) \partial_i v, \partial_i w \rangle \right] dx, \quad (v, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega, \mathbb{R}^2).$$

Owing to (2.4) and (2.5),

$$b_u(v, v) \geq \delta_0 \|v\|_{H^1}^2, \quad v \in H^1(\Omega, \mathbb{R}^2), \quad (2.9)$$

where  $\delta_0 := \min\{\tau\delta_1, \delta_2\}$ , while the boundedness of  $B$  guarantees that

$$|b_u(v, w)| \leq b_* \|v\|_{H^1} \|w\|_{H^1}, \quad (v, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega, \mathbb{R}^2),$$

with

$$b_* := 2 \max_{1 \leq j, k \leq 2} \{|a_{jk}|\} \left( 1 + 2\tau \max_{1 \leq j, k \leq 2} \{\|b_{jk}\|_{\infty}\} \right).$$

We then infer from Lax-Milgram's theorem that there is a unique  $\mathcal{V}[u] \in H^1(\Omega, \mathbb{R}^2)$  such that

$$b_u(\mathcal{V}[u], w) = \int_{\Omega} \langle AU, w \rangle dx, \quad w \in H^1(\Omega, \mathbb{R}^2). \quad (2.10)$$

An immediate consequence of (2.9), (2.10) (with  $w = \mathcal{V}[u]$ ), and Hölder's inequality is the following estimate:

$$\delta_0 \|\mathcal{V}[u]\|_{H^1}^2 \leq b_u(\mathcal{V}[u], \mathcal{V}[u]) \leq \|AU\|_2 \|\mathcal{V}[u]\|_2 \leq \|AU\|_2 \|\mathcal{V}[u]\|_{H^1}.$$

Hence

$$\|\mathcal{V}[u]\|_{H^1} \leq \frac{\|AU\|_2}{\delta_0}. \quad (2.11)$$

We next argue as in the proof of [11, Lemma B.1] to show that the map  $\mathcal{V}$  is continuous and compact from  $L_2(\Omega, \mathbb{R}^2)$  to itself, the proof relying on (2.11), the compactness of the embedding of  $H^1(\Omega, \mathbb{R}^2)$  in  $L_2(\Omega, \mathbb{R}^2)$ , and the continuity and boundedness of  $B$ .

Consider now  $\theta \in [0, 1]$  and a function  $u \in L_2(\Omega, \mathbb{R}^2)$  satisfying  $u = \theta \mathcal{V}[u]$ . Then  $u \in H^1(\Omega, \mathbb{R}^2)$  and, in view of (2.11),

$$\|u\|_2 = \theta \|\mathcal{V}[u]\|_2 \leq \|\mathcal{V}[u]\|_2 \leq \|\mathcal{V}[u]\|_{H^1} \leq \frac{\|AU\|_2}{\delta_0}.$$

Thanks to the above bound and the continuity and compactness properties of the map  $\mathcal{V}$  in  $L_2(\Omega, \mathbb{R}^2)$ , we are in a position to apply Leray-Schauder's fixed point theorem, see [8, Theorem 11.3] for instance, and conclude that the map  $\mathcal{V}$  has a fixed point  $u \in L_2(\Omega, \mathbb{R}^2)$ . Since  $\mathcal{V}$  ranges in  $H^1(\Omega, \mathbb{R}^2)$ , the function  $u$  actually belongs to  $H^1(\Omega, \mathbb{R}^2)$  and satisfies

$$b_u(u, w) = \int_{\Omega} \langle AU, w \rangle dx, \quad w \in H^1(\Omega, \mathbb{R}^2).$$

Finally, given  $v \in H^1(\Omega, \mathbb{R}^2)$ , the function  $w = A^{-1}v$  also belongs to  $H^1(\Omega, \mathbb{R}^2)$  and we infer from the above identity and the symmetry of  $A$  that

$$\int_{\Omega} \langle U, v \rangle dx = \int_{\Omega} \langle AU, w \rangle dx = b_u(u, w) = b_u(u, A^{-1}v)$$

$$= \int_{\Omega} \left[ \langle u, v \rangle + \tau \sum_{i=1}^N \langle B(u) \partial_i u, \partial_i v \rangle \right] dx.$$

We have thus constructed a weak solution  $u \in H^1(\Omega, \mathbb{R}^2)$  to (2.6).

**Step 2.** We now turn to the sign-preserving property (i) and assume that  $U(x) \in [0, \infty)^2$  for a.a.  $x \in \Omega$ . Let  $u \in H^1(\Omega, \mathbb{R}^2)$  be a weak solution to (2.6) and set  $\varphi := -u$ . Then  $(\varphi_{1,+}, \varphi_{2,+})$  belongs to  $H^1(\Omega, \mathbb{R}^2)$  and it follows from (2.6) that

$$\begin{aligned} \int_{\Omega} \left[ \varphi_1 \varphi_{1,+} + \varphi_2 \varphi_{2,+} + \tau \sum_{i=1}^N \sum_{j,k=1}^2 b_{jk}(u) \partial_i \varphi_k \partial_i (\varphi_{j,+}) \right] dx \\ = - \int_{\Omega} (U_1 \varphi_{1,+} + U_2 \varphi_{2,+}) dx \leq 0. \end{aligned} \quad (2.12)$$

We now infer from (2.7) that, for  $1 \leq i \leq N$ ,

$$\begin{aligned} b_{11}(u) \partial_i \varphi_1 \partial_i \varphi_{1,+} &= b_{11}(u) \mathbf{1}_{(-\infty, 0)}(u_1) |\partial_i u_1|^2 \geq 0, \\ b_{12}(u) \partial_i \varphi_2 \partial_i \varphi_{1,+} &= b_{12}(u) \mathbf{1}_{(-\infty, 0)}(u_1) \partial_i u_1 \partial_i u_2 = 0, \\ b_{21}(u) \partial_i \varphi_1 \partial_i \varphi_{2,+} &= b_{21}(u) \mathbf{1}_{(-\infty, 0)}(u_2) \partial_i u_1 \partial_i u_2 = 0, \\ b_{22}(u) \partial_i \varphi_2 \partial_i \varphi_{2,+} &= b_{22}(u) \mathbf{1}_{(-\infty, 0)}(u_2) |\partial_i u_2|^2 \geq 0, \end{aligned}$$

so that the second term on the left-hand side of (2.12) is non-negative. Consequently, (2.12) gives

$$\int_{\Omega} [|\varphi_{1,+}|^2 + |\varphi_{2,+}|^2] dx \leq 0,$$

which implies that  $\varphi_{1,+} = \varphi_{2,+} = 0$  a.e. in  $\Omega$ . Hence,  $u(x) \in [0, \infty)^2$  for a.a.  $x \in \Omega$  as claimed.

**Step 3.** It remains to prove (ii). We thus assume that  $\max\{U_1, U_2\} \leq \rho$  a.e. in  $\Omega$  and consider a weak solution  $u \in H^1(\Omega, \mathbb{R}^2)$  to (2.6). As  $v = ((u_1 - \rho)_+, (u_2 - \rho)_+)$  belongs to  $H^1(\Omega, \mathbb{R}^2)$ , we deduce from (2.6) that

$$\int_{\Omega} \left[ \sum_{j=1}^2 (u_j - U_j) (u_j - \rho)_+ + \tau \sum_{i=1}^N \sum_{j,k=1}^2 b_{jk}(u) \partial_i u_k \partial_i (u_j - \rho)_+ \right] dx = 0.$$

On the one hand,

$$u_j - U_j \geq u_j - \rho \quad \text{a.e. in } \Omega, \quad j = 1, 2,$$

so that

$$(u_j - U_j) (u_j - \rho)_+ \geq (u_j - \rho) (u_j - \rho)_+ = (u_j - \rho)_+^2 \quad \text{a.e. in } \Omega, \quad j = 1, 2.$$

On the other hand, we infer from (2.8) that, for  $1 \leq i \leq N$ ,

$$\begin{aligned} b_{11}(u) \partial_i u_1 \partial_i (u_1 - \rho)_+ &= b_{11}(u) \mathbf{1}_{(\rho, \infty)}(u_1) |\partial_i u_1|^2 \geq 0, \\ b_{12}(u) \partial_i u_2 \partial_i (u_1 - \rho)_+ &= b_{12}(u) \mathbf{1}_{(\rho, \infty)}(u_1) \partial_i u_1 \partial_i u_2 = 0, \\ b_{21}(u) \partial_i u_1 \partial_i (u_2 - \rho)_+ &= b_{21}(u) \mathbf{1}_{(\rho, \infty)}(u_2) \partial_i u_1 \partial_i u_2 = 0, \\ b_{22}(u) \partial_i u_2 \partial_i (u_2 - \rho)_+ &= b_{22}(u) \mathbf{1}_{(\rho, \infty)}(u_2) |\partial_i u_2|^2 \geq 0. \end{aligned}$$



Therefore,

$$\sum_{j=1}^2 \int_{\Omega} (u_j - \rho)_+^2 dx \leq 0,$$

from which we deduce that  $\max\{u_1, u_2\} \leq \rho$  a.e. in  $\Omega$ .  $\square$

**2.2. A regularised system.** We now introduce the two-parameter approximation of (2.1) on which the subsequent analysis relies. Specifically, given  $\rho > 1$ , we define

$$\alpha_{\rho}(z) := \begin{cases} 0 & , & z \leq 0, \\ z & , & 0 \leq z \leq \rho - 1, \\ (\rho - 1)(\rho - z) & , & \rho - 1 \leq z \leq \rho, \\ 0 & , & z \geq \rho, \end{cases}$$

and observe that  $\alpha_{\rho} \in BC(\mathbb{R})$  with

$$0 \leq \alpha_{\rho}(z) \leq \min\{\rho, z_+\}, \quad z \in \mathbb{R}.$$

Next, for  $\varepsilon \in (0, 1)$  and  $X \in \mathbb{R}^2$ , we set

$$M_{\varepsilon}^{\rho}(X) = (m_{\varepsilon, jk}^{\rho}(X))_{1 \leq j, k \leq 2} := \varepsilon I_2 + \lambda_{\varepsilon}((X_{1,+}, X_{2,+})) M^{\rho}(X),$$

where

$$M^{\rho}(X) = (m_{jk}^{\rho}(X))_{1 \leq j, k \leq 2} := \begin{pmatrix} a\alpha_{\rho}(X_1) & b\alpha_{\rho}(X_1) \\ c\alpha_{\rho}(X_2) & d\alpha_{\rho}(X_2) \end{pmatrix}, \quad X \in \mathbb{R}^2, \quad (2.13)$$

and

$$\lambda_{\varepsilon}(X) := \frac{2}{1 + \exp[\varepsilon(X_1 + X_2)]}, \quad X \in \mathbb{R}^2.$$

Note that  $(M^{\rho})_{\rho > 1}$  converges to  $M$ , defined in (1.13), locally uniformly in  $[0, \infty)^2$  as  $\rho \rightarrow \infty$ , while  $(\lambda_{\varepsilon})_{\varepsilon \in (0, 1)}$  converges to 1 locally uniformly in  $\mathbb{R}^2$  as  $\varepsilon \rightarrow 0$ . In fact, for  $R > 0$ ,

$$|\lambda_{\varepsilon}(X) - 1| \leq 2R\varepsilon, \quad X \in [-R, R]^2. \quad (2.14)$$

The outcome of this section is that, given  $\tau > 0$ ,  $\varepsilon \in (0, 1)$ ,  $\rho > 1$ , and  $U \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ , there is a weak solution  $u_{\varepsilon}^{\rho} \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2)$  to

$$u_{\varepsilon}^{\rho} - \tau \sum_{i=1}^N \partial_i (M_{\varepsilon}^{\rho}(u_{\varepsilon}^{\rho}) \partial_i u_{\varepsilon}^{\rho}) = U \quad \text{in } \Omega,$$

which satisfies an appropriate weak version of (2.2), as stated below. The next lemma is actually the building block of the proof of Proposition 2.1.

**Lemma 2.3.** *Given  $\tau > 0$ ,  $U = (F, G) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ ,  $\varepsilon \in (0, 1)$ , and  $\rho \geq \max\{1, \|F\|_{\infty}, \|G\|_{\infty}\}$ , there is a weak solution  $u_{\varepsilon}^{\rho} = (u_{\varepsilon,1}^{\rho}, u_{\varepsilon,2}^{\rho}) \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2)$  to*

$$\int_{\Omega} \left[ \langle u_{\varepsilon}^{\rho}, v \rangle + \tau \sum_{i=1}^N \langle M_{\varepsilon}^{\rho}(u_{\varepsilon}^{\rho}) \partial_i u_{\varepsilon}^{\rho}, \partial_i v \rangle \right] dx = \int_{\Omega} \langle U, v \rangle dx, \quad v \in H^1(\Omega, \mathbb{R}^2), \quad (2.15)$$

which additionally satisfies

$$\max\{\|u_{\varepsilon,1}^{\rho}\|_{\infty}, \|u_{\varepsilon,2}^{\rho}\|_{\infty}\} \leq \rho, \quad (2.16)$$

$$\|u_{\varepsilon}^{\rho}\|_2 \leq C_0 \|U\|_2, \quad (2.17)$$

$$\|\nabla u_\varepsilon^\rho\|_2 \leq C_1(\tau, \varepsilon)\|U\|_2. \quad (2.18)$$

Moreover, given  $n \geq 2$ , there exists a constant  $C(n)$  such that

$$\mathcal{E}_n(u_\varepsilon^\rho) \leq \tau C(n) \frac{\rho^{n-1}}{e^{\varepsilon\rho}} \|\nabla u_\varepsilon^\rho\|_2^2 + \mathcal{E}_n(U). \quad (2.19)$$

*Proof.* Let  $\varepsilon \in (0, 1)$  and  $\rho \geq \max\{1, \|F\|_\infty, \|G\|_\infty\}$ . To deduce the existence result stated in Lemma 2.3 from the already established Lemma 2.2, we first recast (2.15) in the form (2.6). First, owing to the definition of the function  $\alpha_\rho$ , the matrix  $M_\varepsilon^\rho$  lies in  $BC(\mathbb{R}^2, \mathbf{M}_2(\mathbb{R}))$  and satisfies

$$0 \leq m_{\varepsilon, jk}^\rho(X) \leq \varepsilon + 2\rho \max\{a, b, c, d\}, \quad 1 \leq j, k \leq 2, \quad X \in \mathbb{R}^2, \quad (2.20a)$$

as well as

$$\begin{aligned} m_{\varepsilon, 11}^\rho(X) &\geq m_{\varepsilon, 12}^\rho(X) = 0, & X \in (-\infty, 0) \times \mathbb{R}, \\ m_{\varepsilon, 22}^\rho(X) &\geq m_{\varepsilon, 21}^\rho(X) = 0, & X \in \mathbb{R} \times (-\infty, 0). \end{aligned} \quad (2.20b)$$

and

$$\begin{aligned} m_{\varepsilon, 11}^\rho(X) &\geq m_{\varepsilon, 12}^\rho(X) = 0, & X \in (\rho, \infty) \times \mathbb{R}, \\ m_{\varepsilon, 22}^\rho(X) &\geq m_{\varepsilon, 21}^\rho(X) = 0, & X \in \mathbb{R} \times (\rho, \infty). \end{aligned} \quad (2.20c)$$

Next, according to [4], it is natural to use the Hessian matrix of the convex function  $\Phi_2$  to symmetrize (2.15). We thus set

$$S := \frac{bd}{2} D^2 \Phi_2 = \begin{pmatrix} ac & bc \\ bc & bd \end{pmatrix}$$

and observe that  $S$  is symmetric and positive definite by (1.2). In addition, for all  $X \in \mathbb{R}^2$ ,

$$SM_\varepsilon^\rho(X) = \varepsilon S + \lambda_\varepsilon((X_{1,+}, X_{2,+})) SM^\rho(X)$$

with

$$SM^\rho(X) = \begin{pmatrix} a^2 c \alpha_\rho(X_1) + bc^2 \alpha_\rho(X_2) & abc \alpha_\rho(X_1) + bcd \alpha_\rho(X_2) \\ abc \alpha_\rho(X_1) + bcd \alpha_\rho(X_2) & b^2 c \alpha_\rho(X_1) + bd^2 \alpha_\rho(X_2) \end{pmatrix} \in \mathbf{Sym}_2(\mathbb{R}).$$

Since  $\text{tr}(SM^\rho(X)) \geq 0$  and

$$\det(SM^\rho(X)) = \det(S) \det(M^\rho(X)) = bc(ad - bc)^2 \alpha_\rho(X_1) \alpha_\rho(X_2) \geq 0$$

by (1.2), the matrix  $SM^\rho(X)$  is positive semidefinite, so that the matrix  $SM_\varepsilon^\rho(X)$  belongs to  $\mathbf{SPD}_2(\mathbb{R})$  for all  $X \in \mathbb{R}^2$  with

$$\langle SM_\varepsilon^\rho(X) \xi, \xi \rangle \geq \varepsilon \langle S \xi, \xi \rangle \geq \varepsilon \frac{\det(S)}{\text{tr}(S)} |\xi|^2 = \varepsilon \frac{bc(ad - bc)}{ac + bd} |\xi|^2, \quad \xi \in \mathbb{R}^2. \quad (2.20d)$$

According to the properties (2.20), we are now in a position to apply Lemma 2.2 (with  $A = S$  and  $B = M_\varepsilon^\rho$ ) and deduce that there is a solution  $u_\varepsilon^\rho \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2)$  to (2.15) which satisfies (2.16). Moreover, it follows from (2.15) (with  $v = Su_\varepsilon^\rho \in H^1(\Omega, \mathbb{R}^2)$ ), (2.20d), and the positive definiteness of  $S$ ,

$$\langle S \xi, \xi \rangle \geq \frac{bc(ad - bc)}{ac + bd} |\xi|^2, \quad \xi \in \mathbb{R}^2,$$

that

$$\begin{aligned}
\|SU\|_2 \|u_\varepsilon^\rho\|_2 &\geq \int_\Omega \langle SU, u_\varepsilon^\rho \rangle \, dx = \int_\Omega \left[ \langle u_\varepsilon^\rho, Su_\varepsilon^\rho \rangle + \tau \sum_{i=1}^N \langle M_\varepsilon^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i Su_\varepsilon^\rho \rangle \right] \, dx \\
&= \int_\Omega \left[ \langle Su_\varepsilon^\rho, u_\varepsilon^\rho \rangle + \tau \sum_{i=1}^N \langle SM_\varepsilon^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho \rangle \right] \, dx \\
&\geq \frac{bc(ad-bc)}{ac+bd} (\|u_\varepsilon^\rho\|_2^2 + \tau\varepsilon \|\nabla u_\varepsilon^\rho\|_2^2).
\end{aligned}$$

Owing to (1.2), we conclude that the estimates (2.17) and (2.18) are satisfied.

It remains to establish the estimate (2.19). Let  $n \geq 2$ . Since  $u_\varepsilon^\rho \in H^1(\Omega, \mathbb{R}^2) \cap L_\infty(\Omega, \mathbb{R}^2)$ , the vector field  $D\Phi_n(u_\varepsilon^\rho)$  belongs to  $H^1(\Omega, \mathbb{R}^2)$  and we infer from (2.15) (with  $v = D\Phi_n(u_\varepsilon^\rho)$ ) that

$$\int_\Omega \left[ \langle u_\varepsilon^\rho - U, D\Phi_n(u_\varepsilon^\rho) \rangle + \tau \sum_{i=1}^N \langle M_\varepsilon^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i D\Phi_n(u_\varepsilon^\rho) \rangle \right] \, dx = 0. \quad (2.21)$$

On the one hand, the convexity of  $\Phi_n$  implies that

$$\int_\Omega \langle u_\varepsilon^\rho - U, D\Phi_n(u_\varepsilon^\rho) \rangle \, dx \geq \int_\Omega [\Phi_n(u_\varepsilon^\rho) - \Phi_n(U)] \, dx = \mathcal{E}_n(u_\varepsilon^\rho) - \mathcal{E}_n(U). \quad (2.22)$$

On the other hand, using the symmetry and the positive semidefiniteness of the matrix  $D^2\Phi_n(u_\varepsilon^\rho)$ , see Lemma A.2, we have

$$\begin{aligned}
\tau \sum_{i=1}^N \int_\Omega \langle M_\varepsilon^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i D\Phi_n(u_\varepsilon^\rho) \rangle \, dx &= \tau \sum_{i=1}^N \int_\Omega \langle M_\varepsilon^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, D^2\Phi_n(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho \rangle \, dx \\
&= \tau \sum_{i=1}^N \int_\Omega \langle D^2\Phi_n(u_\varepsilon^\rho) M_\varepsilon^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho \rangle \, dx \\
&= \tau\varepsilon \sum_{i=1}^N \int_\Omega \langle D^2\Phi_n(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho \rangle \, dx \\
&\quad + \tau \sum_{i=1}^N \int_\Omega \lambda_\varepsilon(u_\varepsilon^\rho) \langle D^2\Phi_n(u_\varepsilon^\rho) M^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho \rangle \, dx \\
&\geq \tau \sum_{i=1}^N \int_\Omega \lambda_\varepsilon(u_\varepsilon^\rho) \langle D^2\Phi_n(u_\varepsilon^\rho) M^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho \rangle \, dx. \quad (2.23)
\end{aligned}$$

Since  $S_n(u_\varepsilon^\rho) := D^2\Phi_n(u_\varepsilon^\rho)M(u_\varepsilon^\rho)$  is positive semidefinite by Lemma A.3, we further have

$$\begin{aligned}
\tau \sum_{i=1}^N \int_\Omega \lambda_\varepsilon(u_\varepsilon^\rho) \langle D^2\Phi_n(u_\varepsilon^\rho) M^\rho(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho \rangle \, dx \\
= \tau \sum_{i=1}^N \int_\Omega \lambda_\varepsilon(u_\varepsilon^\rho) \langle D^2\Phi_n(u_\varepsilon^\rho) M(u_\varepsilon^\rho) \partial_i u_\varepsilon^\rho, \partial_i u_\varepsilon^\rho \rangle \, dx
\end{aligned}$$

$$\begin{aligned}
& + \tau \sum_{i=1}^N \int_{\Omega} \lambda_{\varepsilon}(u_{\varepsilon}^{\rho}) \langle D^2 \Phi_n(u_{\varepsilon}^{\rho}) [M^{\rho}(u_{\varepsilon}^{\rho}) - M(u_{\varepsilon}^{\rho})] \partial_i u_{\varepsilon}^{\rho}, \partial_i u_{\varepsilon}^{\rho} \rangle dx \\
& \geq \tau \sum_{i=1}^N \int_{\Omega} \lambda_{\varepsilon}(u_{\varepsilon}^{\rho}) \langle D^2 \Phi_n(u_{\varepsilon}^{\rho}) [M^{\rho}(u_{\varepsilon}^{\rho}) - M(u_{\varepsilon}^{\rho})] \partial_i u_{\varepsilon}^{\rho}, \partial_i u_{\varepsilon}^{\rho} \rangle dx. \tag{2.24}
\end{aligned}$$

Taking now advantage of the fact that  $0 \leq u_{\varepsilon,j}^{\rho} \leq \rho$  a.e. in  $\Omega$  for  $j = 1, 2$  by (2.16), we further have

$$\begin{aligned}
& \left| \tau \sum_{i=1}^N \int_{\Omega} \lambda_{\varepsilon}(u_{\varepsilon}^{\rho}) \langle D^2 \Phi_n(u_{\varepsilon}^{\rho}) [M^{\rho}(u_{\varepsilon}^{\rho}) - M(u_{\varepsilon}^{\rho})] \partial_i u_{\varepsilon}^{\rho}, \partial_i u_{\varepsilon}^{\rho} \rangle dx \right| \\
& \leq 2\tau \max\{a, b, c, d\} \|D^2 \Phi_n\|_{L^{\infty}((0,\rho)^2)} \sum_{j=1}^2 \int_{\Omega} \lambda_{\varepsilon}(u_{\varepsilon}^{\rho}) |\alpha_{\rho}(u_{\varepsilon,j}^{\rho}) - u_{\varepsilon,j}^{\rho}| |\nabla u_{\varepsilon}^{\rho}|^2 dx \\
& \leq 4\tau \max\{a, b, c, d\} \kappa_n \rho^{n-2} \sum_{j=1}^2 \int_{\{\rho-1 \leq u_{\varepsilon,j}^{\rho} \leq \rho\}} \frac{|\alpha_{\rho}(u_{\varepsilon,j}^{\rho}) - u_{\varepsilon,j}^{\rho}|}{1 + \exp(\varepsilon u_{\varepsilon,j}^{\rho})} |\nabla u_{\varepsilon}^{\rho}|^2 dx,
\end{aligned}$$

where  $\kappa_n \in \mathbb{R}$  is a positive constant such that

$$|D^2 \Phi_n(X)| \leq \kappa_n (X_1^{n-2} + X_2^{n-2}) \quad \text{for all } X \in [0, \infty)^2.$$

Owing to the definition of  $\alpha_{\rho}$ , we further obtain

$$\begin{aligned}
& \left| \tau \sum_{i=1}^N \int_{\Omega} \lambda_{\varepsilon}(u_{\varepsilon}^{\rho}) \langle D^2 \Phi_n(u_{\varepsilon}^{\rho}) [M^{\rho}(u_{\varepsilon}^{\rho}) - M(u_{\varepsilon}^{\rho})] \partial_i u_{\varepsilon}^{\rho}, \partial_i u_{\varepsilon}^{\rho} \rangle dx \right| \\
& \leq 4\tau \max\{a, b, c, d\} \kappa_n \rho^{n-2} \sum_{j=1}^2 \int_{\{\rho-1 \leq u_{\varepsilon,j}^{\rho} \leq \rho\}} \frac{\rho}{1 + e^{\varepsilon(\rho-1)}} |\nabla u_{\varepsilon}^{\rho}|^2 dx \\
& \leq 8e\tau \max\{a, b, c, d\} \kappa_n \rho^{n-1} e^{-\varepsilon\rho} \|\nabla u_{\varepsilon}^{\rho}\|_2^2. \tag{2.25}
\end{aligned}$$

The desired estimate (2.19) is now a straightforward consequence of the relations (2.21)-(2.25).  $\square$

**2.3. A regularised system:**  $\rho \rightarrow \infty$ . We next study the cluster points as  $\rho \rightarrow \infty$  of the family  $\{u_{\varepsilon}^{\rho} : \rho \geq \max\{1, \|F\|_{\infty}, \|G\|_{\infty}\}\}$  provided in Lemma 2.3, the parameter  $\varepsilon \in (0, 1)$  being held fixed.

**Lemma 2.4.** *Given  $\tau > 0$ ,  $U = (F, G) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ , and  $\varepsilon \in (0, 1)$ , there exist a sequence  $(\rho_l)_{l \geq 1}$  and a function  $u_{\varepsilon} = (u_{\varepsilon,1}, u_{\varepsilon,2}) \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2)$  such that  $\rho_l \rightarrow \infty$  and*

$$u_{\varepsilon}^{\rho_l} \rightarrow u_{\varepsilon} \quad \text{in } L_p(\Omega, \mathbb{R}^2) \text{ for all } p \in [1, \infty) \text{ and pointwise a.e. in } \Omega, \tag{2.26}$$

$$\nabla u_{\varepsilon}^{\rho_l} \rightharpoonup \nabla u_{\varepsilon} \quad \text{in } L_2(\Omega, \mathbb{R}^{2N}). \tag{2.27}$$

Moreover,  $u_{\varepsilon}$  solves the equation

$$\int_{\Omega} \left[ \langle u_{\varepsilon}, v \rangle + \tau \sum_{i=1}^N \langle M_{\varepsilon}(u_{\varepsilon}) \partial_i u_{\varepsilon}, \partial_i v \rangle \right] dx = \int_{\Omega} \langle U, v \rangle dx, \quad v \in H^1(\Omega, \mathbb{R}^2), \tag{2.28}$$

where

$$M_\varepsilon(X) = (m_{\varepsilon,jk}(X))_{1 \leq j,k \leq 2} := \varepsilon I_2 + \lambda_\varepsilon((X_{1,+}, X_{2,+}))M(X),$$

with  $M(X)$  defined in (1.13), and, for each  $n \geq 2$ , we have

$$\mathcal{E}_n(u_\varepsilon) \leq \mathcal{E}_n(U). \quad (2.29)$$

Furthermore,

$$\min \left\{ 1, \frac{c}{d} \right\} \|u_{\varepsilon,1} + u_{\varepsilon,2}\|_\infty \leq \max \left\{ 1, \frac{a}{b} \right\} \|F + G\|_\infty. \quad (2.30)$$

*Proof.* Recalling (2.17)-(2.18), we deduce that  $(u_\varepsilon^\rho)_\rho$  is bounded in  $H^1(\Omega, \mathbb{R}^2)$ . Moreover, since

$$\frac{\varepsilon^n z^n}{n!} \leq e^{\varepsilon z}, \quad z \in [0, \infty), \quad n \geq 1, \quad (2.31)$$

the estimates (2.18) and (2.19), along with Lemma A.4, ensure that  $(u_\varepsilon^\rho)_\rho$  is bounded in  $L_n(\Omega, \mathbb{R}^2)$  for any integer  $n \geq 2$  (with an  $\varepsilon$ -dependent bound). We may then use a Cantor diagonal process, together with Rellich-Kondrachov' theorem and an interpolation argument, to deduce the convergence (2.26) and (2.27) along a sequence  $\rho_l \rightarrow \infty$ , as well as the componentwise non-negativity of  $u_\varepsilon$ .

Since  $\Phi_n$  is convex on  $[0, \infty)^2$  for all  $n \geq 2$ , see Lemma A.2, it follows from (2.18), (2.19), (2.26), and (2.31) that (2.29) holds true. Using once more Lemma A.4, we infer from (2.29) that

$$\|cu_{\varepsilon,1} + du_{\varepsilon,2}\|_n \leq \frac{d}{b} \|aF + bG\|_n$$

for all  $n \geq 2$ . Passing to the limit  $n \rightarrow \infty$  in the above inequality, we deduce that  $u_\varepsilon \in L_\infty(\Omega, \mathbb{R}^2)$  satisfies (2.30).

Let us now consider  $v \in H^1(\Omega, \mathbb{R}^2)$ . Since (2.26) and (2.27) imply that

$$\lim_{l \rightarrow \infty} \int_\Omega \langle u_\varepsilon^{\rho_l}, v \rangle dx = \int_\Omega \langle u_\varepsilon, v \rangle dx \quad \text{and} \quad \lim_{l \rightarrow \infty} \int_\Omega \langle \partial_i u_\varepsilon^{\rho_l}, \partial_i v \rangle dx \rightarrow \int_\Omega \langle \partial_i u_\varepsilon, \partial_i v \rangle dx$$

for  $1 \leq i \leq N$ , the identity (2.28) is satisfied provided that

$$\lim_{l \rightarrow \infty} \int_\Omega \lambda_\varepsilon(u_\varepsilon^{\rho_l}) \langle M^{\rho_l}(u_\varepsilon^{\rho_l}) \partial_i u_\varepsilon^{\rho_l}, \partial_i v \rangle dx = \int_\Omega \lambda_\varepsilon(u_\varepsilon) \langle M(u_\varepsilon) \partial_i u_\varepsilon, \partial_i v \rangle dx \quad (2.32)$$

for each  $1 \leq i \leq N$ . To prove (2.32), we observe that, for  $1 \leq i \leq N$  and  $j \in \{1, 2\}$ ,

$$\int_\Omega \lambda_\varepsilon(u_\varepsilon^{\rho_l}) \langle M^{\rho_l}(u_\varepsilon^{\rho_l}) \partial_i u_\varepsilon^{\rho_l}, \partial_i v \rangle dx = \int_\Omega \lambda_\varepsilon(u_\varepsilon^{\rho_l}) \langle M^{\rho_l}(u_\varepsilon^{\rho_l})^t \partial_i v, \partial_i u_\varepsilon^{\rho_l} \rangle dx \quad (2.33)$$

with

$$\begin{aligned} \left| \lambda_\varepsilon(u_\varepsilon^{\rho_l}) \sum_{k=1}^2 m_{kj}^{\rho_l}(u_\varepsilon^{\rho_l}) \partial_i v_k \right| &\leq 2 \max\{a, b, c, d\} \frac{u_{\varepsilon,1}^{\rho_l} + u_{\varepsilon,2}^{\rho_l}}{1 + \exp[\varepsilon(u_{\varepsilon,1}^{\rho_l} + u_{\varepsilon,2}^{\rho_l})]} |\partial_i v| \\ &\leq \frac{2 \max\{a, b, c, d\}}{\varepsilon} |\partial_i v| \quad \text{a.e. in } \Omega, \end{aligned}$$

by (2.30) and (2.31), and

$$\lim_{l \rightarrow \infty} \lambda_\varepsilon(u_\varepsilon^{\rho_l}) \sum_{k=1}^2 m_{kj}^{\rho_l}(u_\varepsilon^{\rho_l}) \partial_i v_k = \lambda_\varepsilon(u_\varepsilon) \sum_{k=1}^2 m_{kj}(u_\varepsilon) \partial_i v_k \quad \text{a.e. in } \Omega,$$

by (2.13), the pointwise almost everywhere convergence in  $\Omega$  established in (2.26), and the properties of  $\alpha_{\rho_l}$ . Lebesgue's dominated convergence theorem then guarantees that

$$\lim_{l \rightarrow \infty} \left\| \lambda_\varepsilon(u_\varepsilon^{\rho_l}) \sum_{k=1}^2 m_{kj}^{\rho_l}(u_\varepsilon^{\rho_l}) \partial_i v_k - \lambda_\varepsilon(u_\varepsilon) \sum_{k=1}^2 m_{kj}(u_\varepsilon) \partial_i v_k \right\|_2 = 0.$$

Combining the above convergence with (2.27), allows us to pass to the limit as  $l \rightarrow \infty$  in (2.33) and find

$$\begin{aligned} \lim_{l \rightarrow \infty} \int_{\Omega} \lambda_\varepsilon(u_\varepsilon^{\rho_l}) \langle M^{\rho_l}(u_\varepsilon^{\rho_l}) \partial_i u_\varepsilon^{\rho_l}, \partial_i v \rangle \, dx &= \int_{\Omega} \lambda_\varepsilon(u_\varepsilon) \langle M(u_\varepsilon)^t \partial_i v, \partial_i u_\varepsilon \rangle \, dx \\ &= \int_{\Omega} \lambda_\varepsilon(u_\varepsilon) \langle M(u_\varepsilon) \partial_i u_\varepsilon, \partial_i v \rangle \, dx \end{aligned}$$

for  $1 \leq i \leq N$ , which proves (2.32). We have thus shown that  $u_\varepsilon$  solves (2.28) and thereby completed the proof of Lemma 2.4.  $\square$

We next show that the entropy functional  $\mathcal{E}_1$  evaluated at the function  $u_\varepsilon$  identified in Lemma 2.4 is dominated by  $\mathcal{E}_1(U)$  and that the associated dissipation term  $\mathcal{E}_1(U) - \mathcal{E}_1(u_\varepsilon)$  provides a control on the gradient of  $u_\varepsilon$  which is essential when considering the limit  $\varepsilon \rightarrow 0$ .

**Lemma 2.5.** *Let  $\tau > 0$ ,  $U = (F, G) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ , and  $\varepsilon \in (0, 1)$ . The function*

$$u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}) \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2)$$

*identified in Lemma 2.4 satisfies*

$$\mathcal{E}_1(u_\varepsilon) + \frac{\tau}{a} \int_{\Omega} \lambda_\varepsilon(u_\varepsilon) [|\nabla(au_{\varepsilon,1} + \Theta_1 u_{\varepsilon,2})|^2 + \Theta_2 |\nabla u_{\varepsilon,2}|^2] \, dx \leq \mathcal{E}_1(U).$$

*Proof.* Let  $\eta \in (0, 1)$ . Then  $(\ln(u_{\varepsilon,1} + \eta), (b^2/ad) \ln(u_{\varepsilon,2} + \eta)) \in H^1(\Omega, \mathbb{R}^2)$  and we infer from (2.28) that

$$0 = \int_{\Omega} \left[ (u_{\varepsilon,1} - U_1) \ln(u_{\varepsilon,1} + \eta) + \frac{b^2}{ad} (u_{\varepsilon,2} - U_2) \ln(u_{\varepsilon,2} + \eta) \right] \, dx + D(\eta), \quad (2.34)$$

where

$$\begin{aligned} D(\eta) &:= \tau \int_{\Omega} \sum_{i=1}^N (m_{\varepsilon,11}(u_\varepsilon) \partial_i u_{\varepsilon,1} + m_{\varepsilon,12}(u_\varepsilon) \partial_i u_{\varepsilon,2}) \frac{\partial_i u_{\varepsilon,1}}{u_{\varepsilon,1} + \eta} \, dx \\ &\quad + \frac{\tau b^2}{ad} \int_{\Omega} \sum_{i=1}^N (m_{\varepsilon,21}(u_\varepsilon) \partial_i u_{\varepsilon,1} + m_{\varepsilon,22}(u_\varepsilon) \partial_i u_{\varepsilon,2}) \frac{\partial_i u_{\varepsilon,2}}{u_{\varepsilon,2} + \eta} \, dx. \end{aligned}$$

Since  $L(r) = r \ln r - r + 1$  is convex on  $[0, \infty)$  with  $L'(r) = \ln r$ , the first term on the right-hand side of (2.34) can be estimated as follows

$$\begin{aligned} &\int_{\Omega} \left[ (u_{\varepsilon,1} - U_1) \ln(u_{\varepsilon,1} + \eta) + \frac{b^2}{ad} (u_{\varepsilon,2} - U_2) \ln(u_{\varepsilon,2} + \eta) \right] \, dx \\ &\geq \int_{\Omega} \left[ (L(u_{\varepsilon,1} + \eta) - L(U_1 + \eta)) + \frac{b^2}{ad} (L(u_{\varepsilon,2} + \eta) - L(U_2 + \eta)) \right] \, dx \\ &= \mathcal{E}_1((u_{\varepsilon,1} + \eta, u_{\varepsilon,2} + \eta)) - \mathcal{E}_1((U_1 + \eta, U_2 + \eta)). \end{aligned}$$

Using the continuity of  $\Phi_1$  and the boundedness of  $u_\varepsilon$ , see (2.30), we deduce that

$$\liminf_{\eta \rightarrow 0} \int_{\Omega} \left[ (u_{\varepsilon,1} - U_1) \ln(u_{\varepsilon,1} + \eta) + \frac{b^2}{ad} (u_{\varepsilon,2} - U_2) \ln(u_{\varepsilon,2} + \eta) \right] dx \geq \mathcal{E}_1(u_\varepsilon) - \mathcal{E}_1(U). \quad (2.35)$$

Next, recalling the definition of the matrix  $M_\varepsilon$ , see Lemma 2.4, we have

$$\begin{aligned} D(\eta) &= \tau \varepsilon \int_{\Omega} \left( \frac{|\nabla u_{\varepsilon,1}|^2}{u_{\varepsilon,1} + \eta} + \frac{b^2}{ad} \frac{|\nabla u_{\varepsilon,2}|^2}{u_{\varepsilon,2} + \eta} \right) dx \\ &\quad + \frac{\tau}{a} \int_{\Omega} \lambda_\varepsilon(u_\varepsilon) [|\nabla(au_{\varepsilon,1} + \Theta_1 u_{\varepsilon,2})|^2 + \Theta_2 |\nabla u_{\varepsilon,2}|^2] dx \\ &\quad - J_1(\eta) - J_2(\eta), \end{aligned}$$

where

$$\begin{aligned} J_1(\eta) &:= \tau \int_{\Omega} \frac{\eta \lambda_\varepsilon(u_\varepsilon)}{u_{\varepsilon,1} + \eta} \nabla u_{\varepsilon,1} \cdot \nabla(au_{\varepsilon,1} + bu_{\varepsilon,2}) dx, \\ J_2(\eta) &:= \frac{\tau b^2}{ad} \int_{\Omega} \frac{\eta \lambda_\varepsilon(u_\varepsilon)}{u_{\varepsilon,2} + \eta} \nabla u_{\varepsilon,2} \cdot \nabla(cu_{\varepsilon,1} + du_{\varepsilon,2}) dx. \end{aligned}$$

Since  $u_\varepsilon \in H^1(\Omega, \mathbb{R}^2)$  and  $\nabla u_{\varepsilon,j} = 0$  a.e. on the level set  $\{x \in \Omega : u_{\varepsilon,j} = 0\}$  for  $j \in \{1, 2\}$ , we have

$$\begin{aligned} \lim_{\eta \rightarrow 0} \frac{\eta \lambda_\varepsilon(u_\varepsilon)}{u_{\varepsilon,j} + \eta} \nabla u_{\varepsilon,j} &= 0 \quad \text{a.e. in } \Omega, \\ \left| \frac{\eta \lambda_\varepsilon(u_\varepsilon)}{u_{\varepsilon,j} + \eta} \nabla u_{\varepsilon,j} \right| &\leq |\nabla u_{\varepsilon,j}| \quad \text{a.e. in } \Omega. \end{aligned}$$

Lebesgue's dominated convergence theorem ensures now that

$$\lim_{\eta \rightarrow 0} (J_1(\eta) + J_2(\eta)) = 0.$$

This shows that

$$\liminf_{\eta \rightarrow 0} D(\eta) \geq \frac{\tau}{a} \int_{\Omega} \lambda_\varepsilon(u_\varepsilon) [|\nabla(au_{\varepsilon,1} + \Theta_1 u_{\varepsilon,2})|^2 + \Theta_2 |\nabla u_{\varepsilon,2}|^2] dx. \quad (2.36)$$

Passing to the limit  $\eta \rightarrow 0$  in (2.34), we get the desired estimate in view of (2.35) and (2.36).  $\square$

**2.4. A regularised system:  $\varepsilon \rightarrow 0$ .** We complete this section with the proof of Proposition 2.1.

*Proof of Proposition 2.1.* Consider  $\tau > 0$  and  $U = (F, G) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ . Given  $\varepsilon \in (0, 1)$ , let

$$u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}) \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2)$$

denote the weak solution to (2.28) provided by Lemma 2.4. According to (2.30),

$$\max\{\|u_{\varepsilon,1}\|_\infty, \|u_{\varepsilon,2}\|_\infty\} \leq \|u_{\varepsilon,1} + u_{\varepsilon,2}\|_\infty \leq R_0 := \frac{d \max\{a, b\}}{b \min\{c, d\}} \|F + G\|_\infty. \quad (2.37)$$

Hence,

$$\lambda_\varepsilon(u_\varepsilon) \geq \frac{2}{1 + e^{R_0}},$$

a lower bound which, together with Lemma 2.5 and the non-negativity of  $\mathcal{E}_1$ , ensures that

$$(\nabla u_\varepsilon)_{\varepsilon \in (0,1)} \text{ is bounded in } L_2(\Omega, \mathbb{R}^{2N}). \quad (2.38)$$

We now infer from (2.37), (2.38), Rellich-Kondrachov' theorem, an interpolation argument, and a Cantor diagonal process that there exist a function

$$u = (f, g) \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2)$$

and a sequence  $(\varepsilon_l)_{l \geq 1}$ , with  $\varepsilon_l \rightarrow 0$ , such that

$$u_{\varepsilon_l} \rightarrow u \quad \text{in } L_p(\Omega, \mathbb{R}^2) \text{ for all } p \in [1, \infty), \quad (2.39)$$

$$u_{\varepsilon_l} \xrightarrow{*} u \quad \text{in } L_{\infty}(\Omega, \mathbb{R}^2), \quad (2.40)$$

$$\nabla u_{\varepsilon_l} \rightharpoonup \nabla u \quad \text{in } L_2(\Omega, \mathbb{R}^{2N}). \quad (2.41)$$

An immediate consequence of (2.29) and (2.39) is the estimate (2.2). Since  $\sqrt{\lambda_{\varepsilon_l}(u_{\varepsilon_l})} \rightarrow 1$  in  $L_{\infty}(\Omega)$  by (2.14) and (2.37), we conclude together with (2.41) that

$$\begin{aligned} \sqrt{\lambda_{\varepsilon_l}(u_{\varepsilon_l})} \nabla (a u_{\varepsilon_l,1} + \Theta_1 u_{\varepsilon_l,2}) &\rightharpoonup \nabla (a u_1 + \Theta_1 u_2) \quad \text{in } L_2(\Omega, \mathbb{R}^N), \\ \sqrt{\Theta_2 \lambda_{\varepsilon_l}(u_{\varepsilon_l})} \nabla u_{\varepsilon_l,2} &\rightharpoonup \sqrt{\Theta_2} \nabla u_2 \quad \text{in } L_2(\Omega, \mathbb{R}^N). \end{aligned}$$

Moreover, the  $L_{\infty}$ -bound (2.37) and the convergence (2.39) imply that

$$\liminf_{l \rightarrow \infty} \mathcal{E}_1(u_{\varepsilon_l}) \geq \mathcal{E}_1(u),$$

and the estimate (2.3) is now obtained by passing to  $\liminf$  in the inequality reported in Lemma 2.5 (with  $\varepsilon$  replaced by  $\varepsilon_l$ ).

Finally, (2.39), along with (2.37) and the convergence property

$$\lim_{\varepsilon \rightarrow 0} |m_{\varepsilon,jk}(X) - m_{jk}(X)| = 0,$$

which is uniform with respect to  $X \in [0, R_0]^2$  and  $1 \leq j, k \leq 2$ , enables us to use Lebesgue's dominated convergence theorem to show that, for  $v = (\varphi, \psi) \in H^1(\Omega, \mathbb{R}^2)$ ,

$$\lim_{l \rightarrow \infty} \|M_{\varepsilon_l}(u_{\varepsilon_l})^t \partial_i v - M(u)^t \partial_i v\|_2 = 0, \quad 1 \leq i \leq N.$$

Together with (2.39) and (2.41), the above convergence allows us to let  $\varepsilon_l \rightarrow 0$  in (2.28) and conclude that  $u = (f, g)$  satisfies (2.1). This completes the proof of Proposition 2.1.  $\square$

### 3. EXISTENCE OF BOUNDED WEAK SOLUTIONS

This section is devoted to the proof of Theorem 1.1, which relies on rather classical arguments, besides the estimates derived in Proposition 2.1, and proceeds along the lines of the proof of [11, Theorem 1.2]. As a first step, we use Proposition 2.1 to construct a family of piecewise constant functions  $(u^\tau)_{\tau \in (0,1)}$  starting from the initial condition  $(f^{in}, g^{in}) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$ . More precisely, for  $\tau \in (0, 1)$ , we set  $u^\tau(0) := u_0^\tau$  and

$$u^\tau(t) = u_l^\tau, \quad t \in ((l-1)\tau, l\tau], \quad l \in \mathbb{N} \setminus \{0\}, \quad (3.1)$$

where the sequence  $(u_l^\tau)_{l \geq 0}$  is defined as follows:

$$\begin{aligned} u_0^\tau &= u^{in} := (f^{in}, g^{in}) \in L_{\infty,+}(\Omega, \mathbb{R}^2), \\ u_{l+1}^\tau &= (f_{l+1}^\tau, g_{l+1}^\tau) \in H^1(\Omega, \mathbb{R}^2) \cap L_{\infty,+}(\Omega, \mathbb{R}^2) \text{ is the solution to (2.1)} \\ &\text{with } U = u_l^\tau = (f_l^\tau, g_l^\tau) \text{ constructed in Proposition 2.1 for } l \geq 0. \end{aligned} \quad (3.2)$$



In order to establish Theorem 1.1, we show that the family  $(u^\tau)_{\tau \in (0,1)}$  defined in (3.2) converges along a subsequence  $\tau_j \rightarrow 0$  towards a pair  $u = (f, g)$  which fulfills all the requirements of Theorem 1.1.

Below,  $C$  and  $(C_l)_{l \geq 0}$  denote various positive constants depending only on  $(a, b, c, d)$  and  $u^{in}$ . Dependence upon additional parameters will be indicated explicitly.

*Proof of Theorem 1.1.* Let  $\tau \in (0, 1)$  and let  $u^\tau$  be defined in (3.1)-(3.2). Given  $l \geq 0$ , we infer from Proposition 2.1 that

$$\int_{\Omega} \left( f_{l+1}^\tau \varphi + \tau f_{l+1}^\tau \nabla [a f_{l+1}^\tau + b g_{l+1}^\tau] \cdot \nabla \varphi \right) dx = \int_{\Omega} f_l^\tau \varphi dx, \quad \varphi \in H^1(\Omega), \quad (3.3a)$$

$$\int_{\Omega} \left( g_{l+1}^\tau \psi + \tau g_{l+1}^\tau \nabla [c f_{l+1}^\tau + d g_{l+1}^\tau] \cdot \nabla \psi \right) dx = \int_{\Omega} g_l^\tau \psi dx, \quad \psi \in H^1(\Omega). \quad (3.3b)$$

Moreover,

$$\mathcal{E}_n(u_{l+1}^\tau) \leq \mathcal{E}_n(u_l^\tau) \quad \text{for } n \geq 2, \quad (3.4)$$

and we also have

$$\mathcal{E}_1(u_{l+1}^\tau) + \frac{\tau}{a} \int_{\Omega} [|\nabla(a f_{l+1}^\tau + \Theta_1 g_{l+1}^\tau)|^2 + \Theta_2 |\nabla g_{l+1}^\tau|^2] dx \leq \mathcal{E}_1(u_l^\tau). \quad (3.5)$$

It readily follows from (3.1), (3.2), (3.4), and (3.5) that, for  $t > 0$ ,

$$\mathcal{E}_n(u^\tau(t)) \leq \mathcal{E}_n(u^{in}), \quad n \geq 2, \quad (3.6)$$

and

$$\mathcal{E}_1(u^\tau(t)) + \frac{1}{a} \int_0^t \int_{\Omega} [|\nabla(a f^\tau + \Theta_1 g^\tau)|^2 + \Theta_2 |\nabla g^\tau|^2] dx ds \leq \mathcal{E}_1(u^{in}). \quad (3.7)$$

An immediate consequence of (3.6) and Lemma A.4 is the estimate

$$\|f^\tau(t) + g^\tau(t)\|_n \leq \frac{d \max\{a, b\}}{b \min\{c, d\}} \|f^{in} + g^{in}\|_n, \quad n \geq 2, \quad t > 0.$$

Letting  $n \rightarrow \infty$  in the above inequality gives

$$\|f^\tau(t) + g^\tau(t)\|_\infty \leq C_1 := \frac{d \max\{a, b\}}{b \min\{c, d\}} \|f^{in} + g^{in}\|_\infty, \quad t > 0. \quad (3.8)$$

Also, taking advantage of the non-negativity of  $\mathcal{E}_1$ , we deduce from (3.7) that

$$\int_0^t [\|\nabla f^\tau(s)\|_2^2 + \|\nabla g^\tau(s)\|_2^2] ds \leq C_2 := \frac{a^2 + 2(\Theta_2 + \Theta_1^2)}{a\Theta_2} \mathcal{E}_1(u^{in}), \quad t > 0. \quad (3.9)$$

Next, for  $l \geq 1$  and  $t \in ((l-1)\tau, l\tau]$ , we deduce from (3.3a), (3.8), and Hölder's inequality that, for  $\varphi \in H^1(\Omega)$ ,

$$\begin{aligned} \left| \int_{\Omega} (f^\tau(t+\tau) - f^\tau(t)) \varphi dx \right| &= \left| \int_{l\tau}^{(l+1)\tau} \int_{\Omega} f_{l+1}^\tau \nabla [a f_{l+1}^\tau + b g_{l+1}^\tau] \cdot \nabla \varphi dx ds \right| \\ &\leq \int_{l\tau}^{(l+1)\tau} \|f^\tau(s)\|_\infty \|\nabla [a f^\tau(s) + b g^\tau(s)]\|_2 \|\nabla \varphi\|_2 ds \\ &\leq C_1 \|\nabla \varphi\|_2 \int_{l\tau}^{(l+1)\tau} \|\nabla [a f^\tau(s) + b g^\tau(s)]\|_2 ds. \end{aligned}$$

A duality argument then gives

$$\|f^\tau(t + \tau) - f^\tau(t)\|_{(H^1)'} \leq C_1 \int_{l\tau}^{(l+1)\tau} \|\nabla[af^\tau(s) + bg^\tau(s)]\|_2 \, ds$$

for  $t \in ((l-1)\tau, l\tau]$  and  $l \geq 1$ . Now, for  $l_0 \geq 2$  and  $T \in ((l_0-1)\tau, l_0\tau]$ , the above inequality, along with Hölder's inequality, entails that

$$\begin{aligned} \int_0^{T-\tau} \|f^\tau(t + \tau) - f^\tau(t)\|_{(H^1)'}^2 \, dt &\leq \int_0^{(l_0-1)\tau} \|f^\tau(t + \tau) - f^\tau(t)\|_{(H^1)'}^2 \, dt \\ &= \sum_{l=1}^{l_0-1} \int_{(l-1)\tau}^{l\tau} \|f^\tau(t + \tau) - f^\tau(t)\|_{(H^1)'}^2 \, dt \\ &\leq C_1^2 \tau \sum_{l=1}^{l_0-1} \left( \int_{l\tau}^{(l+1)\tau} \|\nabla[af^\tau(s) + bg^\tau(s)]\|_2 \, ds \right)^2 \\ &\leq C_1^2 \tau^2 \sum_{l=1}^{l_0-1} \int_{l\tau}^{(l+1)\tau} \|\nabla[af^\tau(s) + bg^\tau(s)]\|_2^2 \, ds \\ &\leq C_1^2 \tau^2 \int_0^{l_0\tau} \|\nabla[af^\tau(s) + bg^\tau(s)]\|_2^2 \, ds. \end{aligned}$$

We then use (3.9) (with  $t = l_0\tau$ ) and Young's inequality to obtain

$$\begin{aligned} \int_0^{T-\tau} \|f^\tau(t + \tau) - f^\tau(t)\|_{(H^1)'}^2 \, dt &\leq C_1^2 \tau^2 \int_0^{l_0\tau} (2a^2 \|\nabla f^\tau(s)\|_2^2 + 2b^2 \|\nabla g^\tau(s)\|_2^2) \, ds \\ &\leq C_3 \tau^2, \end{aligned} \tag{3.10}$$

with  $C_3 := 2(a^2 + b^2)^2 C_1^2 C_2$ . Similarly,

$$\int_0^{T-\tau} \|g^\tau(t + \tau) - g^\tau(t)\|_{(H^1)'}^2 \, dt \leq C_4 \tau^2, \tag{3.11}$$

with  $C_4 := 2(c^2 + d^2) C_1^2 C_2$ .

According to Rellich-Kondrachov' theorem,  $H^1(\Omega, \mathbb{R}^2)$  is compactly embedded in  $L_2(\Omega, \mathbb{R}^2)$ , while  $L_2(\Omega, \mathbb{R}^2)$  is continuously (and compactly) embedded in  $H^1(\Omega, \mathbb{R}^2)'$ . Gathering (3.8)-(3.11), we infer from [5, Theorem 1] that, for any  $T > 0$ ,

$$(u^\tau)_{\tau \in (0,1)} \text{ is relatively compact in } L_2((0, T) \times \Omega, \mathbb{R}^2). \tag{3.12}$$

Owing to (3.8), (3.9), and (3.12), we may use a Cantor diagonal argument to find a function

$$u = (f, g) \in L_{\infty,+}((0, \infty) \times \Omega, \mathbb{R}^2)$$

and a sequence  $(\tau_m)_{m \geq 1}$ ,  $\tau_m \rightarrow 0$ , such that, for any  $T > 0$  and  $p \in [1, \infty)$ ,

$$\begin{aligned} u^{\tau_m} &\longrightarrow u \quad \text{in } L_p((0, T) \times \Omega, \mathbb{R}^2), \\ u^{\tau_m} &\overset{*}{\rightharpoonup} u \quad \text{in } L_\infty((0, T) \times \Omega, \mathbb{R}^2), \\ u^{\tau_m} &\rightharpoonup u \quad \text{in } L_2((0, T), H^1(\Omega, \mathbb{R}^2)). \end{aligned} \tag{3.13}$$

In addition, the compact embedding of  $L_2(\Omega, \mathbb{R}^2)$  in  $H^1(\Omega, \mathbb{R}^2)'$ , along with (3.6) with  $n = 2$ , (3.10), and (3.11), allows us to apply once more [5, Theorem 1] to conclude that

$$u \in C([0, \infty), H^1(\Omega, \mathbb{R}^2)'). \quad (3.14)$$

Let us now identify the equations solved by the components  $f$  and  $g$  of  $u$ . To this end, let  $\chi \in W_\infty^1([0, \infty))$  be a compactly supported function and  $\varphi \in C^1(\overline{\Omega})$ . In view of (3.3a), classical computations give

$$\begin{aligned} & \int_0^\infty \int_\Omega \frac{\chi(t+\tau) - \chi(t)}{\tau} f^\tau(t) \varphi \, dx dt + \left( \frac{1}{\tau} \int_0^\tau \chi(t) \, dt \right) \int_\Omega f^{in} \varphi \, dx \\ &= \int_0^\infty \int_\Omega \chi(t) f^\tau(t) \nabla [a f^\tau(t) + b g^\tau(t)] \cdot \nabla \varphi \, dx dt. \end{aligned}$$

Taking  $\tau = \tau_m$  in the above identity, it readily follows from (3.13) and the regularity of  $\chi$  and  $\varphi$  that we may pass to the limit as  $m \rightarrow \infty$  and conclude that

$$\begin{aligned} & \int_0^\infty \int_\Omega \frac{d\chi}{dt}(t) f(t, x) \varphi(x) \, dx dt + \chi(0) \int_\Omega f^{in}(x) \varphi(x) \, dx \\ &= \int_0^\infty \int_\Omega \chi(t) f(t, x) \nabla [a f + b g](t, x) \cdot \nabla \varphi(x) \, dx dt. \end{aligned} \quad (3.15)$$

Since  $f \nabla f$  and  $f \nabla g$  belong to  $L_2((0, T) \times \Omega)$  for all  $T > 0$  by (3.13), a density argument ensures that the identity (3.15) is valid for any  $\varphi \in H^1(\Omega)$ . We next use the time continuity (3.14) of  $f$  and a classical approximation argument to show that  $f$  solves (1.9a). A similar argument allows us to derive (1.9b) from (3.3b).

Finally, combining (3.13), (3.14), and a weak lower semicontinuity argument, we may let  $m \rightarrow \infty$  in (3.6), (3.7), and (3.8) with  $\tau = \tau_m$  to show that  $u = (f, g)$  satisfies (1.10), (1.11), and (1.12), thereby completing the proof.  $\square$

#### ACKNOWLEDGMENTS

PhL gratefully acknowledges the hospitality and support of the Fakultät für Mathematik, Universität Regensburg, where part of this work was done.

#### APPENDIX A. THE POLYNOMIALS $\Phi_n$ , $n \geq 2$

Let  $n \geq 2$ . According to the discussion in the introduction, we look for an homogeneous polynomial  $\Phi_n$  of degree  $n$  such that:

(P1)  $\Phi_n$  is convex on  $[0, \infty)^2$ ;

(P2) the matrix  $S_n(X) := D^2 \Phi_n(X) M(X)$  is symmetric and positive semidefinite for  $X \in [0, \infty)^2$ .

We recall that the mobility matrix  $M(X)$  is given by

$$M(X) = (m_{jk}(X))_{1 \leq j, k \leq 2} := \begin{pmatrix} aX_1 & bX_1 \\ cX_2 & dX_2 \end{pmatrix}, \quad X \in \mathbb{R}^2,$$

see (1.13). Specifically, we set

$$\Phi_n(X) := \sum_{j=0}^n a_{j,n} X_1^j X_2^{n-j}, \quad X = (X_1, X_2) \in \mathbb{R}^2, \quad (A.1)$$

with  $a_{j,n}$ ,  $0 \leq j \leq n$ , to be determined in order for properties (P1)-(P2) to be satisfied. We recall that the parameters  $(a, b, c, d)$  are assumed to satisfy (1.2).

**Lemma A.1.** *Set  $a_{0,n} := 1$  and*

$$a_{j,n} := \prod_{k=0}^{j-1} \frac{(n-k)[ak + c(n-k-1)]}{(k+1)[bk + d(n-k-1)]} = \binom{n}{j} \prod_{k=0}^{j-1} \frac{ak + c(n-k-1)}{bk + d(n-k-1)}, \quad 1 \leq j \leq n. \quad (\text{A.2})$$

Then  $a_{j,n} > 0$  for  $0 \leq j \leq n$  and  $S_n(X) = D^2\Phi_n(X)M(X) \in \mathbf{Sym}_2(\mathbb{R})$  for all  $X \in \mathbb{R}^2$ .

*Proof.* Given  $X \in \mathbb{R}^2$ , we compute

$$\begin{aligned} \partial_1^2 \Phi_n(X) &= \sum_{j=1}^{n-1} j(j+1)a_{j+1,n}X_1^{j-1}X_2^{n-j-1} = \sum_{j=0}^{n-2} (j+1)(j+2)a_{j+2,n}X_1^jX_2^{n-j-2}, \\ \partial_1 \partial_2 \Phi_n(X) &= \sum_{j=1}^{n-1} j(n-j)a_{j,n}X_1^{j-1}X_2^{n-j-1} = \sum_{j=0}^{n-2} (j+1)(n-j-1)a_{j+1,n}X_1^jX_2^{n-j-2}, \\ \partial_2^2 \Phi_n(X) &= \sum_{j=0}^{n-2} (n-j)(n-j-1)a_{j,n}X_1^jX_2^{n-j-2}. \end{aligned}$$

It then follows that

$$\begin{aligned} [S_n(X)]_{11} &= aX_1\partial_1^2\Phi_n(X) + cX_2\partial_1\partial_2\Phi_n(X) \\ &= a \sum_{j=1}^{n-1} j(j+1)a_{j+1,n}X_1^jX_2^{n-j-1} + c \sum_{j=0}^{n-2} (j+1)(n-j-1)a_{j+1,n}X_1^jX_2^{n-j-1}, \\ [S_n(X)]_{12} &= bX_1\partial_1^2\Phi_n(X) + dX_2\partial_1\partial_2\Phi_n(X) \\ &= b \sum_{j=1}^{n-1} j(j+1)a_{j+1,n}X_1^jX_2^{n-j-1} + d \sum_{j=0}^{n-2} (j+1)(n-j-1)a_{j+1,n}X_1^jX_2^{n-j-1} \\ &= bn(n-1)a_{n,n}X_1^{n-1} + \sum_{j=1}^{n-2} (j+1)[bj + d(n-j-1)]a_{j+1,n}X_1^jX_2^{n-j-1} \\ &\quad + d(n-1)a_{1,n}X_2^{n-1}, \\ [S_n(X)]_{21} &= aX_1\partial_1\partial_2\Phi_n(X) + cX_2\partial_2^2\Phi_n(X) \\ &= a \sum_{j=1}^{n-1} j(n-j)a_{j,n}X_1^jX_2^{n-j-1} + c \sum_{j=0}^{n-2} (n-j)(n-j-1)a_{j,n}X_1^jX_2^{n-j-1} \\ &= a(n-1)a_{n-1,n}X_1^{n-1} + \sum_{j=1}^{n-2} (n-j)[aj + c(n-j-1)]a_{j,n}X_1^jX_2^{n-j-1} \\ &\quad + cn(n-1)a_{0,n}X_2^{n-1}, \\ [S_n(X)]_{22} &= bX_1\partial_1\partial_2\Phi_n(X) + dX_2\partial_2^2\Phi_n(X) \end{aligned}$$

$$= b \sum_{j=1}^{n-1} j(n-j) a_{j,n} X_1^j X_2^{n-j-1} + d \sum_{j=0}^{n-2} (n-j)(n-j-1) a_{j,n} X_1^j X_2^{n-j-1}.$$

Hence,  $S_n(X)$  is symmetric provided that

$$(j+1)[bj + d(n-j-1)]a_{j+1,n} = (n-j)[aj + c(n-j-1)]a_{j,n}, \quad 0 \leq j \leq n-1,$$

or, equivalently,

$$a_{j+1,n} = \frac{(n-j)[aj + c(n-j-1)]}{(j+1)[bj + d(n-j-1)]} a_{j,n}, \quad 0 \leq j \leq n-1. \quad (\text{A.3})$$

Since  $a_{0,n} = 1$ , the closed form formula (A.2) readily follows from (A.3) and we deduce from (A.2) and the positivity of  $(a, b, c, d)$  that  $a_{j,n} > 0$  for all  $0 \leq j \leq n$ .  $\square$

We next show that  $D^2\Phi_n(X)$  is positive definite for  $X \in [0, \infty)^2 \setminus \{(0, 0)\}$ . This property implies in particular that  $D^2\Phi_n(X)$  is positive semidefinite for  $X \in [0, \infty)^2$ .

**Lemma A.2.** *Let  $\Phi_n$  be the polynomial defined by (A.1) and (A.2). Then  $D^2\Phi_n(X) \in \mathbf{SPD}_2(\mathbb{R})$  for  $X \in [0, \infty)^2 \setminus \{(0, 0)\}$ .*

*Proof.* Given  $X \in [0, \infty)^2$ , it follows from the positivity of the coefficients  $a_{j,n}$ ,  $0 \leq j \leq n$ , of  $\Phi_n$  that

$$\text{tr}(D^2\Phi_n(X)) := \partial_1^2\Phi_n(X) + \partial_2^2\Phi_n(X) \geq 0, \quad X \in [0, \infty)^2.$$

It remains to show that the determinant  $\det(D^2\Phi_n(X))$  is also non-negative. To this end we compute

$$\begin{aligned} \det(D^2\Phi_n(X)) &= \partial_1^2\Phi_n(X)\partial_2^2\Phi_n(X) - [\partial_1\partial_2\Phi_n(X)]^2 \\ &= \sum_{j=0}^{n-2} \sum_{k=0}^{n-2} (j+1)(n-k-1) A_{j,k} X_1^{j+k} X_2^{2n-j-k-4}, \end{aligned} \quad (\text{A.4})$$

where

$$A_{j,k} := (j+2)(n-k)a_{j+2,n}a_{k,n} - (n-j-1)(k+1)a_{j+1,n}a_{k+1,n}, \quad 0 \leq j, k \leq n-2.$$

Using (A.3), we express  $a_{j+2,n}$  and  $a_{k+1,n}$  in terms of  $a_{j+1,n}$  and  $a_{k,n}$ , respectively, to arrive at the following formula

$$\begin{aligned} A_{j,k} &= (n-k)(n-j-1) \left[ \frac{a(j+1) + c(n-j-2)}{b(j+1) + d(n-j-2)} - \frac{ak + c(n-k-1)}{bk + d(n-k-1)} \right] a_{j+1,n} a_{k,n} \\ &= (ad - bc)(n-k)(n-j-1) \frac{(j+1)(n-k-1) - k(n-j-2)}{[b(j+1) + d(n-j-2)][bk + d(n-k-1)]} a_{j+1,n} a_{k,n} \\ &= (ad - bc) \frac{(n-1)(n-k)(n-j-1)(j+1-k)}{\alpha_{j+1,n} \alpha_{k,n}} a_{j+1,n} a_{k,n}, \end{aligned} \quad (\text{A.5})$$

where  $\alpha_{k,n}$  denotes the positive number

$$\alpha_{k,n} := bk + d(n-k-1), \quad 0 \leq k \leq n-1.$$

In particular,

$$A_{k-1,j+1} = -A_{j,k}, \quad 0 \leq j \leq n-3, \quad 1 \leq k \leq n-2. \quad (\text{A.6})$$

It then follows from (A.4) that

$$\begin{aligned}
2 \det(D^2 \Phi_n(X)) &= \sum_{j=0}^{n-2} \sum_{k=0}^{n-2} (j+1)(n-k-1) A_{j,k} X_1^{j+k} X_2^{2n-j-k-4} \\
&\quad + \sum_{l=1}^{n-1} \sum_{i=-1}^{n-3} l(n-i-2) A_{l-1,i+1} X_1^{i+l} X_2^{2n-i-l-4} \\
&= \sum_{j=0}^{n-2} \sum_{k=0}^{n-2} (j+1)(n-k-1) A_{j,k} X_1^{j+k} X_2^{2n-j-k-4} \\
&\quad + \sum_{j=-1}^{n-3} \sum_{k=1}^{n-1} k(n-j-2) A_{k-1,j+1} X_1^{j+k} X_2^{2n-j-k-4} \\
&= \sum_{j=0}^{n-3} \sum_{k=1}^{n-2} (j+1)(n-k-1) A_{j,k} X_1^{j+k} X_2^{2n-j-k-4} \\
&\quad + \sum_{k=0}^{n-2} (n-1)(n-k-1) A_{n-2,k} X_1^{n-2+k} X_2^{n-k-2} \\
&\quad + \sum_{j=0}^{n-3} (j+1)(n-1) A_{j,0} X_1^j X_2^{2n-j-4} \\
&\quad + \sum_{j=0}^{n-3} \sum_{k=1}^{n-2} k(n-j-2) A_{k-1,j+1} X_1^{j+k} X_2^{2n-j-k-4} \\
&\quad + \sum_{k=1}^{n-1} k(n-1) A_{k-1,0} X_1^{k-1} X_2^{2n-k-3} \\
&\quad + \sum_{j=0}^{n-3} (n-1)(n-j-2) A_{n-2,j+1} X_1^{j+n-1} X_2^{n-j-3}.
\end{aligned}$$

According to (1.2) and (A.5),

$$\begin{aligned}
A_{l,0} &= (ad - bc) \frac{n(n-1)(n-1-l)(l+1)}{\alpha_{0,n} \alpha_{l+1,n}} > 0, \quad 0 \leq l \leq n-2, \\
A_{n-2,l} &= (ad - bc) \frac{(n-1)(n-l)(n-1-l)}{\alpha_{n-1,n} \alpha_{l,n}} > 0, \quad 0 \leq l \leq n-2.
\end{aligned}$$

In particular, all the terms in the above identity involving a single sum are non-negative. Therefore, using the symmetry property (A.6) and retaining in the last two sums only the terms corresponding to  $k=1$  and  $j=n-3$ , respectively, we get

$$\begin{aligned}
2 \det(D^2 \Phi_n(X)) &\geq \sum_{j=0}^{n-3} \sum_{k=1}^{n-2} [(j+1)(n-k-1) - k(n-j-2)] A_{j,k} X_1^{j+k} X_2^{2n-j-k-4} \\
&\quad + (n-1) A_{n-2,n-2} X_1^{2n-4} + (n-1) A_{0,0} X_2^{2n-4}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{n-3} \sum_{k=1}^{n-2} (n-1)(j+1-k) A_{j,k} X_1^{j+k} X_2^{2n-j-k-4} \\
&\quad + (n-1) A_{n-2, n-2} X_1^{2n-4} + (n-1) A_{0,0} X_2^{2n-4}.
\end{aligned}$$

Observing that

$$(n-1)(j+1-k) A_{j,k} = (ad-bc) \frac{(n-1)^2 (n-k)(n-j-1)(j+1-k)^2}{\alpha_{j+1,n} \alpha_{k,n}} a_{j+1,n} a_{k,n} \geq 0$$

for  $0 \leq j, k \leq n-2$ , we conclude that

$$2 \det(D^2 \Phi_n(X)) \geq (n-1) A_{n-2, n-2} X_1^{2n-4} + (n-1) A_{0,0} X_2^{2n-4}, \quad X \in [0, \infty)^2. \quad (\text{A.7})$$

Since  $A_{0,0} > 0$  and  $A_{n-2, n-2} > 0$ , we have thus established that, for each  $X \in [0, \infty)^2 \setminus \{(0,0)\}$ , the symmetric matrix  $D^2 \Phi_n(X)$  has non-negative trace and positive determinant, so that it is positive definite.  $\square$

We next turn to the positive definiteness of  $S_n = D^2 \Phi_n M$ .

**Lemma A.3.** *Let  $\Phi_n$  be defined by (A.1) and (A.2). Then  $S_n(X) = D^2 \Phi_n(X) M(X) \in \mathbf{SPD}_2(\mathbb{R})$  for  $X \in (0, \infty)^2$ .*

*Proof.* Let  $X \in (0, \infty)^2$ . On the one hand, by (1.2), (A.7), and the positivity of  $A_{0,0}$  and  $A_{n-2, n-2}$ ,

$$\begin{aligned}
2 \det(S_n(X)) &= 2(ad-bc) X_1 X_2 \det(D^2 \Phi_n(X)) \\
&\geq (ad-bc) X_1 X_2 (n-1) \left[ A_{n-2, n-2} X_1^{2n-4} + (n-1) A_{0,0} X_2^{2n-4} \right] > 0.
\end{aligned}$$

On the other hand, the positivity of  $a_{j,n}$  for  $0 \leq j \leq n$  and (1.2) imply that

$$\operatorname{tr}(S_n(X)) = [S_n(X)]_{11} + [S_n(X)]_{22} > 0.$$

Consequently,  $S_n(X)$  has positive trace and positive determinant, and is thus positive definite as claimed.  $\square$

We end up this section with useful upper and lower bounds for  $\Phi_n$ .

**Lemma A.4.** *Let  $\Phi_n$  be defined by (A.1) and (A.2). Then*

$$\frac{(cX_1 + dX_2)^n}{d^n} \leq \Phi_n(X) \leq \frac{(aX_1 + bX_2)^n}{b^n}, \quad X \in [0, \infty)^2. \quad (\text{A.8})$$

*Proof.* Since the function

$$\chi(z) := \frac{(a-c)z + c}{(b-d)z + d}, \quad z \in [0, 1],$$

is increasing and positive, we deduce from (A.2) that, for  $1 \leq j \leq n$ ,

$$a_{j,n} = \binom{n}{j} \prod_{k=0}^{j-1} \chi\left(\frac{k}{n-1}\right) \leq \binom{n}{j} [\chi(1)]^j = \binom{n}{j} \left(\frac{a}{b}\right)^j$$

and

$$a_{j,n} = \binom{n}{j} \prod_{k=0}^{j-1} \chi\left(\frac{k}{n-1}\right) \geq \binom{n}{j} [\chi(0)]^j = \binom{n}{j} \left(\frac{c}{d}\right)^j.$$

The upper and lower bounds in (A.8) are direct consequences of the above inequalities.  $\square$

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