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2 **Analytical solution and Bayesian inference for interference pumping tests in fractal du-**
3 **al-porosity media**

4

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22 **Abstract**

23 A new analytical solution is developed for interference hydraulic pumping tests in fractured
24 porous media using the dual-porosity concept. Heterogeneous fractured reservoirs are consid-
25 ered with hydrodynamic parameters assumed to follow power-law functions in radial dis-
26 tance. The developed analytical solution is verified by comparison against a finite volume
27 numerical solution. The comparison shows that the numerical solution converges toward the
28 analytical one when the size of the time step decreases. The applicability of the fractal dual-
29 porosity model is then assessed by investigating the identifiability of the parameters from a
30 synthetic interference pumping test with a set of noisy data using Bayesian parameter infer-
31 ence. The results show that if the storage coefficient in the matrix is fixed, the rest of the pa-
32 rameters can be appropriately inferred, otherwise, the identification of the parameters is faced
33 with convergence problems because of equifinality issues.

34 **Keywords:** Fractured porous media, interference pumping test, dual-porosity, fractal media,
35 Bayesian inversion.

36

37 1 INTRODUCTION

38 Simulation of interference pumping tests in porous media allows testing different prospective
39 exploitation scenarios at low costs. These tests consist in pumping a well at a controlled flow
40 rate and measuring the water level response (drawdown) in one or more surrounding observa-
41 tion wells. Pumping tests in fractured porous media have been discussed by several authors
42 (see [1-8], among others). Fractured media are often addressed using the dual-porosity model
43 suggested by Barenblatt *et al.* [9]. The main idea of this model is to consider two overlapping
44 continua: (i) the fractures which carry the major part of water to the pumping well and (ii) the
45 low permeability rock matrix which feeds fluid to the fractures. At each point of the domain,
46 water flow between the fractures and matrix are coupled via a linear exchange term propor-
47 tional to the difference in hydraulic head between the two continua. As stated by De Smedt
48 [8], the simulation of interference pumping tests is often based on numerical models which
49 cannot insure full control of numerical accuracy [10]. Therefore, analytical solutions are high-
50 ly sought-after because (i) they provide valuable insights into **fundamental physics**, as they
51 are free of numerical errors and (ii), although limited to simplified situations; they allow vali-
52 dation of more complex numerical codes.

53 **In the context of interference pumping tests in fractured porous media, an analytical solution**
54 **has been developed for homogenous aquifers by De Smedt [8]. Furthermore, Delay et al. [7]**
55 **developed a numerical solution using the dual-medium approach to describe radial convergent**
56 **flow in heterogeneous fractal media where power laws in space are used for the hydrodynam-**
57 **ic parameters. As a consequence, parameter values decrease with the distance between the**
58 **pumped well and the observed ones. This type of scaling laws inherits from theoretical works**
59 **on fractals [11] and has been used by several authors for single porous media ([12-15], among**
60 **others). Hence, the main objective of this paper is to develop, for the first time, an analytical**
61 **solution for the mathematical model describing interference pumping tests in heterogeneous**

62 **fractal dual media.** Applicability of the model is then assessed by investigating identifiability
63 of the different parameters from a synthetic pumping test with a set of noisy data using Statis-
64 tical Calibration (SC) [16]. **The latter means refining the prior distribution of uncertain pa-**
65 **rameters based on matching simulation outputs with data [17].** In this work, the SC is per-
66 formed with the DREAM_(ZS) software [16] based on the Markov Chain Monte Carlo process
67 (MCMC). **DREAM_(ZS) computes multiple sub-chains in parallel to thoroughly explore the**
68 **parameter space. Taking the last 25% of individuals of the MCMC (when the chains have**
69 **converged) yields multiple sets of parameters that adequately fit the model onto observations.**
70 **These sets can then be used to estimate the updated parameter distributions and therefore the**
71 **best estimates of the parameters and the associate confidence intervals.**

72 **2 MATHEMATICAL MODEL**

73 We assume a horizontal, isotropic, confined and infinite aquifer with a uniform thickness. The
74 pumping well only draws water from the fractures, has a constant rate and fully penetrates the
75 aquifer. Using the dual-porosity concept, the flow in fractured porous media can be described
76 by the following equations [18, 19]:

$$77 \quad S_f \frac{\partial h_f}{\partial t} = \nabla \cdot (K_f \nabla h_f) + \alpha (h_m - h_f) \quad (1)$$

$$78 \quad S_m \frac{\partial h_m}{\partial t} = \alpha (h_f - h_m) \quad (2)$$

79 where indexes m and f refer to the matrix and fracture continua, respectively, $S[L^{-1}]$ is the
80 specific storage, $h[L]$ is the hydraulic head, $K_f[LT^{-1}]$ is the hydraulic conductivity of the
81 fractures, $\alpha[L^{-1}T^{-1}]$ is the exchange rate coefficient between the fractures and the matrix.
82 Following Warren and Root [18], equation (2) assumes pseudo-steady flow in the matrix con-

83 tinuum (term K_m is removed). Assuming radial symmetry, equations (1) and (2) can be for-
 84 mulated as:

$$85 \quad S_f(r) \frac{\partial h_f(r,t)}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[r K_f(r) \frac{\partial h_f(r,t)}{\partial r} \right] + \alpha(r) [h_m(r,t) - h_f(r,t)] \quad (3)$$

$$86 \quad S_m(r) \frac{\partial h_m(r,t)}{\partial t} = \alpha(r) [h_f(r,t) - h_m(r,t)] \quad (4)$$

87 where $r[L]$ is the radial coordinate with the center of the pumping well as origin.

88 The drawdown in an observation well intercepting the fractures can be expressed as follows

$$89 \quad S_f(r) \frac{\partial H_f(r,t)}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r K_f(r) \frac{\partial H_f(r,t)}{\partial r} \right) + \alpha(r) [H_m(r,t) - H_f(r,t)] \quad (5)$$

$$90 \quad S_m(r) \frac{\partial H_m(r,t)}{\partial t} = \alpha(r) [H_f(r,t) - H_m(r,t)] \quad (6)$$

91 where $H_f = h_0 - h_f$ and $H_m = h_0 - h_m$ are respectively the drawdown in the fracture and in the
 92 rock matrix. Initially, a uniform hydraulic head (h_0) is assumed in both the fracture and ma-
 93 trix continua. Therefore, the drawdowns satisfy the following initial conditions:

$$94 \quad H_f(r,0) = H_m(r,0) = 0 \quad (7)$$

95 Far away from the pumping well, the drawdowns are assumed to be zero

$$96 \quad \lim_{r \rightarrow \infty} H_f(r,t) = \lim_{r \rightarrow \infty} H_m(r,t) = 0 \quad (8)$$

97 The limit where the radius approaches zero corresponds to the well withdrawal rate

$$98 \quad \lim_{r \rightarrow 0} \left[-2\pi r B K_f(r) \frac{\partial H_f(r,t)}{\partial r} \right] = Q_0, \quad t > 0 \quad (9)$$

99 where B is the thickness of the aquifer and Q_0 is the pumping flow rate.

100 **In the sequel, we limit our investigation for the semi-analytical solution of equations (5-9) to**
 101 **the case of a confined reservoir of a fractal structure with a single well in an unbounded me-**

102 dium as described in [7, 12]. Note that more complex fractal approaches have been developed
 103 in the case of many wells with variable rates [20] and by considering transient flow from
 104 block to fissure and skin between the fissure and the block [21].

105 In this work, following Delay *et al.* [7], we consider dual porosity media in which the hydro-
 106 dynamic parameters are power-law functions in space (decreasing functions of the radial dis-
 107 tance r between the pumping well and the observed one). Moreover, for sake of simplicity,
 108 we assume the same functions for the exchange rate coefficient α and for the storage coeffi-
 109 cients S_f and S_m . Hence, the values of the hydraulic parameters vary with respect to radial
 110 distance according to the following power-law functions:

$$111 \quad \begin{aligned} K_f(r) &= K_{f_0} r^{-a}; & S_f(r) &= S_{f_0} r^{-b} \\ S_m(r) &= S_{m_0} r^{-b}; & \alpha(r) &= \alpha_0 r^{-b} \end{aligned} \quad (10)$$

112 where the power-law exponents a and b are assumed to be positive.

113 3 ANALYTICAL SOLUTION

114 3.1 Analytical solution in the Laplace domain

115 In this section, we follow the strategy developed by De Smedt [8] to derive the analytical so-
 116 lution of the system formed by equations (5)-(10). The solution is derived using the Laplace
 117 transform technique with respect to time variable t . The Laplace transform of a time-
 118 dependent function y is

$$119 \quad \bar{y}(p) = L[y(t); t \rightarrow p] = \int_0^{\infty} y(t) e^{-pt} dt \quad (11)$$

120 Applying the Laplace transform to (5)-(6) and using (10) leads to

$$121 \quad pS_{f_0} r^{-b} \bar{H}_f(r, p) = \frac{K_{f_0}}{r} \frac{\partial}{\partial r} \left[r^{1-a} \frac{\partial \bar{H}_f(r, p)}{\partial r} \right] + \alpha_0 r^{-b} [\bar{H}_m(r, p) - \bar{H}_f(r, p)] \quad (12)$$

$$122 \quad pS_{m_0} \bar{H}_m(r, p) = \alpha_0 [\bar{H}_f(r, p) - \bar{H}_m(r, p)] \quad (13)$$

123 Equation (13) can be rearranged

$$124 \quad \bar{H}_m(r, p) = \frac{\alpha_0 / S_{m_0}}{p + \alpha_0 / S_{m_0}} \bar{H}_f(r, p) \quad (14)$$

125 Substituting (14) into (12) yields

$$126 \quad r^2 \frac{\partial^2 \bar{H}_f}{\partial r^2} + (1-a)r \frac{\partial \bar{H}_f}{\partial r} - \frac{S_{f_0}}{K_{f_0}} \left[p + \frac{(\alpha_0 / S_{f_0}) p}{p + \alpha_0 / S_{m_0}} \right] r^{a-b+2} \bar{H}_f = 0 \quad (15)$$

127 Equation (15) is a second order ordinary differential equation; its general solution is of the
 128 form (cf. Polyanin and Zaitsev [22], page 159, equation (127))

$$129 \quad \bar{H}_f(r, p) = r^{\frac{a}{2}} \left\{ C_1 J_\nu \left[\frac{2iS_{f_0}^{1/2} r^{n/2}}{nK_{f_0}^{1/2}} \sqrt{A(p)} \right] + C_2 Y_\nu \left[\frac{2iS_{f_0}^{1/2} r^{n/2}}{nK_{f_0}^{1/2}} \sqrt{A(p)} \right] \right\} \quad (16)$$

130 where $n = a - b + 2$, $\nu = \frac{a}{n}$, $A(p) = p + \frac{(\alpha_0 / S_{f_0}) p}{p + \alpha_0 / S_{m_0}}$, $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions

131 of the first and second kinds, respectively. The parameters C_1 and C_2 are constant that can be
 132 calculated using boundary conditions.

133 In the Laplace domain, the boundary conditions become

$$134 \quad \lim_{r \rightarrow \infty} \bar{H}_f(r, p) = 0 \quad (17)$$

135 and

$$136 \quad \lim_{r \rightarrow 0} \left[-2\pi B K_{f_0} r^{1-a} \frac{\partial \bar{H}_f(r, p)}{\partial r} \right] = \frac{Q_0}{p} \quad (18)$$

137 The solution of (15), taking into account the boundary conditions (17) and (18) is

$$138 \quad \bar{H}_f(r, p) = \frac{2r^{\frac{a}{2}}}{n^{1-\nu} \pi \Gamma(1-\nu) B K_{f_0}^{1-\nu/2} S_{f_0}^{\nu/2}} \frac{\bar{Q}(p) K_\nu \left[\frac{2S_{f_0}^{1/2} r^{n/2}}{nK_{f_0}^{1/2}} \sqrt{p + \frac{(\alpha_0 / S_{f_0}) p}{p + \alpha_0 / S_{m_0}}} \right]}{\left[\sqrt{p + \frac{(\alpha_0 / S_{f_0}) p}{p + \alpha_0 / S_{m_0}}} \right]^\nu} \quad (19)$$

139 where K_ν is the modified Bessel function of the second kind of order ν . More details about
 140 the derivation of the solution (19) are given in the Appendix.

141

142 3.2 Analytical solution in the time domain

143 The solution in the time domain $H_f(r, t)$ is sought using the convolution theorem of the La-
 144 place transform [23]

$$145 \quad L \left[\int_0^t y(\tau, t-\tau) d\tau; t \rightarrow p \right] = L \left\{ L \left[y(t_1, t_2); t_2 \rightarrow p \right]; t_1 \rightarrow p \right\} \quad (20)$$

146 Hence, equation (19) is written as

$$147 \quad \bar{H}_f(r, p) = \bar{H}_f(r; p_1, p_2) = \frac{2Q_0 r^{\frac{a}{2}}}{n^{1-\nu} \pi \Gamma(1-\nu) B K_{f_0}^{1-\nu/2} S_{f_0}^{\nu/2}} \frac{K_\nu \left[\frac{2S_{f_0}^{1/2} r^{n/2}}{nK_{f_0}^{1/2}} \sqrt{p_1 + \frac{(\alpha_0/S_{f_0}) p_2}{p_2 + \alpha_0/S_{m_0}}} \right]}{p_2 \left[\sqrt{p_1 + \frac{(\alpha_0/S_{f_0}) p_2}{p_2 + \alpha_0/S_{m_0}}} \right]^\nu} \quad (21)$$

148 where $p_1 = p_2 = p$ but different indexes are used to indicate that the inverse transformation is
 149 performed in two steps.

150 The different indexes used in (21) indicate that the inverse Laplace transform is developed in
 151 two steps. In a first step, the inverse Laplace transform with respect to the variable p_1 is ex-
 152 pressed as follows

$$153 \quad L^{-1} \left[\bar{H}_f(r; p_1, p_2); p_1 \rightarrow t_1 \right] = \frac{n^\nu Q_0 e^{-\frac{S_{f_0} r^n}{2n^2 K_{f_0} t_1}} \cdot W_{\frac{1-\nu}{2}, \frac{\nu}{2}} \left(\frac{S_{f_0} r^n}{n^2 K_{f_0} t_1} \right)}{\pi \Gamma(1-\nu) B K_{f_0}^{\frac{1-\nu}{2}} S_{f_0}^{\frac{1+\nu}{2}} r^{\frac{2-b}{2}}} \cdot \frac{e^{-\frac{(\alpha_0/S_{f_0}) t_1 p_2}{p_2 + \alpha_0/S_{m_0}}}}{p_2} \quad (22)$$

154 where t_1 is the time variable obtained by the inverse Laplace transform with respect to the
 155 variable p_1 and $W_{\kappa, \mu}(z)$ is the Whittaker function (Abramowitz and Stegun [24] p. 505).

156 In the second step, the inverse Laplace transform with respect to p_2 is obtained using the in-

157 verse transform of the J -function [25] based on the formula $L^{-1}\left[\frac{1}{p}e^{-\frac{\alpha p}{p+\beta}}; p \rightarrow t\right] = J(\alpha, \beta t)$

158 which yields

$$159 \quad \bar{H}_f(r; p_1, p_2) = \frac{n^\nu Q_0 e^{-\frac{S_{f_0} r^n}{2n^2 K_{f_0} t_1}} \cdot t_1^{\frac{\nu-1}{2}} \cdot W_{\frac{1-\nu}{2}, \frac{\nu}{2}}\left(\frac{S_{f_0} r^n}{n^2 K_{f_0} t_1}\right) \cdot J\left(\frac{\alpha_0 t_1}{S_{f_0}}, \frac{\alpha_0 t_2}{S_{m_0}}\right)}{2\pi\Gamma(1-\nu)BK_{f_0}^{\frac{1-\nu}{2}}S_{f_0}^{\frac{1+\nu}{2}}r^{\frac{2-b}{2}}} \quad (23)$$

160 The Bessel integral J -function is defined as follows [8, 26]

$$161 \quad J(a, b) = 1 - e^{-b} \int_0^a e^{-\lambda} I_0(2\sqrt{b\lambda}) d\lambda = e^{-a-b} \sum_{n=0}^{\infty} \frac{b^n}{n!} \sum_{m=0}^n \frac{a^m}{m!} \quad (24)$$

162 In view of (20) and (23), the analytical expression of $H_f(r, t)$ is

$$163 \quad H_f(r, t) = \frac{n^\nu Q_0}{2\pi\Gamma(1-\nu)BK_{f_0}^{\frac{1-\nu}{2}}S_{f_0}^{\frac{1+\nu}{2}}r^{\frac{2-b}{2}}} \int_0^t \tau^{\frac{\nu-1}{2}} \cdot e^{-\frac{S_{f_0} r^n}{2n^2 K_{f_0} \tau}} \cdot W_{\frac{1-\nu}{2}, \frac{\nu}{2}}\left(\frac{S_{f_0} r^n}{n^2 K_{f_0} \tau}\right) \cdot J\left(\frac{\alpha_0 \tau}{S_{f_0}}, \frac{\alpha_0 (t-\tau)}{S_{m_0}}\right) d\tau \quad (25)$$

165 A similar expression can be obtained for $H_m(r, t)$ using the convolution theorem of the La-
166 place transform for equation (14) and using equation (19).

167 In the case of homogeneous media ($a = b = 0$), we have $W_{\frac{1}{2}, 0}(z) = z^{\frac{1}{2}} e^{-\frac{z}{2}}$ and (25) reduces to

168 the De Smedt [8] solution:

$$169 \quad H_f(r, t) = \frac{Q_0}{4\pi BK_{f_0}} \int_0^t \tau^{-1} e^{-\frac{S_{f_0} r^2}{4K_{f_0} \tau}} \cdot J\left(\frac{\alpha_0 \tau}{S_{f_0}}, \frac{\alpha_0 (t-\tau)}{S_{m_0}}\right) d\tau \quad (26)$$

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173 4 COMPARISON TO NUMERICAL SOLUTIONS

174 We numerically solve two interference pumping tests dealing with homogeneous and fractal
175 dual media. For each test, the numerical solution is compared against the corresponding ana-
176 lytical solution using (25). The numerical solution of the system (5)-(10) is developed using
177 the standard Finite Volume (FV) method. The domain of size $L=10,000m$ is discretized us-
178 ing a uniform spatial discretization of size $\Delta r = 5$ m. A similar solution is obtained using a
179 finer spatial discretization of size $\Delta r = 1$ m which shows the weak sensitivity of the numerical
180 solution to the spatial discretization. The duration of the simulation is 2×10^6 s. Three time
181 step sizes are investigated with $\Delta t = 10^5$ s, $\Delta t = 5 \times 10^4$ s and $\Delta t = 10^4$ s. The pumping flow rate
182 is $Q_0 = 5$ m³/h. The initial head in the aquifer is $h_0 = 100$ m and the aquifer thickness is
183 $B = 100$ m. The specific storage of the fractures and rock matrix are respectively $S_{f_0} = 5 \times 10^{-6}$
184 m⁻¹ and $S_{m_0} = 10^{-4}$ m⁻¹. The hydraulic conductivity of the fractures is $K_{f_0} = 10^{-4}$ m/s and the
185 rate of exchange between the matrix and the fractures is $\alpha_0 = 5 \times 10^{-11}$ m⁻¹s⁻¹.

186 Figure 1 depicts the drawdown in both the fracture and matrix at 10m and 100m. The draw-
187 down in the fissure starts at early times because storage of the fissures is immediately effec-
188 tive at the start of the pumping test, while storage of the matrix needs much more time to be-
189 come noticeable. A flattening of the drawdown curve in the fissure can be observed at inter-
190 mediate pumping time which is typical for double-porosity behavior because of the transition
191 from fracture to matrix water storage release.

192 The numerical FV solution is compared to the analytical solution (25) in the case of homoge-
193 neous ($a = b = 0$) and fractal ($a = 0.6$ and $b = 0.4$) dual media. The comparison is performed
194 at two observation wells located respectively at distances $r_1 = 10$ m and $r_2 = 100$ m from the
195 pumping well. Significant discrepancies can be observed between analytical and numerical

196 solutions when the latter is evaluated using a large time step size (Figure 2). The drawdown in
197 the case of fractal dual media is larger than in homogeneous dual media because of the reduc-
198 tion of the hydrodynamic parameters with the distance r from the pumping well. In fact, in
199 the case of fractal dual media, the permeability of the fractures is proportional to $r^{-0.6}$ where-
200 as the exchange rate and the specific storage of both the fractures and the matrix are propor-
201 tional to $r^{-0.4}$. Thereby, the discrepancy between analytical and numerical solutions (Figure
202 2) is more pronounced with dual fractal media than with homogeneous dual media. For both
203 homogeneous and fractal dual media, the numerical solution converges toward the analytical
204 solution (25) when the size of the time step decreases (Figure 2). Note that the reduction of
205 the permeability of the fractures with respect to the distance r is also responsible, via the
206 Whittaker function in equation (25), of the delay for the stabilisation of the solution in the
207 fractal case (Figure 2 b) compared to the homogeneous case (Figure 2a).

208 5 BAYESIAN INVERSION OF A PUMPING TEST IN FRACTAL DUAL MEDIA

209 In practical applications, most of the hydraulic parameters cannot be measured directly and
210 should be estimated using an inversion procedure. In the case of fractal dual media, the inver-
211 sion can be challenging because of model equifinality, implying that different parameter com-
212 binations can fit the model responses to data. To assess the applicability of the fractal dual
213 porosity model, the identifiability of the parameters of the analytical solution (25) is assessed
214 using a synthetic interference pumping test with a set of noisy data. Parameter identification is
215 performed in a Bayesian framework where the prior knowledge about the parameters and the
216 observed data are merged to define the joint posterior probability distribution function (pdf) of
217 the parameters. In this work, the pdf is performed using the DREAM_(ZS) software [16] based
218 on the Markov Chain Monte Carlo (MCMC) sampler. DREAM_(ZS) generates random se-
219 quences of parameter sets that asymptotically converge toward the target joint posterior dis-

220 tribution [27]. Thus, if the number of runs is sufficiently high, the generated samples allows
 221 exploring the entire parameter space of the posterior distribution of the parameters and pro-
 222 vides the pairwise parameter correlations and the uncertainty of model predictions.

223 The Bayes theorem states that the probability density function of the model parameters condi-
 224 tioned onto data can be expressed as:

$$225 \quad p(\xi | \mathbf{y}_{mes}) \propto p(\mathbf{y}_{mes} | \xi) p(\xi), \quad (27)$$

226 where $p(\xi | \mathbf{y}_{mes})$ is the likelihood function measuring how well the model fits the observa-
 227 tions \mathbf{y}_{mes} , and $p(\xi)$ is the prior information about the parameter before the observations are
 228 made. In this work, a Gaussian distribution defines the likelihood function because the *obser-*
 229 *vements* are simulated and corrupted with Gaussian errors. Hence, the parameter posterior dis-
 230 tribution is expressed as:

$$231 \quad p(\xi | \mathbf{y}_{mes}) \propto \sigma_h^{-N_h} \exp\left(-\frac{SS_h(\xi)}{2\sigma_h^2}\right), \quad (28)$$

232 where $SS_h(\xi)$, are the sums of the squared differences between the observed and modeled
 233 drawdowns. For instance, $SS_h(\xi) = \sum_{k=1}^{N_h} (h_{mes}^{(k)} - h_{mod}^{(k)}(\xi))^2$, which includes the observed $h_{mes}^{(k)}$
 234 and predicted $h_{mod}^{(k)}$ drawdowns at time t_k for the number of observations N_h .

235 A synthetic pumping test is generated by running the analytical solution using the reference
 236 parameter values given in Table 1 for K_{f_0} , S_{f_0} , α_0 , a and b . As in Fahs *et al.* [28], the stor-
 237 age coefficient in the matrix S_{m_0} is fixed in this study to be 10^{-4} m^{-1} .

238 Delay et al [7] showed that results of calibration improved when several drawdown curves at
 239 several locations are inverted at once. In the sequel, the analytical solution is run for a long
 240 period of 2×10^7 s to calculate the drawdown at four observation wells located respectively at
 241 $r_1 = 1 \text{ m}$, $r_2 = 10 \text{ m}$, $r_3 = 100 \text{ m}$ and $r_4 = 500 \text{ m}$ from the pumping well. Four datasets of 2000

242 values each (corresponding to the drawdown at the observation wells measured each 10^4 s)
243 are used as target responses for model inversion.

244 Gaussian noise with a standard deviation of 0.01m and a mean of zero was added to the target
245 responses. We assume that all hydraulic parameters have uniform prior distributions over the
246 ranges given in Table 1. As recommended in Vrugt *et al.* [29], we consider that the posterior
247 distribution is stationary if the Gelman and Ruban [30] criterion is less than 1.2 and that the
248 chains are not autocorrelated.

249 The MCMC method was terminated after 15000 model runs. The convergence was reached at
250 around 10000 model runs. Figure 3 shows the results of the identification based on the last
251 25% of individuals of the MCMC sampler (when the chains have converged). The "on-
252 diagonal" plots in these figures display the inferred parameter distributions, whereas the "off-
253 diagonal" plots represent the pairwise correlations in the MCMC sample. If the drawings are
254 independent, non-sloping scatterplots should be observed. However, if a good value of a giv-
255 en parameter is conditioned by the value of another parameter, then their pairwise scatterplot
256 should show a narrow sloping stripe. The results of Figure 3 show that the five parameters
257 K_{f_0} , S_{f_0} , α_0 , a and b are appropriately estimated; they have almost symmetric bell-shaped
258 posterior distributions and have strongly narrowed their prior intervals. The scatter plots in
259 this figure show a moderate correlation between the parameters a and b (a correlation factor
260 of 0.61), a high correlation between the parameters S_{f_0} and b (a correlation factor of 0.79)
261 and a very high correlation between the parameters K_{f_0} and a (a correlation factor of 0.98).

262 The mean and 95% confidence intervals of the samples that adequately fit the model onto
263 observations are reported in Table 2. From this table, we can see that the parameters K_{f_0} , a
264 and b are well identified with respectively a mean value of $0.99 \times 10^{-5} \text{ ms}^{-1}$, 0.299 and 0.2.
265 These values are almost identical to the reference values of Table 1. The uncertainty on these

266 parameters is small since the 95% confidence intervals are respectively $[0.9-1.1]\times 10^{-5} \text{ ms}^{-1}$,
267 $[0.28-0.32]$ and $[0.17-0.23]$. The parameters S_{f_0} and α_0 are less well identified. Indeed, alt-
268 hough their mean values (respectively $5.03\times 10^{-5} \text{ m}^{-1}$ and $3.06\times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$) are very close to
269 the reference values, their posterior uncertainty remains quite large since their confidence
270 intervals are respectively $[3.2-7.6]\times 10^{-5} \text{ m}^{-1}$ and $[2.1-4.5]\times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$.
271 Finally note that the identification procedure does not converge if the storage coefficient in
272 the matrix S_{m_0} is involved in the inversion procedure (not fixed). In that case, the identifica-
273 tion of the six parameters K_{f_0} , S_{f_0} , S_{m_0} , α_0 , a and b was not possible. Indeed, in this case,
274 the MCMC algorithm was faced to convergence problem because of **equifinality** issues.

275

276 **6 CONCLUSION**

277 A new analytical solution for interference hydraulic pumping tests in fractal fractured porous
278 media is developed using the dual-porosity concept and power laws in space for all hydrody-
279 namic parameters. The developed analytical solution is compared to a numerical one obtained
280 with the FV method using different time step sizes. Discrepancies are observed between ana-
281 lytical and numerical solutions when the latter is used with large time steps. These discrepan-
282 cies are more pronounced with dual fractal media than with homogeneous dual media. When
283 the size of the time step decreases, the numerical solution converges toward the analytical one
284 for both homogeneous and fractal dual media.

285 The applicability of the fractal dual-porosity model was then assessed by investigating identi-
286 fiability of the hydraulic parameters from a synthetic interference pumping test with a set of
287 noisy data using the MCMC sampler. The results show that if the storage coefficient in the

288 matrix is fixed, the rest of the parameters can be appropriately identified, whereas conver-
289 gence problems can be encountered if all the hydraulic parameters are to be estimated.

290

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296

297 **Appendix: Derivation of the Analytical Solution in the Laplace Domain**

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299 Using the boundary condition at infinity (17) and the following asymptotic expansions for

300 large arguments ($|z| \rightarrow \infty$) of J_ν and Y_ν [22], $J_\nu(z) = \sqrt{2/\pi z} \{P(\nu, z) \cos \chi - Q(\nu, z) \sin \chi\}$

301 and $Y_\nu(z) = \sqrt{2/\pi z} \{P(\nu, z) \cos \chi + Q(\nu, z) \sin \chi\}$ for $|\arg z| < \pi$ with $\chi = z - \left(\frac{1}{2}\nu + \frac{1}{4}\right)\pi$,

302 $P(\nu, z) = \sum_{k=0}^{\infty} (-1)^k \frac{(\nu, 2k)}{(2z)^{2k}}$ and $Q(\nu, z) = \sum_{k=0}^{\infty} (-1)^k \frac{(\nu, 2k+1)}{(2z)^{2k+1}}$, equation (16) gives

303 $C_1 + iC_2 = 0$. Therefore, this latter can be written as

$$304 \quad \bar{H}_f(r, p) = C_1 r^{\frac{a}{2}} H_\nu^{(1)} \left[\frac{2iS_{f_0}^{1/2} r^{n/2}}{nK_{f_0}^{1/2}} \sqrt{A(p)} \right] \quad (\text{A.1})$$

305 where $H_\nu^{(1)}$ is the Hankel function of the first kind defined by $H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z)$ for

306 any complex number z .

307 The boundary condition at the well (18) is then used to find C_1 . Note that for $\nu > 0$ the

308 Hankel function $H_\nu^{(1)} \sim -\frac{i}{\pi} \Gamma(\nu) \left(\frac{1}{2}z\right)^{-\nu}$ as z approaches zero [22], where Γ is the Gamma

309 function. Using this expansion we can write

$$310 \quad r^{1-a} \frac{\partial H_f(r, t)}{\partial r} \sim \frac{i^{1-\nu} S_{f_0}^{\nu/2} C_1}{2\pi K_{f_0}^{\nu/2}} \left\{ -r^{-a} n^\nu \left[\sqrt{A(p)} \right]^{-\nu} \{a\Gamma(\nu) - n\Gamma(1+\nu)\} + n^{1-\nu} \left[\sqrt{A(p)} \right]^\nu \Gamma(1-\nu) \right\}$$

$$311 \quad (\text{A.2})$$

312 as r approaches zero. Using the property of the Gamma function $\Gamma(1+\nu) = \nu\Gamma(\nu)$ for $\nu > 0$,
 313 the first term of right hand side of equation (A.2) vanishes since $n\nu = a$. Substituting (A.2)

314 into (18) we get $C_1 = -\frac{\bar{Q}(p)}{i^{1-\nu} n^{1-\nu} \Gamma(1-\nu) \mathbf{BK}_{f_0}^{1-\nu/2} \mathcal{S}_{f_0}^{\nu/2} [\sqrt{A(p)}]^\nu}$.

315 Finally, inserting the value of C_1 into (A.1) and using the relationship

316 $K_\nu(z) = \frac{1}{2} \pi i e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}\left(ze^{\frac{1}{2}\pi i}\right)$ for $-\pi < \arg z \leq \frac{1}{2}\pi$, where K_ν is the modified Bessel func-

317 tion of the second kind of order ν , we get the solution in the Laplace domain as defined by
 318 (19).

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413 diagonal scatterplots represent the value of a parameter α versus that of a parameter β in the
414 best solutions inferred by MCMC.

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422 **Table 1**

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Parameters	Lower bounds	Upper bounds	Reference values
K_{f_0} [ms ⁻¹]	5×10^{-7}	5×10^{-4}	1×10^{-5}
S_{f_0} [m ⁻¹]	10^{-6}	10^{-3}	5×10^{-5}
α_0 [m ⁻¹ s ⁻¹]	10^{-12}	10^{-9}	3×10^{-11}
a [-]	0.1	0.5	0.3
b [-]	0.1	0.5	0.2

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429 **Table 2**

430

Parameters	Mean	95% Conf. Int.
K_{f_0} [ms^{-1}]	0.99×10^{-5}	$[0.9-1.1] \times 10^{-5}$
S_{f_0} [m^{-1}]	5.03×10^{-5}	$[3.2-7.6] \times 10^{-5}$
α_0 [$\text{m}^{-1}\text{s}^{-1}$]	3.06×10^{-11}	$[2.1-4.5] \times 10^{-11}$
a [-]	0.299	[0.28-0.32]
b [-]	0.2	[0.17-0.23]

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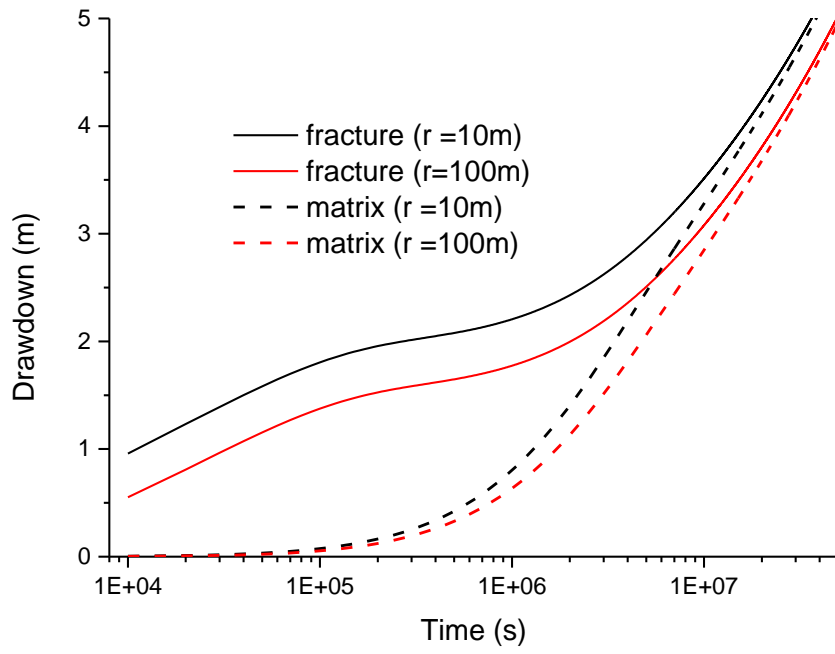
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442 **Figure 1**

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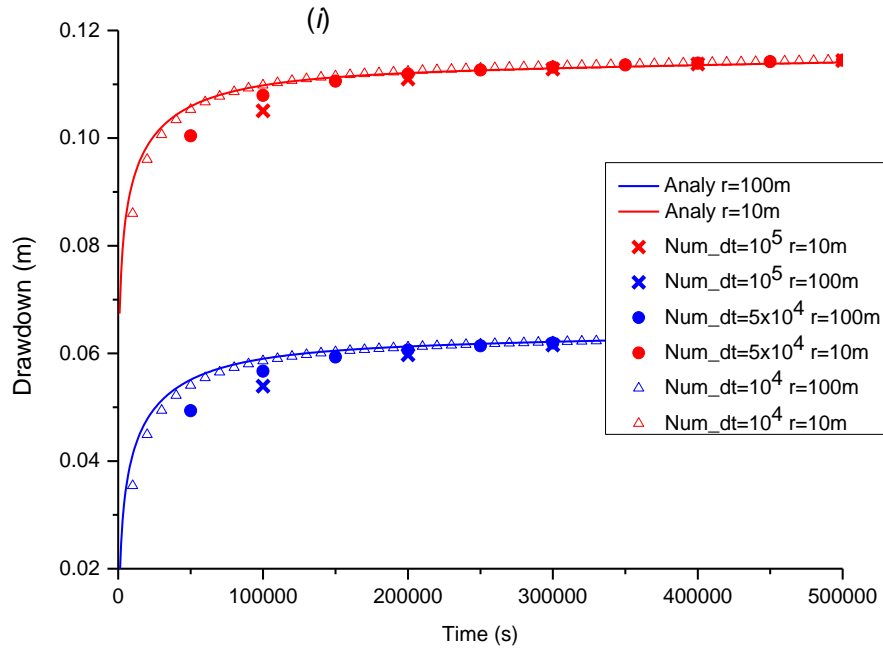


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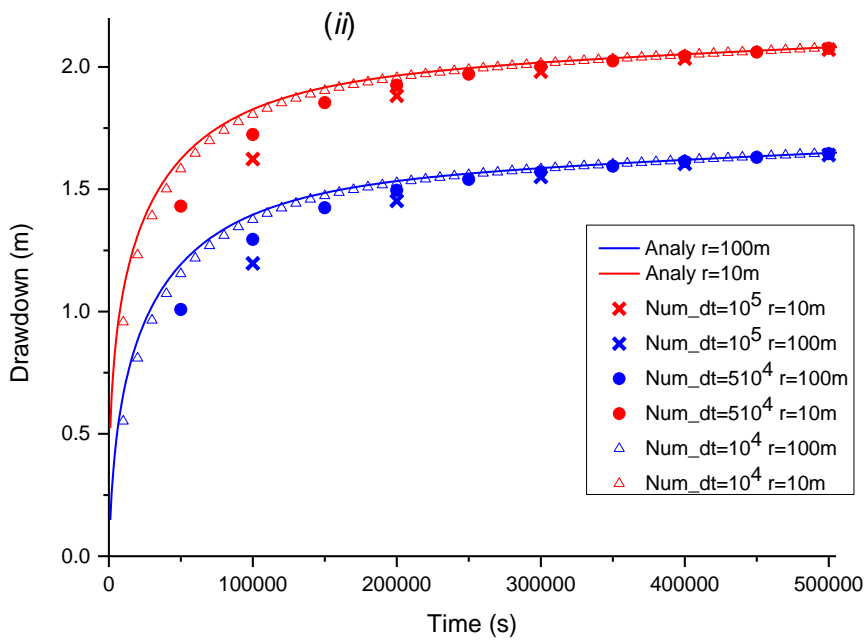
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447 **Figure 2**



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Figure 3

