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► **To cite this version:**

Indranil Biswas, Sorin Dumitrescu, Sebastian Heller. BRANCHED $SL(r, C)$ -OPERS. 2022. hal-03526980

HAL Id: hal-03526980

<https://hal.science/hal-03526980>

Preprint submitted on 14 Jan 2022

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BRANCHED $SL(r, \mathbb{C})$ -OPERS

INDRANIL BISWAS, SORIN DUMITRESCU, AND SEBASTIAN HELLER

ABSTRACT. Branched projective structures were introduced by Mandelbaum [21], [22], and opers were introduced by Beilinson and Drinfeld [2], [3]. We define the branched analog of $SL(r, \mathbb{C})$ -opers and investigate their properties. For the usual $SL(r, \mathbb{C})$ -opers, the underlying holomorphic vector bundle is independent of the opers. For the branched $SL(r, \mathbb{C})$ -opers, the underlying holomorphic vector bundle actually depends on the oper. Given a branched $SL(r, \mathbb{C})$ -oper, we associate to it another holomorphic vector bundle equipped with a logarithmic connection. This holomorphic vector bundle supporting a logarithmic connection does not depend on the branched oper. We characterize the branched $SL(r, \mathbb{C})$ -opers in terms of the logarithmic connections on this fixed holomorphic vector bundle.

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1. INTRODUCTION

Opers were introduced by Beilinson and Drinfeld [2], [3]. For this they were motivated by the earlier works of Drinfeld and Sokolov [10], [11]. Given a semisimple complex Lie group G , a G -oper on a compact Riemann surface X consists of

2010 *Mathematics Subject Classification.* 53B15, 14H60, 14D21.

Key words and phrases. Opers, logarithmic connection, residue, branch structure.

- a holomorphic principal G -bundle P on X equipped with a holomorphic connection ∇ , and
- a holomorphic reduction of the structure group of P to a Borel subgroup of G ,

such that the reduction satisfies an analogue of the Griffiths transversality condition with respect to this connection ∇ , and moreover the second fundamental form of the above reduction of structure group to a Borel subgroup, for the connection ∇ , satisfies certain nondegeneracy conditions. Opers are useful in diverse topics, for example in geometric Langlands correspondence, nonabelian Hodge theory, some branches of mathematics physics, differential equations et cetera; see [4], [12], [17], [15], [16], [8], [13], [14], [20], [23], [7] and references therein.

An $\mathrm{SL}(2, \mathbb{C})$ -oper on a Riemann surface X corresponds to a complex projective structure on X . We recall that a complex projective structure on X is given by a covering of X by holomorphic coordinate charts such that all the transition functions are restrictions of Möbius transformations [19]. During the early seventies, Mandelbaum introduced, and studied, the notion of a branched complex projective structure. Locally, they are like a singular complex projective structure obtained by pulling back an usual complex projective structure through a holomorphic ramified covering map. See [6] for a recent study of branched complex projective structures; for higher dimensional complex manifolds, a branched analog of the more general notion of holomorphic Cartan geometries was recently introduced in [5].

Our aim here is to introduce, and investigate, the branched analog of $\mathrm{SL}(r, \mathbb{C})$ -opers.

Let X be a compact connected Riemann surface, and let $S \subset X$ be a finite subset.

A *branched* $\mathrm{SL}(r, \mathbb{C})$ -oper on X , with branching over S , is given by a triple of the form (V, \mathcal{F}, D) , where

- V is a rank r holomorphic vector bundle on X such that $\bigwedge^r V = \mathcal{O}_X$,
- \mathcal{F} is a filtration of holomorphic subbundles

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = V \quad (1.1)$$

with $\mathrm{rank}(F_i) = i$, for all $1 \leq i \leq r$, and

- D is a holomorphic connection on V such that $D(F_i) \subset F_{i+1} \otimes K_X$, for all $1 \leq i \leq r-1$, and D induces the trivial connection on $\bigwedge^r V$.

Moreover, for all $1 \leq i \leq r-1$, the second fundamental form of F_i , for the holomorphic connection D , vanishes exactly on S (see Definition 2.1).

We recall that for an usual $\mathrm{SL}(r, \mathbb{C})$ -oper, the second fundamental forms of the filtration \mathcal{F} are required to be nonzero everywhere.

For usual $\mathrm{SL}(r, \mathbb{C})$ -opers, the underlying vector bundle V supporting the holomorphic connection is a fixed one; neither its isomorphism class, nor the filtration \mathcal{F} , depends on the oper. However, for branched $\mathrm{SL}(r, \mathbb{C})$ -opers the underlying vector bundle V supporting the holomorphic connection depends on the oper. Although each graded piece F_{i+1}/F_i for the filtration \mathcal{F} in (1.1) is actually independent of the branched $\mathrm{SL}(r, \mathbb{C})$ -oper.

Denote the above quotient line bundle F_r/F_{r-1} by Q . We prove that a branched $\mathrm{SL}(r, \mathbb{C})$ -oper produces a logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$, where $J^{r-1}(Q)$ is the $(r-1)$ -th jet bundle of Q ; see Proposition 3.5. Given the logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ associated to a branched $\mathrm{SL}(r, \mathbb{C})$ -oper, the branched $\mathrm{SL}(r, \mathbb{C})$ -oper can actually be recovered back using the Hecke transformations on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$. The residues, and also the local properties of these logarithmic connections, are studied. Based on these, we characterize all logarithmic connections on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ that arise from branched $\mathrm{SL}(r, \mathbb{C})$ -opers; see Theorem 6.1.

The branched $\mathrm{SL}(r, \mathbb{C})$ -opers are closely related with the $\mathrm{SL}(r, \mathbb{C})$ -opers with regular singularity introduced by Beilinson and Drinfeld; this is explained in Section 6.1, which was communicated to us by Edward Frenkel.

Once the branched $\mathrm{SL}(r, \mathbb{C})$ -opers have been defined, it is straight-forward to define the branched orthogonal and branched symplectic opers. We have omitted this exercise.

2. BRANCHED $\mathrm{SL}(r)$ -OPERS

Let X be a compact connected Riemann surface of genus g . The holomorphic cotangent and tangent bundles of X are denoted by K_X and TX respectively. The sheaf of holomorphic functions on X will be denoted by \mathcal{O}_X .

Let V be a holomorphic vector bundle on X . A *logarithmic connection* on V singular over an effective divisor S is a holomorphic differential operator

$$D : V \longrightarrow V \otimes K_X \otimes \mathcal{O}_X(S)$$

satisfying the Leibniz identity which states that

$$D(fs) = fD(s) + s \otimes df \tag{2.1}$$

for any locally defined holomorphic function f on X and any locally defined holomorphic section s of V . If S is the zero divisor, then D is called a *holomorphic connection* [1]. Any logarithmic connection on V is integrable (same as flat) because $\Omega_X^2 = 0$. A criterion of Weil and Atiyah says that V admits a holomorphic connection if and only if the degree of every indecomposable component of V is zero [1, p. 203, Theorem 10], [24]. In particular, if V is indecomposable, and $\mathrm{degree}(V) = 0$, then V admits a holomorphic connection [1, p. 203, Proposition 19].

Let \mathbf{D} be a logarithmic connection on V . For a holomorphic subbundle $F \subset V$, consider the following composition of homomorphisms

$$F \hookrightarrow V \xrightarrow{\mathbf{D}} V \otimes K_X \otimes \mathcal{O}_X(S) \xrightarrow{q_0 \otimes \mathrm{Id}_{K_X \otimes \mathcal{O}_X(S)}} (V/F) \otimes K_X \otimes \mathcal{O}_X(S), \tag{2.2}$$

where $q_0 : V \longrightarrow V/F$ is the natural quotient map. From (2.1) it follows that this composition of homomorphisms is actually \mathcal{O}_X -linear, and hence it produces a holomorphic section

$$\mathrm{SF}(\mathbf{D}, F) \in H^0(X, \mathrm{Hom}(F, V/F) \otimes K_X \otimes \mathcal{O}_X(S)) = H^0(X, F^* \otimes (V/F) \otimes K_X \otimes \mathcal{O}_X(S)), \tag{2.3}$$

which is known as the *second fundamental form* of F for the logarithmic connection \mathbf{D} .

Assuming that V admits a holomorphic connection, let D be a holomorphic connection on V . Then the composition of homomorphisms in (2.2) simplifies as

$$F \hookrightarrow V \xrightarrow{D} V \otimes K_X \xrightarrow{q_0 \otimes \text{Id}_{K_X}} (V/F) \otimes K_X, \quad (2.4)$$

and it gives the following analogue of (2.3)

$$\text{SF}(D, F) \in H^0(X, \text{Hom}(F, V/F) \otimes K_X) = H^0(X, F^* \otimes K_X \otimes (V/F)), \quad (2.5)$$

which is known as the *second fundamental form* of F for the holomorphic connection D .

The rank of V is denoted by r ; assume that $r \geq 2$. Let

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = V$$

be a filtration of holomorphic subbundles such that for all $1 \leq i \leq r-1$,

- $\text{rank}(F_i) = i$, and
- $D(F_i) \subset F_{i+1} \otimes K_X$, where D as before is a holomorphic connection on V .

Consequently, the second fundamental form $\text{SF}(D, F_i)$ in (2.5) satisfies the following condition:

$$\text{SF}(D, F_i)(F_i) \subset (F_{i+1}/F_i) \otimes K_X \subset (V/F_i) \otimes K_X$$

for all $1 \leq i \leq r-1$. Therefore, $\text{SF}(D, F_i)$ produces a holomorphic homomorphism

$$\text{SF}(D, i) : F_i/F_{i-1} \longrightarrow (F_{i+1}/F_i) \otimes K_X \quad (2.6)$$

for every $1 \leq i \leq r-1$.

Assume that there is holomorphic line bundle \mathbf{L}_0 on X such that the line bundle $\text{Hom}(F_i/F_{i-1}, (F_{i+1}/F_i) \otimes K_X) = (F_{i+1}/F_i) \otimes (F_i/F_{i-1})^* \otimes K_X$ is holomorphically isomorphic to \mathbf{L}_0 for all $1 \leq i \leq r-1$. This implies that

$$F_{i+1}/F_i = (F_i/F_{i-1}) \otimes (\mathbf{L}_0 \otimes TX) = (F_i/F_{i-1}) \otimes \mathbf{L}, \quad (2.7)$$

where $\mathbf{L} := \mathbf{L}_0 \otimes TX$. Then we have

$$\det V := \bigwedge^r V = \bigotimes_{j=1}^r F_j/F_{j-1} = F_1^{\otimes r} \otimes \mathbf{L}^{\otimes \frac{r(r-1)}{2}} = (F_r/F_{r-1})^{\otimes r} \otimes \left(\mathbf{L}^{\otimes \frac{r(r-1)}{2}} \right)^*. \quad (2.8)$$

Fix a reduced effective divisor

$$S := \sum_{k=1}^d x_k \quad (2.9)$$

on X of degree $d \geq 0$; so $\{x_1, \dots, x_d\}$ are distinct points of X . We will introduce branched $\text{SL}(r, \mathbb{C})$ -opers on X with branching over S .

If r is an even integer, we assume that d is also even. If r is an even integer, fix a holomorphic line bundle \mathcal{L} on X of degree $1 + \frac{d}{2} - g$ (recall that d is an even integer in this case) such that

$$\mathcal{L}^{\otimes 2} = TX \otimes \mathcal{O}_X(S); \quad (2.10)$$

also, fix a holomorphic isomorphism of $\mathcal{L}^{\otimes 2}$ with $TX \otimes \mathcal{O}_X(S)$. Note that if S is the zero divisor, then the dual \mathcal{L}^* is a theta characteristic on X .

Definition 2.1. A *branched $\mathrm{SL}(r, \mathbb{C})$ -oper* over X with *branching* over S is a triple

$$(V, \mathcal{F}, D),$$

where

- V is a rank r holomorphic vector bundle on X such that $\bigwedge^r V = \mathcal{O}_X$,
- \mathcal{F} is a filtration of holomorphic subbundles

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = V$$

with $\mathrm{rank}(F_i) = i$, for all $1 \leq i \leq r$, and

- D is a holomorphic connection on V ,

such that the following five conditions hold:

- (1) if r is even, then $F_1 = \left(\mathcal{L}^{\otimes(r-1)}\right)^*$, where \mathcal{L} is the line bundle in (2.10), and
if r is odd, then $F_1 = \left((TX \otimes \mathcal{O}_X(S))^{\otimes \frac{r-1}{2}}\right)^*$,
- (2) $F_{j+1}/F_j = F_1 \otimes (TX \otimes \mathcal{O}_X(S))^{\otimes j}$, for all $0 \leq j \leq r-1$,
- (3) the connection on $\bigwedge^r V = \mathcal{O}_X$ induced by the connection D on V coincides with the trivial connection given by the de Rham differential d on \mathcal{O}_X ,
- (4) $D(F_i) \subset F_{i+1} \otimes K_X$, for all $1 \leq i \leq r-1$, and
- (5) the homomorphism in (2.6)

$$\mathrm{SF}(D, i) : F_i/F_{i-1} \longrightarrow (F_{i+1}/F_i) \otimes K_X$$

$$= (F_i/F_{i-1}) \otimes TX \otimes \mathcal{O}_X(S) \otimes K_X = (F_i/F_{i-1}) \otimes \mathcal{O}_X(S)$$

coincides with the natural inclusion map of F_i/F_{i-1} in $(F_i/F_{i-1}) \otimes \mathcal{O}_X(S)$ (recall that S is an effective divisor).

Remark 2.2. Note that the above condition that $F_{j+1}/F_j = F_1 \otimes (TX \otimes \mathcal{O}_X(S))^{\otimes j}$ implies that $(F_{i+1}/F_i) = (F_i/F_{i-1}) \otimes TX \otimes \mathcal{O}_X(S)$. The fifth condition that $\mathrm{SF}(D, i)$ is the natural inclusion map coincides with the condition that $\mathrm{SF}(D, i)$ is given by tensoring with the section of $\mathcal{O}_X(S)$ defined by the constant function 1 on X .

Remark 2.3. The holomorphic line bundle \mathbf{L}_0 in (2.7) is $\mathcal{O}_X(S)$ in Definition 2.1. So \mathbf{L} in (2.8) is $\mathcal{O}_X(S) \otimes TX$ in Definition 2.1. The second condition in Definition 2.1 that $F_{j+1}/F_j = F_1 \otimes (TX \otimes \mathcal{O}_X(S))^{\otimes j}$ is obtained from (2.7), and the first condition in Definition 2.1 is motivated by (2.8), because $\bigwedge^r V = \mathcal{O}_X$.

When S in (2.9) is the zero divisor, a branched $\mathrm{SL}(r, \mathbb{C})$ -oper is an usual $\mathrm{SL}(r, \mathbb{C})$ -oper [2], [3]. When $S = 0$, the underlying holomorphic vector bundle V of an $\mathrm{SL}(r, \mathbb{C})$ -oper does not depend on the oper. In this regular case, the vector bundle V is the jet bundle $J^{r-1}\left((TX)^{\otimes \frac{r-1}{2}}\right)$ if r is odd, and V is $J^{r-1}(\mathcal{L}^{\otimes(r-1)})$, with $\mathcal{L}^2 = TX$, if r is even.

Contrary to the situation in the regular case (meaning usual $\mathrm{SL}(r, \mathbb{C})$ -opers), when $\mathrm{degree}(S) = d > 0$, the underlying holomorphic vector bundle V of a branched $\mathrm{SL}(r, \mathbb{C})$ -oper, in general, depends on the branched oper. Indeed, while the successive quotients for the filtration \mathcal{F} of V , given in Definition 2.1, are clearly independent of the branched

$\mathrm{SL}(r, \mathbb{C})$ -oper, the holomorphic vector bundle V itself depends on the branched oper. In Section 3 we will show that a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) with branching over S gives rise to a logarithmic connection on a certain holomorphic vector bundle over X obtained by performing Hecke transformation on V over the points of S . This holomorphic vector bundle, on which the logarithmic connection resides, turns out to be actually independent of the branched $\mathrm{SL}(r, \mathbb{C})$ -oper.

Take a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) over X with branching over S . Let D^* be the holomorphic connection on the dual vector bundle V^* induced by the connection D . The filtration \mathcal{F} of V

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = V$$

produces the following filtration of holomorphic subbundles of V^* :

$$V^* = F_r^* \twoheadrightarrow F_{r-1}^* \twoheadrightarrow \cdots \twoheadrightarrow F_2^* \twoheadrightarrow F_1^* \twoheadrightarrow F_0^* = 0;$$

the subbundles of V^* giving the filtration are the kernels of the projections $V^* \twoheadrightarrow F_i^*$. This filtration by holomorphic subbundles of V^* will be denoted by \mathcal{F}^* .

Lemma 2.4. *The above triple $(V^*, \mathcal{F}^*, D^*)$ is a branched $\mathrm{SL}(r, \mathbb{C})$ -oper over X with branching over S .*

Proof. As in (2.4), take a holomorphic subbundle $F \subset V$. So we have the holomorphic subbundle $(V/F)^* \subset V^*$. Note that the quotient bundle $V^*/((V/F)^*)$ is identified with F^* . Let

$$\begin{aligned} \mathrm{SF}(D^*, (V/F)^*) &\in H^0(X, \mathrm{Hom}((V/F)^*, V^*/((V/F)^*)) \otimes K_X) \\ &= H^0(X, \mathrm{Hom}((V/F)^*, F^*) \otimes K_X) = H^0(X, (V/F) \otimes F^* \otimes K_X) \end{aligned}$$

be the second fundamental form of $(V/F)^* \subset V^*$ for the connection D^* on V^* (see (2.5)). Then it is straightforward to check that

$$\mathrm{SF}(D^*, (V/F)^*) = \mathrm{SF}(D, F), \quad (2.11)$$

where $\mathrm{SF}(D, F)$ is constructed in (2.5). The lemma is a straightforward consequence of (2.11). \square

2.1. Examples. We now give some examples of branched $\mathrm{SL}(r, \mathbb{C})$ -opers.

Let

$$\varpi : X \longrightarrow Y$$

be a nonconstant holomorphic map between compact connected Riemann surface such that all the branch points of ϖ have branch number 1, and let (V, \mathcal{F}, D) be an usual $\mathrm{SL}(r, \mathbb{C})$ -oper on Y . Then the pullback $(\varpi^*V, \varpi^*\mathcal{F}, \varpi^*D)$ is a branched $\mathrm{SL}(r, \mathbb{C})$ -oper on X . The branching divisor for $(\varpi^*V, \varpi^*\mathcal{F}, \varpi^*D)$ coincides with the branching divisor for the map ϖ .

Let $\varpi : X \longrightarrow \mathbb{CP}^1$ be any nonconstant holomorphic map such that all the branch points of ϖ have branch number 1. Then pulling back the standard $\mathrm{SL}(2, \mathbb{C})$ -oper on \mathbb{CP}^1 by ϖ we get a branched $\mathrm{SL}(2, \mathbb{C})$ -oper on X .

More generally, let V be a rank two holomorphic vector bundle on X with $\bigwedge^2 V = \mathcal{O}_X$, and let D be a holomorphic connection on V such that the holomorphic connection on $\bigwedge^2 V$ induced by D coincides with the holomorphic connection on \mathcal{O}_X given by the de Rham differential d . Let $L \subset V$ be a holomorphic line subbundle satisfying the following two conditions:

- (1) the second fundamental form $\mathrm{SF}(D, L)$ is not identically zero, and
- (2) all the zeros of $\mathrm{SF}(D, L)$ are of order one.

Then (V, L, D) is a branched $\mathrm{SL}(2, \mathbb{C})$ -oper on X . The branching divisor for (V, L, D) coincides with the divisor on which $\mathrm{SF}(D, L)$ vanishes. The previous example corresponds to the special case where $V = \mathcal{O}_X^{\oplus 2}$ and D is the trivial connection on it.

Let (V, L, D) be a branched $\mathrm{SL}(2, \mathbb{C})$ -oper on X . Then the holomorphic line subbundle $L \subset V$ produces a filtration $\{F_i\}_{i=1}^r$ of the symmetric product $\mathrm{Sym}^{r-1}(V)$ whose i -th term F_i is the image of $L^{\otimes(r-i)} \otimes V^{\otimes(i-1)}$ in $\mathrm{Sym}^{r-1}(V)$. The triple $(\mathrm{Sym}^{r-1}(V), \{F_i\}_{i=1}^r, \tilde{D})$, where \tilde{D} is the holomorphic connection on $\mathrm{Sym}^{r-1}(V)$ induced by D , is a branched $\mathrm{SL}(r, \mathbb{C})$ -oper on X . Its branching divisor coincides with that of (V, L, D) .

3. LOGARITHMIC CONNECTION FROM BRANCHED $\mathrm{SL}(r)$ -OPERS

3.1. Homomorphism to a jet bundle. Let

$$(V, \mathcal{F}, D) \tag{3.1}$$

be a branched $\mathrm{SL}(r, \mathbb{C})$ -oper over X , with branching over S , where \mathcal{F} stands for the filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = V \tag{3.2}$$

of holomorphic subbundles of V .

We will construct another filtered holomorphic vector bundle, filtered by holomorphic subbundles, from the branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) in (3.1).

For notational convenience, let

$$Q := F_r/F_{r-1} = F_1 \otimes (TX \otimes \mathcal{O}_X(S))^{\otimes(r-1)} \tag{3.3}$$

denote the quotient line bundle in (3.2); the second statement in Definition 2.1 gives the above isomorphism. We will construct a holomorphic homomorphism

$$\Phi : V \longrightarrow J^{r-1}(Q) \tag{3.4}$$

to the $(r-1)$ -th order jet bundle of the line bundle Q in (3.3).

Let

$$q : V \longrightarrow Q = V/F_{r-1} \tag{3.5}$$

be the natural quotient map. Take any $x \in X$ and also take any $v \in V_x$. Consider a simply connected open neighborhood $U \subset X$ of x , and denote by

$$\tilde{v} \in H^0(U, V)$$

the unique flat section of $V|_U$, for the connection D , such that $\tilde{v}(x) = v$. Restricting the section $q(\tilde{v}) \in H^0(U, Q)$ to the $(r-1)$ -th order infinitesimal neighborhood of x , where q is the quotient map in (3.5), we get an element

$$\tilde{v}' := q(\tilde{v})|_{rx} \in J^{r-1}(Q)_x,$$

where rx is the nonreduced divisor with multiplicity r . The map Φ in (3.4) sends v to this element $\tilde{v}' \in J^{r-1}(Q)_x$ constructed from v using the connection D . The homomorphism Φ is evidently holomorphic.

We have the natural short exact sequence of jet bundles

$$0 \longrightarrow Q \otimes K_X^{\otimes k} \longrightarrow J^k(Q) \longrightarrow J^{k-1}(Q) \longrightarrow 0 \quad (3.6)$$

for all $k \geq 1$. For $0 \leq j \leq r$, let H_j be the kernel of the projection $J^{r-1}(Q) \longrightarrow J^{r-1-j}(Q)$ in (3.6); we use the convention that $J^k(W) = 0$ if $k < 0$. So $\text{rank}(H_j) = j$, and we have the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow H_j \longrightarrow J^{r-1}(Q) \longrightarrow J^{r-1-j}(Q) \longrightarrow 0 \quad (3.7)$$

on X . From (3.7) it follows that the quotient H_j/H_{j-1} coincides with the kernel of the projection $J^{r-j}(Q) \longrightarrow J^{r-j-1}(Q)$. Therefore, from (3.6) it follows that

$$H_j/H_{j-1} = Q \otimes K_X^{\otimes(r-j)} \quad (3.8)$$

for all $1 \leq j \leq r$.

Let

$$X_0 := X \setminus \{x_1, \dots, x_d\} = X \setminus S \subset X \quad (3.9)$$

be the complement of S in (2.9).

The triple (V, \mathcal{F}, D) in (3.1) defines an usual $\text{SL}(r, \mathbb{C})$ -oper over the open subset $X_0 \subset X$ in (3.9). The following proposition is well-known for the usual $\text{SL}(r, \mathbb{C})$ -opers.

Proposition 3.1. *The restriction $\Phi|_{X_0}$ of the homomorphism in (3.4) to X_0 is an isomorphism $V|_{X_0} \xrightarrow{\sim} J^{r-1}(Q)|_{X_0}$.*

For all $0 \leq j \leq r$, the homomorphism $\Phi|_{X_0}$ takes the subbundle $F_j|_{X_0}$ (defined in (3.2)) isomorphically to the subbundle $H_j|_{X_0}$ (constructed in (3.7)).

From Proposition 3.1 it follows that when S is the zero divisor, then

$$F_r = V = J^{r-1}(Q).$$

In particular, the holomorphic vector bundle V does not depend on the $\text{SL}(r, \mathbb{C})$ -oper, when $S = 0$.

The following is a consequence of Proposition 3.1.

Corollary 3.2. *The homomorphism Φ in (3.4) satisfies the condition*

$$\Phi(F_j) \subset H_j$$

for all $0 \leq j \leq r$, where F_j and H_j are defined in (3.2) and (3.7) respectively.

Proof. Since $\Phi|_{X_0}(F_j|_{X_0}) \subset H_j|_{X_0}$ (see Proposition 3.1), and X_0 is a dense subset of X , we conclude that $\Phi(F_j) \subset H_j$. \square

From Corollary 3.2 it follows that Φ produces a grading preserving holomorphic homomorphism from the graded vector bundle $\bigoplus_{j=1}^r F_j/F_{j-1}$ in (3.2) to the graded vector bundle $\bigoplus_{j=1}^r H_j/H_{j-1}$ in (3.7). For any $1 \leq j \leq r$, let

$$\Phi_j : F_j/F_{j-1} \longrightarrow H_j/H_{j-1} = Q \otimes K_X^{\otimes(r-j)} \quad (3.10)$$

be the homomorphism induced by Φ ; see (3.8) for the isomorphism in (3.10).

For any $1 \leq j \leq r$, note that $Q \otimes K_X^{\otimes(r-j)} \otimes \mathcal{O}_X(-(r-j)S)$ is actually a subsheaf of $Q \otimes K_X^{\otimes(r-j)}$ because S is an effective divisor. The following proposition identifies this subsheaf $Q \otimes K_X^{\otimes(r-j)} \otimes \mathcal{O}_X(-(r-j)S)$ with the image of Φ_j .

Proposition 3.3. *For each $1 \leq j \leq r$, the image of the homomorphism Φ_j in (3.10) is the subsheaf*

$$Q \otimes K_X^{\otimes(r-j)} \otimes \mathcal{O}_X(-(r-j)S) \subset Q \otimes K_X^{\otimes(r-j)} = H_j/H_{j-1}.$$

Proof. Recall the homomorphisms $\mathrm{SF}(D, i)$ in statement (5) of Definition 2.1. Take any $1 \leq k \leq r-1$. Consider the composition of homomorphisms

$$\begin{aligned} & (\mathrm{SF}(D, r-1) \otimes \mathrm{Id}_{K_X^{\otimes(r-1-k)}}) \circ \cdots \circ (\mathrm{SF}(D, k+1) \otimes \mathrm{Id}_{K_X}) \circ \mathrm{SF}(D, k) : F_k/F_{k-1} \\ & \longrightarrow (F_r/F_{r-1}) \otimes K_X^{\otimes(r-k)} = Q \otimes K_X^{\otimes(r-k)} = (F_k/F_{k-1}) \otimes \mathcal{O}_X((r-k)S); \end{aligned} \quad (3.11)$$

note that since $(F_{i+1}/F_i) \otimes K_X = (F_i/F_{i-1}) \otimes \mathcal{O}_X(S)$ for all $1 \leq i \leq r-1$ (see Remark 2.2) it follows that $(F_r/F_{r-1}) \otimes K_X^{\otimes(r-k)} = (F_k/F_{k-1}) \otimes \mathcal{O}_X((r-k)S)$. From statement (5) in Definition 2.1 we know that the composition of homomorphisms in (3.11) coincides with the natural inclusion map

$$F_k/F_{k-1} \hookrightarrow (F_k/F_{k-1}) \otimes \mathcal{O}_X((r-k)S).$$

In other words, if $\mathbf{1} \in H^0(X, \mathcal{O}_X((r-k)S))$ is the section given by the constant function 1 on X , then the composition of homomorphisms in (3.11) coincides with the homomorphism

$$\mathrm{Id}_{F_k/F_{k-1}} \otimes \mathbf{1} : F_k/F_{k-1} \longrightarrow (F_k/F_{k-1}) \otimes \mathcal{O}_X((r-k)S).$$

On the other hand, we have $Q \otimes K_X^{\otimes(r-k)} = H_k/H_{k-1}$ (see (3.8)). Therefore, the image of the composition of homomorphisms in (3.11) is

$$(H_k/H_{k-1}) \otimes \mathcal{O}_X(-(r-k)S) = Q \otimes K_X^{\otimes(r-k)} \otimes \mathcal{O}_X(-(r-k)S) \subset Q \otimes K_X^{\otimes(r-k)}.$$

This completes the proof. \square

For $0 \leq j \leq r$, define

$$\widehat{H}_j := H_j \otimes \mathcal{O}_X(-(r-1)S) \subset J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S),$$

where H_j is constructed in (3.7). So the filtration $\{H_j\}_{j=0}^r$ of $J^{r-1}(Q)$ produces a filtration of holomorphic subbundles

$$0 = \widehat{H}_0 \subset \widehat{H}_1 \subset \cdots \subset \widehat{H}_{r-1} \subset \widehat{H}_r = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \quad (3.12)$$

of $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$.

The next result is deduced using Proposition 3.3 and Corollary 3.2.

Corollary 3.4. *The following inclusions hold:*

$$J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \subset V \xrightarrow{\Phi} J^{r-1}(Q),$$

where the inclusion map $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \hookrightarrow J^{r-1}(Q)$ given by the composition of inclusions is the natural inclusion map. Moreover, the inclusion map $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \hookrightarrow V$ takes the subsheaf \widehat{H}_j in (3.12) into the subsheaf F_j in (3.2), and the inclusion map $V \xrightarrow{\Phi} J^{r-1}(Q)$ takes F_j into H_j , for all $0 \leq j \leq r$.

Proof. By Proposition 3.3, the image of the homomorphism Φ_j in (3.10) is the subsheaf

$$Q \otimes K_X^{\otimes(r-j)} \otimes \mathcal{O}_X(-(r-j)S) \subset Q \otimes K_X^{\otimes(r-j)} = H_j/H_{j-1}.$$

Consequently, we have the following inclusions:

$$\begin{aligned} \widehat{H}_j/\widehat{H}_{j-1} &= (H_j/H_{j-1}) \otimes \mathcal{O}_X(-(r-1)S) \hookrightarrow F_j/F_{j-1} \\ &= (H_j/H_{j-1}) \otimes \mathcal{O}_X(-(r-j)S) \hookrightarrow H_j/H_{j-1}. \end{aligned} \quad (3.13)$$

In view of Corollary 3.2, from these inclusion maps it follows that the homomorphism Φ in (3.4) satisfies the following:

$$\widehat{H}_r := J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \subset \Phi(V) \subset J^{r-1}(Q). \quad (3.14)$$

We identify the coherent analytic sheaf V with the coherent analytic sheaf $\Phi(V)$ using the isomorphism Φ between them.

The inclusion map $\widehat{H}_r = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \hookrightarrow \Phi(V)$ in (3.14) takes \widehat{H}_j in (3.12) into $\Phi(F_j)$. Also, the inclusion map $\Phi(V) \hookrightarrow J^{r-1}(Q)$ in (3.14) takes $\Phi(F_j)$ into the subbundle $H_j \subset J^{r-1}(Q)$ in (3.7). \square

In view of Corollary 3.4, the filtration of holomorphic subbundles in (3.12) fits in the following filtration of coherent analytic sheaves:

$$0 = \widehat{H}_0 \subset \widehat{H}_1 \subset \cdots \subset \widehat{H}_{r-1} \subset \widehat{H}_r = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \subset V. \quad (3.15)$$

3.2. A logarithmic connection. Take a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) as in (3.1). The following proposition shows that the holomorphic connection D produces a logarithmic connection on the holomorphic vector bundle $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$.

Proposition 3.5. *The holomorphic connection $D : V \rightarrow V \otimes K_X$ in (3.1) sends the subsheaf $\widehat{H}_r := J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \subset V$ in (3.15) to the subsheaf*

$$(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-2)S) \otimes K_X) \cap (V \otimes K_X) \subset V \otimes K_X.$$

In other words, D produces a logarithmic connection, singular over S , on the holomorphic vector bundle $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$.

Proof. Recall that $V|_{X_0} = J^{r-1}(Q)|_{X_0} = (J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-2)S))|_{X_0}$, where X_0 is the open subset in (3.9). So we need to investigate D only around the points of S . Take any point

$$x' \in S.$$

Fix a holomorphic splitting of the filtration $\{F_i\}_{i=0}^r$ of V in (3.2) over a sufficiently small analytic neighborhood U of x' ; this subset U is chosen such that $U \cap S = x'$. So we have a holomorphic isomorphism

$$V|_U \xrightarrow{\sim} \bigoplus_{i=1}^r (F_i/F_{i-1})|_U =: \bigoplus_{i=1}^r \mathcal{F}_i \quad (3.16)$$

using the notation $\mathcal{F}_i := (F_i/F_{i-1})|_U$. We would express the holomorphic connection $D|_U$ in (3.1) in terms of the decomposition in (3.16). From the fifth condition in Definition 2.1 it follows that $D|_U$ has the following expression in terms of the decomposition in (3.16):

$$D|_U = \begin{pmatrix} D_1 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \cdots & \alpha_{1,r-2} & \alpha_{1,r-1} & \alpha_{1,r} \\ \gamma_1 & D_2 & \alpha_{2,3} & \alpha_{2,4} & \cdots & \alpha_{2,r-2} & \alpha_{2,r-1} & \alpha_{2,r} \\ 0 & \gamma_2 & D_3 & \alpha_{3,4} & \cdots & \alpha_{3,r-2} & \alpha_{3,r-1} & \alpha_{3,r} \\ 0 & 0 & \gamma_3 & D_4 & \cdots & \alpha_{4,r-2} & \alpha_{4,r-1} & \alpha_{4,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & D_{r-2} & \alpha_{r-2,r-1} & \alpha_{r-2,r} \\ 0 & 0 & 0 & 0 & \cdots & \gamma_{r-2} & D_{r-1} & \alpha_{r-1,r} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{r-1} & D_r \end{pmatrix} \quad (3.17)$$

where D_i (the entry at the $i \times i$ -th position of the matrix) is a holomorphic connection on \mathcal{F}_i , and γ_i (the entry at the $(i+1) \times i$ -th position) is a section

$$\gamma_i \in H^0(U, \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_{i+1}) \otimes (K_X|_U) \otimes \mathcal{O}_U(-x')); \quad (3.18)$$

in other words, γ_i is a holomorphic homomorphism $\mathcal{F}_i \rightarrow \mathcal{F}_{i+1} \otimes (K_X|_U)$ that vanishes at the point $x' \in U$. For any $j > i$, we have

$$\alpha_{i,j} \in H^0(U, \mathrm{Hom}(\mathcal{F}_j, \mathcal{F}_i) \otimes (K_X|_U)).$$

The entry at the $i \times j$ -th position of the matrix in (3.17) is zero if $i > j + 1$. It may be mentioned that by choosing the splitting in (3.16) carefully it is possible to make $\alpha_{i,j} = 0$ for all $j > i$; but this will not be needed here.

The decomposition of $V|_U$ in (3.16) produces a holomorphic decomposition of the vector bundle $(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))|_U$. To see this, consider the intersection of coherent analytic subsheaves of $V|_U$

$$\mathcal{G}_i := \mathcal{F}_i \cap (J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))|_U = \mathcal{F}_i \cap (J^{r-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x')) \subset V|_U; \quad (3.19)$$

note that both \mathcal{F}_i and $J^{r-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x')$ are subsheaves of $V|_U$ (see Corollary 3.4 and (3.16)). Then we have a holomorphic decomposition of $J^{r-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x')$

$$J^{r-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x') = \bigoplus_{i=1}^r \mathcal{G}_i \quad (3.20)$$

into a direct sum of holomorphic line bundles on U . Indeed, the natural homomorphism of coherent analytic sheaves

$$\bigoplus_{i=1}^r \mathcal{G}_i \longrightarrow J^{r-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x')$$

is clearly surjective; it is also injective because it is injective over the open subset $U \setminus \{x'\} \subset U$ and the coherent analytic sheaf $\bigoplus_{i=1}^r \mathcal{G}_i$ is torsionfree. From (3.13) it follows immediately that

- $\mathcal{G}_i = \mathcal{F}_i \otimes \mathcal{O}_U(-(i-1)x')$, and
- the inclusion map $\mathcal{G}_i \hookrightarrow \mathcal{F}_i$ (see (3.19)) coincides with the natural inclusion map

$$\mathcal{F}_i \otimes \mathcal{O}_U(-(i-1)x') \hookrightarrow \mathcal{F}_i.$$

Now it is straight-forward to check that the connection operator $D|_U$ in (3.17) on $\bigoplus_{i=1}^r \mathcal{F}_i$ produces a holomorphic differential operator

$$\bigoplus_{i=1}^r \mathcal{G}_i \longrightarrow \left(\bigoplus_{i=1}^r \mathcal{G}_i \right) \otimes (K_X|_U) \otimes \mathcal{O}_U(x').$$

To see this, first note that the section γ_i in (3.17) produces a holomorphic homomorphism

$$\mathcal{G}_i \longrightarrow \mathcal{G}_{i+1} \otimes (K_X|_U)$$

because the homomorphism $\gamma_i \in H^0(U, \text{Hom}(\mathcal{F}_i, \mathcal{F}_{i+1}) \otimes (K_X|_U))$ in (3.18) vanishes at the point x' . Secondly, for any $j > i$, the section $\alpha_{i,j}$ in (3.17) produces a section

$$\tilde{\alpha}_{i,j} \in H^0(U, \text{Hom}(\mathcal{G}_j, \mathcal{G}_i) \otimes (K_X|_U) \otimes \mathcal{O}_U(-(j-i)x')), \quad (3.21)$$

in particular, $\tilde{\alpha}_{i,j} \in H^0(U, \text{Hom}(\mathcal{G}_j, \mathcal{G}_i) \otimes (K_X|_U))$. Thirdly, the connection operator

$$D_i : \mathcal{F}_i \longrightarrow \mathcal{F}_i \otimes (K_X|_U)$$

in (3.17) produces a first order holomorphic differential operator

$$\mathcal{G}_i \longrightarrow \mathcal{G}_i \otimes (K_X|_U) \otimes \mathcal{O}_U(x')$$

that satisfies the Leibniz identity, because D_i satisfies the Leibniz identity.

Consequently, D sends the subsheaf $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \subset V$ in (3.15) to the subsheaf

$$(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-2)S) \otimes K_X) \bigcap (V \otimes K_X) \subset V \otimes K_X.$$

Since

$$J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-2)S) \otimes K_X = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \otimes K_X \otimes \mathcal{O}_X(S),$$

this implies that D produces a logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$. \square

Remark 3.6. Consider the holomorphic connection D_i in (3.17) on the holomorphic vector bundle \mathcal{F}_i on U . The connection on $\mathcal{G}_i := \mathcal{F}_i \bigcap (J^{r-1}(Q|_U) \otimes \mathcal{O}_U(-(r-1)x'))$ induced by D_i is singular at x' if $i \geq 2$. Therefore, the singular locus of the logarithmic connections on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ constructed in Proposition 3.5 is exactly S .

Take any branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) as in (3.1). Let \mathcal{D} denote the logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ constructed in Proposition 3.5 from (V, \mathcal{F}, D) . The following is a straight-forward consequence of the construction of \mathcal{D} .

Corollary 3.7.

- (1) *The logarithmic connection \mathcal{D} on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ constructed in Proposition 3.5 satisfies the condition*

$$\mathcal{D}(\widehat{H}_i) \subset \widehat{H}_{i+1} \otimes K_X \otimes \mathcal{O}_X(S)$$

for all $1 \leq i \leq r-1$, where $\{\widehat{H}_j\}_{j=0}^r$ is the filtration of $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ in (3.12).

- (2) *The second fundamental form of the subbundle*

$$\widehat{H}_i|_{X_0} \subset (J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))|_{X_0}$$

for the holomorphic connection $\mathcal{D}|_{X_0}$ is nowhere zero on X_0 for all $1 \leq i \leq r-1$, where X_0 is the open subset in (3.9).

Proof. Over the open subset X_0 in (3.9), we have

$$J^{r-1}(Q)|_{X_0} = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)|_{X_0} = V|_{X_0}$$

(see Proposition 3.1). This isomorphism takes $F_j|_{X_0} \subset V|_{X_0}$ isomorphically to $H_j|_{X_0} = \widehat{H}_j|_{X_0}$ (see Proposition 3.1). It also takes the holomorphic connection $D|_{X_0}$ to $\mathcal{D}|_{X_0}$; this is an immediate consequence of the construction of \mathcal{D} . Since X_0 is dense in X , the first statement now follows from the facts that $D(F_i) \subset F_{i+1} \otimes K_X$ for all $1 \leq i \leq r-1$ (see the fourth statement in Definition 2.1), while the second statement follows from the fact that the second fundamental form of the subbundle $F_i \subset V$, for the holomorphic connection D , is nowhere zero on X_0 (see the fifth statement in Definition 2.1). \square

For any $1 \leq i \leq r-1$, let

$$\mathrm{SF}(\mathcal{D}, \widehat{H}_i) \in H^0(X, \mathrm{Hom}(\widehat{H}_i, \widehat{H}_r/\widehat{H}_i) \otimes K_X \otimes \mathcal{O}_X(S))$$

be the second fundamental form of the subbundle \widehat{H}_i in (3.12) for the logarithmic connection \mathcal{D} on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) = \widehat{H}_r$ constructed in Proposition 3.5; see (2.3) for the second fundamental form. From Corollary 3.7(1) we know that $\mathrm{SF}(\mathcal{D}, \widehat{H}_i)$ lies in the image of the natural inclusion map

$$H^0(X, \mathrm{Hom}(\widehat{H}_i, \widehat{H}_{i+1}/\widehat{H}_i) \otimes K_X \otimes \mathcal{O}_X(S)) \hookrightarrow H^0(X, \mathrm{Hom}(\widehat{H}_i, \widehat{H}_r/\widehat{H}_i) \otimes K_X \otimes \mathcal{O}_X(S))$$

given by the inclusion map $\widehat{H}_{i+1} \hookrightarrow \widehat{H}_r$. So we get a homomorphism

$$\mathrm{SF}(\mathcal{D}, i) \in H^0(X, \mathrm{Hom}(\widehat{H}_i/\widehat{H}_{i-1}, \widehat{H}_r/\widehat{H}_i) \otimes K_X \otimes \mathcal{O}_X(S)) \quad (3.22)$$

for every $1 \leq i \leq r-1$ given by these $\mathrm{SF}(\mathcal{D}, \widehat{H}_j)$.

For the filtration of E given in (3.12), from (3.13) and (3.8) it follows immediately that

$$\widehat{H}_j/\widehat{H}_{j-1} = Q \otimes K_X^{\otimes(r-j)} \otimes \mathcal{O}_X(-(r-1)S) \quad (3.23)$$

for all $1 \leq j \leq r$. Therefore, $\text{SF}(\mathcal{D}, i)$ in (3.22) is a holomorphic section

$$\text{SF}(\mathcal{D}, i) \in H^0(X, \mathcal{O}_X(S)) \quad (3.24)$$

for every $1 \leq i \leq r - 1$.

The following lemma is a refinement of Corollary 3.7(2).

Lemma 3.8. *The holomorphic section $\text{SF}(\mathcal{D}, i)$ in (3.24) coincides with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X .*

Proof. In the proof of Corollary 3.7 it was observed that

$$(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S), \{\widehat{H}_i\}_{i=1}^r, \mathcal{D})|_{X_0} = (V, \{F_i\}_{i=1}^r, D)|_{X_0},$$

where X_0 the open subset X_0 in (3.9). Therefore, the lemma follows from the fifth statement in Definition 2.1 and Remark 2.2. \square

Using (3.6) it is deduced that

$$\det J^{r-1}(Q) := \bigwedge^r J^{r-1}(Q) = Q^{\otimes r} \otimes K_X^{\otimes r(r-1)/2}.$$

Therefore, from (3.3) it follows that

$$\det J^{r-1}(Q) = F_1^{\otimes r} \otimes (TX)^{\otimes r(r-1)/2} \otimes \mathcal{O}_X(r(r-1)S).$$

Now the expression of F_1 in the first statement in Definition 2.1 gives that

$$\det J^{r-1}(Q) = \mathcal{O}_X\left(\frac{r(r-1)}{2}S\right).$$

Hence we have the following:

$$\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)) = \mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right). \quad (3.25)$$

Take any branched $\text{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) as in (3.1). We recall that the holomorphic connection on $\bigwedge^r V = \mathcal{O}_X$ induced by D coincides with the holomorphic connection on \mathcal{O}_X defined by the de Rham differential d (see the third statement in Definition 2.1). Let \mathcal{D} denote the logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ constructed in Proposition 3.5 from (V, \mathcal{F}, D) . Since \mathcal{D} is given by D , we have the following:

Lemma 3.9. *The logarithmic connection on $\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ induced by the logarithmic connection \mathcal{D} on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ coincides with the logarithmic connection on $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ given by the de Rham differential d , once $\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ is identified with $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ using (3.25).*

In Section 6 we will determine all the logarithmic connections on the rank r vector bundle $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ that arise from branched $\text{SL}(r, \mathbb{C})$ -opers on X with branching over S .

4. RESIDUES

Take a point $y \in S$. The fiber of $K_X \otimes \mathcal{O}_X(S)$ over y is identified with \mathbb{C} by the Poincaré adjunction formula [18, p. 146]. To explain this isomorphism

$$(K_X \otimes \mathcal{O}_X(S))_y \xrightarrow{\sim} \mathbb{C}, \quad (4.1)$$

let z be a holomorphic coordinate function on X defined on an analytic open neighborhood of y such that $z(y) = 0$. Then we have an isomorphism $\mathbb{C} \rightarrow (K_X \otimes \mathcal{O}_X(S))_y$ that sends any $c \in \mathbb{C}$ to $c \cdot \frac{dz}{z}(y) \in (K_X \otimes \mathcal{O}_X(S))_y$. It is straightforward to check that this map $\mathbb{C} \rightarrow (K_X \otimes \mathcal{O}_X(S))_y$ is independent of the choice the holomorphic coordinate function z .

Let $D_W : W \rightarrow W \otimes K_X \otimes \mathcal{O}_X(S)$ be a logarithmic connection on a holomorphic vector bundle W on X . Consider the composition of homomorphisms

$$W \xrightarrow{D_W} W \otimes K_X \otimes \mathcal{O}_X(S) \rightarrow (W \otimes K_X \otimes \mathcal{O}_X(S))_y = W_y;$$

it is evidently \mathcal{O}_X -linear. Therefore, this composition of homomorphisms is given by an endomorphism

$$\mathrm{Res}(D_W, y) : W_y \rightarrow W_y;$$

this endomorphism is called the *residue* of D_W at y .

For convenience we use the notation

$$E := \widehat{H}_r = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S) \quad (4.2)$$

(see (3.12)).

Take a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) as in (3.1). As before, let \mathcal{D} denote the logarithmic connection on E (see (4.2)) constructed in Proposition 3.5 from (V, \mathcal{F}, D) . Recall from Remark 3.6 that the singular locus of \mathcal{D} is exactly S . The following lemma describes the residues of \mathcal{D} .

Lemma 4.1. *For any point $x' \in S$, let*

$$\mathrm{Res}(\mathcal{D}, x') \in \mathrm{End}(E_{x'})$$

be the residue of \mathcal{D} at x' . Then the eigenvalues of $\mathrm{Res}(\mathcal{D}, x')$ are the integers $\{0, 1, \dots, r-2, r-1\}$, and the multiplicity of each of them is one. For any $0 \leq i \leq r-1$, the eigenspace of $\mathrm{Res}(\mathcal{D}, x')$ for the eigenvalue i is contained in the subspace

$$(\widehat{H}_{i+1})_{x'} \subset E_{x'}$$

(see (3.12) and (4.2)).

Proof. Let L be a holomorphic line bundle on the open neighborhood $U \subset X$ of x' (as in (3.16)) equipped with a holomorphic connection D_L . Then for any integer k , the differential operator $D_L|_{U \setminus \{x'\}}$ on $L|_{U \setminus \{x'\}} = (L \otimes \mathcal{O}_U(kx'))|_{U \setminus \{x'\}}$ extends to a logarithmic connection on $L \otimes \mathcal{O}_U(kx')$ over U . In fact it coincides with the logarithmic connection on $L \otimes \mathcal{O}_U(kx')$ given by the holomorphic connection D_L on L and the logarithmic connection on $\mathcal{O}_U(kx')$ defined by the de Rham differential d . The residue, at x' , of this logarithmic connection on $L \otimes \mathcal{O}_U(kx')$ is $-k$; indeed, this follows immediately from the fact that the

residue of the logarithmic connection on $\mathcal{O}_U(kx')$ defined by the de Rham differential d is $-k$. In view of this, the lemma follows from the expression of $D|_U$ in (3.17) in terms of the direct sum of line bundles \mathcal{G}_i in (3.20). Recall the observation in the proof of Proposition 3.5 that the section γ_i in (3.18) produces a holomorphic homomorphism

$$\mathcal{G}_i \longrightarrow \mathcal{G}_{i+1} \otimes (K_X|_U).$$

Therefore, γ_i does not contribute to the residue $\text{Res}(\mathcal{D}, x')$. Since $\tilde{\alpha}_{i,j}$ in (3.21) does not have any pole as a homomorphism from \mathcal{G}_j to $\mathcal{G}_i \otimes (K_X|_U)$, it also does not contribute to the residue $\text{Res}(\mathcal{D}, x')$.

Therefore, the residue of \mathcal{D} at x' is given by the residues on the logarithmic connections on \mathcal{G}_j , $1 \leq j \leq r$, induced by the holomorphic connections D_j on \mathcal{F}_j in (3.17). From the above observation on the residue of the logarithmic connection on $L \otimes \mathcal{O}_U(kx')$ we know that the residues on this logarithmic connection \mathcal{G}_j is $j - 1$. This proves the lemma. \square

We will end this section on residue by noting some general properties of it.

Let D_W be a logarithmic connection, on a holomorphic vector bundle $W \rightarrow X$, which is singular over $y \in X$. Assume that the residue $\text{Res}(D_W, y)$ of D_W at y is semisimple, meaning $\text{Res}(D_W, y)$ is diagonalizable. Let $\lambda_1, \dots, \lambda_b$ be the eigenvalues of $\text{Res}(D_W, y)$ (they need not be of multiplicity one). For any $1 \leq i \leq b$, let

$$W_y^i \subset W_y$$

be the eigenspace of $\text{Res}(D_W, y) \in \text{End}(W_y)$ for the eigenvalue λ_i . For any given $1 \leq k < b$, consider the following natural homomorphisms

$$W \longrightarrow W_y \longrightarrow W_y / \left(\bigoplus_{i=1}^k W_y^i \right);$$

both are natural quotient maps. The kernel of this composition of homomorphisms will be denoted by \widetilde{W} . So \widetilde{W} is a torsion-free coherent analytic sheaf on X that fits in the following short exact sequence of coherent analytic sheaves on X :

$$0 \longrightarrow \widetilde{W} \xrightarrow{\phi} W \xrightarrow{q_1} W_y / \left(\bigoplus_{i=1}^k W_y^i \right) \longrightarrow 0. \quad (4.3)$$

The following is straight-forward to check.

Lemma 4.2. *The logarithmic connection $D_W : W \rightarrow W \otimes K_X \otimes \mathcal{O}_X(y)$ sends the subsheaf $\widetilde{W} \xrightarrow{\phi} W$ to $\widetilde{W} \otimes K_X \otimes \mathcal{O}_X(y)$. Hence D_W induces a logarithmic connection on \widetilde{W} .*

The logarithmic connection on \widetilde{W} induced by D_W will be denoted by \widetilde{D} .

We will now describe the residue of \widetilde{D} at the singular point y .

Let

$$0 \longrightarrow \mathrm{kernel}(\phi(y)) \longrightarrow \widetilde{W}_y \xrightarrow{\phi(y)} W_y \xrightarrow{q_1} \mathrm{cokernel}(\phi(y)) = W_y / \left(\bigoplus_{i=1}^k W_y^i \right) \longrightarrow 0 \quad (4.4)$$

be the exact sequence of vector spaces obtained by restricting, to the point y , the short exact sequence of coherent analytic sheaves in (4.3). Note that the homomorphism of fibers of vector bundles corresponding to an injective homomorphism of coherent analytic sheaves need not be injective, so $\mathrm{kernel}(\phi(y))$ may be nonzero.

We will now show that there is a canonical isomorphism

$$\left(\bigoplus_{i=k+1}^b W_y^i \right) \otimes (K_X)_y \xrightarrow{\sim} \mathrm{kernel}(\phi(y)). \quad (4.5)$$

To prove (4.5), take any $w \in \left(\bigoplus_{i=k+1}^b W_y^i \right) \otimes (K_X)_y$. Using the isomorphism $(K_X)_y = \mathcal{O}_Y(-y)_y$ (see (4.1)), we have

$$w \in \left(\bigoplus_{i=k+1}^b W_y^i \right) \otimes \mathcal{O}_Y(-y)_y = \bigoplus_{i=k+1}^b (W^i \otimes \mathcal{O}_Y(-y))_y.$$

Now take a homomorphic section defined on an analytic open neighborhood $U \subset X$

$$s \in H^0(U, (W|_U) \otimes \mathcal{O}_U(-y))$$

such that $s(y) = w$. We note that $q_1(s) = 0$, where q_1 is the projection in (4.3). Therefore, from (4.3) it follows that s is the image of a holomorphic section of $\widetilde{W}|_U$ under the homomorphism ϕ in (4.3). Let

$$\tilde{s} \in H^0(U, \widetilde{W}|_U)$$

be the unique holomorphic section such that $\phi(\tilde{s}) = s$. It can be shown that the evaluation

$$\tilde{s}(y) \in \widetilde{W}_y$$

is independent of the choice of the above section $s \in H^0(U, (W|_U) \otimes \mathcal{O}_U(-y))$ satisfying $s(y) = w$. Indeed, for another holomorphic section

$$t \in H^0(U, (W|_U) \otimes \mathcal{O}_U(-y))$$

with $t(y) = w$, we have

$$s - t \in H^0(U, (W|_U) \otimes \mathcal{O}_U(-2y)),$$

and hence

$$\tilde{s} - \tilde{t} \in H^0(U, (\widetilde{W}|_U) \otimes \mathcal{O}_U(-y)), \quad (4.6)$$

where $\tilde{t} \in H^0(U, \widetilde{W}|_U)$ is the unique section for which $\phi(\tilde{t}) = t$. From (4.6) it follows immediately that $\tilde{s}(y) = \tilde{t}(y)$, and hence $\tilde{s}(y) \in \widetilde{W}_y$ is independent of the choice of the section $s \in H^0(U, (W|_U) \otimes \mathcal{O}_U(-y))$ satisfying $s(y) = w$.

For the homomorphism $\phi(y)$ in (4.4) we have

$$\phi(y)(\tilde{s}(y)) = 0,$$

because $\phi(\tilde{s}) = s \in H^0(U, (W|_U) \otimes \mathcal{O}_U(-y))$. Therefore, from (4.4) we conclude that

$$\tilde{s}(y) \in \text{kernel}(\phi(y)).$$

The isomorphism in (4.5) sends any

$$w \in \left(\bigoplus_{i=k+1}^b W_y^i \right) \otimes (K_X)_y$$

to $\tilde{s}(y) \in \text{kernel}(\phi(y))$ constructed above from it.

The following lemma is a straight-forward consequence of the construction of residue of a logarithmic connection.

Lemma 4.3. *The residue $\text{Res}(\tilde{D}, y)$ of the logarithmic connection on \tilde{D} on \tilde{W} (see Lemma 4.2) has the following properties:*

- (1) *The eigenvalues of $\text{Res}(\tilde{D}, y)$ are $\{\lambda_i\}_{i=1}^k \cup \{\lambda_i + 1\}_{i=k+1}^b$.*
- (2) *For any $k+1 \leq i \leq b$, the eigenspace of $\text{Res}(\tilde{D}, y)$ for the eigenvalue $\lambda_i + 1$ is the subspace $W_y^i \otimes (K_X)_y \subset \tilde{W}_y$ (see (4.5) and (4.4)).*
- (3) *For any $1 \leq i \leq k$, the eigenspace of $\text{Res}(\tilde{D}, y)$ for the eigenvalue λ_i is taken isomorphically to the eigenspace W_y^i by the homomorphism $\phi(y)$ in (4.4).*

5. LOCAL MONODROMY

A logarithmic connection has local monodromy around a singular point of the connection. The two holomorphic vector bundles V and $E := J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ are identified over X_0 (3.9) (see Corollary 3.4), and this identification takes the holomorphic connection $\mathcal{D}|_{X_0}$ on $E|_{X_0}$ in Lemma 4.1 to the holomorphic connection $D|_{X_0}$ on $V|_{X_0}$ in (3.1). Now $D|_{X_0}$ does not have local monodromy around any point $x' \in S$ because D is a holomorphic connection on V . Hence the logarithmic connection \mathcal{D} has trivial local monodromy around every point of S , as well. In this section we will reformulate this condition of vanishing of local monodromies.

Let

$$\mathbb{D} : E \longrightarrow E \otimes K_X \otimes \mathcal{O}_X(S) \tag{5.1}$$

be a logarithmic connection on $E := J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ singular over S that satisfies the following four conditions:

- (1) $\mathbb{D}(\hat{H}_i) \subset \hat{H}_{i+1} \otimes K_X \otimes \mathcal{O}_X(S)$, for all $1 \leq i \leq r-1$, where $\{\hat{H}_j\}_{j=0}^r$ is the filtration of $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ in (3.12). This implies that the second fundamental forms of the subbundles \hat{H}_j produce a homomorphism

$$\text{SF}(\mathbb{D}, i) \in H^0(X, \mathcal{O}_X(S))$$

as in (3.24) for every $1 \leq i \leq r-1$.

- (2) For all $1 \leq i \leq r-1$, the above section $\text{SF}(\mathbb{D}, i) \in H^0(X, \mathcal{O}_X(S))$ coincides with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X .

- (3) For every $x' \in S$, the eigenvalues of $\mathrm{Res}(\mathbb{D}, x')$ are the integers $\{0, 1, \dots, r-2, r-1\}$. Note that the multiplicity of each of the eigenvalues is one.
- (4) For all $0 \leq i \leq r-1$ and every $x' \in S$, the eigenspace of $\mathrm{Res}(\mathbb{D}, x')$ for the eigenvalue i is contained in the subspace

$$(\widehat{H}_{i+1})_{x'} \subset E_{x'}$$

(see (3.12) and (4.2)).

In other words, \mathbb{D} shares all the properties of \mathcal{D} stated in Lemma 4.1, Corollary 3.7 and Lemma 3.8.

Let

$$L_{x'}(i) \subset E_{x'} \quad (5.2)$$

be the eigenline for the eigenvalue $0 \leq i \leq r-1$ of the residue $\mathrm{Res}(\mathbb{D}, x')$ at $x' \in S$. So we have a decomposition of the fiber $E_{x'}$

$$E_{x'} = \bigoplus_{i=0}^{r-1} L_{x'}(i), \quad (5.3)$$

where $L_{x'}(i)$ is the subspace in (5.2). The above condition that the eigenspace of $\mathrm{Res}(\mathbb{D}, x')$ for the eigenvalue i is contained in the subspace $(\widehat{H}_{i+1})_{x'} \subset E_{x'}$ implies that

$$(\widehat{H}_{j+1})_{x'} = \bigoplus_{i=0}^j L_{x'}(i)$$

for all $0 \leq j \leq r-1$; so we have

$$L_{x'}(j) = (\widehat{H}_{j+1})_{x'} / (\widehat{H}_j)_{x'}. \quad (5.4)$$

We recall from (3.23) that for the filtration of E given in (3.12),

$$\widehat{H}_j / \widehat{H}_{j-1} = Q \otimes K_X^{\otimes(r-j)} \otimes \mathcal{O}_X(-(r-1)S)$$

for all $1 \leq j \leq r$.

Take any point $x' \in S$, and also take any integer $2 \leq j \leq r$. We will construct, from \mathbb{D} , an element

$$\begin{aligned} M_j(\mathbb{D}, x') &\in \mathrm{Hom}((\widehat{H}_j / \widehat{H}_{j-1})_{x'} \otimes (T_{x'}X)^{\otimes(j-1)}, (\widehat{H}_{j-1} / \widehat{H}_{j-2})_{x'} \otimes (T_{x'}X)^{\otimes(j-2)}) \quad (5.5) \\ &= \mathrm{Hom}((\widehat{H}_j / \widehat{H}_{j-1})_{x'}, (\widehat{H}_{j-1} / \widehat{H}_{j-2})_{x'}) \otimes (K_X)_{x'} = (K_X^{\otimes 2})_{x'}, \end{aligned}$$

where $T_{x'}X$ is the holomorphic tangent space to X at x' ; see (3.23) for the above equality

$$\mathrm{Hom}((\widehat{H}_j / \widehat{H}_{j-1})_{x'}, (\widehat{H}_{j-1} / \widehat{H}_{j-2})_{x'}) = (K_X)_{x'}$$

in (5.5).

The fiber $\mathcal{O}_X((j-1)S)_{x'}$ is identified with $(T_{x'}X)^{\otimes(j-1)}$ using the Poincaré adjunction formula (see (4.1)). This produces an isomorphism

$$\eta : (\widehat{H}_j / \widehat{H}_{j-1})_{x'} \otimes \mathcal{O}_X((j-1)S)_{x'} \xrightarrow{\sim} (\widehat{H}_j / \widehat{H}_{j-1})_{x'} \otimes (T_{x'}X)^{\otimes(j-1)} \quad (5.6)$$

Take any

$$w \in (\widehat{H}_j / \widehat{H}_{j-1})_{x'} \otimes (T_{x'}X)^{\otimes(j-1)}. \quad (5.7)$$

Using (5.4) and η (constructed in (5.6)) we have

$$(\widehat{H}_j/\widehat{H}_{j-1})_{x'} \otimes (T_{x'}X)^{\otimes(j-1)} = L_{x'}(j-1) \otimes \mathcal{O}_X((j-1)x')_{x'}.$$

Let

$$w' \in L_{x'}(j-1) \otimes \mathcal{O}_X((j-1)x')_{x'}$$

be the element that corresponds to w (see (5.7)) by this isomorphism.

Now we choose a holomorphic section

$$\tilde{w} \in H^0(U, (E|_U) \otimes \mathcal{O}_U((j-1)x')) , \quad (5.8)$$

defined on some sufficiently small analytic neighborhood $U \subset X$ of x' , such that

$$\tilde{w}(x') = w'.$$

Let $\widehat{\mathbb{D}}$ be the logarithmic connection on $(E|_U) \otimes \mathcal{O}_U((j-1)x')$ induced by the logarithmic connection $\mathbb{D}|_U$ on $E|_U$ and the logarithmic connection on $\mathcal{O}_U((j-1)x')$ given by the de Rham differential d . Consider the residue $\text{Res}(\widehat{\mathbb{D}}, x') \in \text{End}(E_{x'} \otimes \mathcal{O}_U((j-1)x')_{x'})$ of the logarithmic connection $\widehat{\mathbb{D}}$ at the point x' . It can be shown that the line

$$L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'} \subset E_{x'} \otimes \mathcal{O}_U((j-1)x')_{x'}$$

(see (5.3)) is contained in the eigenspace of $\text{Res}(\widehat{\mathbb{D}}, x')$ for the eigenvalue 0. Indeed, $\text{Res}(\mathbb{D}, x')$ acts on $L_{x'}(j-1)$ as multiplication by $j-1$ and the logarithmic connection on $\mathcal{O}_U((j-1)x')$ given by the de Rham differential d has the property that its residue at x' is $1-j$. Hence $L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'}$ is contained in the eigenspace of $\text{Res}(\widehat{\mathbb{D}}, x')$ for the eigenvalue 0. Actually, $L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'}$ is the eigenspace of $\text{Res}(\widehat{\mathbb{D}}, x')$ for the eigenvalue 0.

Since the section \tilde{w} in (5.8) satisfies the condition $\tilde{w}(x') \in L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'}$, from the above property of $\text{Res}(\widehat{\mathbb{D}}, x')$ that $L_{x'}(j-1) \otimes \mathcal{O}_U((j-1)x')_{x'}$ is contained in the eigenspace of $\text{Res}(\widehat{\mathbb{D}}, x')$ for the eigenvalue 0 it follows that

$$\mathbb{D}(\tilde{w}) \in H^0(U, (E|_U) \otimes K_U \otimes \mathcal{O}_U((j-1)x')) , \quad (5.9)$$

where $K_U := K_X|_U$.

The decomposition of $E_{x'}$ in (5.3) gives a decomposition

$$E_{x'} \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} = \bigoplus_{i=0}^{r-1} L_{x'}(i) \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} . \quad (5.10)$$

Let

$$\beta_{j-2}(w) \in L_{x'}(j-2) \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} \quad (5.11)$$

be the component of $\mathbb{D}(\tilde{w})(x') \in (E \otimes K_U \otimes \mathcal{O}_U((j-1)x'))_{x'}$ (see (5.9)) in

$$L_{x'}(j-2) \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} \subset E_{x'} \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'}$$

with respect to the decomposition in (5.10). Since $L_{x'}(j-2) = (\widehat{H}_{j-1})_{x'}/(\widehat{H}_{j-2})_{x'}$ (see (5.4)), and $\mathcal{O}_U((j-1)x')_{x'} = (T_{x'}X)^{\otimes(j-1)}$ (see (4.1)), the element $\beta_{j-2}(w)$ in (5.11) is also an element

$$\beta_{j-2}(w) \in (\widehat{H}_{j-1}/\widehat{H}_{j-2})_{x'} \otimes (T_{x'}X)^{\otimes(j-2)}. \quad (5.12)$$

The map $M_j(\mathbb{D}, x')$ in (5.5) sends the element w in (5.7) to $\beta_{j-2}(w)$ constructed in (5.12). But we need to show that this map is well-defined in the sense that $\beta_{j-2}(w)$ depends only on w , in other words, $\beta_{j-2}(w)$ is independent of the choice of the section \tilde{w} in (5.8). The following lemma shows that $\beta_{j-2}(w)$ depends only on w .

Lemma 5.1. *The element $\beta_{j-2}(w)$ constructed in (5.12) does not depend on the choice of the section \tilde{w} in (5.8).*

Proof. We may replace \tilde{w} in (5.8) by $\tilde{w} + t$, where

$$t \in H^0(U, (E|_U) \otimes \mathcal{O}_U((j-1)x'))$$

with $t(x') = 0$, where E is defined in (4.2). Let

$$\beta_{j-2}^t(w) \in (\widehat{H}_{j-1}/\widehat{H}_{j-2})_{x'} \otimes (T_{x'}X)^{\otimes(j-2)}$$

be the element constructed as in (5.12) after substituting $\tilde{w} + t$ in place of \tilde{w} in the construction of $\beta_{j-2}(w)$. To prove the lemma we need to show that

$$\beta_{j-2}^t(w) = \beta_{j-2}(w). \quad (5.13)$$

To prove (5.13), first note that

$$t \in H^0(U, (E|_U) \otimes \mathcal{O}_U((j-2)x')) \subset H^0(U, (E|_U) \otimes \mathcal{O}_U((j-1)x')) \quad (5.14)$$

because of the given condition that $t(x') = 0$.

Let $\widehat{\mathbb{D}}_1$ be the logarithmic connection on $(E|_U) \otimes \mathcal{O}_U((j-2)x')$ given by the logarithmic connection $\mathbb{D}|_U$ on $E|_U$ and the logarithmic connection on $\mathcal{O}_U((j-2)x')$ given by the de Rham differential d . We note that $\widehat{\mathbb{D}}_1$ is simply the restriction of the logarithmic connection $\widehat{\mathbb{D}}$ to the subsheaf

$$(E|_U) \otimes \mathcal{O}_U((j-2)x') \subset (E|_U) \otimes \mathcal{O}_U((j-1)x').$$

From (5.14) we have

$$\widehat{\mathbb{D}}_1(t) \in H^0(U, (E|_U) \otimes K_U \otimes \mathcal{O}_U((j-1)x')).$$

Using the isomorphism in (4.1), the evaluation, at x' , of this section $\widehat{\mathbb{D}}_1(t)$ is considered as an element of

$$\widehat{\mathbb{D}}_1(t)(x') \in E_{x'} \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} = E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'}. \quad (5.15)$$

The decomposition in (5.3) produces a decomposition

$$E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'} = \bigoplus_{i=0}^{r-1} L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'}. \quad (5.16)$$

The residue of the logarithmic connection $\widehat{\mathbb{D}}_1$ at x'

$$\mathrm{Res}(\widehat{\mathbb{D}}_1, x') \in \mathrm{End}(E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'})$$

preserves the decomposition in (5.16). Moreover, $\mathrm{Res}(\widehat{\mathbb{D}}_1, x')$ acts on the subspace

$$L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'} \subset E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'}$$

in (5.16) as multiplication by $i - j + 2$. Indeed, the residue $\text{Res}(\mathbb{D}, x')$ acts on $L_{x'}(i)$ as multiplication by i (see (5.2)), and the residue, at x' , of the logarithmic connection on $\mathcal{O}_U((j-2)x')$ given by the de Rham differential d is $2 - j$. Consequently, $\text{Res}(\widehat{\mathbb{D}}_1, x')$ acts on $L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'}$ as multiplication by $i - j + 2$. This implies that

$$\text{Res}(\widehat{\mathbb{D}}_1, x')(E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'}) \subset \bigoplus_{i \in \{0, \dots, r-1\} \setminus \{j-2\}} L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'}, \quad (5.17)$$

and $\text{kernel}(\text{Res}(\widehat{\mathbb{D}}_1, x')) = L_{x'}(j-2) \otimes \mathcal{O}_U((j-2)x')_{x'}$.

On the other hand, the evaluation $\widehat{\mathbb{D}}_1(t)(x') \in E_{x'} \otimes \mathcal{O}_U((j-2)x')_{x'}$ in (5.15) satisfies the identity

$$\widehat{\mathbb{D}}_1(t)(x') = \text{Res}(\widehat{\mathbb{D}}_1, x')((t)(x')).$$

Therefore, from (5.17) it follows that

$$\widehat{\mathbb{D}}_1(t)(x') \in \bigoplus_{i \in \{0, \dots, r-1\} \setminus \{j-2\}} L_{x'}(i) \otimes \mathcal{O}_U((j-2)x')_{x'}. \quad (5.18)$$

Recall that $\beta_{j-2}(w)$ in (5.11) is the component of $\mathbb{D}(\tilde{w})(x')$ in

$$L_{x'}(j-2) \otimes (K_U \otimes \mathcal{O}_U((j-1)x'))_{x'} = L_{x'}(j-2) \otimes \mathcal{O}_U((j-2)x')_{x'}$$

with respect to the decomposition in (5.10). Therefore, from (5.18) it follows immediately that the equality in (5.13) holds. As noted before, (5.13) completes the proof. \square

The map $M_j(\mathbb{D}, x')$ in (5.5) is defined by sending any element w as in (5.7) to $\beta_{j-2}(w)$ constructed in (5.12) from w .

Remark 5.2. Let D' be a logarithmic connection singular at a point x , such that residue at x is semisimple. If λ is an eigenvalue of the local monodromy of D' around x , then $\lambda = \exp(2\pi\sqrt{-1}b)$, where b is an eigenvalue of the residue $\text{Res}(D', x)$ [9]. For any point $x' \in S$, the eigenvalues of the residue $\text{Res}(\mathbb{D}, x')$ of the connection \mathbb{D} in (5.1) are integers. Therefore, we conclude that 1 is the only eigenvalue of the local monodromy of \mathbb{D} around the point x' . In other words, the local monodromy of \mathbb{D} around x' is a unipotent automorphism. This local monodromy is given by

$$(M_2(\mathbb{D}, x'), M_3(\mathbb{D}, x'), \dots, M_r(\mathbb{D}, x')) \in ((K_X^{\otimes 2})_{x'})^{\oplus(r-1)},$$

where the elements $M_j(\mathbb{D}, x')$ are constructed in (5.5).

6. CHARACTERIZING THE LOGARITHMIC CONNECTIONS

As in (5.1), let \mathbb{D} be a logarithmic connection on $E := J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ singular over S and satisfying the following four conditions:

- (1) $\mathbb{D}(\widehat{H}_i) \subset \widehat{H}_{i+1} \otimes K_X \otimes \mathcal{O}_X(S)$, for all $1 \leq i \leq r-1$, where $\{\widehat{H}_j\}_{j=0}^r$ is the filtration of $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ in (3.12). This implies that the second fundamental forms of the subbundles \widehat{H}_j produce a homomorphism

$$\text{SF}(\mathbb{D}, i) \in H^0(X, \mathcal{O}_X(S)) \quad (6.1)$$

as in (3.24) for every $1 \leq i \leq r-1$.

- (2) For all $1 \leq i \leq r-1$, the section $\mathrm{SF}(\mathbb{D}, i)$ in (6.1) coincides with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X .
- (3) For every $x' \in S$, the eigenvalues of $\mathrm{Res}(\mathbb{D}, x')$ are the integers $\{0, 1, \dots, r-2, r-1\}$, with the multiplicity of each of them being one.
- (4) For all $0 \leq i \leq r-1$ and every $x' \in S$, the eigenspace of $\mathrm{Res}(\mathbb{D}, x')$ for the eigenvalue i is contained in the subspace $(\widehat{H}_{i+1})_{x'} \subset E_{x'}$ in (4.2).

Theorem 6.1. *There is a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) such that the above logarithmic connection \mathbb{D} on $E := J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ coincides with the logarithmic connection on E associated to (V, \mathcal{F}, D) by Proposition 3.5 if and only if the following two conditions hold:*

- (1) *The logarithmic connection on $\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ induced by the logarithmic connection \mathbb{D} on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ coincides with the logarithmic connection on $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ given by the de Rham differential d , once $\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ is identified with $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ using (3.25).*
- (2) *$M_j(\mathbb{D}, x') = 0$, for all $2 \leq j \leq r$ and every $x' \in S$, where $M_j(\mathbb{D}, x')$ are constructed in (5.5).*

Proof. If there is a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) such that \mathbb{D} coincides with the logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ associated to (V, \mathcal{F}, D) by Proposition 3.5, then from Lemma 3.9 we know that the logarithmic connection on $\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ induced by \mathbb{D} on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ coincides with the logarithmic connection on $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ given by the de Rham differential d , once $\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ is identified with $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ using (3.25).

Therefore, we assume that the logarithmic connection on $\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ induced by \mathbb{D} on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ coincides with the logarithmic connection on $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ given by the de Rham differential d , after $\det(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S))$ is identified with $\mathcal{O}_X\left(-\frac{r(r-1)}{2}S\right)$ using (3.25).

To prove the theorem we need to show the following: There is a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) such that \mathbb{D} coincides with the logarithmic connection on $E = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ associated to (V, \mathcal{F}, D) by Proposition 3.5 if and only if $M_j(\mathbb{D}, x') = 0$ for all $2 \leq j \leq r$.

In Proposition 3.5 we constructed a logarithmic connection on E from a branched $\mathrm{SL}(r, \mathbb{C})$ -oper. The above statement will be proved by establishing an inverse of this construction in Proposition 3.5.

Consider the holomorphic vector bundle

$$\mathcal{W} := J^{r-1}(Q) = E \otimes \mathcal{O}_X((r-1)S) \tag{6.2}$$

on X (see (4.2)). The logarithmic connection \mathbb{D} on E and the logarithmic connection on $\mathcal{O}_X((r-1)S)$ given by the de Rham differential d together produce a logarithmic

connection on the holomorphic vector bundle \mathcal{W} in (6.2). Let

$$\tilde{\mathbb{D}} : \mathcal{W} \longrightarrow \mathcal{W} \otimes K_X \otimes \mathcal{O}_X(S) \quad (6.3)$$

be this logarithmic connection on \mathcal{W} given by \mathbb{D} and the de Rham differential d . At any point $x' \in S$, the residue of the logarithmic connection on $\mathcal{O}_X((r-1)S)$ given by the de Rham differential d is $1-r$. On the other hand, the eigenvalues of $\text{Res}(\mathbb{D}, x')$ are given to be $\{0, 1, \dots, r-2, r-1\}$. Therefore, the eigenvalues of the residue $\text{Res}(\tilde{\mathbb{D}}, x')$ of $\tilde{\mathbb{D}}$ at x' are $\{1-r, 2-r, \dots, -1, 0\}$. We note that the multiplicity of every eigenvalue of $\text{Res}(\tilde{\mathbb{D}}, x')$ is one.

For each $x' \in S$, let

$$\ell_{x'}(0) \subset \mathcal{W}_{x'}$$

be the eigenspace, for the eigenvalue 0, of $\text{Res}(\tilde{\mathbb{D}}, x')$; so $\ell_{x'}(0)$ is a line in $\mathcal{W}_{x'}$. Let \mathcal{W}_1 be the holomorphic vector bundle of rank r on X defined by the following short exact sequence of coherent analytic sheaves on X :

$$0 \longrightarrow \mathcal{W}_1 \longrightarrow \mathcal{W} \longrightarrow \bigoplus_{x' \in S} \mathcal{W}_{x'}/\ell_{x'}(0) \longrightarrow 0. \quad (6.4)$$

From Lemma 4.2 we know that the logarithmic connection $\mathbb{D} : \mathcal{W} \longrightarrow \mathcal{W} \otimes K_X \otimes \mathcal{O}_X(S)$ preserves the subsheaf \mathcal{W}_1 in (6.4). Let

$$\mathbb{D}_1 : \mathcal{W}_1 \longrightarrow \mathcal{W}_1 \otimes K_X \otimes \mathcal{O}_X(S)$$

be the logarithmic connection on \mathcal{W}_1 induced by \mathbb{D} . Since the eigenvalues of $\text{Res}(\mathbb{D}, x')$ are $\{0, -1, \dots, 2-r, 1-r\}$, and $\ell_{x'}(0)$ is the eigenspace of $\text{Res}(\mathbb{D}, x')$ for the eigenvalue 0, from Lemma 4.3 it follows that the eigenvalues of $\text{Res}(\mathbb{D}_1, x')$ are $\{0, -1, \dots, 2-r\}$. The eigenvalue 0 of $\text{Res}(\mathbb{D}_1, x')$ has multiplicity two, while the rest of the eigenvalues of $\text{Res}(\mathbb{D}_1, x')$ are of multiplicity one.

For each $x' \in S$, let

$$H_1(x') \subset (\mathcal{W}_1)_{x'}$$

be the eigenspace of $\text{Res}(\mathbb{D}_1, x') \in \text{End}((\mathcal{W}_1)_{x'})$ for the eigenvalue 0. As noted above, we have $\dim H_1(x') = 2$. Imitating (6.4) we define \mathcal{W}_2 . More precisely, let \mathcal{W}_2 be the holomorphic vector bundle of rank r on X defined by the following short exact sequence of coherent analytic sheaves on X :

$$0 \longrightarrow \mathcal{W}_2 \longrightarrow \mathcal{W}_1 \longrightarrow \bigoplus_{x' \in S} (\mathcal{W}_1)_{x'}/H_1(x') \longrightarrow 0. \quad (6.5)$$

From Lemma 4.2 we know that the logarithmic connection $\mathbb{D}_1 : \mathcal{W}_1 \longrightarrow \mathcal{W}_1 \otimes K_X \otimes \mathcal{O}_X(S)$ preserves the subsheaf \mathcal{W}_2 (6.5). Let

$$\mathbb{D}_2 : \mathcal{W}_2 \longrightarrow \mathcal{W}_2 \otimes K_X \otimes \mathcal{O}_X(S)$$

be the logarithmic connection on \mathcal{W}_2 induced by \mathbb{D}_1 . Since the eigenvalues of $\text{Res}(\mathbb{D}_1, x')$ are $\{0, -1, \dots, 2-r\}$, from Lemma 4.3 it follows that the eigenvalues of $\text{Res}(\mathbb{D}_2, x')$ are $\{0, -1, \dots, 3-r\}$. The eigenvalue 0 of $\text{Res}(\mathbb{D}_2, x')$ has multiplicity three.

For each $x' \in S$, let $H_2(x') \subset (\mathcal{W}_2)_{x'}$ be the eigenspace of $\mathrm{Res}(\mathbb{D}_2, x')$ for the eigenvalue 0. Define the holomorphic vector bundle \mathcal{W}_3 by the short exact sequence of coherent analytic sheaves

$$0 \longrightarrow \mathcal{W}_3 \longrightarrow \mathcal{W}_2 \longrightarrow \bigoplus_{x' \in S} (\mathcal{W}_2)_{x'} / H_2(x') \longrightarrow 0.$$

We now proceed inductively. To explain this, for $2 \leq j \leq r-2$, suppose that we have constructed a holomorphic vector bundle \mathcal{W}_j , and a logarithmic connection \mathbb{D}_j on it, such that the following conditions hold:

- For each $x' \in S$, the eigenvalues of $\mathrm{Res}(\mathbb{D}_j, x')$ are $\{0, -1, \dots, j+1-r\}$.
- The multiplicity of the eigenvalue zero of $\mathrm{Res}(\mathbb{D}_j, x')$ is $j+1$.

Let

$$H_j(x') \subset (\mathcal{W}_j)_{x'}$$

be the eigenspace of $\mathrm{Res}(\mathbb{D}_j, x')$ for the eigenvalue 0. Then define the holomorphic vector bundle \mathcal{W}_{j+1} by the short exact sequence of coherent analytic sheaves

$$0 \longrightarrow \mathcal{W}_{j+1} \longrightarrow \mathcal{W}_j \longrightarrow \bigoplus_{x' \in S} (\mathcal{W}_j)_{x'} / H_j(x') \longrightarrow 0. \quad (6.6)$$

From Lemma 4.2 we know that the logarithmic connection \mathbb{D}_j on \mathcal{W}_j preserves the subsheaf $\mathcal{W}_{j+1} \subset \mathcal{W}_j$ in (6.6); the logarithmic connection on \mathcal{W}_{j+1} induced by \mathbb{D}_j is denoted by \mathbb{D}_{j+1} . Since the eigenvalues of $\mathrm{Res}(\mathbb{D}_j, x')$ are $\{0, -1, \dots, j+1-r\}$, from Lemma 4.3 it follows that the eigenvalues of $\mathrm{Res}(\mathbb{D}_{j+1}, x')$ are $\{0, -1, \dots, j+2-r\}$. The eigenvalue 0 of $\mathrm{Res}(\mathbb{D}_{j+1}, x')$ has multiplicity $j+2$.

Proceeding inductively, we finally obtain the following:

- (1) a holomorphic vector bundle \mathcal{W}_{r-1} on X of rank r , and
- (2) a logarithmic connection \mathbb{D}_{r-1} on \mathcal{W}_{r-1} whose singular locus is contained in S , and for each point $x' \in S$, the residue

$$\mathrm{Res}(\mathbb{D}_{r-1}, x') \in \mathrm{End}((\mathcal{W}_{r-1})_{x'}) \quad (6.7)$$

is nilpotent (meaning, zero is the only eigenvalue of it).

The next step in the proof of the theorem is to prove the following proposition.

Proposition 6.2. *Take any point $x' \in S$. Then*

$$\mathrm{Res}(\mathbb{D}_{r-1}, x') = 0$$

(see (6.7)) if and only if $M_j(\mathbb{D}, x') = 0$, for all $2 \leq j \leq r$, where $M_j(\mathbb{D}, x')$ are constructed in (5.5).

Proof. Consider the holomorphic vector bundle \mathcal{W} in (6.2). Let

$$\varphi : \mathcal{W}_{r-1} \longrightarrow \mathcal{W} \quad (6.8)$$

be the following composition of homomorphisms

$$\mathcal{W}_{r-1} \longrightarrow \mathcal{W}_{r-2} \longrightarrow \cdots \longrightarrow \mathcal{W}_2 \longrightarrow \mathcal{W}_1 \longrightarrow \mathcal{W};$$

see (6.6), (6.5) and (6.4) for the above homomorphisms. We note that the homomorphism φ in (6.8) is an isomorphism over the open subset X_0 in (3.9). We will now show that any holomorphic subbundle $\mathcal{V} \subset \mathcal{W}$ produces a holomorphic subbundle of \mathcal{W}_{r-1} . To prove this, let

$$\mathcal{V}' \subset \mathcal{W}_{r-1}$$

be the coherent analytic subsheaf uniquely defined by the following condition: A holomorphic section $\sigma \in H^0(U, \mathcal{W}_{r-1})$ over some analytic open subset $U \subset X$ is a section of \mathcal{V}' if and only if the restriction of $\varphi(\sigma)$ to the complement $U \setminus (U \cap S)$ is a section of the subbundle $\mathcal{V} \subset \mathcal{W}$. It is straight-forward to check that \mathcal{V}' is a holomorphic subbundle of \mathcal{W}_{r-1} .

For $0 \leq j \leq r$, consider the holomorphic subbundle

$$H_j \subset J^{r-1}(Q) = \mathcal{W}$$

(see (3.7) and (6.2)). Let

$$\mathcal{E}_j \subset \mathcal{W}_{r-1}$$

be the holomorphic subbundle corresponding to H_j . So we have the filtration of holomorphic subbundles

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \mathcal{W}_{r-1} \quad (6.9)$$

of \mathcal{W}_{r-1} . Note that we have

$$\varphi(\mathcal{E}_j) \subset H_j \quad (6.10)$$

for all $0 \leq j \leq r$, where φ is the homomorphism in (6.8). Therefore, φ produces a homomorphism

$$\varphi_j : \mathcal{E}_j/\mathcal{E}_{j-1} \longrightarrow H_j/H_{j-1} = Q \otimes K_X^{\otimes(r-j)} \quad (6.11)$$

for all $1 \leq j \leq r$; see (3.8) for the isomorphism in (6.11).

Recall that the logarithmic connection \mathbb{D} on $E = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ satisfies the following condition: For any $x' \in S$ and any $0 \leq i \leq r-1$, the eigenspace of $\text{Res}(\mathbb{D}, x')$ for the eigenvalue i is contained in the subspace $(\widehat{H}_{i+1})_{x'} \subset E_{x'}$ (see (3.12) and (4.2)). This condition implies that the logarithmic connection $\widetilde{\mathbb{D}}$ on \mathcal{W} in (6.3) has the following property: For any $x' \in S$ and any $0 \leq i \leq r-1$, the eigenspace of $\text{Res}(\widetilde{\mathbb{D}}, x')$ for the eigenvalue $i-r+1$ is contained in the subspace $(H_{i+1})_{x'} \subset \mathcal{W}_{x'}$ in (3.7). Using this and (6.10) it follows that $\text{Res}(\mathbb{D}_{r-1}, x')$ in (6.7) preserves the filtration of subspaces

$$0 = (\mathcal{E}_0)_{x'} \subset (\mathcal{E}_1)_{x'} \subset (\mathcal{E}_2)_{x'} \subset \cdots \subset (\mathcal{E}_{r-1})_{x'} \subset (\mathcal{E}_r)_{x'} = (\mathcal{W}_{r-1})_{x'} \quad (6.12)$$

obtained from (6.9). Since $\text{Res}(\mathbb{D}_{r-1}, x')$ is a nilpotent endomorphism, and it preserves the filtration in (6.12), we conclude that

$$\text{Res}(\mathbb{D}_{r-1}, x')((\mathcal{E}_i)_{x'}) \subset (\mathcal{E}_{i-1})_{x'}$$

for all $1 \leq i \leq r$. It also follows from the construction of \mathcal{W}_{r-1} that

$$\text{Res}(\mathbb{D}_{r-1}, x')((\mathcal{E}_i)_{x'}) \cap (\mathcal{E}_{i-2})_{x'} = 0$$

for all $2 \leq i \leq r$.

The above observations on $\text{Res}(\mathbb{D}_{r-1}, x')$ combine together to give the following:

For every $x \in S$, the residue $\mathrm{Res}(\mathbb{D}_{r-1}, x')$ gives an element

$$\mathbb{R}(\mathbb{D}_{r-1}, x') \in \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathcal{E}_{i+1}/\mathcal{E}_i, \mathcal{E}_i/\mathcal{E}_{i-1})_{x'}. \quad (6.13)$$

This $\mathbb{R}(\mathbb{D}_{r-1}, x')$ has the property that $\mathrm{Res}(\mathbb{D}_{r-1}, x') = 0$ if and only if $\mathbb{R}(\mathbb{D}_{r-1}, x') = 0$.

From (6.11) and the construction of \mathcal{W}_{r-1} it follows that

$$\mathcal{E}_i/\mathcal{E}_{i-1} = (H_i/H_{i-1}) \otimes \mathcal{O}_X(-(r-i)S) = Q \otimes K_X^{\otimes(r-i)} \otimes \mathcal{O}_X(-(r-i)S)$$

for all $1 \leq i \leq r$. From this we conclude that

$$\mathcal{E}_{i+1}/\mathcal{E}_i = (\mathcal{E}_i/\mathcal{E}_{i-1}) \otimes TX \otimes \mathcal{O}_X(S). \quad (6.14)$$

The isomorphism in (6.14) implies that for any $x' \in S$ and all $1 \leq i \leq k-1$, we have

$$\mathrm{Hom}(\mathcal{E}_{i+1}/\mathcal{E}_i, \mathcal{E}_i/\mathcal{E}_{i-1})_{x'} = (K_X \otimes \mathcal{O}_X(-S))_{x'} = (K_X^{\otimes 2})_{x'}$$

(see (4.1) for the last isomorphism). Therefore, $\mathbb{R}(\mathbb{D}_{r-1}, x')$ in (6.13) can be considered as an element

$$\mathbb{R}(\mathbb{D}_{r-1}, x') \in \bigoplus_{i=1}^{r-1} (K_X^{\otimes 2})_{x'} = ((K_X^{\otimes 2})_{x'})^{\oplus(r-1)}.$$

For any $1 \leq i \leq r-1$, the element of $(K_X^{\otimes 2})_{x'}$ in the i -th component of $\mathbb{R}(\mathbb{D}_{r-1}, x')$, with respect to the above decomposition, coincides with $M_{i+1}(\mathbb{D}, x')$ constructed in (5.5). It was noted earlier that $\mathrm{Res}(\mathbb{D}_{r-1}, x') = 0$ if and only if $\mathbb{R}(\mathbb{D}_{r-1}, x') = 0$. Therefore, the proof of the proposition is complete. \square

Continuing with the proof of Theorem 6.1, first assume that there is a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) such that \mathbb{D} coincides with the logarithmic connection on $E = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ associated to (V, \mathcal{F}, D) by Proposition 3.5.

It is straight-forward to check that the construction of the triple $(\mathcal{W}_{r-1}, \{\mathcal{E}_i\}_{i=0}^r, \mathbb{D}_{r-1})$ from $(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S), \mathbb{D})$ is the inverse of the construction of $(J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S), \mathbb{D})$ from (V, \mathcal{F}, D) . More precisely, $(\mathcal{W}_{r-1}, \{\mathcal{E}_i\}_{i=0}^r, \mathbb{D}_{r-1})$ coincides with (V, \mathcal{F}, D) . In particular, \mathbb{D}_{r-1} is a holomorphic connection on \mathcal{W}_{r-1} , as D is a holomorphic connection on V . Now from Proposition 6.2 we conclude that $M_j(\mathbb{D}, x') = 0$ for all $2 \leq j \leq r$ and every $x' \in S$.

To prove the converse, assume that

$$M_j(\mathbb{D}, x') = 0 \quad (6.15)$$

for all $2 \leq j \leq r$ and every $x' \in S$, where $M_j(\mathbb{D}, x')$ are constructed in (5.5). We will show that there is a branched $\mathrm{SL}(r, \mathbb{C})$ -oper (V, \mathcal{F}, D) such that \mathbb{D} coincides with the logarithmic connection on $E = J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ associated to (V, \mathcal{F}, D) by Proposition 3.5.

Since (6.15) holds, from Proposition 6.2 we know that the logarithmic connection \mathbb{D}_{r-1} on \mathcal{W}_{r-1} is actually a holomorphic connection. Consider the filtration $\{\mathcal{E}_i\}_{i=0}^r$ of \mathcal{W}_{r-1} in (6.9). It can be shown that

$$\mathbb{D}_{r-1}(\mathcal{E}_i) \subset \mathcal{E}_{i+1} \otimes K_X \quad (6.16)$$

for all $0 \leq i \leq r-1$. Indeed, we have

$$(\mathcal{W}_{r-1}, \{\mathcal{E}_i\}_{i=0}^r, \mathbb{D}_{r-1})|_{X_0} = (J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S), \{\widehat{H}_i\}_{i=0}^r, \mathbb{D}) \quad (6.17)$$

over the nonempty open subset X_0 in (3.9); the filtration $\{\widehat{H}_i\}_{i=0}^r$ is constructed in (3.12). So (6.16) follows from the given condition that

$$\mathbb{D}(\widehat{H}_i) \subset \widehat{H}_{i+1} \otimes K_X \otimes \mathcal{O}_X(S)$$

for all $0 \leq i \leq r-1$.

In view of (6.16), the second fundamental form of $\mathcal{E}_i \subset \mathcal{W}_{r-1}$ for the holomorphic connection \mathbb{D}_{r-1} produces a homomorphism

$$\Psi_i \in H^0(X, \text{Hom}(\mathcal{E}_i/\mathcal{E}_{i-1}, \mathcal{E}_{i+1}/\mathcal{E}_i) \otimes K_X)$$

for every $1 \leq i \leq r-1$. Now using the isomorphism in (6.14) we conclude that

$$\Psi_i \in H^0(X, \mathcal{O}_X(S)). \quad (6.18)$$

Recall the given condition that for all $1 \leq i \leq r-1$, the section $\text{SF}(\mathbb{D}, i)$ in (6.1) is given by the constant function 1 on X . Therefore, from the isomorphism in (6.17) we conclude that the section Ψ_i in (6.18) coincides with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X . From this it follows that

$$(\mathcal{W}_{r-1}, \{\mathcal{E}_i\}_{i=0}^r, \mathbb{D}_{r-1})$$

is a branched $\text{SL}(r, \mathbb{C})$ -oper. The logarithmic connection on $J^{r-1}(Q) \otimes \mathcal{O}_X(-(r-1)S)$ that corresponds to the branched $\text{SL}(r, \mathbb{C})$ -oper $(\mathcal{W}_{r-1}, \{\mathcal{E}_i\}_{i=0}^r, \mathbb{D}_{r-1})$ by Proposition 3.5 coincides with \mathbb{D} , because the construction $(\mathcal{W}_{r-1}, \{\mathcal{E}_i\}_{i=0}^r, \mathbb{D}_{r-1})$ from \mathbb{D} is the inverse of the construction in Proposition 3.5. This completes the proof. \square

6.1. $\text{SL}(r, \mathbb{C})$ -opers with regular singularity and branched $\text{SL}(r, \mathbb{C})$ -opers. This subsection was communicated to us by Edward Frenkel.

In this subsection we relate the branched $\text{SL}(r, \mathbb{C})$ -opers with the $\text{SL}(r, \mathbb{C})$ -opers with regular singularity and trivial monodromy introduced by Frenkel and Gaitsgory in [15, Section 2.9].

Let G be a connected simple affine algebraic group over \mathbb{C} , and let X be a smooth projective algebraic curve over \mathbb{C} . As before, S is a finite subset of X . Beilinson and Drinfeld have introduced in [2, Section 4] and [3, Section 3.8] the notion of a G -oper on X with *regular singularity* at S . We recall that it is an ordinary G -oper on the complement of S in X , whose restriction to the formal punctured disc $D_{x_k}^\times$ at each point $x_k \in S$ is a G -oper on the punctured disc with regular singularity; the details are in [3, Section 3.8.8] (see also [15, Section 2.4]). Further, Beilinson and Drinfeld have defined the *residue* of such an oper. This residue is a point of the geometric invariant theoretic quotient $\mathfrak{g}/G \simeq \mathfrak{h}/W$, where \mathfrak{g} is the Lie algebra of G , $\mathfrak{h} \subset \mathfrak{g}$ is its Cartan subalgebra, and W is the corresponding Weyl group $N_G(T)/T$.

Denote by ϖ the natural projection $\mathfrak{h} \rightarrow \mathfrak{h}/W$. For a dominant integral coweight $\check{\lambda} \in \mathfrak{h}$, denote by

$$\text{Op}_{\mathfrak{g}}^{\text{RS}, \varpi(-\check{\lambda}-\check{\rho})}$$

the space of G -opers with regular singularity on a punctured disc with fixed residue $\varpi(-\check{\lambda} - \check{\rho})$.

Frenkel and Gaitsgory have defined in [15, Section 2.9], following Beilinson and Drinfeld (unpublished), a subspace

$$\mathrm{Op}_{\mathfrak{g}}^{\check{\lambda}, \mathrm{reg}} \subset \mathrm{Op}_{\mathfrak{g}}^{\mathrm{RS}, \varpi(-\check{\lambda} - \check{\rho})} \quad (6.19)$$

consisting of all those G -opers whose monodromy around the origin is trivial. So, in particular, $\mathrm{Op}_{\mathfrak{g}}^{0, \mathrm{reg}}$ is naturally identified with the space of regular G -opers on the disc. It may be mentioned that opers of this kind play an important rôle in the geometric Langlands correspondence (see [16] and [14, Section 9.6]).

The following proposition is due to E. Frenkel.

Proposition 6.3. *There is a natural isomorphism between the space of all branched $\mathrm{SL}(r, \mathbb{C})$ -opers on X with branching over S and the space of all $\mathrm{SL}(r, \mathbb{C})$ -opers on X with regular singularity at S satisfying the following two conditions:*

- (1) *the restriction to the punctured formal disc around each point $x_k \in S$ has residue $\varpi(-2\check{\rho})$, and*
- (2) *this oper belongs to the subspace $\mathrm{Op}_{\mathfrak{g}}^{\check{\rho}, \mathrm{reg}}$ in (6.19).*

The proof of Proposition 6.3 follows directly by comparing Definition 2.1 and Theorem 6.1 with the definition of the subspace $\mathrm{Op}_{\mathfrak{g}}^{\check{\rho}, \mathrm{reg}}$ in [15].

ACKNOWLEDGEMENTS

We are very grateful to Edward Frenkel for communicating Section 6.1.

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