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# ON THE MONODROMY MAP FOR THE LOGARITHMIC DIFFERENTIAL SYSTEMS

MARIAN APRODU, INDRANIL BISWAS, SORIN DUMITRESCU, AND SEBASTIAN HELLER

ABSTRACT. We study the monodromy map for logarithmic  $\mathfrak{g}$ -differential systems over an oriented surface  $S_0$  of genus  $g$ , with  $\mathfrak{g}$  being the Lie algebra of a complex reductive affine algebraic group  $G$ . These logarithmic  $\mathfrak{g}$ -differential systems are triples of the form  $(X, D, \Phi)$ , where  $(X, D) \in \mathcal{T}_{g,d}$  is an element of the Teichmüller space of complex structures on  $S_0$  with  $d \geq 1$  ordered marked points  $D \subset S_0 = X$  and  $\Phi$  is a logarithmic connection on the trivial holomorphic principal  $G$ -bundle  $X \times G$  over  $X$  whose polar part is contained in the divisor  $D$ . We prove that the monodromy map from the space of logarithmic  $\mathfrak{g}$ -differential systems to the character variety of  $G$ -representations of the fundamental group of  $S_0 \setminus D$  is an immersion at the generic point, in the following two cases:

- (1)  $g \geq 2$ ,  $d \geq 1$ , and  $\dim_{\mathbb{C}} G \geq d + 2$ ;
- (2)  $g = 1$  and  $\dim_{\mathbb{C}} G \geq d$ .

The above monodromy map is nowhere an immersion in the following two cases:

- (1)  $g = 0$  and  $d \geq 4$ ;
- (2)  $g \geq 1$  and  $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$ .

This extends to the logarithmic case the main results in [CDHL], [BD] dealing with nonsingular holomorphic  $\mathfrak{g}$ -differential systems (which corresponds to the case of  $d = 0$ ).

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## 1. INTRODUCTION

The study of the Riemann-Hilbert mapping, which associates to a flat (algebraic or holomorphic) connection its monodromy morphism from the fundamental group is a classical topic in algebraic and analytical geometry (see, for instance, [De], [Ka] and references therein).

We recall the set-up and results of [CDHL] and [BD], the predecessors of this paper. Let  $G$  be a connected reductive affine algebraic group defined over  $\mathbb{C}$ , with  $\dim G > 0$ , and let  $\mathfrak{g}$  be the Lie algebra of  $G$ . A  $\mathfrak{g}$ -differential system is a pair of the form  $(X, \Phi)$ , where  $X$  is a complex structure on a compact oriented smooth surface  $S_0$  of genus  $g$ , and  $\Phi$  is a holomorphic connection on the trivial holomorphic principal  $G$ -bundle  $X \times G$  over the Riemann surface  $X$ . A  $\mathfrak{g}$ -differential system  $(X, \Phi)$  is called irreducible if  $\Phi$  is not induced by a holomorphic connection on  $X \times P$  for some proper parabolic subgroup  $P$  of  $G$ . Since any holomorphic connection on a Riemann surface is flat, associating the monodromy representation to a holomorphic connection we obtain a map from the space of irreducible  $\mathfrak{g}$ -differential systems to the irreducible  $G$ -character variety  $\text{Hom}(\pi_1(S_0), G)^{\text{ir}}/G$ . This monodromy map is actually holomorphic.

The main result of [CDHL] says that, if  $g = 2$ , this Riemann-Hilbert monodromy map is a local diffeomorphism from the space of irreducible  $\mathfrak{g}$ -differential systems into the irreducible  $G$ -character variety, for  $G = \text{SL}(2, \mathbb{C})$ . Being inspired by [CDHL], in [BD] it was shown that, for all  $g \geq 2$ , the above monodromy map is an immersion on an open dense subset of the space of irreducible  $\mathfrak{g}$ -differential systems, for all reductive groups  $G$  with  $\dim_{\mathbb{C}} G \geq 3$ .

Our aim here is to study the Riemann-Hilbert monodromy mapping for logarithmic  $\mathfrak{g}$ -differential systems, where  $\mathfrak{g}$  is as above. These logarithmic  $\mathfrak{g}$ -differential systems are defined by triples of the form  $(X, D, \Phi)$ , where  $(X, D) \in \mathcal{T}_{g,d}$  is an element of the Teichmüller space of complex structures on  $S_0$  with  $d$  ordered marked points  $D \subset S_0 = X$  (see Section 3), and  $\Phi$  is a logarithmic connection on the trivial holomorphic principal  $G$ -bundle  $X \times G$  over  $X$  whose polar part is contained in the divisor  $D$ .

We prove the following (see Theorem 4.4):

**Theorem 1.1.** *Assume that  $3g - 3 + d > 0$  and  $d \geq 1$ . The Riemann-Hilbert monodromy mapping from the above space of irreducible logarithmic  $\mathfrak{g}$ -differential systems to the character variety of irreducible  $G$ -representations of the fundamental group of  $S_0 \setminus D$  is an immersion at the generic point in the following two cases:*

- (1)  $g \geq 2$  and  $\dim_{\mathbb{C}} G \geq d + 2$ ;
- (2)  $g = 1$  and  $\dim_{\mathbb{C}} G \geq d$ .

*The Riemann-Hilbert monodromy mapping from the above space of irreducible logarithmic  $\mathfrak{g}$ -differential systems to the character variety of irreducible  $G$ -representations of the fundamental group of  $S_0 \setminus D$  is nowhere an immersion in the following two cases:*

- (1)  $g = 0$ ;

(2)  $g \geq 1$  and  $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$  (in particular, when  $g = 1$  and  $\dim_{\mathbb{C}} G < d$ ).

We note that Theorem 1.1 gives a complete answer only when  $g = 0$  or  $g = 1$ . For given  $g \geq 2$  and  $G$ , there are finitely many cases of  $d$  that are not addressed in Theorem 1.1. When  $g = 1$  and  $d = 0$ , from the first part of Theorem 1.1 it follows that the monodromy mapping from the space of irreducible logarithmic  $\mathfrak{g}$ -differential systems is an immersion at the generic point; see Remark 4.6.

Theorem 1.1, extends to the class of logarithmic  $\mathfrak{g}$ -differential systems, the main result in [BD] which deals with the nonsingular holomorphic  $\mathfrak{g}$ -differential systems (corresponding to the case  $d = 0$ ). Notice that the hypothesis  $3g - 3 + d > 0$  in Theorem 1.1 implies that the above Teichmüller space  $\mathcal{T}_{g,d}$  has positive dimension.

Given a reductive complex affine algebraic group  $G_0$ , by setting  $G$  to be the product group  $G_0^m$ ,  $m \geq 1$ , we can make its dimension arbitrarily large.

The proof of Theorem 1.1 is based on a transversality result in the moduli space  $\mathcal{B}_G$  of quadruples of the form  $(X, D, E_G, \Phi)$ , where

- $(X, D) \in \mathcal{T}_{g,d}$ ,
- $E_G$  is a holomorphic principal  $G$ -bundle on  $X$  such that  $E_G$  is topologically trivial, and
- $\Phi$  is a logarithmic connection on  $E_G$  whose polar part is contained in  $D$ .

A key ingredient of this transversality condition is proved in Lemma 4.7 which is an adaptation to the logarithmic case of Theorem 1.1 in [Gi] (where its proof is attributed to R. Lazarsfeld).

The article is organized as follows. Sections 2 and 3 are preparatory: they introduce the concept of logarithmic connections on holomorphic principal bundles, the above moduli space  $\mathcal{B}_G$  of quadruples  $(X, D, E_G, \Phi)$  and the  $G$ -character variety. We describe the infinitesimal deformation space of quadruples (the tangent space of  $\mathcal{B}_G$ ) as the first hypercohomology group of a certain 2-term complex (see Proposition 3.2). Section 4 is devoted to the proof of the main result (Theorem 4.4) and deals with the transversality, in the tangent space of  $\mathcal{B}_G$ , between the isomonodromy foliation and the subspace of logarithmic  $\mathfrak{g}$ -differential systems. This transversality condition, which is equivalent to the monodromy map being an immersion on the space logarithmic of  $\mathfrak{g}$ -differential systems, is proved by combining a criteria given in Lemma 4.3 (also Proposition 4.5), with Lemma 4.7 (dealing with the case  $g \geq 3$ ) and Lemma 4.9 (dealing with the case of  $g = 2$ ).

## 2. THE LOGARITHMIC ATIYAH BUNDLE

Let  $X$  be a compact connected Riemann surface. Let

$$D := \{x_1, \dots, x_d\} \subset X \tag{2.1}$$

be  $d$  distinct points, with  $d \geq 2$ . For notational convenience, the divisor  $x_1 + \dots + x_d$  of degree  $d$  on  $X$  will also be denoted by  $D$ . For a holomorphic vector bundle  $V$  on  $X$ , the holomorphic vector bundles  $V \otimes \mathcal{O}_X(D)$  and  $V \otimes \mathcal{O}_X(-D)$  will be denoted by  $V(D)$

and  $V(-D)$  respectively. The holomorphic tangent and cotangent bundles of  $X$  will be denoted by  $TX$  and  $K_X$  respectively.

Let  $G$  be a connected complex affine algebraic group with  $\dim G > 0$ . The Lie algebra of  $G$  will be denoted by  $\mathfrak{g}$ . Let

$$p : E_G \longrightarrow X \quad (2.2)$$

be a holomorphic principal  $G$ -bundle over  $X$ . The action of  $G$  on  $E_G$  produces an action of  $G$  on the holomorphic tangent bundle  $TE_G$  of  $E_G$ . The quotient

$$\text{At}(E_G) := (TE_G)/G \longrightarrow X \quad (2.3)$$

is the Atiyah bundle for  $E_G$  [At]. Let  $dp : TE_G \longrightarrow p^*TX$  be the differential of the map  $p$  in (2.2). Let

$$\text{ad}(E_G) := \text{kernel}(dp)/G \subset (TE_G)/G \quad (2.4)$$

be the adjoint bundle for  $E_G$ . Note that this holomorphic vector bundle  $\text{kernel}(dp)$  is identified with the trivial holomorphic vector bundle  $E_G \times \mathfrak{g} \longrightarrow E_G$  using the action of  $G$  on  $E_G$ . Hence  $\text{ad}(E_G)$  coincides with the vector bundle  $E_G \times^G \mathfrak{g} \longrightarrow X$  associated to  $E_G$  for the adjoint action of  $G$  on  $\mathfrak{g}$ .

Thus we have a short exact sequence of holomorphic vector bundles on  $X$

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G) \xrightarrow{d'p} TX \longrightarrow 0, \quad (2.5)$$

where  $\text{At}(E_G)$  is defined in (2.3), and the projection  $d'p$  is induced by  $dp$ ; the sequence in (2.5) is known as the Atiyah exact sequence. Define

$$\text{At}(E_G)(-\log D) := (d'p)^{-1}(TX(-D)) \subset \text{At}(E_G), \quad (2.6)$$

where  $d'p$  is the homomorphism in (2.5). So, from (2.5) we have the *logarithmic Atiyah exact sequence*

$$0 \longrightarrow \text{ad}(E_G) \xrightarrow{\iota_0} \text{At}(E_G)(-\log D) \xrightarrow{\widehat{dp}} TX(-D) \longrightarrow 0, \quad (2.7)$$

where  $\widehat{dp}$  is the restriction of the homomorphism  $d'p$  to  $\text{At}(E_G)(-\log D)$ , and  $\iota_0$  is given by the homomorphism  $\text{ad}(E_G) \longrightarrow \text{At}(E_G)$  in (2.5). We have the following commutative diagram of homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{ad}(E_G) & \xrightarrow{\iota_0} & \text{At}(E_G)(-\log D) & \xrightarrow{\widehat{dp}} & TX(-D) \longrightarrow 0 \\ & & \parallel & & \downarrow \iota & & \downarrow \iota' \\ 0 & \longrightarrow & \text{ad}(E_G) & \longrightarrow & \text{At}(E_G) & \xrightarrow{d'p} & TX \longrightarrow 0 \end{array} \quad (2.8)$$

where  $\iota$  and  $\iota'$  are the natural inclusion maps.

A *logarithmic connection* on  $E_G$  with polar part in  $D$  is a holomorphic homomorphism

$$\Phi : TX(-D) \longrightarrow \text{At}(E_G)(-\log D)$$

such that

$$\widehat{dp} \circ \Phi = \text{Id}_{TX(-D)}, \quad (2.9)$$

where  $\widehat{dp}$  is the surjective homomorphism in (2.7).

Since we have  $\iota' \circ \widehat{dp} = (d'p) \circ \iota$  (see (2.8)), and  $\iota'(y)(TX(-D)_y) = 0$  for every point  $y \in D$  in (2.1), for a logarithmic connection  $\Phi$  on  $E_G$ , from (2.9) we have

$$\iota' \circ \widehat{dp} \circ \Phi(TX(-D)_y) = \iota'(y)(TX(-D)_y) = 0$$

for every  $y \in D$ . Consequently, from the commutativity of (2.8) we conclude that  $(d'p) \circ \iota \circ \Phi(TX(-D)_y) = 0$ . This implies that

$$\iota \circ \Phi(TX(-D)_y) \subset \text{ad}(E_G)_y \subset \text{At}(E_G)_y \quad (2.10)$$

(see (2.8)). On the other hand,  $TX(-D)_y = \mathbb{C}$  by the Poincaré adjunction formula [GH, p. 146]; for any holomorphic coordinate function  $z$  on  $X$  around  $y$  with  $z(y) = 0$ , the map  $\mathbb{C} \rightarrow TX(-D)_y$  defined by  $\lambda \mapsto (\lambda \frac{dz}{z})(y)$  is actually independent of the choice of the coordinate function  $z$ . The element

$$(\iota \circ \Phi)(y)(1) \in \text{ad}(E_G)_y$$

(see (2.10)) is called the *residue* of  $\Phi$  at  $y$ ; see [De].

Fixing  $X$ , the infinitesimal deformations of the principal  $G$ -bundle  $E_G$  are parametrized by  $H^1(X, \text{ad}(E_G))$  [Do].

We recall that the infinitesimal deformations of the  $d$ -pointed Riemann surface  $(X, D)$  are parametrized by  $H^1(X, TX(-D))$ . The infinitesimal deformations of the above triple  $(X, D, E_G)$  are parametrized by  $H^1(X, \text{At}(E_G)(-\log D))$  [BHH], [Ch1], [Ch2], [Hu], [Do].

The following lemma is standard (see [BHH, Section 2.2] and [Hu]).

**Lemma 2.1.**

- (1) *The homomorphism of cohomologies*

$$\widehat{dp}_* : H^1(X, \text{At}(E_G)(-\log D)) \rightarrow H^1(X, TX(-D)),$$

*induced by the projection  $\widehat{dp}$  in (2.7), corresponds to the forgetful map from the infinitesimal deformations of the triple  $(X, D, E_G)$  to the infinitesimal deformations of the pair  $(X, D)$  obtained by simply forgetting the principal  $G$ -bundle.*

- (2) *The homomorphism of cohomologies*

$$\iota_{0*} : H^1(X, \text{ad}(E_G)) \rightarrow H^1(X, \text{At}(E_G)(-\log D)),$$

*induced by the homomorphism  $\iota_0$  in (2.7), coincides with the map from the infinitesimal deformations of the principal  $G$ -bundle  $E_G$  to the infinitesimal deformations of the triple  $(X, D, E_G)$  obtained by keeping the pair  $(X, D)$  fixed.*

### 3. LOGARITHMIC CONNECTIONS AND ISOMONODROMY

**3.1. Logarithmic Atiyah bundle.** Since  $\text{At}(E_G) := (TE_G)/G$  (see (2.3)), the subsheaf  $\text{At}(E_G)(-\log D) \subset \text{At}(E_G)$  corresponds to a subsheaf of the sheaf of  $G$ -invariant holomorphic vector fields on  $E_G$ . We will have occasions to use the following description of this subsheaf of the sheaf of  $G$ -invariant holomorphic vector fields on  $E_G$ .

Let

$$\tilde{D} := p^{-1}(D) \subset E_G$$

be the divisor, where  $p$  is the projection in (2.2). Let

$$TE_G(-\log \tilde{D}) \subset TE_G$$

be the corresponding logarithmic tangent bundle. We recall that this subsheaf is characterized by the following property: A holomorphic vector field  $v$ , defined on an open subset  $U \subset E_G$ , is a section of  $TE_G(-\log \tilde{D})$  if and only if for every holomorphic function  $f$  on  $U$  that vanishes on  $\tilde{D} \cap U$ , the function  $v(f)$  also vanishes on  $\tilde{D} \cap U$ . Since the divisor  $\tilde{D}$  is smooth, it follows that  $TE_G(-\log \tilde{D})$  is a locally free  $\mathcal{O}_{E_G}$ -submodule of  $TE_G$ . Consequently,  $TE_G(-\log \tilde{D})$  is a holomorphic vector bundle on  $E_G$ . The above characterizing property of  $TE_G(-\log \tilde{D})$  immediately implies that the Lie bracket operation of locally defined holomorphic vector fields on  $E_G$  preserves the subsheaf  $TE_G(-\log \tilde{D})$ .

To describe  $TE_G(-\log \tilde{D})$  locally, take a point  $x \in \tilde{D}$ . Let  $(z_1, z_2, \dots, z_m)$  be holomorphic coordinate functions on  $E_G$  defined around  $x$  such that  $z_1 = p \circ z$  for some holomorphic coordinate function  $z$  on  $X$  around  $p(x)$ , and also  $z_i(x) = 0$  for all  $1 \leq i \leq m$ ; here  $p$  denotes the projection in (2.2). Then  $TE_G(-\log \tilde{D})$  around  $x$  is generated by the holomorphic vector fields  $z_1 \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_m}$ .

The action of  $G$  on  $TE_G$ , induced by the action of  $G$  on  $E_G$ , actually preserves the subsheaf  $TE_G(-\log \tilde{D})$ . It is now straightforward to check that

$$\text{At}(E_G)(-\log D) = TE_G(-\log \tilde{D})/G. \quad (3.1)$$

Let

$$\Phi : TX(-D) \longrightarrow \text{At}(E_G)(-\log D) \quad (3.2)$$

be a logarithmic connection on  $E_G$ . Let

$$\tilde{\Phi} : \text{At}(E_G)(-\log D) \longrightarrow \text{ad}(E_G) \quad (3.3)$$

be the holomorphic homomorphism uniquely determined by the following conditions:

- (1)  $\tilde{\Phi} \circ \iota_0 = \text{Id}_{\text{ad}(E_G)}$ , where  $\iota_0$  is the injective homomorphism in (2.7), and
- (2)  $\text{kernel}(\tilde{\Phi}) = \Phi(TX(-D))$ .

In view of (2.4) and (3.1), the homomorphism  $\tilde{\Phi}$  in (3.3) produces a  $G$ -invariant surjective holomorphic homomorphism

$$\Phi'_0 : TE_G(-\log \tilde{D}) \longrightarrow \text{kernel}(dp), \quad (3.4)$$

where  $p$  is the projection in (2.2).

Let  $w$  be a holomorphic vector field on an open subset  $U \subset X$  that vanishes on  $U \cap D$ . In view of (3.1), the section  $\Phi(w)$  of  $\text{At}(E_G)(-\log D)|_U$  corresponds to a unique  $G$ -invariant holomorphic section of  $TE_G(-\log \tilde{D})|_{p^{-1}(U)}$  satisfying the condition that

$$dp(\Phi(w)) = p^*w$$

(as sections of  $p^*TX$ ); let

$$\Phi(w)' \in H^0(p^{-1}(U), TE_G(-\log \tilde{D})|_{p^{-1}(U)}) \quad (3.5)$$

denote this section constructed from  $w$ .

**Lemma 3.1.** *Let  $v$  be a  $G$ -invariant holomorphic section of  $TE_G(-\log \tilde{D})|_{p^{-1}(U)}$ . Then the following three hold:*

(1) *The holomorphic section  $\Phi'_0([\Phi(w)', v])$  of  $\ker(dp)$  is  $G$ -invariant, where  $\Phi'_0$  is the homomorphism in (3.4) and  $\Phi(w)'$  is the section of  $TE_G(-\log \tilde{D})|_{p^{-1}(U)}$  constructed above from  $w$ .*

(2) *For every holomorphic function  $h$  on  $U$ ,*

$$\Phi'_0([\Phi(h \cdot w)', v]) = (h \circ p) \cdot \Phi'_0([\Phi(w)', v]).$$

(3) *If  $v = \Phi(v_1)'$  for some holomorphic section  $v_1$  of  $T(-D)|_U$ , then*

$$\Phi'_0([\Phi(w)', v]) = 0.$$

*Proof.* As noted before, the Lie bracket operation of locally defined holomorphic vector fields on  $E_G$  preserves the subsheaf  $TE_G(-\log \tilde{D})$ . Since the homomorphism  $\Phi'_0$  in (3.4) is  $G$ -invariant, and  $\Phi(w)'$  is  $G$ -invariant, while  $v$  is given to be  $G$ -invariant, it follows that  $\Phi'_0([\Phi(w)', v])$  is also  $G$ -invariant.

To prove the second statement, consider the identity

$$[\Phi(h \cdot w)', v] = (h \circ p) \cdot [\Phi(w)', v] - v(h \circ p) \cdot \Phi(w)'.$$

Since  $\Phi'_0(\Phi(w)') = 0$ , where  $\Phi'_0$  is the homomorphism in (3.4), the second statement follows from this identity.

The third statement follows from the fact that any holomorphic one-dimensional distribution is integrable.  $\square$

In view of (2.4) and (3.1), from Lemma 3.1(2) we get a homomorphism

$$\tilde{\Phi} : TX(-D) \otimes \text{At}(E_G)(-\log D) \longrightarrow \text{ad}(E_G).$$

Then the homomorphism

$$\tilde{\Phi} \otimes \text{Id}_{TX(-D)^*} : \text{At}(E_G)(-\log D)TX(-D) \otimes TX(-D)^* \longrightarrow \text{ad}(E_G) \otimes TX(-D)^*$$

produces a homomorphism

$$\hat{\Phi} : \text{At}(E_G)(-\log D) \longrightarrow \text{ad}(E_G) \otimes TX(-D)^* = \text{ad}(E_G) \otimes K_X(D) \quad (3.6)$$

using the duality pairing  $TX(-D) \otimes TX(-D)^* \longrightarrow \mathcal{O}_X$ .

Let  $\mathcal{C}_\bullet$  be the 2-term complex of sheaves on  $X$

$$\mathcal{C}_\bullet : \mathcal{C}_0 := \text{At}(E_G)(-\log D) \xrightarrow{\hat{\Phi}} \mathcal{C}_1 := \text{ad}(E_G) \otimes K_X(D), \quad (3.7)$$

where  $\hat{\Phi}$  is the homomorphism constructed in (3.6), and  $\mathcal{C}_i$  is at the  $i$ -th position.

From Lemma 3.1(3) we know that

$$\hat{\Phi} \circ \Phi = 0.$$



Consequently, the logarithmic connection  $\Phi$  in (3.2) produces a homomorphism of complexes

$$\Phi^C : TX(-D) \longrightarrow \mathcal{C}_\bullet, \quad (3.8)$$

where  $TX(-D)$  is the one-term complex concentrated at the 0-th position, and  $\mathcal{C}_\bullet$  is the complex in (3.7). In other words, we have the commutative diagram

$$\begin{array}{ccccc} TX(-D) & : & TX(-D) & \longrightarrow & 0 \\ \Phi^C \downarrow & & \downarrow \Phi & & \downarrow \\ \mathcal{C}_\bullet & : & \mathcal{C}_0 & \xrightarrow{\hat{\Phi}} & \mathcal{C}_1 \end{array} \quad (3.9)$$

**3.2. Character variety.** Let  $\mathcal{T}_{g,d}$  denote the Teichmüller space of compact connected Riemann surfaces of genus  $g$  with  $d$  ordered marked points, where  $d \geq 1$ . We will always assume that  $3g - 3 + d > 0$ . This  $\mathcal{T}_{g,d}$  is a complex manifold of dimension  $3g - 3 + d$ . We recall a description of  $\mathcal{T}_{g,d}$  which will be used here. Let  $S_0$  be an oriented  $C^\infty$  surface of genus  $g$ , and let  $D_0 \subset S_0$  be  $d$  ordered distinct points. Let  $\mathbf{C}(S_0)$  denote the space of all  $C^\infty$  complex structures on  $S_0$  compatible with the given orientation of  $S_0$ . Let  $\text{Diff}_{D_0}^0(S_0)$  denote the group of all orientation preserving diffeomorphisms

$$\beta : S_0 \longrightarrow S_0$$

such that

- $\beta(x) = x$  for every  $x \in D_0$ , and
- $\beta$  is homotopic to the identity map of  $S_0$  through a continuous family of diffeomorphisms  $\beta_t$  of  $S_0$ ,  $0 \leq t \leq 1$ , such that  $\beta_t(x) = x$  for all  $t$  and all  $x \in D_0$ .

The group  $\text{Diff}_{D_0}^0(S_0)$  acts on  $\mathbf{C}(S_0)$  by pushing forward complex structures using diffeomorphisms. Then we have

$$\mathcal{T}_{g,d} = \mathbf{C}(S_0) / \text{Diff}_{D_0}^0(S_0).$$

We now assume the complex connected affine algebraic group  $G$  to be reductive. The complement  $S_0 \setminus D_0$  will be denoted by  $S'_0$ . Let

$$\mathcal{R}_G(S'_0) := \text{Hom}^{\text{ir}}(\pi_1(S'_0), G) / G \quad (3.10)$$

be the irreducible  $G$ -character variety for  $S'_0$ ; the space  $\text{Hom}^{\text{ir}}(\pi_1(S'_0), G)$  consists of all homomorphisms  $\gamma : \pi_1(S'_0) \longrightarrow G$  such that  $\gamma(\pi_1(S'_0))$  is not contained in any proper parabolic subgroup of  $G$ . We note that  $\mathcal{R}_G(S'_0)$  does not depend on the choice of the base point needed to define the fundamental group of  $S'_0$ . Since  $\pi_1(S'_0)$  is finitely presented, the complex algebraic structure of  $G$  produces a complex algebraic structure on  $\mathcal{R}_G(S'_0)$ , so  $\mathcal{R}_G(S'_0)$  is a complex affine variety. It is in fact a smooth complex orbifold. We have

$$\dim \mathcal{R}_G(S'_0) = (2g + d - 1) \cdot \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} [G, G]. \quad (3.11)$$

For more details of the above dimension count the reader is referred to [Go] and [Si, Proposition 49] (to which the monodromy around the poles should be added).

**3.3. Monodromy of logarithmic connections.** A logarithmic connection  $\Phi$  on a holomorphic principal  $G$ -bundle  $E_G \rightarrow X$  is called *irreducible* if there is no holomorphic reduction of structure group  $E_P \subset E_G|_{X \setminus D}$  to some proper parabolic subgroup  $P \subset G$ , over the open subset  $X \setminus D \subset X$ , such that  $\Phi$  is induced by a holomorphic connection on  $E_P$ .

The above definition of irreducibility needs a clarification, because in the special case where  $E_G$  is the trivial holomorphic principal  $G$ -bundle, and  $D$  is the zero divisor — so the logarithmic connection  $\Phi$  is holomorphic, meaning it has no poles — this definition of irreducibility is, a priori, weaker than the definition, given in the Introduction, of irreducible holomorphic  $\mathfrak{g}$ -differential systems. More precisely, in the definition, given in the Introduction, of irreducible holomorphic  $\mathfrak{g}$ -differential systems, the principal  $P$ -bundle is required to be the trivial bundle  $X \times P \rightarrow X$ , while the above definition does not impose any other condition on  $E_P$  apart from the condition that the logarithmic connection  $\Phi$  is induced by a logarithmic connection on  $E_P$ . We will show the following:

Let  $\Phi$  be a holomorphic connection on the trivial principal  $G$ -bundle

$$\mathcal{E}_G^0 := X \times G \rightarrow X,$$

and let  $E_P \subset \mathcal{E}_G^0$  be a holomorphic reduction of structure group to  $P$  over  $X$ , such that  $\Phi$  is induced by a holomorphic connection on  $E_P$ . Then  $E_P$  is the trivial principal  $P$ -bundle  $X \times P \rightarrow X$ .

To prove the above statement, first note that a holomorphic reduction of structure group  $E_P \subset \mathcal{E}_G^0$  to  $P$  is given by a holomorphic map  $\phi : X \rightarrow G/P$ . For this map  $\phi$ , we have

$$\phi^*T(G/P) = \text{ad}(\mathcal{E}_G^0)/\text{ad}(E_P). \quad (3.12)$$

If  $E_P$  admits a holomorphic connection  $\Phi_P$ , then  $\Phi_P$  induces holomorphic connections on both  $\text{ad}(\mathcal{E}_G^0)$  and  $\text{ad}(E_P)$ . This implies that

$$\text{degree}(\text{ad}(\mathcal{E}_G^0)) = 0 = \text{degree}(\text{ad}(E_P)),$$

and hence from (3.12) it follows that

$$\text{degree}(\phi^*T(G/P)) = 0. \quad (3.13)$$

Since the anti-canonical line bundle  $K_{G/P}^{-1}$  on  $G/P$  is ample, from (3.13) we conclude that  $\phi$  is a constant map. Consequently,  $P$  is the trivial principal  $P$ -bundle  $X \times P \rightarrow X$ .

Let

$$\varphi : \mathcal{B}_G \rightarrow \mathcal{T}_{g,d} \quad (3.14)$$

be the moduli space of irreducible logarithmic connections on topologically trivializable holomorphic principal  $G$ -bundles. So  $\mathcal{B}_G$  is the moduli space of quadruples of the form  $(X, D, E_G, \Phi)$ , where

- $(X, D) \in \mathcal{T}_{g,d}$ ,
- $E_G$  is a holomorphic principal  $G$ -bundle on  $X$  such that  $E_G$  is topologically trivial, and

- $\Phi$  is an irreducible logarithmic connection on  $E_G$  whose polar part is contained in  $D$ .

The map  $\varphi$  in (3.14) sends any  $(X, D, E_G, \Phi)$  to the pair  $(X, D)$ . The moduli space  $\mathcal{B}_G$  is a smooth complex orbifold.

Any logarithmic connection on a Riemann surface  $X$  is flat because  $\bigwedge^2(TX)^* = 0$  (consequently, its curvature 2-form vanishes identically). So considering monodromy representation of logarithmic connections, we get a holomorphic map

$$\theta : \mathcal{B}_G \longrightarrow \mathcal{R}_G(S'_0), \quad (3.15)$$

where  $\mathcal{R}_G(S'_0)$  is constructed in (3.10).

We will prove that a logarithmic connection is irreducible if and only if the corresponding monodromy representation is irreducible.

First, let  $E_G$  be a holomorphic principal  $G$ -bundle on  $X$  equipped with a logarithmic connection  $\Phi$ , such that  $\Phi$  is *not* irreducible. So there is a proper parabolic subgroup  $P \subset G$ , a holomorphic reduction of the structure group of  $E_G|_{X \setminus D}$  to  $P$ , given by a subbundle  $E_P \subset E_G|_{X \setminus D}$ , and a holomorphic connection  $\Phi_P$  on  $E_P$ , such that the logarithmic connection on  $E_G$  induced by  $\Phi_P$  coincide with  $\Phi$ . Since the monodromy of  $\Phi_P$  coincides with the monodromy of  $\Phi$ , the monodromy of  $\Phi$  is contained in  $P$ , and hence the monodromy representation for  $\Phi$  is not irreducible. To prove the converse, let  $\Phi$  be an irreducible logarithmic connection on a holomorphic principal  $G$ -bundle  $E_G$  on  $X$ . Take a point  $x_0 \in X \setminus D$  and fix a point  $z_0 \in (E_G)_{x_0}$  in the fiber of  $E_G$  over  $x_0$ . Taking parallel translations of  $z_0$  along all possible homotopy classes of loops based at  $x_0$  we get the monodromy representation

$$H_\Phi : \pi_1(X \setminus D, x_0) \longrightarrow G$$

of  $\Phi$ . Assume that the image of  $H_\Phi$  is contained in a parabolic subgroup  $P \subsetneq G$ . Let  $\mathcal{S} \subset E_G|_{X \setminus D}$  be the subset obtained by taking parallel translations of  $z_0$  along all possible homotopy classes of paths starting at  $x_0$ . Then

$$E_P := \mathcal{S}P \subset E_G|_{X \setminus D}$$

(recall that  $G$  acts on  $E_G$ ) is a holomorphic reduction of the structure group of  $E_G|_{X \setminus D}$  to  $P$  over  $X \setminus D$ . The logarithmic connection  $\Phi$  produces a holomorphic connection on the holomorphic principal  $P$ -bundle  $E_P$ , which in turn induces  $\Phi$ . Consequently, the logarithmic connection  $\Phi$  is not irreducible. Thus, a logarithmic connection is irreducible if and only if the corresponding monodromy representation is irreducible.

**3.4. Isomonodromy.** Let

$$d\theta : T\mathcal{B}_G \longrightarrow \theta^*T\mathcal{R}_G(S'_0)$$

be the differential of the map  $\theta$  in (3.15). The map  $\theta$  is a holomorphic submersion, meaning  $d\theta$  is surjective. The kernel of  $d\theta$

$$\mathcal{I} := \text{kernel}(d\theta) \subset T\mathcal{B}_G \quad (3.16)$$

is a holomorphic foliation on  $\mathcal{B}_G$ ; it is known as the *isomonodromy foliation*.

For any point  $(X, D) \in \mathcal{T}_{g,d}$ , the restriction of  $\theta$  to  $\varphi^{-1}((X, D))$ , where  $\varphi$  is the projection in (3.14), is a holomorphic local diffeomorphism. Consequently, for any point  $z \in \mathcal{B}_G$ , the differential of  $\varphi$

$$d\varphi(z) : T_z\mathcal{B}_G \longrightarrow T_{\varphi(z)}\mathcal{T}_{g,d},$$

when restricted to the subspace  $\mathcal{I}_z \subset T_z\mathcal{B}_G$  in (3.16), produces an isomorphism

$$\mathcal{I}_z \xrightarrow{\sim} T_{\varphi(z)}\mathcal{T}_{g,d}.$$

Therefore, there is a unique holomorphic homomorphism

$$\mathbb{L} : \varphi^*T\mathcal{T}_{g,d} \longrightarrow T\mathcal{B}_G \tag{3.17}$$

such that

- $d\varphi \circ \mathbb{L} = \text{Id}_{\varphi^*T\mathcal{T}_{g,d}}$ , and
- $\mathbb{L}(\varphi^*T\mathcal{T}_{g,d}) \subset \mathcal{I}$ , where  $\mathcal{I}$  is constructed in (3.16).

Since for any point  $(X, D) \in \mathcal{T}_{g,d}$ , the restriction of  $\theta$  to  $\varphi^{-1}((X, D))$  is a holomorphic local diffeomorphism, it follows that  $\mathbb{L}$  actually satisfies the condition that

$$\mathbb{L}(\varphi^*T\mathcal{T}_{g,d}) = \mathcal{I}. \tag{3.18}$$

**Proposition 3.2.** *Take any point  $z = (X, D, E_G, \Phi) \in \mathcal{B}_G$ .*

- (1) *The tangent space to  $\mathcal{B}_G$  at  $z$  is the first hypercohomology*

$$T_z\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet),$$

*where  $\mathcal{C}_\bullet$  is the complex in (3.7).*

- (2) *The homomorphism*

$$\mathbb{L}(z) : T_{\varphi(z)}\mathcal{T}_{g,d} = H^1(X, TX(-D)) \longrightarrow T_z\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet)$$

*in (3.17) coincides with the homomorphism of hypercohomologies*

$$\Phi_*^C : H^1(X, TX(-D)) \longrightarrow \mathbb{H}^1(X, \mathcal{C}_\bullet)$$

*induced by the homomorphism  $\Phi^C$  in (3.8).*

For the proof of Proposition 3.2 the reader is referred to [Ko, Proposition 3.8] (Proposition 3.4 of the arxiv version of [Ko]), [Ch2, p. 1417, Proposition 5.1], [Ch1], [In] and [BS].

#### 4. MONODROMY MAP ON LOGARITHMIC DIFFERENTIAL SYSTEMS

**4.1. Logarithmic differential systems.** As before,  $G$  is a connected complex reductive affine algebraic group with  $\dim G > 0$ , and

$$d_s := \dim_{\mathbb{C}}[G, G]. \tag{4.1}$$

Consider the moduli space  $\mathcal{B}_G$  in (3.14). Let

$$\mathbb{T}(G) \subset \mathcal{B}_G \tag{4.2}$$

be the locus of all  $(X, D, E_G, \Phi)$  such that the holomorphic principal  $G$ -bundle  $E_G$  on  $X$  is holomorphically trivial. Note that  $E_G$  is topologically trivial by the definition of  $\mathcal{B}_G$ ; also by the definition of  $\mathcal{B}_G$  the logarithmic connection  $\Phi$  is irreducible. The subset  $\mathbb{T}(G)$  in (4.2) is a complex subspace.

**Proposition 4.1.** *The complex space  $\mathbb{T}(G)$  in (4.2) is a complex orbifold of dimension  $(g + d - 1) \cdot \dim_{\mathbb{C}} G - d_s + 3g - 3 + d$ , where  $d_s$  is defined in (4.1).*

*Proof.* Let  $\varpi : C_{g,d} \rightarrow \mathcal{T}_{g,d}$  be the universal Riemann surface equipped with the universal divisor  $\mathcal{D} \subset C_{g,d}$  of relative degree  $d$  over  $\mathcal{T}_{g,d}$ . Let  $\mathcal{K} \rightarrow C_{g,d}$  be the relative holomorphic cotangent bundle for the projection  $\varpi$ . Let  $\varphi' : \mathbb{T}(G) \rightarrow \mathcal{T}_{g,d}$  be the restriction of the map  $\varphi$  in (3.14).

For any Riemann surface  $X$ , the space of all logarithmic connections on the trivial holomorphic principal  $G$ -bundle  $X \times G \rightarrow X$  with polar part contained in  $D \subset X$  is the vector space  $H^0(X, K_X(D)) \otimes \mathfrak{g}$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ . Consequently,  $\mathbb{T}(G)$  is the quotient of an open dense subset of the total space of  $\varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(\mathcal{D})) \otimes \mathfrak{g}$  by the adjoint action of  $G$ ; the group  $G$  acts trivially on  $\varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(\mathcal{D}))$  and it has the adjoint action on  $\mathfrak{g}$ .

Take a point

$$\mathbf{w} \in \varpi_*(\mathcal{K} \otimes \mathcal{O}_{C_{g,d}}(\mathcal{D})) \otimes \mathfrak{g}$$

that defines an irreducible logarithmic connection on the trivial principal  $G$ -bundle. The adjoint action of the center of  $G$  on  $\mathfrak{g}$  is trivial. The isotropy subgroup of  $[G, G]$  for the action of  $[G, G]$  on  $\mathbf{w}$  is a finite subgroup of  $[G, G]$ . The proposition follows from these.  $\square$

Let

$$\widehat{\theta} : \mathbb{T}(G) \rightarrow \mathcal{R}_G(S'_0), \quad (4.3)$$

be the restriction to  $\mathbb{T}(G) \subset \mathcal{B}_G$  of the monodromy map  $\theta$  in (3.15). We are interested in the following question: When is the map  $\widehat{\theta}$  an immersion over an open dense subset of  $\mathbb{T}(G)$ ?

**Remark 4.2.** In [BD] it was proved that  $\widehat{\theta}$  is an immersion over an open dense subset of  $\mathbb{T}(G)$ , if  $g \geq 2$ ,  $d = 0$  and  $\dim_{\mathbb{C}} G \geq 3$ . From this it can be deduced that  $\widehat{\theta}$  is an immersion over an open dense subset of  $\mathbb{T}(G)$ , if  $g \geq 2$ ,  $d = 1$  and  $\dim_{\mathbb{C}} G \geq 3$ . To see this, first note that there is no logarithmic one-form on a compact Riemann surface  $X$  with exactly one pole, because the residue has to be zero. So the space  $\mathbb{T}(G)$  in (4.3) for  $d = 1$  coincides with  $\mathbb{T}(G)$  for  $d = 0$ . On the other hand, the natural map from the space  $\mathcal{R}_G(S'_0)$  in (4.3) for  $d = 0$  to the space  $\mathcal{R}_G(S'_0)$  for  $d = 1$  is an embedding; this natural map corresponds to restricting any flat  $G$ -connection on  $S_0$  to the open subset  $S'_0 = S_0 \setminus D_0$  of it. Therefore, we conclude that the map  $\widehat{\theta}$  is an immersion over an open dense subset of  $\mathbb{T}(G)$ , if  $g \geq 2$ ,  $d = 1$  and  $\dim_{\mathbb{C}} G \geq 3$ .

**4.2. The main theorem.** We first state a lemma of linear algebra which will be used in the proof of Theorem 4.4.

**Lemma 4.3.** *Let  $\beta : V \rightarrow W$  be a linear map between two finite dimensional complex vector spaces. Let  $S_1$  and  $S_2$  be two subspaces of  $V$  such that*

- (1)  $\ker(\beta) \subset S_1$ , and
- (2) *the homomorphism  $\beta|_{S_2} : S_2 \rightarrow W$  is injective.*

*Then  $\dim S_1 \cap S_2 = \dim \beta(S_1) \cap \beta(S_2)$ .*

*Proof.* Since  $\beta|_{S_2}$  is injective, the restriction  $\beta|_{S_1 \cap S_2}$  is injective. For any  $v \in S_2$  with  $\beta(v) \in \beta(S_1)$ , there is an element  $w \in S_1$  such that  $\beta(v) = \beta(w)$ . But then  $v - w \in \ker(\beta) \subset S_1$ , and hence  $v \in S_1$ . Consequently, the restriction  $\beta|_{S_1 \cap S_2}$  is realized as an isomorphism between  $S_1 \cap S_2$  and  $\beta(S_1) \cap \beta(S_2)$ .  $\square$

**Theorem 4.4.** *Assume that  $3g - 3 + d > 0$  and  $d \geq 1$ . The map  $\hat{\theta}$  in (4.3) is an immersion over a nonempty open dense subset of  $\mathbb{T}(G)$  in the following two cases:*

- (1)  $g \geq 2$  and  $\dim_{\mathbb{C}} G \geq d + 2$ ;
- (2)  $g = 1$  and  $\dim_{\mathbb{C}} G \geq d$ .

*The map  $\hat{\theta}$  in (4.3) is nowhere an immersion in the following two cases:*

- (1)  $g = 0$ ;
- (2)  $g \geq 1$  and  $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$ .

*Proof.* First assume that  $g = 0$ . The trivial holomorphic principal  $G$ -bundle over  $\mathbb{CP}^1$  is rigid [Ra], [Ha]. In other words, in any holomorphic family of holomorphic principal  $G$ -bundle over  $\mathbb{CP}^1$ , parametrized by a complex manifold  $Z$ , the locus of points of  $Z$  over which the principal  $G$ -bundle on  $\mathbb{CP}^1$  is holomorphically trivial is an open subset of  $Z$ . Therefore, the map  $\hat{\theta}$  in (4.3) is nowhere an immersion. We note that this also follows from the fact that

$$\dim \mathbb{T}(G) - \mathcal{R}_G(S'_0) = d - 3 > 0$$

if  $g = 0$  (see (3.11) and Proposition 4.1).

So we assume that  $g \geq 1$ .

If  $g \geq 1$  and  $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$ , then from (3.11) and Proposition 4.1 we have

$$\dim \mathbb{T}(G) - \mathcal{R}_G(S'_0) = 3g - 3 + d - g \cdot \dim_{\mathbb{C}} G > 0.$$

Hence the map  $\hat{\theta}$  in (4.3) is nowhere an immersion in this case also.

So we assume that at least one of the following two holds:

- (1)  $g \geq 2$  and  $\dim_{\mathbb{C}} G \geq d + 2$ ;
- (2)  $g = 1$  and  $\dim_{\mathbb{C}} G \geq d$ .

The map  $\widehat{\theta}$  in (4.3) is an immersion over the subset of  $\mathbb{T}(G)$  over which the homomorphism

$$\bigwedge^e d\widehat{\theta} : \bigwedge^e T\mathbb{T}(G) \longrightarrow \widehat{\theta}^* \bigwedge^e T\mathcal{R}_G(S'_0)$$

is fiber-wise nonzero, where  $e = \dim_{\mathbb{C}} \mathbb{T}(G)$  and  $d\widehat{\theta}$  is the differential of the map  $\widehat{\theta}$ . Therefore, to prove the theorem, it suffices to show that there is a point  $z \in \mathbb{T}(G)$  such that the differential at  $z$

$$d\widehat{\theta}(z) : T_z\mathbb{T}(G) \longrightarrow T_{\widehat{\theta}(z)}\mathcal{R}_G(S'_0) \quad (4.4)$$

is injective; recall that  $\mathbb{T}(G)$  is irreducible.

Take a point

$$z = (X, D, E_G, \Phi) \in \mathbb{T}(G). \quad (4.5)$$

We recall that  $\mathcal{I}(z) = \ker((d\theta)(z))$  (see (3.16)). The homomorphism  $d\widehat{\theta}(z)$  (see (4.4)) is injective if and only if

$$\mathcal{I}(z) \cap T_z\mathbb{T}(G) = 0; \quad (4.6)$$

note that both  $\mathcal{I}(z)$  and  $T_z\mathbb{T}(G)$  are subspaces of the tangent space  $T_z\mathcal{B}_G$ .

We will use Lemma 4.3 to prove that (4.6) holds when  $z$  is chosen suitably.

We recall from Proposition 3.2(1) that  $T_z\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet)$ , where  $\mathcal{C}_\bullet$  is the complex in (3.7). Also, recall that the infinitesimal deformations of the triple  $(X, D, E_G)$  are parametrized by  $H^1(X, \text{At}(E_G)(-\log D))$ . Let

$$\rho : \mathbb{H}^1(X, \mathcal{C}_\bullet) \longrightarrow H^1(X, \text{At}(E_G)(-\log D)) \quad (4.7)$$

be the forgetful map that sends any infinitesimal deformation of the quadruple

$$z = (X, D, E_G, \Phi)$$

in (4.5) to the infinitesimal deformation of the triple  $(X, D, E_G)$  obtained from it by simply forgetting the logarithmic connection. We shall describe  $\rho$  explicitly.

Let  $\mathcal{A}_\bullet$  be the one-term complex with  $\text{At}(E_G)(-\log D)$  at the 0-th position.

Consider the homomorphism  $\mathcal{H}$  of complexes

$$\begin{array}{ccc} \mathcal{C}_\bullet & : & \text{At}(E_G)(-\log D) \xrightarrow{\widehat{\Phi}} \text{ad}(E_G) \otimes K_X(D) \\ \mathcal{H} \downarrow & & \parallel \qquad \qquad \qquad \downarrow \\ \mathcal{A}_\bullet & : & \text{At}(E_G)(-\log D) \longrightarrow 0 \end{array} \quad (4.8)$$

where  $\mathcal{C}_\bullet$  is the complex in (3.7). Let

$$\mathcal{H}_* : \mathbb{H}^1(X, \mathcal{C}_\bullet) \longrightarrow \mathbb{H}^1(X, \mathcal{A}_\bullet) = H^1(X, \text{At}(E_G)(-\log D)) \quad (4.9)$$

be the homomorphism of hypercohomologies induced by this homomorphism of complexes. Then the homomorphism  $\rho$  in (4.7) coincides with  $\mathcal{H}_*$  in (4.9).

In Lemma 4.3, set  $V = \mathbb{H}^1(X, \mathcal{C}_\bullet)$ ,  $W = H^1(X, \text{At}(E_G)(-\log D))$ ,  $\beta = \mathcal{H}_*$  (see (4.9)),  $S_1 = T_z\mathbb{T}(G)$  (see (4.2)) and  $S_2 = \mathcal{I}(z)$  (see (3.16)).

We will show that the hypotheses in Lemma 4.3 are satisfied.

**Proposition 4.5.** *For the above data, the two conditions in Lemma 4.3 hold.*

*Proof of Proposition 4.5.* The first condition in Lemma 4.3 says that

$$\ker(\mathcal{H}_*) \subset T_z\mathbb{T}(G). \quad (4.10)$$

To prove (4.10), we will identify the kernel of  $\mathcal{H}_*$ . For this, observe that the homomorphism of complexes  $\mathcal{H}$  in (4.8) fits in the following short exact sequence of complexes:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}'_{\bullet} & : & 0 & \longrightarrow & \mathrm{ad}(E_G) \otimes K_X(D) \\ \downarrow & & \downarrow & & \parallel \\ \mathcal{C}_{\bullet} & : & \mathrm{At}(E_G)(-\log D) & \xrightarrow{\widehat{\Phi}} & \mathrm{ad}(E_G) \otimes K_X(D) \\ \mathcal{H} \downarrow & & \parallel & & \downarrow \\ \mathcal{A}_{\bullet} & : & \mathrm{At}(E_G)(-\log D) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

This short exact sequence of complexes yields the following long exact sequence of hypercohomologies:

$$\begin{aligned} \longrightarrow \mathbb{H}^1(X, \mathcal{A}'_{\bullet}) &= H^0(X, \mathrm{ad}(E_G) \otimes K_X(D)) \xrightarrow{\nu} \mathbb{H}^1(X, \mathcal{C}_{\bullet}) \\ &\xrightarrow{\mathcal{H}_*} \mathbb{H}^1(X, \mathcal{A}_{\bullet}) = H^1(X, \mathrm{At}(E_G)(-\log D)) \longrightarrow \cdots \end{aligned}$$

The above homomorphism  $\nu$  corresponds to moving the holomorphic connection on the trivializable holomorphic principal  $G$ -bundle  $E_G$ , keeping the triple  $(X, D, E_G)$  fixed. This immediately implies that (4.10) holds.

The second condition in Lemma 4.3 says that the restriction of the homomorphism  $\mathcal{H}_*$  to  $\mathcal{I}(z)$

$$\mathcal{H}_*|_{\mathcal{I}(z)} : \mathcal{I}(z) \longrightarrow H^1(X, \mathrm{At}(E_G)(-\log D)) \quad (4.11)$$

is injective.

To prove that the homomorphism in (4.11) is injective, from (3.18) we conclude that  $\mathcal{H}_*|_{\mathcal{I}(z)}$  is injective if the composition of homomorphisms

$$H^1(X, TX(-D)) \xrightarrow{\mathbb{L}(z)} \mathbb{H}^1(X, \mathcal{C}_{\bullet}) \xrightarrow{\mathcal{H}_*} H^1(X, \mathrm{At}(E_G)(-\log D)) \quad (4.12)$$

is injective, where  $\mathbb{L}(z)$  is the homomorphism in (3.17). From Proposition 3.2(2) we know that  $\mathbb{L}(z) = \Phi_*^C$ . Therefore, the composition of homomorphisms in (4.12) coincides with the homomorphism of cohomologies

$$\Phi_* : H^1(X, TX(-D)) \longrightarrow H^1(X, \mathrm{At}(E_G)(-\log D))$$

induced by the logarithmic connection  $\Phi : TX(-D) \longrightarrow \mathrm{At}(E_G)(-\log D)$  in (4.5). But from the definition of a logarithmic connection we know that

- the homomorphism  $\Phi$  is fiber-wise injective, and
- $\Phi(TX(-D))$  is a direct summand of  $\mathrm{At}(E_G)(-\log D)$ .



Consequently, the above homomorphism  $\Phi_*$  is injective. Hence the composition of homomorphisms in (4.12) is injective. This implies that the homomorphism in (4.11) is injective. This completes the proof of Proposition 4.5.  $\square$

Continuing with the proof of Theorem 4.4, in view of Proposition 4.5, from Lemma 4.3 we conclude that the statement in (4.6) is equivalent to the following statement:

$$\mathcal{H}_*(\mathcal{I}(z)) \cap \mathcal{H}_*(T_z\mathbb{T}(G)) = 0, \quad (4.13)$$

where  $\mathcal{H}_*$  is the homomorphism in (4.9).

Fix a holomorphic trivialization of the principal  $G$ -bundle  $E_G$  in (4.5). Using it we will identify  $E_G$  with the trivial holomorphic principal  $G$ -bundle  $X \times G \rightarrow X$ . So  $\text{ad}(E_G)$  is the trivial holomorphic vector bundle  $X \times \mathfrak{g} \rightarrow X$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ , and also

$$\text{At}(E_G)(-\log D) = \text{ad}(E_G) \oplus TX(-D) = X \times \mathfrak{g} \oplus TX(-D). \quad (4.14)$$

Let  $\Phi_0$  be the trivial logarithmic (in fact it is holomorphic) connection on the trivial holomorphic principal  $G$ -bundle  $X \times G \rightarrow X$ . Note that the trivial holomorphic connection on  $E_G$  does not depend on the choice of the trivialization of  $E_G$ . The homomorphism

$$TX(-D) \rightarrow \text{At}(E_G)(-\log D)$$

that defines  $\Phi_0$  coincides with the inclusion map

$$TX(-D) \hookrightarrow \text{ad}(E_G) \oplus TX(-D) = \text{At}(E_G)(-\log D)$$

(see (4.14)). So we have

$$\Phi = \Phi_0 + \delta, \quad (4.15)$$

where

$$\delta \in H^0(X, K_X(D) \otimes \mathfrak{g}) = H^0(X, K_X(D)) \otimes \mathfrak{g};$$

recall that  $\text{ad}(E_G) = X \times \mathfrak{g}$ .

Consider the infinitesimal deformations of the triple  $(X, D, E_G)$  in (4.5) such the principal  $G$ -bundle remains trivial, but the pair  $(X, D)$  moves. These correspond to the image of the homomorphism

$$H^1(X, TX(-D)) \rightarrow H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))$$

(see (4.14) for the decomposition) given by the identity map of  $H^1(X, TX(-D))$  and the zero map of  $H^1(X, TX(-D))$  to  $H^1(X, \text{ad}(E_G))$ . In other words, these correspond to the image of the homomorphism of cohomologies

$$H^1(X, TX(-D)) \rightarrow H^1(X, \text{At}(E_G)(-\log D))$$

induced by the inclusion map  $TX(-D) \hookrightarrow \text{ad}(E_G) \oplus TX(-D)$  which is defined using (4.14).

Consequently, the subspace in (4.13)

$$\mathcal{H}_*(T_z\mathbb{T}(G)) \subset H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))$$

coincides with the subspace

$$\begin{aligned} 0 \oplus H^1(X, TX(-D)) &= H^1(X, TX(-D)) \subset H^1(X, \text{At}(E_G)(-\log D)) \\ &= H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)). \end{aligned}$$

Consider the section  $\delta$  in (4.15). Using the natural duality pairing

$$TX(-D) \otimes K_X(D) \longrightarrow \mathcal{O}_X$$

it produces a homomorphism

$$\widehat{\delta} : TX(-D) \longrightarrow \mathcal{O}_X \otimes \mathfrak{g} = \text{ad}(E_G). \quad (4.16)$$

Let

$$\widehat{\delta}_* : H^1(X, TX(-D)) \longrightarrow H^1(X, \mathcal{O}_X) \otimes \mathfrak{g} = H^1(X, \text{ad}(E_G)) \quad (4.17)$$

be the homomorphism of cohomologies induced by  $\widehat{\delta}$  in (4.16).

We will now show that the subspace in (4.13)

$$\mathcal{H}_*(\mathcal{I}(z)) \subset H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))$$

coincides with the subspace

$$\{(\widehat{\delta}_*(v), v) \mid v \in H^1(X, TX(-D))\} \subset H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)),$$

where  $\widehat{\delta}_*$  is the homomorphism in (4.17).

To prove this, let

$$\iota_{0*} : H^1(X, \text{ad}(E_G)) \longrightarrow H^1(X, \text{At}(E_G)(-\log D)) \quad (4.18)$$

be the homomorphism of cohomologies induced by the homomorphism  $\iota_0$  of sheaves in (2.7). We note that  $\iota_{0*}$  coincides with the natural map that sends any infinitesimal deformation of  $E_G$  (keeping  $(X, D)$  fixed) to the corresponding infinitesimal deformation of  $(X, D, E_G)$  where only  $E_G$  is moving.

Consider the homomorphism  $\Phi^C$  in (3.8) constructed from the connection  $\Phi$ . Let

$$\Phi^{0,C} : TX(-D) \longrightarrow \mathcal{C}_\bullet^0 \quad (4.19)$$

be the homomorphism as in (3.8) constructed for the trivial connection  $\Phi_0$  on  $E_G$ ; here  $\mathcal{C}_\bullet^0$  is the complex as in (3.7) for the trivial connection  $\Phi_0$ . From (4.15) it follows immediately that

$$\mathcal{H} \circ \Phi^C - \mathcal{H}^0 \circ \Phi^{0,C} = \iota_0 \circ \widehat{\delta}, \quad (4.20)$$

where  $\widehat{\delta}$ ,  $\mathcal{H}$  and  $\iota_0$  are the homomorphisms in (4.16), (4.8) and (2.7) respectively, while  $\mathcal{H}^0$  is the homomorphism for the trivial holomorphic connection  $\Phi_0$  constructed as in (4.8) (by substituting  $\Phi_0$  in place of  $\Phi$  in the construction of  $\mathcal{H}$ ). As in Proposition 3.2(2), let

$$\Phi_*^{0,C} : H^1(X, TX(-D)) \longrightarrow \mathbb{H}^1(X, \mathcal{C}_\bullet^0) \quad (4.21)$$

be the homomorphism of hypercohomologies induced by  $\Phi^{0,C}$  in (4.19). From (4.20) we conclude that

$$\mathcal{H}_* \circ \Phi_*^C - \mathcal{H}_*^0 \circ \Phi_*^{0,C} = \iota_{0*} \circ \widehat{\delta}_*, \quad (4.22)$$

where  $\widehat{\delta}_*$ ,  $\mathcal{H}_*$ ,  $\Phi_*^{0,C}$ ,  $\iota_{0*}$  and  $\Phi_*^C$  are the homomorphisms in (4.17), (4.9), (4.21), (4.18) and Proposition 3.2(2) respectively, and

$$\mathcal{H}_*^0 : \mathbb{H}^1(X, \mathcal{C}_\bullet^0) \longrightarrow H^1(X, \text{At}(E_G)(-\log D)) \quad (4.23)$$

is the homomorphism of hypercohomologies induced by the homomorphism  $\mathcal{H}^0$  in (4.20). Note that both sides of (4.22) are actually homomorphisms from  $H^1(X, TX(-D))$  to  $H^1(X, \text{At}(E_G)(-\log D))$ . Also, note that from the decomposition in (4.14) it follows immediately that the homomorphism  $\iota_{0*}$  in (4.22) is injective. In fact, the decomposition in (4.14) realizes  $H^1(X, \text{ad}(E_G))$  as a direct summand of  $H^1(X, \text{At}(E_G)(-\log D))$ .

Now consider the homomorphism

$$\mathbb{L}(z) : T_{(X,D)}\mathcal{T}_{g,d} \longrightarrow T_z\mathcal{B}_G$$

in (3.17) constructed for the connection  $\Phi$  in the expression of  $z$  in (4.5). Let

$$\mathbb{L}^0 : T_{(X,D)}\mathcal{T}_{g,d} \longrightarrow T_{(X,D,E_G,\Phi_0)}\mathcal{B}_G$$

be the homomorphism as in (3.17) constructed for the trivial connection  $\Phi_0$ . From Proposition 3.2(2) we know that

$$\mathcal{H}_* \circ \mathbb{L} - \mathcal{H}_*^0 \circ \mathbb{L}^0 = \mathcal{H}_* \circ \Phi_*^C - \mathcal{H}_*^0 \circ \Phi_*^{0,C}, \quad (4.24)$$

where  $\mathcal{H}_*$  and  $\mathcal{H}_*^0$  are the homomorphisms in (4.9) and (4.23) respectively; recall that  $T_z\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet)$  and  $T_{(X,D,E_G,\Phi_0)}\mathcal{B}_G = \mathbb{H}^1(X, \mathcal{C}_\bullet^0)$ .

Combining (4.22) and (4.24) it follows that

$$\mathcal{H}_* \circ \mathbb{L} - \mathcal{H}_*^0 \circ \mathbb{L}^0 = \iota_{0*} \circ \widehat{\delta}_*, \quad (4.25)$$

where  $\widehat{\delta}_*$  is the homomorphism in (4.17). It was noted earlier that the decomposition in (4.14) realizes  $H^1(X, \text{ad}(E_G))$  as a direct summand of  $H^1(X, \text{At}(E_G)(-\log D))$ .

Therefore, from (4.25) and (3.18) we conclude that the subspace in (4.13)

$$\mathcal{H}_*(\mathcal{I}(z)) \subset H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))$$

(see (4.14) for the above decomposition) coincides with the subspace

$$\{(\widehat{\delta}_*(v), v) \mid v \in H^1(X, TX(-D))\} \subset H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)).$$

On the other hand, it was shown earlier the subspace in (4.13)

$$\mathcal{H}_*(T_z\mathbb{T}(G)) \subset H^1(X, \text{At}(E_G)(-\log D)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D))$$

coincides with the subspace

$$0 \oplus H^1(X, TX(-D)) = H^1(X, TX(-D)) \subset H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)).$$

Combining these two we obtain an isomorphism

$$\eta : \text{kernel}(\widehat{\delta}_*) \xrightarrow{\sim} \mathcal{H}_*(\mathcal{I}(z)) \cap \mathcal{H}_*(T_z\mathbb{T}(G)) \quad (4.26)$$

that sends any  $v \in \text{kernel}(\widehat{\delta}_*) \subset H^1(X, TX(-D))$  to

$$(0, v) \in H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX(-D)) = H^1(X, \text{At}(E_G)(-\log D)).$$

Consequently, (4.13) holds if and only if we have

$$\text{kernel}(\widehat{\delta}_*) = 0, \quad (4.27)$$

where  $\widehat{\delta}_*$  is the homomorphism constructed in (4.17).

Take any subspace

$$V \subset H^0(X, K_X(D)).$$

Let  $H^1(X, TX(-D)) \otimes V \rightarrow H^1(X, \mathcal{O}_X)$  be the homomorphism constructed using the duality pairing  $TX(-D) \otimes K_X(D) \rightarrow \mathcal{O}_X$ . Let

$$F_V : H^1(X, TX(-D)) \rightarrow H^1(X, \mathcal{O}_X) \otimes V^* \quad (4.28)$$

be the homomorphism given by it. From the construction of  $\widehat{\delta}_*$  in (4.17) we see that

$$\text{kernel}(\widehat{\delta}_*) = \text{kernel}(F_V),$$

where  $V \subset H^0(X, K_X(D))$  is the image of the homomorphism

$$H_\delta : \mathfrak{g}^* \rightarrow H^0(X, K_X(D)) \quad (4.29)$$

given by  $\delta$  in (4.15); note that since  $\delta \in H^0(X, K_X(D)) \otimes \mathfrak{g}$ , it produces a homomorphism  $H_\delta$  as in (4.29) by sending any  $w \in \mathfrak{g}^*$  to  $w(\delta) \in H^0(X, K_X(D))$ . Consequently, (4.27) holds if and only if

$$\text{kernel}(F_{H_\delta(\mathfrak{g}^*)}) = 0, \quad (4.30)$$

where  $H_\delta$  and  $F_{H_\delta(\mathfrak{g}^*)}$  are the homomorphism constructed in (4.29) and (4.28) respectively.

It is evident that there is an element  $z = (X, D, E_G, \Phi) \in \mathbb{T}(G)$  such that (4.30) holds if and only if there is a subspace  $V \subset H^0(X, K_X(D))$ , with  $\dim V \leq \dim \mathfrak{g}$ , satisfying the condition that the homomorphism  $F_V$  in (4.28) is injective. Indeed, choosing a homomorphism

$$\delta' : \mathfrak{g}^* \rightarrow H^0(X, K_X(D))$$

for which  $V \subset \delta'(\mathfrak{g}^*)$ , consider the element  $\delta \in H^0(X, K_X(D)) \otimes \mathfrak{g}$  given by  $\delta'$ . Then the logarithmic connection  $(X, D, X \times G, \Phi_0 + \delta)$  satisfies (4.30), where  $\Phi_0$  is the trivial holomorphic connection on  $X \times G \rightarrow X$ .

First assume that  $g = 1$  (hence, by hypothesis,  $d \geq 1$ ) and  $\dim_{\mathbb{C}} G \geq d$ . This implies that

$$\dim H^0(X, K_X(D)) = d \leq \dim_{\mathbb{C}} G.$$

So in this case there is a subspace  $V \subset H^0(X, K_X(D))$ , with  $\dim V \leq \dim \mathfrak{g}$ , for which the homomorphism  $F_V$  in (4.28) is injective, if the homomorphism

$$F_{H^0(X, K_X(D))} : H^1(X, TX(-D)) \rightarrow H^1(X, \mathcal{O}_X) \otimes H^0(X, K_X(D))^* \quad (4.31)$$

is injective; if the homomorphism in (4.31) is injective, then we may take  $V$  to be  $H^0(X, K_X(D))$  itself and the homomorphism  $F_V$  is injective.

The homomorphism in (4.31) is injective if the dual homomorphism

$$F_{H^0(X, K_X(D))}^* : H^0(X, K_X) \otimes H^0(X, K_X(D)) \rightarrow H^0(X, K_X^{\otimes 2}(D)) \quad (4.32)$$

is surjective. Now the homomorphism in (4.32) is injective because  $\dim H^0(X, K_X) = 1$ . On the other hand, we have

$$\dim H^0(X, K_X(D)) = d = \dim H^0(X, K_X^{\otimes 2}(D)),$$

so the homomorphism in (4.32) is an isomorphism, in particular, it is surjective. This proves the theorem when  $g = 1$  and  $\dim_{\mathbb{C}} G \geq d$ .

Now assume that  $g \geq 2$  and  $\dim_{\mathbb{C}} G \geq d + 2$ .

Since  $\dim \mathfrak{g} \geq d + 2$ , we conclude that there is an element  $z = (X, D, E_G, \Phi) \in \mathbb{T}(G)$  such that (4.30) holds if there is a subspace  $V \subset H^0(X, K_X(D))$ , with  $\dim V \leq d + 2$ , for which the homomorphism  $F_V$  in (4.28) is injective. From Lemma 4.7 and Lemma 4.9 (see also Remark 4.10) it follows that such a subspace  $V$  exists. This completes the proof of the theorem.  $\square$

**Remark 4.6.** From Theorem 4.4 it follows that when  $g = 1$  and  $d = 0$ , the map  $\widehat{\theta}$  in (4.3) is an immersion over a nonempty open dense subset of  $\mathbb{T}(G)$ . Indeed, from Remark 4.2 we know that  $\mathbb{T}(G)$  for  $d = 0$  coincides with  $\mathbb{T}(G)$  for  $d = 1$ . On the other hand,  $\mathcal{R}_G(S'_0)$  for  $d = 0$  is embedded into  $\mathcal{R}_G(S'_0)$  for  $d = 1$ . From Theorem 4.4 we know that the map  $\widehat{\theta}$  in (4.3) is an immersion over a nonempty open dense subset of  $\mathbb{T}(G)$  if  $g = 1$  and  $d = 1$ . Therefore, the same holds when  $g = 1$  and  $d = 0$ . Recall that  $\dim G > 0$ .

In view of Remark 4.2, we assume that  $d > 1$  when  $g > 1$ .

**Lemma 4.7.** *Take integers  $g > 1$  and  $d > 1$ . Then for any compact connected non-hyperelliptic Riemann surface  $X$  of genus  $g \geq 3$ , and any effective divisor  $D$  on  $X$  of degree  $d$ , there exists a subspace  $W \subset H^0(X, K_X(D))$ , with  $\dim W = d + 2$ , such that the homomorphism constructed in (4.28)*

$$F_W : H^1(X, TX(-D)) \longrightarrow H^1(X, \mathcal{O}_X) \otimes W^*$$

is injective.

*Proof.* For a compact Riemann surface  $X$  of genus  $g$ , and an effective divisor  $D$  on  $X$  of degree  $d$ , denote the holomorphic line bundle  $K_X^{\otimes 2} \otimes \mathcal{O}_X(D)$  by  $K_X^2(D)$ . For any subspace  $V \subset H^0(X, K_X(D))$ , let

$$F_V^* : H^0(X, K_X) \otimes V \longrightarrow K^0(X, K_X^2(D))$$

be the dual of the homomorphism  $F_V$  in (4.28).

We need to show that there is a  $W$  with  $\dim W = d + 2$  such that the above homomorphism

$$F_W^* : H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2(D)) \tag{4.33}$$

is surjective.

Consider the natural homomorphism

$$J : H^0(X, K_X) \otimes H^0(X, K_X(D)) \longrightarrow H^0(X, K_X^2(D)). \tag{4.34}$$

We will now show that under our assumptions, the homomorphism  $J$  in (4.34) is surjective. To this end, we apply [Gr, Theorem (4.e.1)] and see that it suffices to prove that

$$h^1(X, \mathcal{O}_X(D)) \leq g - 2. \quad (4.35)$$

When  $D$  is non-special, (4.35) evidently holds. So we suppose that  $D$  is special. In order to prove (4.35), first assume that  $d \geq 4$ . Then Clifford's theorem (see [GH, p. 251]) says that  $h^0(X, \mathcal{O}_X(D)) \leq d/2 + 1$ . Now using Riemann-Roch theorem we get that

$$d + 1 - g = h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) \leq \frac{d}{2} + 1 - h^1(X, \mathcal{O}_X(D)).$$

This implies that (4.35) holds, and hence  $J$  is surjective in this case by [Gr, Theorem (4.e.1)].

Assume now that  $d = 2$  or  $d = 3$ . Since  $X$  is not hyperelliptic, if  $d = 2$ , then we have  $h^0(X, \mathcal{O}_X(D)) = 1$ . If  $d = 3$ , Clifford's theorem implies that  $h^0(X, \mathcal{O}_X(D)) \leq 2$ . Then the Riemann-Roch theorem implies that (4.35) holds in both these cases. Applying [Gr, Theorem (4.e.1)], we infer that  $J$  is surjective in these cases as well.

Consequently, we have obtained the surjectivity of the map  $J$  in (4.34) for any pair  $(X, D)$  as in the lemma.

From the commutative diagram

$$\begin{array}{ccc} H^0(X, K_X) \otimes H^0(X, K_X(D)) & \xrightarrow{J} & H^0(X, K_X^2(D)) \\ \downarrow & & \downarrow \\ H^0(X, K_X) \otimes H^0(X, K_X(D))/H^0(X, K_X) & \longrightarrow & H^0(X, K_X^2(D))/H^0(X, K_X^2) \end{array}$$

we notice that the surjectivity of  $J$  implies the surjectivity of the map

$$H^0(X, K_X) \otimes (H^0(X, K_X(D))/H^0(X, K_X)) \longrightarrow H^0(X, K_X^2(D))/H^0(X, K_X^2).$$

Consider  $U \subset H^0(X, K_X(D))$  of dimension  $(d - 1)$  such that  $U \cap H^0(X, K_X) = \{0\}$  inside  $H^0(X, K_X(D))$ . Then the map

$$U \longrightarrow H^0(X, K_X(D))/H^0(X, K_X)$$

is an isomorphism and hence the induced map

$$H^0(X, K_X) \otimes U \longrightarrow H^0(X, K_X^2(D))/H^0(X, K_X^2) \quad (4.36)$$

is surjective.

On the other hand, since  $X$  is non-hyperelliptic, [Gi, Theorem 1.1] (whose proof is attributed to Lazarsfeld) shows that for a general subspace  $W_0 \subset H^0(X, K_X)$  of dimension 3 the multiplication map

$$H^0(X, K_X) \otimes W_0 \longrightarrow H^0(X, K_X^2) \quad (4.37)$$

is surjective. Set

$$W = W_0 \oplus U \subset H^0(X, K_X(D)).$$

The surjectivity of the maps in (4.36) and (4.37) implies the surjectivity of

$$F^*W : H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2(D))$$

which concludes the proof.  $\square$

**Remark 4.8.** Lemma 4.7 is optimal in the following sense. If  $W \subset H^0(X, K_X(D))$  is a subspace such that the intersection  $W \cap H^0(X, K_X)$  inside  $H^0(X, K_X(D))$  is at least three-dimensional and  $H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2(D))$  is surjective, then  $\dim W \geq d+2$ . This claim is easily obtained by reverting the argument in the proof of Lemma 4.7.

Lemma 4.7 excluded the case of  $g = 2$ . This is dealt with separately below.

**Lemma 4.9.** *Let  $X$  be a compact connected Riemann surface of genus two, and let  $D$  be an effective divisor of degree  $d > 1$  such that  $D \notin |K_X|$ . Then the multiplication map*

$$H^0(X, K_X) \otimes H^0(X, K_X(D)) \longrightarrow H^0(X, K_X^2(D))$$

*is surjective.*

*Proof.* We start with the short exact sequence

$$0 \longrightarrow TX \longrightarrow H^0(X, K_X) \otimes \mathcal{O}_X \longrightarrow K_X \longrightarrow 0,$$

twist it by  $K_X(D)$  and take the corresponding long exact sequence of cohomologies

$$H^0(X, K_X) \otimes H^0(X, K_X(D)) \longrightarrow H^0(X, K_X^2(D)) \longrightarrow H^1(X, \mathcal{O}_X(D)) \longrightarrow \dots$$

By the hypothesis, we have  $H^1(X, \mathcal{O}_X(D)) = 0$  and hence from this exact sequence of cohomologies it follows that the multiplication map is surjective.  $\square$

**Remark 4.10.** Note that, under the hypotheses of Lemma 4.9, the Riemann-Roch theorem implies that  $h^0(X, K_X(D)) = d+1 < d+2$ .

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#### REFERENCES

- [At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.
- [BD] I. Biswas and S. Dumitrescu, The monodromy map from differential systems to character variety is generically immersive, [arxiv.org/abs/2002.05927](https://arxiv.org/abs/2002.05927).
- [BHH] I. Biswas, V. Heu and J. Hurtubise, Isomonodromic deformations of logarithmic connections and stability, *Math. Ann.* **366** (2016), 121–140.

- [BS] C. L. Bremer and D. S. Sage, Isomonodromic deformations of connections with singularities of parahoric formal type, *Comm. Math. Phys.* **313** (2012), 175–208.
- [CDHL] G. Calsamiglia, B. Deroin, V. Heu and F. Loray, The Riemann-Hilbert mapping for  $\mathfrak{sl}_2$  systems over genus two curves, *Bull. Soc. Math. France* **147** (2019), 159–195.
- [Ch1] T. Chen, The associated map of the nonabelian Gauss–Manin connection, Thesis (Ph.D.)–University of Pennsylvania (2012), <https://www.math.upenn.edu/grad/dissertations/ChenThesis.pdf>.
- [Ch2] T. Chen, The associated map of the nonabelian Gauss–Manin connection, *Cent. Eur. Jour. Math.* **10** (2012), 1407–1421.
- [De] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.
- [Do] I. F. Donin, Construction of a versal family of deformations for holomorphic bundles over a compact complex space, *Math. USSR Sb.* **23** (1974), 405–416.
- [Gi] D. Gieseker, A lattice version of the KP equation, *Acta Math.* **168** (1992), 219–248.
- [Go] W. M. Goldman, The symplectic nature of fundamental groups of surfaces, *Adv. in Math.* **54** (1984), 200–225.
- [Gr] M. L. Green, Koszul cohomology and geometry of projective varieties, *J. Differential Geometry* **19** (1984), 125–171.
- [GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.
- [Ha] G. Harder, Halbeinfache Gruppenschemata über vollständigen Kurven, *Invent. Math.* **6** (1968), 107–149.
- [Hu] L. Huang, On joint moduli spaces, *Math. Ann.* **302** (1995), 61–79.
- [In] M. Inaba, Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence, *Jour. Alg. Geom.* **22** (2013), 407–480.
- [Ka] N. M. Katz, An overview of Deligne’s work on Hilbert’s twenty-first problem, *Mathematical developments arising from Hilbert problems* (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), pp. 537–557. Amer. Math. Soc., Providence, R. I., 1976.
- [Ko] A. Komyo, Hamiltonian structures of isomonodromic deformations on moduli spaces of parabolic connections, *Jour. Math. Soc. Japan* (to appear), arXiv:1611.03601.
- [Ra] A. Ramanathan, Deformations of principal bundles on the projective line, *Invent. Math.* **71** (1983), 165–191.
- [Si] A. S. Sikora, Character varieties, *Trans. Amer. Math. Soc.* **364** (2012), 173–208.

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