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# Convex maps on $\mathbb{R}^n$ and positive definite matrices

Jean-Christophe Bourin and Jingjing Shao

**Abstract.** We obtain several convexity statements involving positive definite matrices. In particular, if  $A, B, X, Y$  are invertible matrices and  $A, B$  are positive, we show that the map

$$(s, t) \mapsto \text{Tr} \log (X^* A^s X + Y^* B^t Y)$$

is jointly convex on  $\mathbb{R}^2$ . This is related to some exotic matrix Hölder inequalities such as

$$\left\| \sinh \left( \sum_{i=1}^m A_i B_i \right) \right\| \leq \left\| \sinh \left( \sum_{i=1}^m A_i^p \right) \right\|^{1/p} \left\| \sinh \left( \sum_{i=1}^m B_i^q \right) \right\|^{1/q}$$

for all positives matrices  $A_i, B_i$ , such that  $A_i B_i = B_i A_i$ , conjugate exponents  $p, q$  and unitarily invariant norms  $\| \cdot \|$ . Our approach to obtain these results consists in studying the behaviour of some functionals along the geodesics of the Riemannian manifold of positive definite matrices.

*Keywords.* Matrix inequalities, Matrix geometric mean, Majorization, Positive linear maps.

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## 1 Convex and log-convex maps

This short note aims to point out some convex maps involving positive definite matrices. We denote by  $\mathbb{M}_n$  the space of  $n$ -by- $n$  matrices with complex entries, and by  $\mathbb{P}_n$  its positive definite cone. A non-negative, continuous function  $f(t)$  defined on  $[0, \infty)$  is geometrically convex if  $f(\sqrt{ab}) \leq \sqrt{f(a)f(b)}$  for all  $a, b > 0$ , equivalently if  $\log f(e^t)$  is convex on  $\mathbb{R}$ . Note that a function  $\varphi(t)$  on  $(0, \infty)$  satisfies the geometric-arithmetic convexity inequality

$$\varphi(\sqrt{ab}) \leq \frac{\varphi(a) + \varphi(b)}{2}, \quad a, b > 0,$$

if and only if  $e^{\varphi(t)}$  is geometrically convex, equivalently  $\varphi(e^t)$  is convex on  $\mathbb{R}$ . This convexity property can be extended to the matrix setting as follows.

**Theorem 1.1.** *Let  $\varphi(t)$  be a nondecreasing function defined on  $(0, \infty)$  such that  $\varphi(e^t)$  is convex. Let  $A_i \in \mathbb{P}_n$  and  $X_i \in \mathbb{M}_n$  be invertible,  $i = 1, \dots, m$ . Then, the map*

$$(t_1, \dots, t_m) \mapsto \text{Tr} \varphi \left( \sum_{i=1}^m X_i^* A_i^{t_i} X_i \right)$$

*is jointly convex on  $\mathbb{R}^m$ .*

Letting  $\varphi(t) = \log t$ , we get the statement of the Abstract. Theorem 1.1 can be derived from the following more general log-convexity theorem. Recall that a symmetric norm on  $\mathbb{M}_n$  satisfies  $\|UAV\| = \|A\|$  for all  $A \in \mathbb{M}_n$  and all unitary matrices  $U, V \in \mathbb{M}_n$ . We denote by  $\mathbb{M}_n^+$  the positive semi-definite cone of  $\mathbb{M}_n$ . A positive linear map  $\Phi : \mathbb{M}_n \mapsto \mathbb{M}_d$  satisfies  $\Phi(\mathbb{M}_n^+) \subset \mathbb{M}_d^+$ . A classical example is the Schur multiplier  $A \mapsto Z \circ A$  with  $Z \in \mathbb{M}_n^+$ .

**Theorem 1.2.** *Let  $A_i \in \mathbb{M}_n^+$  and  $X_i \in \mathbb{M}_n$ ,  $i = 1, \dots, m$ , and let  $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_d$  be a positive linear map. Then, for all symmetric norms and all non-decreasing geometrically convex function  $g(t)$ , the map*

$$(t_1, \dots, t_m) \mapsto \left\| g \left( \Phi \left( \sum_{i=1}^m X_i^* A_i^{t_i} X_i \right) \right) \right\|$$

*is jointly log-convex on  $\mathbb{R}^m$ .*

We will prove in the next section these two theorems. Here are some special cases of Theorem 1.2.

**Corollary 1.3.** *Let  $A, Z \in \mathbb{M}_n^+$ . Then, for all symmetric norms and all non-decreasing geometrically convex function  $g(t)$ ,*

$$\|g(Z \circ I)\|^2 \leq \|g(Z \circ A)\| \cdot \|g(Z \circ A^{-1})\|.$$

**Corollary 1.4.** *Let  $A_i \in \mathbb{M}_n^+$  and  $X_i \in \mathbb{M}_n$ ,  $i = 1, \dots, m$ . Then, for all symmetric norms and all non-decreasing geometrically convex function  $g(t)$ ,*

$$\left\| g \left( \sum_{i=1}^m X_i^* X_i \right) \right\|^2 \leq \left\| g \left( \sum_{i=1}^m X_i^* A_i X_i \right) \right\| \cdot \left\| g \left( \sum_{i=1}^m X_i^* A_i^{-1} X_i \right) \right\|.$$

**Corollary 1.5.** *Let  $A_i \in \mathbb{M}_n^+$  and  $\lambda_i > 0$ ,  $i = 1, \dots, m$ , such that  $\sum_{i=1}^m \lambda_i = 1$ . let  $p > 1$  and  $p^{-1} + q^{-1} = 1$ . Then, for all symmetric norms and all non-decreasing geometrically convex function  $g(t)$ ,*

$$\left\| g \left( \sum_{i=1}^m \lambda_i A_i \right) \right\| \leq \|g(I)\|^{1/q} \cdot \left\| g \left( \sum_{i=1}^m \lambda_i A_i^p \right) \right\|^{1/p}.$$

If  $f(t)$  and  $g(t)$  are geometrically convex then so are  $f(t) + g(t)$ ,  $\max\{f(t), g(t)\}$ ,  $f(t)g(t)$ ,  $e^{f(t)}$  and  $f^\alpha(t)$  for all  $\alpha > 0$ . Hence the above results may be applied to a large class of functions, for instance

$$g(t) = \sum_{k=1}^p c_k t^{\alpha_k}, \quad c_k > 0, \quad \alpha_k \geq 0$$

or

$$g(t) = \max\{c, \beta t^\alpha\}, \quad c, \alpha, \beta \geq 0.$$

Some interesting examples of geometrically convex (also called multiplicatively convex) functions defined on a sub-interval of the positive half-line are given in [3]. These functions can be used to obtain exotic matrix inequalities. A recent study [2] of a two variables log-convex functional have provided many classical and new matrix inequalities.

## 2 Geodesics and log-majorization

The space  $\mathbb{P}_n$  of  $n$ -by- $n$  positive definite matrices is a symmetric Riemannian manifold. There exists a unique geodesic joining two distinct points  $A, B \in \mathbb{P}_n$ , that can be parametrized as

$$t \mapsto A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \quad t \in (-\infty, \infty). \quad (2.1)$$

In particular, the middle point between  $A$  and  $B$  is  $A\#_{1/2}B$ , the geometric mean, often merely denoted as  $A\#B$ . For a general  $t$ , especially when  $t \in (0, 1)$ ,  $A\#_t B$  is a weighed geometric mean. We refer to [1] for a background on the geometric mean and  $\mathbb{P}_n$ .

Given  $S, T \in \mathbb{M}_n^+$ , the weak log-majorization relation  $S \prec_{w\log} T$  means that

$$\prod_{j=1}^k \lambda_j(S) \leq \prod_{j=1}^k \lambda_j(T)$$

for all  $k = 1, \dots, n$ , where  $\lambda_1(\cdot) \geq \dots \geq \lambda_n(\cdot)$  stand for the eigenvalues arranged in nonincreasing order. We denote by  $S^\downarrow$  the diagonal matrix with the eigenvalues  $\lambda_1(S), \dots, \lambda_n(S)$  down to the diagonal.

**Theorem 2.1.** *Let  $A_i, B_i \in \mathbb{P}_n$ ,  $i = 1, \dots, m$  and let  $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_d$  be a positive linear map. Then, for all symmetric norms and all non-decreasing geometrically convex function  $g(t)$ , the map*

$$(t_1, \dots, t_m) \mapsto \left\| g \left( \Phi \left( \sum_{i=1}^m A_i \#_{t_i} B_i \right) \right) \right\|$$

is jointly log-convex on  $\mathbb{R}^m$ .

*Proof.* Let  $A, B \in \mathbb{P}_n$  and let  $\Psi : \mathbb{M}_n \rightarrow \mathbb{M}_d$  be a positive linear map. We first prove the single variable case of the theorem by showing that the function

$$t \mapsto \|g(\Psi(A\#_t B))\| \quad (2.2)$$

is log convex on  $(-\infty, \infty)$ . From Ando's operator inequality

$$\Psi(A\#B) \leq \Psi(A)\#\Psi(B)$$

and the relation  $\Psi(A)\# \Psi(B) = \Psi(A)^{1/2}V\Psi(B)^{1/2}$  for some unitary  $V \in \mathbb{M}_d$ , we infer by Horn's inequality, the weak log-majorization

$$\Psi(A\#B) \prec_{w\log} \Psi(A)^{1/2\downarrow}\Psi(B)^{1/2\downarrow}$$

Since  $g(t)$  is geometrically convex, we have  $g(e^{(a+b)/2}) \leq \sqrt{g(e^a)g(e^b)} \leq (g(e^a)+g(e^b))/2$ . Hence  $t \mapsto g(e^t)$  is a nondecreasing convex function on  $(-\infty, \infty)$ . The above weak log-majorization then ensures that

$$g(\Psi(A\#B)) \prec_w g(\Psi(A)^{1/2\downarrow}\Psi(B)^{1/2\downarrow})$$

and using that  $g(t)$  is geometrically convex, we infer

$$g(\Psi(A\#B)) \prec_w g(\Psi(A))^{1/2\downarrow} g(\Psi(B))^{1/2\downarrow}.$$

This weak majorization says that

$$\|g(\Psi(A\#B))\| \leq \|g(\Psi(A))^{1/2\downarrow} g(\Psi(B))^{1/2\downarrow}\|$$

for all symmetric norms. The Cauchy-Schwarz inequality for symmetric norms yields

$$\|g(\Psi(A\#B))\| \leq \|g(\Psi(A))\|^{1/2} \|g(\Psi(B))\|^{1/2}.$$

Since  $A\#_{(s+t)/2}B = (A\#_sB)\#(A\#_tB)$ , we get

$$\|g(\Psi(A\#_{(s+t)/2}B))\| \leq \|g(\Psi(A\#_sB))\|^{1/2} \|g(\Psi(A\#_tB))\|^{1/2}, \quad (2.3)$$

for all  $s, t \in (-\infty, \infty)$ , thus (2.2) is a log-convex function.

We turn to the severable variables case. Let  $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_d$  be a positive linear map, and let  $A_i, B_i \in \mathbb{P}_n$ ,  $i = 1, \dots, m$ . Consider the two block diagonal matrices in  $\mathbb{M}_m(\mathbb{M}_n)$ ,

$$A = A_1\#_{s_1}B_1 \oplus \cdots \oplus A_m\#_{s_m}B_m, \quad B = A_1\#_{t_1}B_1 \oplus \cdots \oplus A_m\#_{t_m}B_m,$$

so that

$$A\#_{1/2}B = A_1\#_{\frac{s_1+t_1}{2}}B_1 \oplus \cdots \oplus A_m\#_{\frac{s_m+t_m}{2}}B_m.$$

Define the positive linear map  $\Psi : \mathbb{M}_m(\mathbb{M}_n) \rightarrow \mathbb{M}_n$ ,

$$\Psi([A_{i,j}]) := \Phi \left( \sum_{i=1}^m A_{i,i} \right).$$

From (2.3) with  $s = 0$ , and  $t = 1$ , we get

$$\left\| g \left( \Phi \left( \sum_{i=1}^m A_i\#_{\frac{s_i+t_i}{2}}B_i \right) \right) \right\| \leq \left\| g \left( \Phi \left( \sum_{i=1}^m A_i\#_{s_i}B_i \right) \right) \right\|^{1/2} \left\| g \left( \Phi \left( \sum_{i=1}^m A_i\#_{t_i}B_i \right) \right) \right\|^{1/2}$$

which completes the proof.  $\square$

**Corollary 2.2.** Let  $\varphi(t)$  be a nondecreasing function defined on  $(0, \infty)$ . Suppose that  $\exp \varphi(t)$  is geometrically convex and let  $A_i, B_i \in \mathbb{P}_n$ ,  $i = 1, \dots, m$ . Then, the map

$$(t_1, \dots, t_m) \mapsto \operatorname{Tr} \varphi \left( \sum_{i=1}^m A_i \#_{t_i} B_i \right)$$

is jointly convex on  $\mathbb{R}^m$ .

*Proof.* Let  $\varphi(t) = \log g(t)$ , where  $g(t)$  is geometrically convex. Since  $g^\alpha(t)$  is also geometrically convex for all  $\alpha > 0$ , Theorem 2.1 with the normalized trace norm shows that the map

$$(t_1, \dots, t_m) \mapsto \frac{1}{n} \operatorname{Tr} g^\alpha \left( \sum_{i=1}^m A_i \#_{t_i} B_i \right)$$

is jointly log-convex, and so is

$$(t_1, \dots, t_m) \mapsto \left\{ \frac{1}{n} \operatorname{Tr} g^\alpha \left( \sum_{i=1}^m A_i \#_{t_i} B_i \right) \right\}^{1/\alpha}.$$

Letting  $\alpha \searrow 0$ , we infer that the map

$$(t_1, \dots, t_m) \mapsto \det^{1/n} g \left( \sum_{i=1}^m A_i \#_{t_i} B_i \right)$$

is jointly log-convex. Thus the map

$$(t_1, \dots, t_m) \mapsto \log \det g \left( \sum_{i=1}^m A_i \#_{t_i} B_i \right) = \operatorname{Tr} \varphi \left( \sum_{i=1}^m A_i \#_{t_i} B_i \right)$$

is jointly convex. □

Theorem 2.1 can be regarded as a generalized Hölder inequality. This is more transparent for a single variable and pairs of commuting operators. Note that for two commuting positive definite matrices,  $A \#_t B = A^{1-t} B^t$ . Letting  $t = q^{-1}$  ( $= 0p^{-1} + 1q^{-1}$ ) and using Theorem 2.1 yields our next and last corollary.

**Corollary 2.3.** Let  $A_i, B_i \in \mathbb{M}_n^+$  such that  $A_i B_i = B_i A_i$ ,  $i = 1, \dots, m$ . Let  $p > 1$  and  $p^{-1} + q^{-1} = 1$ . Then, for all symmetric norms and all non-decreasing geometrically convex function  $g(t)$ ,

$$\left\| g \left( \sum_{i=1}^m A_i B_i \right) \right\| \leq \left\| g \left( \sum_{i=1}^m A_i^p \right) \right\|^{1/p} \cdot \left\| g \left( \sum_{i=1}^m B_i^q \right) \right\|^{1/q}.$$

Choosing  $g(t) = \sinh t$ , we recapture the Hölder inequality of the Abstract.

We close the paper by showing that Theorem 2.1 is equivalent to Theorem 1.2 (and similarly for Corollary 2.2 and Theorem 1.1). To this end, first note that by a limit argument we may assume that, in Theorem 1.2,  $X_i$  and  $A_i$  are invertible,  $i = 1, \dots, m$ . Then, using the polar decomposition  $X_i = U|X_i|$ , observe that

$$X_i^* A^{t_i} X_i = |X_i|(U^* A U)^{t_i} |X_i| = C \#_{t_i} D$$

with  $C = |X_i|^2$  and  $D = |X_i|U^* A U|X_i| = X_i^* A X_i$ .

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