Numerical range and positive block matrices

Jean-Christophe Bourin, Antoine Mhanna

Numerical range and positive block matrices

Jean-Christophe Bourin* and Eun-Young Lee†

Abstract. We obtain several norm and eigenvalue inequalities for positive matrices partitioned into four blocks. The results involve the numerical range $W(X)$ of the off-diagonal block X, especially the distance d from 0 to $W(X)$. A special consequence is an estimate,

$$\operatorname{diam} W\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) - \operatorname{diam} W\left(\frac{A+B}{2}\right) \geq 2d,$$

between the diameters of the numerical ranges for the full matrix and its partial trace.

Keywords. Numerical range, Partitioned matrices, norm inequalities.

2010 mathematics subject classification. 15A60, 47A12, 47A30, 15A42.

1 The width of the numerical range

Let $\mathbb{M}_n$ denote the space of n-by-n complex matrices, and let $\langle u, v\rangle = u^*v$ be the canonical inner product of $\mathbb{C}^n$, linear in the second variable. The numerical range of $X \in \mathbb{M}_n$ is defined as

$$W(X) = \{\langle h, Xh\rangle \ : \ \|h\| = 1\}.$$

The Hausdorff-Toeplitz theorem states that $W(X)$ is a compact convex set containing the spectrum of X. In case of a normal matrix, the numerical range is precisely the convex hull of the spectrum. The symbol $\|\cdot\|$ will also denote any symmetric norm on $\mathbb{M}_{2n}$. Such a norm is also called a unitarily invariant norm. It satisfies the unitary invariance property $\|UTV\| = \|T\|$ for all $T \in \mathbb{M}_{2n}$ and all unitary matrices $U, V \in \mathbb{M}_{2n}$, and it induces a symmetric norm on $\mathbb{M}_n$ in an obvious way, by considering $\mathbb{M}_n$ as the upper left corner of $\mathbb{M}_{2n}$ completed with some zero entries.

A positive matrix means a Hermitian positive semi-definite matrix. It has been pointed out [7] that the width of $W(X)$ contributes to an estimate of the norm of a partitioned positive matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$. In Matrix Analysis, positive matrices partitioned into four blocks are a fundamental tool and these matrices are also of basic importance in applications, especially in Quantum Information Theory. The main theorem of [7] reads as follows.

*Funded by the ANR Projet (No. ANR-19-CE40-0002) and by the French Investissements d'Avenir program, project ISITE-BFC (contract ANR-15-IDEX-03).

†This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A3B07043682)

Theorem 1.1. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$. *Suppose that* $W(X)$ *has the width* ω. *Then, for all symmetric norms,*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq \|A + B + \omega I\|.$$

Here I stands for the identity matrix and the width of $W(X)$ is the smallest distance between two parallel straight lines such that the strip between these two lines contains $W(X)$. Hence the partial trace $A + B$ may be used to give an upper bound for the norms of the full block-matrix. This note will provide a lower bound, stated in Section 2, and several consequences.

Theorem 1.1 is the first inequality involving the width of the numerical range; classical results rather deal with the numerical radius, $w(X) = \max\{|z| : z \in W(X)\}$. Our new lower bound will also have an unusual feature as it involves the distance from 0 to the numerical range, $\operatorname{dist}(0, W(X)) = \min\{|z| : z \in W(X)\}$. For a background on the numerical range we refer to [12], where the term of Field of values is used. Some very interesting inequalities for the numerical radius can be found in [11], [13], and in the recent article [8].

In case of Hermitian off-diagonal blocks, Theorem 1.1 holds with $w = 0$. More generally, if $X = aI + bH$ for some $a, b \in \mathbb{C}$ and some Hermitian matrix H, we have $\omega = 0$ as $W(X)$ is a line segment. This special case of the theorem was first shown by Mhanna [14]. In particular, if the off-diagonal blocks are normal two-by-two matrices, then we can take $\omega = 0$. This does not hold any longer for three-by-three normal matrices, a detailed study of this phenomenon is given in [10] and [9].

For Hermitian off-diagonal blocks, a stronger statement than Theorem 1.1 with $w = 0$ holds. The following decomposition was shown in [6, Theorem 2.2].

Theorem 1.2. *Let* $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ *be a positive matrix partitioned into four Hermitian blocks in* $\mathbb{M}_n$. *Then, for some pair of unitary matrices* $U, V \in \mathbb{M}_{2n}$,

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} = \frac{1}{2} \left\{ U \begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & A+B \end{bmatrix} V^* \right\}.$$

For decompositions of positive matrices partitioned into a larger number of blocks, see [5]. We close this section by recalling some facts on symmetric norms, classical text books such as [2], [12] and [15] are good references.

A symmetric norm on $\mathbb{M}_n$, can be defined by its restriction to the positive cone $\mathbb{M}_n^+$. Symmetric norms on $\mathbb{M}_n^+$ are characterized by three properties:

(i) $\|\lambda A\| = \lambda \|A\|$ for all $A \in \mathbb{M}_n^+$ and all $\lambda \geq 0$,

(ii) $\|UAU^*\|$ for all $A \in \mathbb{M}_n^+$ and all unitaries $U \in \mathbb{M}_n$,

(iii) $\|A\| \leq \|A + B\| \leq \|A\| + \|B\|$ for all A, $B \in \mathbb{M}_n^+$.

Let $\lambda_1^{\downarrow}(A) \geq \cdots \geq \lambda_n^{\downarrow}(A)$ stand for the eigenvalues of $A \in \mathbb{M}_n^+$ arranged in non-increasing order. Then, the Ky Fan k-norms,

$$\|A\|_{(k)} = \sum_{j=1}^{k} \lambda_j^{\downarrow}(A)$$

are symmetric norms, $k = 1, \ldots, n$. Thus $\|A\|_{(1)}$ is the operator norm, usually denoted by $\|A\|_\infty$ while $\|A\|_{(n)}$ is the trace norm, usually written $\|A\|_1$. For $A, B \in \mathbb{M}_n^+$, the following conditions are equivalent:

(a) $\|A\|_{(k)} \leq \|B\|_{(k)}$ for all $k = 1, \ldots, n$,

(b) $\|A\| \leq \|B\|$ for all symmetric norms,

(c) The vector of the eigenvalues of A is dominated by a convex combination of permutations of the vector of the eigenvalues of B, equivalently,

$$A \leq \sum_{i=1}^{n+1} \alpha_i U_i B U_i^*$$

for some unitary matrices U_i and some scalars $\alpha_i \geq 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$.

When these conditions hold (especially when explicitly stated as (a)) one says that A is weakly majorized by B and one writes $A \prec_w B$. If furthemore in (a) one has the equality $\|A\|_{(n)} = \|B\|_{(n)}$, that it is A and B have the same trace, then A is majorized by B, written $A \prec B$. Thus $A \prec B$ means that (c) holds with the equality sign: A is in the convex hull of the unitary orbit of B. Theorem 1.2 is a special majorization.

A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ is called doubly stochastic if Φ preserves positivity, identity, and trace. For all $A \in \mathbb{M}_n^+$, we then have $\Phi(A) \prec A$, see the last section of Ando's survey [1].

2 The distance from 0 to the numerical range

We state our main result and infer several corollaries. The proof of the theorem is postponed to Section 3.

Theorem 2.1. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \operatorname{dist}(0, W(X))$. *Then, for all symmetric norms,*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \geq \left\| \left(\frac{A+B}{2} + dI \right) \oplus \left(\frac{A+B}{2} - dI \right) \right\|.$$

Here, the direct sum is a standard notation for block-diagonal matrices

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}.$$

Since we have equality for the trace, Theorem 2.1 is a majorization relation. We have $(A+B)/2 \geq dI$, otherwise, the trace norm of the left-hand side would be strictly smaller than the right-hand side one, a contradiction with the theorem.

By a basic principle of majorization, Theorem 2.1 is equivalent to some trace inequalities.

Corollary 2.2. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \operatorname{dist}(0, W(X))$. *Then, for every convex function* $g : [0,\infty) \to (-\infty,\infty)$,

$$\operatorname{Tr} g\left(\frac{A+B}{2} + dI\right) + \operatorname{Tr} g\left(\frac{A+B}{2} - dI\right) \leq \operatorname{Tr} g\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right).$$

Symmetric norms $\|\cdot\|$ on $\mathbb{M}_n^+$ are the homogeneous, unitarily invariant, convex functionals. The concave counterpart, the symmetric anti-norms $\|\cdot\|_!$, have been introduced and studied in papers [3] and [4, Section 4]. We recall their basic properties, parallel to those of symmetric norms given at the end of Section 1. Symmetric anti-norms on $\mathbb{M}_n^+$ are continuous functionals characterized by three properties:

(i) $\|\lambda A\|_! = \lambda \|A\|_!$ for all $A \in \mathbb{M}_n^+$ and all $\lambda \geq 0$,

(ii) $\|UAU^*\|_!$ for all $A \in \mathbb{M}_n^+$ and all unitaries $U \in \mathbb{M}_n$,

(iii) $\|A+B\|_! \geq \|A\|_! + \|B\|_!$ for all A, $B \in \mathbb{M}_n^+$.

Let $\lambda_1^\uparrow(A) \leq \cdots \leq \lambda_n^\uparrow(A)$ stand for the eigenvalues of $A \in \mathbb{M}_n^+$ arranged in non-decreasing order. Then, the Ky Fan k-anti-norms,

$$\|A\|_{(k)!} = \sum_{j=1}^{k} \lambda_j^\uparrow(A)$$

are symmetric anti-norms, $k = 1, \ldots, n$. The following conditions are equivalent:

(a) $\|A\|_{(k)!} \geq \|B\|_{(k)!}$ for all $k = 1, \ldots, n$,

(b) $\|A\|_! \geq \|B\|_!$ for all symmetric anti-norms,

(c) The vector of the eigenvalues of A dominates some convex combination of permutations of the vector of the eigenvalues of B, equivalently,

$$A \geq \sum_{i=1}^{n+1} \alpha_i U_i B U_i^*$$

for some unitary matrices U_i and some scalars $\alpha_i \geq 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$.

The continuity assumption is not essential, but deleting it would lead to rather strange functionals which are not continuous on the boundary of $\mathbb{M}_n^+$, such as $\|A\|_! := \operatorname{Tr} A$ if A is invertible and $\|A\|_! := 0$ if A is not invertible.

Note that the trace norm is both a symmetric norm and a symmetric anti-norm and that the majorization $A \prec B$ in $\mathbb{M}_n^+$ also entails that $\|A\|_! \geq \|B\|_!$ for all symmetric anti-norms. Thus Theorem 2.1 is equivalent to the following statement:

Corollary 2.3. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$*, let* $d = \operatorname{dist}(0, W(X))$*. Then, for all symmetric anti-norms,*

$$\left\| \left(\frac{A+B}{2} + dI \right) \oplus \left(\frac{A+B}{2} - dI \right) \right\|_! \geq \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_!.$$

Corollary 2.4. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \operatorname{dist}(0, W(X))$*. Then,*

$$\lambda_1^{\downarrow} \left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \lambda_1^{\downarrow} \left(\frac{A+B}{2} \right) \geq d$$

and

$$\lambda_1^{\uparrow} \left(\frac{A+B}{2} \right) - \lambda_1^{\uparrow} \left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \geq d.$$

Proof. The first inequality follows from Therorem 2.1 applied to the symmetric norm $A \mapsto \lambda_1^{\downarrow}(A)$ (the operator norm on the positive cone), while the second inequality follows from Corollary 2.3 applied to the anti-norm $A \mapsto \lambda_1^{\uparrow}(A)$ □

By adding these two inequalities we get an estimate for the spread of the matrices, i.e., for the diameter of the numerical ranges.

Corollary 2.5. *For every positive matrix partitioned into four blocks of same size,*

$$\operatorname{diam} W \left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \operatorname{diam} W \left(\frac{A+B}{2} \right) \geq 2d,$$

where d *is the distance from* 0 *to* $W(X)$*.*

Of course

$$\operatorname{diam} W \left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \geq \operatorname{diam} W \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) \geq \operatorname{diam} W \left(\frac{A+B}{2} \right),$$

however the ratio

$$\rho = \frac{1}{2d} \left\{ \operatorname{diam} W \left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \operatorname{diam} W \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) \right\}$$

can be arbitrarily small as shown by the following example where the blocks are in $\mathbb{M}_2$,

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \end{bmatrix},$$

and by noting that ρ then takes the value α^{-1} which tends to 0 as $\alpha \to \infty$.

The Minkowki inequality for positive m-by-m matrices,

$$\det{}^{1/m}(A+B) \geq \det{}^{1/m}(A) + \det{}^{1/m}(B),$$

shows that the functional $A \mapsto \det^{1/m}(A)$ is a symmetric anti-norm on $\mathbb{M}_m^+$. For this anti-norm Theorem 2.1 reads as:

Corollary 2.6. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$ *and let* $d = \operatorname{dist}(0, W(X))$. *Then,*

$$\det\left\{\left(\frac{A+B}{2}\right)^2 - d^2 I\right\} \geq \det\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right).$$

Letting $X = 0$, we recapture a basic property: the determinant is a log-concave map on the positive cone of $\mathbb{M}_n$. Hence Corollary 2.6 refines this property.

By a basic principle of majorization, Corollary 2.3 is equivalent to the following seemingly more general statement.

Corollary 2.7. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$, *let* $d = \operatorname{dist}(0, W(X))$, *and let* $f(t)$ *be a nonnegative concave function on* $[0, \infty)$. *Then,*

$$\left\| f\left(\frac{A+B}{2} + dI\right) \oplus f\left(\frac{A+B}{2} - dI\right) \right\|_! \geq \left\| f\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) \right\|_!$$

for all symmetric anti-norms.

3 Proof of Theorem 2.1

We want to show the majorization in $\mathbb{M}_{2n}^+$

$$\begin{bmatrix} \frac{A+B}{2} + dI & 0 \\ 0 & \frac{A+B}{2} - dI \end{bmatrix} \prec \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \tag{3.1}$$

where $d = \operatorname{dist}(0, W(X)$. We use two lemmas, the first one might belong to folklore.

Lemma 3.1. *Let* $\{A_k\}_{k=1}^m$ *and* $\{B_k\}_{k=1}^m$ *be two families of* r*-by-*r *positive matrices such that* $A_k \prec B_k$ *for each* k. *Then,*

$$\oplus_{k=1}^m A_k \prec \oplus_{k=1}^m B_k.$$

Proof. Let p_k denote any integer such that $0 \leq p_k \leq m$, $k = 1, \ldots, m$. With this notation, we then have, for each integer $p = 1, \ldots, mr$,

$$\begin{aligned}\sum_{j=1}^{p} \lambda_j^{\downarrow}(\oplus_{k=1}^{m} A_k) &= \max_{p_1+p_2+\cdots+p_m=p} \sum_{k=1}^{m} \sum_{j=1}^{p_k} \lambda_j^{\downarrow}(A_k) \\ &\leq \max_{p_1+p_2+\cdots+p_m=p} \sum_{k=1}^{m} \sum_{j=1}^{p_k} \lambda_j^{\downarrow}(B_k) \\ &= \sum_{j=1}^{p} \lambda_j^{\downarrow}(\oplus_{k=1}^{m} B_k)\end{aligned}$$

with equality for $p = mr$. □

Lemma 3.2. *Let $X, Y \in \mathbb{M}_n^+$ and let $\delta > 0$ be such that $X \geq Y \geq \delta I$. Then,*

$$\begin{bmatrix} X + \delta I & 0 \\ 0 & X - \delta I \end{bmatrix} \prec \begin{bmatrix} X + Y & 0 \\ 0 & X - Y \end{bmatrix}.$$

Proof. Let $\{e_k\}_{k=1}^n$ be an orthonormal basis of $\mathbb{C}^n$ and define two n-by-n diagonal positive matrices

$$D_+ = \operatorname{diag}(\langle e_1, (X+Y)e_1\rangle, \ldots, \langle e_n, (X+Y)e_n\rangle)$$

and

$$D_- = \operatorname{diag}(\langle e_1, (X-Y)e_1\rangle, \ldots, \langle e_n, (X-Y)e_n\rangle).$$

Since extracting a diagonal is a doubly stochastic map (a pinching), we have

$$\begin{bmatrix} D_+ & 0 \\ 0 & D_- \end{bmatrix} \prec \begin{bmatrix} X + Y & 0 \\ 0 & X - Y \end{bmatrix}. \tag{3.2}$$

Now, choose the basis $\{e_k\}_{k=1}^n$ as a basis of eigenvectors for X, $\lambda_k^{\downarrow}(X) = \langle e_k, Xe_k\rangle$, and observe that the majorization in $\mathbb{M}_2^+$,

$$\begin{pmatrix} \lambda_k^{\downarrow}(X) + \delta & 0 \\ 0 & \lambda_k^{\downarrow}(X) - \delta \end{pmatrix} \prec \begin{pmatrix} \langle e_k, (X+Y)e_k\rangle & 0 \\ 0 & \langle e_k, (X-Y)e_k\rangle \end{pmatrix},$$

holds for every k. Applying Lemma 3.1 then shows that

$$\bigoplus_{k=1}^{n} \begin{pmatrix} \lambda_k^{\downarrow}(X) + \delta & 0 \\ 0 & \lambda_k^{\downarrow}(X) - \delta \end{pmatrix} \prec \bigoplus_{k=1}^{n} \begin{pmatrix} \langle e_k, (X+Y)e_k\rangle & 0 \\ 0 & \langle e_k, (X-Y)e_k\rangle \end{pmatrix}.$$

This means that

$$\begin{bmatrix} X + \delta I & 0 \\ 0 & X - \delta I \end{bmatrix} \prec \begin{bmatrix} D_+ & 0 \\ 0 & D_- \end{bmatrix}$$

and we may combine this majorization with (3.2) to complete the proof. □

We turn to the proof of (3.1).

Proof. Suppose first that $d = 0$, that is $0 \in W(X)$. Note that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \prec \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \tag{3.3}$$

as the operation of taking the block diagonal is doubly stochastic.

Using the unitary congruence with

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \tag{3.4}$$

we observe that

$$J \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} J^* = \begin{bmatrix} \frac{A+B}{2} & \frac{A-B}{2} \\ \frac{A-B}{2} & \frac{A+B}{2} \end{bmatrix}$$

Hence we have

$$\begin{bmatrix} \frac{A+B}{2} & 0 \\ 0 & \frac{A+B}{2} \end{bmatrix} \prec \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and combining with (3.3) establishes (3.1) for the case $d = 0$.

Now assume that $d > 0$, that is $0 \notin W(X)$. Using the unitary congruence implemented by

$$\begin{bmatrix} I & 0 \\ 0 & e^{-i\theta} I \end{bmatrix}$$

we may replace the right hand side of (3.1) with

$$\begin{bmatrix} A & e^{i\theta} X \\ e^{-i\theta} X^* & B \end{bmatrix}$$

Thanks to the rotation property $W(e^{i\theta} X) = e^{i\theta} W(X)$, by choosing the adequate θ, we may then and do assume that $W(X)$ lies the half-plane of $\mathbb{C}$ consiting of complex numbers with real parts greater or equal than d,

$$W(X) \subset \{ z = x + iy \ : \ x \geq d \}.$$

The projection property for the real part of the numerical range, $\operatorname{Re} W(X) = W(\operatorname{Re} X)$ with $\operatorname{Re} X = (X + X^*)/2$, then ensures that

$$\operatorname{Re} X \geq dI.$$

Now, using again a unitary congruence with (3.4), wet get

$$J \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} J^* = \begin{bmatrix} \frac{A+B}{2} - \operatorname{Re} X & * \\ * & \frac{A+B}{2} + \operatorname{Re} X \end{bmatrix}$$

where $*$ stands for unspecified entries. Hence

$$\begin{bmatrix} \frac{A+B}{2} - \operatorname{Re} X & 0 \\ 0 & \frac{A+B}{2} + \operatorname{Re} X \end{bmatrix} \prec \begin{bmatrix} A & X \\ X^* & B \end{bmatrix},$$

equivalently,

$$\begin{bmatrix} \frac{A+B}{2} + \operatorname{Re} X & 0 \\ 0 & \frac{A+B}{2} - \operatorname{Re} X \end{bmatrix} \prec \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

and applying Lemma 3.2 then yields (3.1). □

References

[1] T. Ando, Majorization, doubly stochastic matrices, and comparison of eigenvalues, *Linear Algebra Appl.* 118 (1989), 163–248.

[2] R. Bhatia, Matrix Analysis, Gradutate Texts in Mathematics, Springer, New-York, 1996.

[3] J.-C. Bourin and F. Hiai, Norm and anti-norm inequalities for positive semi-definite matrices, *Internat. J. Math.* 22 (2011), 1121–1138.

[4] J.-C. Bourin and F. Hiai, Jensen and Minkowski inequalities for operator means and anti-norms, *Linear Algebra Appl.* 456 (2014), 22–53.

[5] J.-C. Bourin and E.-Y. Lee, Decomposition and partial trace of positive matrices with Hermitian blocks, Int. J. Math. 24 (2013) 1350010.

[6] J.-C. Bourin, E.-Y. Lee and M. Lin, Positive matrices partitioned into a small number of Hermitian blocks,*Linear Algebra Appl.* 438 (2013) 2591–2598.

[7] J.-C. Bourin and A. Mhanna, Positive block matrices and numerical ranges, *C. R. Acad. Sci. Paris*, 355 no. 10 (2017) 1077–1081.

[8] B. E. Cain, Improved inequalities for the numerical radius: when inverse commutes with the norm. Bull. Aust. Math. Soc. 97 (2018), no. 2, 293–296.

[9] M. Gumus, J. Liu, S. Raouafi, T.-Y. Tam, Positive semi-definite 2×2 block matrices and norm inequalities, *Linear Algebra Appl.* 551 (2018) 83–91.

[10] T. Hayashi, On a norm inequality for a positive block-matrix, Linear Algebra Appl. 566 (2019), 86–97.

[11] J.A.R. Holbrook, Multiplicative properties of the numerical radius in operator theory, J. Reine Angew. Math. 237 (1969), 166–174.

[12] R. Horn and C.R. Johnson Topics in matrix analysis. Corrected reprint of the 1991 original. Cambridge University Press, Cambridge, 1994.

[13] F. Kittaneh, Numerical radius inequalities for Hilbert space operators. Studia Math. 168 (2005), no. 1, 73–80.

[14] A. Mhanna, On symmetric norm inequalities and positive definite block-matrices, *Math. Ineq. Appl.*, 21 (2018), no. 1, 133–138.

[15] B. Simon, Trace ideals and their applications. Second edition. Mathematical Surveys and Monographs, 120, American Mathematical Society, Providence, RI, 2005.

Jean-Christophe Bourin
Laboratoire de mathématiques de Besançon, UMR n^o 6623, CNRS,
Université de Bourgogne Franche-Comté
Email: jcbourin@univ-fcomte.fr

Eun-Young Lee
Department of mathematics, KNU-Center for Nonlinear Dynamics,
Kyungpook National University,
Daegu 702-701, Korea.
Email: eylee89@knu.ac.kr