Numerical range and positive block matrices
Jean-Christophe Bourin, Eun-Young Lee

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Abstract. We obtain several norm and eigenvalue inequalities for positive matrices partitioned into four blocks. The results involve the numerical range $W(X)$ of the off-diagonal block $X$, especially the distance $d$ from 0 to $W(X)$. A special consequence is an estimate,

$$\text{diam} W \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \text{diam} W \left( \frac{A + B}{2} \right) \geq 2d,$$

between the diameters of the numerical ranges for the full matrix and its partial trace.

Keywords. Numerical range, Partitioned matrices, norm inequalities.

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1 The width of the numerical range

Let $\mathbb{M}_n$ denote the space of $n$-by-$n$ complex matrices, and let $\langle u, v \rangle = u^*v$ be the canonical inner product of $\mathbb{C}^n$, linear in the second variable. The numerical range of $X \in \mathbb{M}_n$ is defined as

$$W(X) = \{ \langle h, Xh \rangle : \|h\| = 1 \}.$$ 

The Hausdorff-Toeplitz theorem states that $W(X)$ is a compact convex set containing the spectrum of $X$. In case of a normal matrix, the numerical range is precisely the convex hull of the spectrum. The symbol $\| \cdot \|$ will also denote any symmetric norm on $\mathbb{M}_{2n}$. Such a norm is also called a unitarily invariant norm. It satisfies the unitary invariance property $\|UTV\| = \|T\|$ for all $T \in \mathbb{M}_{2n}$ and all unitary matrices $U, V \in \mathbb{M}_{2n}$, and it induces a symmetric norm on $\mathbb{M}_n$ in an obvious way, by considering $\mathbb{M}_n$ as the upper left corner of $\mathbb{M}_{2n}$ completed with some zero entries.

A positive matrix means a Hermitian positive semi-definite matrix. It has been pointed out [7] that the width of $W(X)$ contributes to an estimate of the norm of a partitioned positive matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$. In Matrix Analysis, positive matrices partitioned into four blocks are a fundamental tool and these matrices are also of basic importance in applications, especially in Quantum Information Theory. The main theorem of [7] reads as follows.

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Theorem 1.1. Let \( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \) be a positive matrix partitioned into four blocks in \( \mathbb{M}_n \). Suppose that \( W(X) \) has the width \( \omega \). Then, for all symmetric norms,

\[
\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq \| A + B + \omega I \|
\]

Here \( I \) stands for the identity matrix and the width of \( W(X) \) is the smallest distance between two parallel straight lines such that the strip between these two lines contains \( W(X) \). Hence the partial trace \( A + B \) may be used to give an upper bound for the norms of the full block-matrix. This note will provide a lower bound, stated in Section 2, and several consequences.

Theorem 1.1 is the first inequality involving the width of the numerical range; classical results rather deal with the numerical radius, \( w(X) = \max\{|z| : z \in W(X)\} \). Our new lower bound will also have an unusual feature as it involves the distance from 0 to the numerical range, \( \text{dist}(0, W(X)) = \min\{|z| : z \in W(X)\} \). For a background on the numerical range we refer to [12], where the term of Field of values is used. Some very interesting inequalities for the numerical radius can be found in [11], [13], and in the recent article [8].

In case of Hermitian off-diagonal blocks, Theorem 1.1 holds with \( w = 0 \). More generally, if \( X = aI + bH \) for some \( a, b \in \mathbb{C} \) and some Hermitian matrix \( H \), we have \( \omega = 0 \) as \( W(X) \) is a line segment. This special case of the theorem was first shown by Mhanna [14]. In particular, if the off-diagonal blocks are normal two-by-two matrices, then we can take \( \omega = 0 \). This does not hold any longer for three-by-three normal matrices, a detailed study of this phenomenon is given in [10] and [9].

For Hermitian off-diagonal blocks, a stronger statement than Theorem 1.1 with \( w = 0 \) holds. The following decomposition was shown in [6, Theorem 2.2].

Theorem 1.2. Let \( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \) be a positive matrix partitioned into four Hermitian blocks in \( \mathbb{M}_n \). Then, for some pair of unitary matrices \( U, V \in \mathbb{M}_{2n} \),

\[
\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = \frac{1}{2} \left\{ U \begin{bmatrix} A + B & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & A + B \end{bmatrix} V^* \right\}.
\]

For decompositions of positive matrices partitioned into a larger number of blocks, see [5]. We close this section by recalling some facts on symmetric norms, classical textbooks such as [2], [12] and [15] are good references.

A symmetric norm on \( \mathbb{M}_n \), can be defined by its restriction to the positive cone \( \mathbb{M}_n^+ \). Symmetric norms on \( \mathbb{M}_n^+ \) are characterized by three properties:

(i) \( \| \lambda A \| = \lambda \| A \| \) for all \( A \in \mathbb{M}_n^+ \) and all \( \lambda \geq 0 \),

(ii) \( \| UAU^* \| \) for all \( A \in \mathbb{M}_n^+ \) and all unitaries \( U \in \mathbb{M}_n \),

(iii) \( \| A \| \leq \| A + B \| \leq \| A \| + \| B \| \) for all \( A, B \in \mathbb{M}_n^+ \).
Let $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ stand for the eigenvalues of $A \in M_n^+$ arranged in non-increasing order. Then, the Ky Fan $k$-norms,

$$\|A\|_{(k)} = \sum_{j=1}^k \lambda_j(A)$$

are symmetric norms, $k = 1, \ldots, n$. Thus $\|A\|_{(1)}$ is the operator norm, usually denoted by $\|A\|_\infty$ while $\|A\|_{(n)}$ is the trace norm, usually written $\|A\|_1$. For $A, B \in M_n^+$, the following conditions are equivalent:

(a) $\|A\|_{(k)} \leq \|B\|_{(k)}$ for all $k = 1, \ldots, n$,

(b) $\|A\| \leq \|B\|$ for all symmetric norms,

(c) The vector of the eigenvalues of $A$ is dominated by a convex combination of permutations of the vector of the eigenvalues of $B$, equivalently,

$$A \leq \sum_{i=1}^{n+1} \alpha_i U_i B U_i^*$$

for some unitary matrices $U_i$ and some scalars $\alpha_i \geq 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$.

When these conditions hold (especially when explicitly stated as (a)) one says that $A$ is weakly majorized by $B$ and one writes $A \prec_w B$. If furthermore in (a) one has the equality $\|A\|_{(n)} = \|B\|_{(n)}$, that is $A$ and $B$ have the same trace, then $A$ is majorized by $B$, written $A \prec B$. Thus $A \prec B$ means that (c) holds with the equality sign: $A$ is in the convex hull of the unitary orbit of $B$. Theorem 1.2 is a special majorization.

A linear map $\Phi : M_n \to M_n$ is called doubly stochastic if $\Phi$ preserves positivity, identity, and trace. For all $A \in M_n^+$, we then have $\Phi(A) \prec A$, see the last section of Ando's survey [1].

2 The distance from 0 to the numerical range

We state our main result and infer several corollaries. The proof of the theorem is postponed to Section 3.

**Theorem 2.1.** Let $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ be a positive matrix partitioned into four blocks in $M_n$ and let $d = \text{dist}(0, W(X))$. Then, for all symmetric norms,

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \geq \left\| \left( \frac{A + B}{2} + dI \right) \oplus \left( \frac{A + B}{2} - dI \right) \right\|.$$

Here, the direct sum is a standard notation for block-diagonal matrices

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}.$$
Since we have equality for the trace, Theorem 2.1 is a majorization relation. We have \((A + B)/2 \geq dI\), otherwise, the trace norm of the left-hand side would be strictly smaller than the right-hand side one, a contradiction with the theorem.

By a basic principle of majorization, Theorem 2.1 is equivalent to some trace inequalities.

**Corollary 2.2.** Let \(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\) be a positive matrix partitioned into four blocks in \(\mathbb{M}_n\) and let \(d = \text{dist}(0, W(X))\). Then, for every convex function \(g : [0, \infty) \rightarrow (-\infty, \infty)\),

\[
\text{Tr} \ g \left( \frac{A + B}{2} + dI \right) + \text{Tr} \ g \left( \frac{A + B}{2} - dI \right) \leq \text{Tr} \ g \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) .
\]

Symmetric norms \(\| \cdot \|\) on \(\mathbb{M}_n^+\) are the homogeneous, unitarily invariant, convex functionals. The concave counterpart, the symmetric anti-norms \(\| \cdot \|!\), have been introduced and studied in papers [3] and [4, Section 4]. We recall their basic properties, parallel to those of symmetric norms given at the end of Section 1. Symmetric anti-norms on \(\mathbb{M}_n^+\) are continuous functionals characterized by three properties:

(i) \(\| \lambda A \|! = \lambda \| A \|!\) for all \(A \in \mathbb{M}_n^+\) and all \(\lambda \geq 0\),

(ii) \(\| UAU^* \|!\) for all \(A \in \mathbb{M}_n^+\) and all unitaries \(U \in \mathbb{M}_n\),

(iii) \(\| A + B \|! \geq \| A \|! + \| B \|!\) for all \(A, B \in \mathbb{M}_n^+\).

Let \(\lambda_1^n(A) \leq \cdots \leq \lambda_n^n(A)\) stand for the eigenvalues of \(A \in \mathbb{M}_n^+\) arranged in non-decreasing order. Then, the Ky Fan \(k\)-anti-norms,

\[
\| A \|_{(k)}! = \sum_{j=1}^{k} \lambda_j^n(A)
\]

are symmetric anti-norms, \(k = 1, \ldots, n\). The following conditions are equivalent:

(a) \(\| A \|_{(k)}! \geq \| B \|_{(k)}!\) for all \(k = 1, \ldots, n\),

(b) \(\| A \|! \geq \| B \|!\) for all symmetric anti-norms,

(c) The vector of the eigenvalues of \(A\) dominates some convex combination of permutations of the vector of the eigenvalues of \(B\), equivalently,

\[
A \geq \sum_{i=1}^{n+1} \alpha_i U_i B U_i^*
\]

for some unitary matrices \(U_i\) and some scalars \(\alpha_i \geq 0\) such that \(\sum_{i=1}^{n+1} \alpha_i = 1\).
The continuity assumption is not essential, but deleting it would lead to rather strange functionals which are not continuous on the boundary of $\mathbb{M}_n^+$, such as $\|A\|_! := \text{Tr} A$ if $A$ is invertible and $\|A\|_! := 0$ if $A$ is not invertible.

Note that the trace norm is both a symmetric norm and a symmetric anti-norm and that the majorization $A \prec B$ in $\mathbb{M}_n^+$ also entails that $\|A\|_! \geq \|B\|_!$ for all symmetric anti-norms. Thus Theorem 2.1 is equivalent to the following statement:

**Corollary 2.3.** Let $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ be a positive matrix partitioned into four blocks in $\mathbb{M}_n$, let $d = \text{dist}(0, W(X))$. Then, for all symmetric anti-norms,

$$\left\| \left( \frac{A+B}{2} + dI \right) \oplus \left( \frac{A+B}{2} - dI \right) \right\|_! \geq \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_! .$$

**Corollary 2.4.** Let $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ be a positive matrix partitioned into four blocks in $\mathbb{M}_n$ and let $d = \text{dist}(0, W(X))$. Then,

$$\lambda_1^+ \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \lambda_1^+ \left( \frac{A+B}{2} \right) \geq d$$

and

$$\lambda_1^+ \left( \frac{A+B}{2} \right) - \lambda_1^+ \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \geq d.$$

**Proof.** The first inequality follows from Theorem 2.1 applied to the symmetric norm $A \mapsto \lambda_1^+(A)$ (the operator norm on the positive cone), while the second inequality follows from Corollary 2.3 applied to the anti-norm $A \mapsto \lambda_1^+(A)$.

By adding these two inequalities we get an estimate for the spread of the matrices, i.e., for the diameter of the numerical ranges.

**Corollary 2.5.** For every positive matrix partitioned into four blocks of same size,

$$\text{diam} W \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \text{diam} W \left( \frac{A+B}{2} \right) \geq 2d,$$

where $d$ is the distance from 0 to $W(X)$.

Of course

$$\text{diam} W \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \geq \text{diam} W \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) \geq \text{diam} W \left( \frac{A+B}{2} \right),$$

however the ratio

$$\rho = \frac{1}{2d} \left\{ \text{diam} W \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \text{diam} W \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) \right\}$$
can be arbitrarily small as shown by the following example where the blocks are in $\mathbb{M}_2$,

\[
\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} = \begin{bmatrix}
(\alpha & 0 \\
0 & \alpha^{-1}
\end{bmatrix} \begin{bmatrix}
(1 & 0) \\
0 & 1
\end{bmatrix} \begin{bmatrix}
(\alpha^{-1} & 0) \\
0 & \alpha
\end{bmatrix},
\]

and by noting that $\rho$ then takes the value $\alpha^{-1}$ which tends to 0 as $\alpha \to \infty$.

The Minkowki inequality for positive $m$-by-$m$ matrices,

\[\det^{1/m}(A + B) \geq \det^{1/m}(A) + \det^{1/m}(B),\]

shows that the functional $A \mapsto \det^{1/m}(A)$ is a symmetric anti-norm on $\mathbb{M}_m^+$. For this anti-norm Theorem 2.1 reads as:

**Corollary 2.6.** Let \[\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}\] be a positive matrix partitioned into four blocks in $\mathbb{M}_n$, and let $d = \text{dist}(0, W(X))$. Then,

\[\det \left\{ \left( \frac{A + B}{2} \right)^2 - d^2 I \right\} \geq \det \left( \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} \right).\]

Letting $X = 0$, we recapture a basic property: the determinant is a log-concave map on the positive cone of $\mathbb{M}_n$. Hence Corollary 2.6 refines this property.

By a basic principle of majorization, Corollary 2.3 is equivalent to the following seemingly more general statement.

**Corollary 2.7.** Let \[\begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}\] be a positive matrix partitioned into four blocks in $\mathbb{M}_n$, let $d = \text{dist}(0, W(X))$, and let $f(t)$ be a nonnegative concave function on $[0, \infty)$. Then,

\[\| f \left( \frac{A + B}{2} + dI \right) \oplus f \left( \frac{A + B}{2} - dI \right) \|_1 \geq \| f \left( \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} \right) \|_1\]

for all symmetric anti-norms.

### 3 Proof of Theorem 2.1

We want to show the majorization in $\mathbb{M}_{2n}^+$

\[
\begin{bmatrix}
\frac{A + B}{2} + dI & 0 \\
0 & \frac{A + B}{2} - dI
\end{bmatrix} \preceq \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}
\]

(3.1)

where $d = \text{dist}(0, W(X))$. We use two lemmas, the first one might belong to folklore.

**Lemma 3.1.** Let \(\{A_k\}_{k=1}^m\) and \(\{B_k\}_{k=1}^m\) be two families of $r$-by-$r$ positive matrices such that $A_k \prec B_k$ for each $k$. Then,

\[\oplus_{k=1}^m A_k \prec \oplus_{k=1}^m B_k.\]
Proof. Let \( p_k \) denote any integer such that \( 0 \leq p_k \leq m \), \( k = 1, \ldots, m \). With this notation, we then have, for each integer \( p = 1, \ldots, mr \),

\[
\sum_{j=1}^{p} \lambda_j^+ \left( \bigoplus_{k=1}^{m} A_k \right) = \max_{p_1 + p_2 + \cdots + p_m = p} \sum_{j=1}^{m} \sum_{k=1}^{p_k} \lambda_j^+ (A_k)
\leq \max_{p_1 + p_2 + \cdots + p_m = p} \sum_{j=1}^{m} \sum_{k=1}^{p_k} \lambda_j^+ (B_k)
= \sum_{j=1}^{p} \lambda_j^+ \left( \bigoplus_{k=1}^{m} B_k \right)
\]

with equality for \( p = mr \).

Lemma 3.2. Let \( X, Y \in \mathbb{M}_n^+ \) and let \( \delta > 0 \) be such that \( X \geq Y \geq \delta I \). Then,

\[
\begin{bmatrix}
X + \delta I & 0 \\
0 & X - \delta I
\end{bmatrix} \prec \begin{bmatrix}
X + Y & 0 \\
0 & X - Y
\end{bmatrix}.
\]

Proof. Let \( \{e_k\}_{k=1}^{n} \) be an orthonormal basis of \( \mathbb{C}^n \) and define two \( n \)-by-\( n \) diagonal positive matrices

\[
D_+ = \text{diag}(\langle e_1, (X + Y)e_1 \rangle, \ldots, \langle e_n, (X + Y)e_n \rangle)
\]

and

\[
D_- = \text{diag}(\langle e_1, (X - Y)e_1 \rangle, \ldots, \langle e_n, (X - Y)e_n \rangle).
\]

Since extracting a diagonal is a doubly stochastic map (a pinching), we have

\[
\begin{bmatrix}
D_+ & 0 \\
0 & D_-
\end{bmatrix} \prec \begin{bmatrix}
X + Y & 0 \\
0 & X - Y
\end{bmatrix}.
\]

This means that

\[
\begin{bmatrix}
X + \delta I & 0 \\
0 & X - \delta I
\end{bmatrix} \prec \begin{bmatrix}
D_+ & 0 \\
0 & D_-
\end{bmatrix}
\]

and we may combine this majorization with (3.2) to complete the proof. \qed
We turn to the proof of (3.1).

Proof. Suppose first that \( d = 0 \), that is \( 0 \in W(X) \). Note that

\[
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix} \prec \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}
\]

(3.3)
as the operation of taking the block diagonal is doubly stochastic.

Using the unitary congruence with

\[
J = \frac{1}{\sqrt{2}} \begin{bmatrix}
I & -I \\
I & I
\end{bmatrix}
\]

(3.4)
we observe that

\[
J \begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix} J^* = \begin{bmatrix}
\frac{A+B}{2} & \frac{A-B}{2} \\
\frac{A-B}{2} & \frac{A+B}{2}
\end{bmatrix}
\]

Hence we have

\[
\begin{bmatrix}
\frac{A+B}{2} & 0 \\
0 & \frac{A+B}{2}
\end{bmatrix} \prec \begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\]

and combining with (3.3) establishes (3.1) for the case \( d = 0 \).

Now assume that \( d > 0 \), that is \( 0 \not\in W(X) \). Using the unitary congruence implemented by

\[
\begin{bmatrix}
I & 0 \\
0 & e^{-i\theta} I
\end{bmatrix}
\]

we may replace the right hand side of (3.1) with

\[
\begin{bmatrix}
A & e^{i\theta} X \\
e^{-i\theta} X^* & B
\end{bmatrix}
\]

Thanks to the rotation property \( W(e^{i\theta} X) = e^{i\theta} W(X) \), by choosing the adequate \( \theta \), we may then and do assume that \( W(X) \) lies the half-plane of \( \mathbb{C} \) consisting of complex numbers with real parts greater or equal than \( d \),

\[
W(X) \subset \{ z = x + iy : x \geq d \}.
\]

The projection property for the real part of the numerical range, \( \text{Re} W(X) = W(\text{Re} X) \) with \( \text{Re} X = (X + X^*)/2 \), then ensures that

\[
\text{Re} X \geq d I.
\]

Now, using again a unitary congruence with (3.4), we get

\[
J \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix} J^* = \begin{bmatrix}
\frac{A+B}{2} - \text{Re} X & * \\
* & \frac{A+B}{2} + \text{Re} X
\end{bmatrix}
\]

where * stands for unspecified entries. Hence

\[
\begin{bmatrix}
\frac{A+B}{2} - \text{Re} X & 0 \\
0 & \frac{A+B}{2} + \text{Re} X
\end{bmatrix} \prec \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}
\]
equivalently,

\[
\begin{bmatrix}
\frac{A+B}{2} + \text{Re} X & 0 \\
0 & \frac{A+B}{2} - \text{Re} X
\end{bmatrix} \prec \begin{bmatrix}
A & X \\
X^* & B
\end{bmatrix}
\]

and applying Lemma 3.2 then yields (3.1). \( \square \)
References


Jean-Christophe Bourin
Laboratoire de mathématiques de Besançon, UMR n° 6623, CNRS,
Université de Bourgogne Franche-Comté
Email: jcbourin@univ-fcomte.fr
Eun-Young Lee
Department of mathematics, KNU-Center for Nonlinear Dynamics,
Kyungpook National University,
Daegu 702-701, Korea.
Email: eylee89@knu.ac.kr