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Clarkson-McCarthy inequalities with unitary and isometry orbits

Jean-Christophe Bourin* and Eun-Young Lee†

Abstract. A refinement of a trace inequality of McCarthy establishing the uniform convexity of the Schatten p -classes for $p > 2$ is proved: if A, B are two n -by- n matrices, then there exists some pair of n -by- n unitary matrices U, V such that

$$U \left| \frac{A+B}{2} \right|^p U^* + V \left| \frac{A-B}{2} \right|^p V^* \leq \frac{|A|^p + |B|^p}{2}.$$

A similar statement holds for compact Hilbert space operators. Another improvement of McCarthy's inequality is given via the new operator parallelogram law,

$$|A+B|^2 \oplus |A-B|^2 = U_0(|A|^2 + |B|^2)U_0^* + V_0(|A|^2 + |B|^2)V_0^*$$

for some pair of $2n$ -by- n isometry matrices U_0, V_0 .

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1 Introduction

Let \mathbb{M}_n denote the space of complex $n \times n$ matrices and let \mathbb{M}_n^+ stand for the positive (semi-definite) cone. In [8] McCarthy established a Clarkson type inequality for the Schatten p -norms $\|\cdot\|_p$ and $A, B \in \mathbb{M}_n$,

$$2(\|A\|_p^p + \|B\|_p^p) \leq \|A+B\|_p^p + \|A-B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p), \quad p \geq 2. \quad (1.1)$$

For $0 < p \leq 2$, these inequalities are reversed. In the commutative (or scalar) case, these inequalities are almost trivial. Note that the left hand side inequality is equivalent to the right hand side one by $A = X + Y$, $B = X - Y$. Those inequalities for $p \geq 2$ show that the unit ball for the Schatten p -norm is uniformly convex, this is more readily apparent when written as

$$\left\| \frac{A+B}{2} \right\|_p^p + \left\| \frac{A-B}{2} \right\|_p^p \leq \frac{\|A\|_p^p + \|B\|_p^p}{2}, \quad p \geq 2, \quad (1.2)$$

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i.e.,

$$\mathrm{Tr} \left| \frac{A+B}{2} \right|^p + \mathrm{Tr} \left| \frac{A-B}{2} \right|^p \leq \frac{\mathrm{Tr} |A|^p + \mathrm{Tr} |B|^p}{2}, \quad p \geq 2. \quad (1.3)$$

Thus, if $\|A\|_p = \|B\|_p = 1$ and $\|A - B\|_p = \varepsilon$,

$$\left\| \frac{A+B}{2} \right\|_p \leq (1 - (\varepsilon/2)^p)^{1/p}, \quad p \geq 2,$$

which estimates the uniform convexity modulus of the Schatten p -classes for $p \geq 2$. There also exists a Clarkson type inequality showing the uniform convexity of the Schatten p -classes in case of $1 < p < 2$. This case is not as simple as the case $p > 2$ and a Three Lines Theorem argument is required. It seems that no real analytic proof are known (the original proof given by McCarthy collapses, see [6], p. 297)

The Clarkson-McCarthy inequalities (1.1) have been nicely extended to a large class of unitarily invariant norms by Bhatia and Holbrook [1]. Recall that a unitarily invariant norm on \mathbb{M}_n , also called a symmetric norm, satisfies $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. Another remarkable generalization due to Hirzallah and Kittaneh [7] states that

$$2 \| |A|^p + |B|^p \| \leq \| |A+B|^p + |A-B|^p \| \leq 2^{p-1} \| |A|^p + |B|^p \|, \quad p \geq 2,$$

for all symmetric norms, and these inequalities are reversed for $0 < p \leq 2$. This double inequality can be rewritten in a similar form as (1.2) and, thanks to Fan's dominance principle, this can be subsumed as the weak majorization

$$\sum_{j=1}^k \lambda_j^\downarrow \left(\left| \frac{A+B}{2} \right|^p + \left| \frac{A-B}{2} \right|^p \right) \leq \sum_{j=1}^k \lambda_j^\downarrow \left(\frac{|A|^p + |B|^p}{2} \right), \quad k = 1, 2, \dots \quad (1.4)$$

where $\lambda_1^\downarrow(X) \geq \lambda_2^\downarrow(X) \geq \dots \geq \lambda_n^\downarrow(X)$ stand for the eigenvalues of $X \in \mathbb{M}_n^+$. For $k = n$, we recapture (1.3), hence (1.4) is a considerable improvement of (1.2).

We will show in the next two sections some refinements of (1.2). These results yields some eigenvalue estimates which complete (1.4). The last section show how to modify the statements of Section 2 when dealing with operators on an infinite dimensional Hilbert space.

2 Unitary orbits

The operator parallelogram law does hold, for every pair $A, B \in \mathbb{M}_n$,

$$|A+B|^2 + |A-B|^2 = 2(|A|^2 + |B|^2), \quad (2.1)$$

equivalently,

$$\left| \frac{A+B}{2} \right|^2 + \left| \frac{A-B}{2} \right|^2 = \frac{|A|^2 + |B|^2}{2}.$$

We will state two theorems which show for $p \neq 2$ how this equality is modified as operator inequalities involving unitary orbits. Taking traces we recapture (1.1)-(1.2).

Theorem 2.1. *Let $A, B \in \mathbb{M}_n$ and $p > 2$. Then there exists two unitarie $U, V \in \mathbb{M}_n$ such that*

$$U \left| \frac{A+B}{2} \right|^p U^* + V \left| \frac{A-B}{2} \right|^p V^* \leq \frac{|A|^p + |B|^p}{2}.$$

Proof. Note that

$$\left| \frac{A+B}{2} \right|^p = \left(\frac{|A|^2 + |B|^2 + A^*B + B^*A}{4} \right)^{p/2}$$

and

$$\left| \frac{A-B}{2} \right|^p = \left(\frac{|A|^2 + |B|^2 - (A^*B + B^*A)}{4} \right)^{p/2}.$$

Now, recall [5, Corollary 3.2]: Given two positive matrices X, Y and a monotone convex function $g(t)$ defined on $[0, \infty)$ such that $g(0) \leq 0$, we have

$$g(X+Y) \geq U_0 g(X) U_0^* + V_0 g(Y) V_0^* \quad (2.2)$$

for some pair of unitary matrices U_0 and V_0 . Applying this to $g(t) = t^{p/2}$,

$$X = \frac{|A|^2 + |B|^2 + A^*B + B^*A}{4}$$

and

$$Y = \frac{|A|^2 + |B|^2 - (A^*B + B^*A)}{4},$$

we obtain

$$\left(\frac{|A|^2 + |B|^2}{2} \right)^{p/2} \geq U_0 \left| \frac{A+B}{2} \right|^p U_0^* + V_0 \left| \frac{A-B}{2} \right|^p V_0^*. \quad (2.3)$$

Next, recall [5, Corollary 2.2]: Given two positive matrices X, Y and a monotone convex function $g(t)$ defined on $[0, \infty)$, we have

$$\frac{g(X) + g(Y)}{2} \geq W g \left(\frac{X+Y}{2} \right) W^* \quad (2.4)$$

for some unitary matrix W . Applying this to $g(t) = t^{p/2}$, $X = |A|^2$ and $Y = |B|^2$, we get

$$\frac{|A|^p + |B|^p}{2} \geq W \left(\frac{|A|^2 + |B|^2}{2} \right)^{p/2} W^*. \quad (2.5)$$

Combining (2.3) and (2.5) completes the proof with $U = WU_0$ and $V = WV_0$. \square

Corollary 2.2. *Let $A, B \in \mathbb{M}_n$ and $p > 2$. Then, for all $k = 1, 2, \dots, n$,*

$$\sum_{j=1}^k \lambda_j^\uparrow \left(\frac{|A|^p + |B|^p}{2} \right) \geq \sum_{j=1}^k \lambda_j^\uparrow \left(\left| \frac{A+B}{2} \right|^p \right) + \sum_{j=1}^k \lambda_j^\uparrow \left(\left| \frac{A-B}{2} \right|^p \right).$$

Corollary 2.3. *Let $A, B \in \mathbb{M}_n$ and $p > 2$. Then, for all $k = 1, 2, \dots, n$,*

$$\left\{ \prod_{j=1}^k \lambda_j^\uparrow \left(\frac{|A|^p + |B|^p}{2} \right) \right\}^{1/k} \geq \left\{ \prod_{j=1}^k \lambda_j^\uparrow \left(\left| \frac{A+B}{2} \right|^p \right) \right\}^{1/k} + \left\{ \prod_{j=1}^k \lambda_j^\uparrow \left(\left| \frac{A-B}{2} \right|^p \right) \right\}^{1/k}.$$

Here $\lambda_1^\uparrow(X) \leq \lambda_2^\uparrow(X) \leq \dots \leq \lambda_n^\uparrow(X)$ stand for the eigenvalues of $X \in \mathbb{M}_n^+$ arranged in the nondecreasing order. These two corollaries follow from the theorem and the fact that the functionals on \mathbb{M}_n^+

$$X \mapsto \sum_{j=1}^k \lambda_j^\uparrow(X)$$

and

$$X \mapsto \left\{ \prod_{j=1}^k \lambda_j^\uparrow(X) \right\}^{1/k}$$

are two basic examples of symmetric anti-norms, see [3].

The next corollary follows from Theorem 2.1 combined with a classical inequality of Weyl for the eigenvalues of the sum of two Hermitian matrices.

Corollary 2.4. *Let $A, B \in \mathbb{M}_n$ and $p > 2$. Then, for all $j, k \in \{0, \dots, n-1\}$ such that $j+k+1 \leq n$,*

$$\lambda_{j+1}^\downarrow \left(\frac{|A|^p + |B|^p}{2} \right) \geq \lambda_{j+k+1}^\downarrow \left(\left| \frac{A+B}{2} \right|^p \right) + \lambda_{k+1}^\uparrow \left(\left| \frac{A-B}{2} \right|^p \right).$$

For a monotone concave function $g(t)$ defined on $[0, \infty)$ such that $g(0) \geq 0$, the inequalities (2.2) and (2.4) are reversed. Applying this to $g(t) = t^{q/2}$, $2 > q > 0$, the same proof than that of Theorem 2.1 gives the following statement.

Theorem 2.5. *Let $A, B \in \mathbb{M}_n$ and $2 > q > 0$. Then, for some unitaries $U, V \in \mathbb{M}_n$,*

$$U \left| \frac{A+B}{2} \right|^q U^* + V \left| \frac{A-B}{2} \right|^q V^* \geq \frac{|A|^q + |B|^q}{2}.$$

By using Weyl's inequality, this theorem yields an interesting eigenvalue estimate.

Corollary 2.6. *Let $A, B \in \mathbb{M}_n$ and $2 > q > 0$. Then, for all $j, k \in \{0, \dots, n-1\}$ such that $j+k+1 \leq n$,*

$$\lambda_{j+k+1}^\downarrow \left(\frac{|A|^q + |B|^q}{2} \right) \leq \lambda_{j+1}^\downarrow \left(\left| \frac{A+B}{2} \right|^q \right) + \lambda_{k+1}^\downarrow \left(\left| \frac{A-B}{2} \right|^q \right).$$

3 Another parallelogram law

We shall point out a new operator parallelogram law where the usual sum in (2.1) is replaced by a direct sum. A matrix $V \in \mathbb{M}_{m,n}$, the space of $m \times n$ matrices, is called an *isometry* whenever V^*V is the identity of \mathbb{M}_n .

Theorem 3.1. *Let $A, B \in \mathbb{M}_n$. Then, for some isometries $U, V \in \mathbb{M}_{2n,n}$,*

$$|A + B|^2 \oplus |A - B|^2 = U(|A|^2 + |B|^2)U^* + V(|A|^2 + |B|^2)V^*$$

Proof. Note that

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

is unitarily equivalent to

$$\begin{pmatrix} A + B & 0 \\ 0 & A - B \end{pmatrix}$$

via the unitary congruence implemented by

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

Thus $|A + B|^2 \oplus |A - B|^2$ is unitarily equivalent to

$$\left| \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right|^2 = \begin{pmatrix} |A|^2 + |B|^2 & A^*B + B^*A \\ A^*B + B^*A & |A|^2 + |B|^2 \end{pmatrix}. \quad (3.1)$$

Now, recall [5, Lemma 3.4]: Given any positive matrix partitioned in four n -by- n blocks, we can find two unitary matrices $U, V \in \mathbb{M}_{2n}$ such that

$$\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} = U \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U^* + V_0 \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix} V_0^*.$$

Applying this to (3.1) we then have two unitary matrices $U, V \in \mathbb{M}_{2n}$ such that

$$|A + B|^2 \oplus |A - B|^2 = U \begin{pmatrix} |A|^2 + |B|^2 & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & |A|^2 + |B|^2 \end{pmatrix} V^*$$

which is a statement equivalent to our theorem. □

The next corollary is a matrix version of the identity $|z|^2 = x^2 + y^2$ for complex numbers $z = x + iy$. It follows from Theorem 3.1 applied to $A = X$ and $B = iY$.

Corollary 3.2. *Let $Z \in \mathbb{M}_n$ with Cartesian decomposition $Z = X + iY$. Then, for some isometries $U, V \in \mathbb{M}_{2n,n}$,*

$$|Z|^2 \oplus |Z|^2 = U(X^2 + Y^2)U^* + V(X^2 + Y^2)V^*.$$

From this and the reverse form of (2.2) for $g(t) = \sqrt{t}$ we see that the scalar identity $|z| = \sqrt{x^2 + y^2}$ is transformed in the matrix setting as an operator inequality.

Corollary 3.3. *Let $Z \in \mathbb{M}_n$ with Cartesian decomposition $Z = X + iY$. Then, for some isometries $U, V \in \mathbb{M}_{2n,n}$,*

$$|Z| \oplus |Z| \leq U\sqrt{X^2 + Y^2}U^* + V\sqrt{X^2 + Y^2}V^*.$$

By applying twice the inequality (2.2) to the convex function $g(t) = t^{p/2}$, our parallelogram law yields the next corollary.

Corollary 3.4. *Let $A, B \in \mathbb{M}_n$ and $p > 2$. Then, for some isometries $U_0, V_0, U_1, V_1 \in \mathbb{M}_{2n,n}$,*

$$|A + B|^p \oplus |A - B|^p \geq U_0|A|^pU_0^* + V_0|B|^pV_0^* + U_1|A|^pU_1^* + V_1|B|^pV_1^*.$$

If one uses the concave function $g(t) = t^{q/2}$, we obtain a reversed inequality.

Corollary 3.5. *Let $A, B \in \mathbb{M}_n$ and $2 > q > 0$. Then, for some isometries $U_0, V_0, U_1, V_1 \in \mathbb{M}_{2n,n}$,*

$$|A + B|^q \oplus |A - B|^q \leq U_0|A|^qU_0^* + V_0|B|^qV_0^* + U_1|A|^qU_1^* + V_1|B|^qV_1^*.$$

Our parallelogram law yields another extension of the Clarkson-McCarthy trace inequality (1.2).

Theorem 3.6. *Let $A, B \in \mathbb{M}_n$ and $p > 2$. Then, for some isometries $U, V \in \mathbb{M}_{2n,n}$,*

$$\left| \frac{A+B}{2} \right|^p \oplus \left| \frac{A-B}{2} \right|^p \leq \frac{1}{2} \left\{ U \frac{|A|^p + |B|^p}{2} U^* + V \frac{|A|^p + |B|^p}{2} V^* \right\}.$$

Proof. Theorem 3.1 says that

$$\left| \frac{A+B}{2} \right|^2 \oplus \left| \frac{A-B}{2} \right|^2 = \frac{1}{2} \left\{ U \frac{|A|^2 + |B|^2}{2} U^* + V \frac{|A|^2 + |B|^2}{2} V^* \right\}$$

and applying twice inequality (2.4) with the convex function $g(t) = t^{p/2}$ completes the proof. \square

The simplest form of Weyl's inequality, $\lambda_{2j+1}^\downarrow(S+T) \leq \lambda_{j+1}^\downarrow(S) + \lambda_{j+1}^\downarrow(T)$ for $S, T \in \mathbb{M}_n^+$, and Theorem 3.6 provide one more Clarkson-McCarthy type eigenvalue estimate.

Corollary 3.7. *Let $A, B \in \mathbb{M}_n$ and $p > 2$. Then, for all $j \in \{0, \dots, n-1\}$,*

$$\lambda_{2j+1}^\downarrow \left(\left| \frac{A+B}{2} \right|^p \oplus \left| \frac{A-B}{2} \right|^p \right) \leq \lambda_{j+1}^\downarrow \left(\frac{|A|^p + |B|^p}{2} \right).$$

The reverse form of Theorem 3.6 occurs for $2 > q > 0$.

Theorem 3.8. *Let $A, B \in \mathbb{M}_n$ and $2 > q > 0$. Then, for some isometries $U, V \in \mathbb{M}_{2n,n}$,*

$$\left| \frac{A+B}{2} \right|^q \oplus \left| \frac{A-B}{2} \right|^q \geq \frac{1}{2} \left\{ U \frac{|A|^q + |B|^q}{2} U^* + V \frac{|A|^q + |B|^q}{2} V^* \right\}.$$

4 Infinite dimension

The Schatten classes are usually defined as classes of compact operators on an infinite dimensional, separable Hilbert space. In this setting, a slight modification of the previous theorems is necessary. The correct statements for compact operators require to use partial isometries rather than unitary operators.

Let $g(t)$ be an increasing continuous function defined on $[0, \infty)$ such that $g(0) > 0$. Then the functional calculus $g : \mathbb{K}^+ \rightarrow \mathbb{K}^+$, $X \mapsto g(X)$ is norm continuous. Thus, if X_n , $n \geq 1$, is a sequence of finite rank operators in \mathbb{K}^+ converging to X , then we have

$$\lambda_j^\downarrow(X_n) \rightarrow \lambda_j^\downarrow(X), \quad j = 1, 2, \dots$$

and

$$\lambda_j^\downarrow(g(X_n)) \rightarrow \lambda_j^\downarrow(g(X)), \quad j = 1, 2, \dots$$

as $n \rightarrow \infty$. Now let $X, Y \in \mathbb{K}^+$ and pick two sequences of finite rank positive operators X_n and Y_n such that $X_n \rightarrow X$ and $Y_n \rightarrow Y$. Since we deal with finite rank operators, if $g(t)$ is convex, (2.4) yields

$$\frac{g(X_n) + g(Y_n)}{2} \geq Wg\left(\frac{X_n + Y_n}{2}\right)W^*$$

for some unitary operator W . This says that

$$\lambda_j^\downarrow\left(\frac{g(X_n) + g(Y_n)}{2}\right) \geq \lambda_j^\downarrow\left(g\left(\frac{X_n + Y_n}{2}\right)\right), \quad j = 1, 2, \dots,$$

and, letting $n \rightarrow \infty$, we get

$$\lambda_j^\downarrow\left(\frac{g(X) + g(Y)}{2}\right) \geq \lambda_j^\downarrow\left(g\left(\frac{X + Y}{2}\right)\right), \quad j = 1, 2, \dots,$$

which exactly says that

$$\frac{g(X) + g(Y)}{2} \geq Wg\left(\frac{X + Y}{2}\right)W^*$$

for some partial isometry W such that $\text{supp}(W) = \text{supp}(X + Y)$.

Therefore, for compact operators, Equation (2.4) of the proof of Theorem 2.1 still holds with a partial isometry W such that $\text{supp}(W) = \text{supp}(X + Y)$. A similar statement holds for Equation (2.2): Given $X, Y \in \mathbb{K}^+$ and an increasing, strictly convex function $g(t)$ such that $g(0) > 0$, it is shown in [2, Theorem 2.1(i)] that

$$g(X + Y) \geq U_0g(X)U_0^* + V_0g(Y)V_0^*$$

for some partial isometries U_0, V_0 such that $\text{supp}(U_0) = \text{supp}(X)$ and $\text{supp}(V_0) = \text{supp}(Y)$. Hence, we may brought the proof of Theorem 2.1 to the setting of compact operators and we obtain the following refinement of the Clarkson-McCarthy inequalities for Schatten p -classes, $p > 2$.

Theorem 4.1. *Let $A, B \in \mathbb{K}$ and $p > 2$. Then there exist two partial isometries $U, V \in \mathbb{B}$ such that $\text{supp}(U) = \text{supp}(A + B)$, $\text{supp}(V) = \text{supp}(A - B)$, and*

$$U \left| \frac{A+B}{2} \right|^p U^* + V \left| \frac{A-B}{2} \right|^p V^* \leq \frac{|A|^p + |B|^p}{2}.$$

We may also give some versions of Theorems 2.1 and 2.5 for measurable operators affiliated to a semi-finite von Neumann algebra (we refer to [6] for a background on measurable operators). We close the paper by stating such a result for a type II_1 factor. We do not give details, we just mention that the result follows from [4, Lemma 3.3] and [4, Theorem 3.2].

We denote by $\overline{\mathcal{N}}$ the space of measurable operators affiliated to a type II_1 factor \mathcal{N} .

Theorem 4.2. *Let $A, B \in \overline{\mathcal{N}}$ and $2 > q > 0$. Then, for every $\varepsilon > 0$, there exist two unitaries $U, V \in \mathcal{N}$ such that*

$$U \left| \frac{A+B}{2} \right|^q U^* + V \left| \frac{A-B}{2} \right|^q V^* \geq \frac{|A|^q + |B|^q}{2} - \varepsilon I.$$

For $q > 2$, the reverse inequality holds, with an εI term instead of $-\varepsilon I$.

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