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Matrix inequalities from a two variables functional

Jean-Christophe Bourin and Eun-Young Lee*

Abstract. We introduce a two variables norm functional and establish its joint log-convexity. This entails and improves many remarkable matrix inequalities, most of them related to the log-majorization theorem of Araki. In particular: *if A is a positive semidefinite matrix and N is a normal matrix, $p \geq 1$ and Φ is a sub-unital positive linear map, then $|A\Phi(N)A|^p$ is weakly log-majorized by $A^p\Phi(|N|^p)A^p$.* This far extension of Araki's theorem (when Φ is the identity and N is positive) complements some recent results of Hiai and contains several special interesting cases such as a triangle inequality for normal operators and some extensions of the Golden-Thompson trace inequality. Some applications to Schur products are also obtained.

Keywords. Matrix inequalities, Majorization, Positive linear maps, Schur products.

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1 Log-majorization and log-convexity

Matrices are regarded as non-commutative extensions of scalars and functions. Since matrices do not commute in general, most scalars identities cannot be brought to the matrix setting, however they sometimes have a matrix version, which is not longer an identity but an inequality. These kind of inequalities are of fundamental importance in our understanding of the noncommutative world of matrices. A famous, fifty years old example of such an inequality is the Golden-Thompson trace inequality: for Hermitian n -by- n matrices S and T ,

$$\mathrm{Tr} e^{S+T} \leq \mathrm{Tr} e^{S/2} e^T e^{S/2}.$$

A decade after, Lieb and Thirring [14] obtained a stronger, remarkable trace inequality: for all positive semidefinite n -by- n matrices $A, B \in \mathbb{M}_n^+$ and all integers $p \geq 1$,

$$\mathrm{Tr} (ABA)^p \leq \mathrm{Tr} A^p B^p A^p. \quad (1.1)$$

This was finally extended some fifteen years later by Araki [1] as a very important theorem in matrix analysis and its applications. Given $X, Y \in \mathbb{M}_n^+$, we write $X \prec_{w\log} Y$ when the series of n inequalities holds,

$$\prod_{j=1}^k \lambda_j(X) \leq \prod_{j=1}^k \lambda_j(Y)$$

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for $k = 1, \dots, n$, where $\lambda_j(\cdot)$ stands for the eigenvalues arranged in decreasing order. If further equality occurs for $k = n$, we write $X \prec_{\log} Y$. Araki's theorem considerably strengthens the Lieb-Thirring trace inequality as the beautiful log-majorization

$$(ABA)^p \prec_{\log} A^p B^p A^p \quad (1.2)$$

for all real numbers $p \geq 1$. In particular, this ensures (1.1) for all $p \geq 1$.

Log- and weak log-majorization relations play a fundamental role in matrix analysis, a basic one for normal operators $X, Y \in \mathbb{M}_n$ asserts that

$$|X + Y| \prec_{w\log} |X| + |Y|. \quad (1.3)$$

This useful version of the triangle inequality belongs to the folklore and is a byproduct of Horn's inequalities, see the proof of [6, Corollary 1.4].

This article aims to provide new matrix inequalities containing (1.2) and (1.3). These inequalities are given in Section 2. The first part dealing with positive operators is closely related to a recent paper of Hiai [11]. The second part of Section 2 considers normal operators and contains our main theorem (Theorem 2.9), mentioned in the Abstract.

Our main idea, and technical tool, is Theorem 1.2 below. It establishes the log-convexity of a two variables functional. Fixing one variable in this functional yields a generalization of (1.2) involving a third matrix $Z \in \mathbb{M}_n$, of the form

$$(AZ^*BZA)^p \prec_{w\log} A^p Z^* B^p Z A^p.$$

We will also derive the following weak log-majorization which contains both (1.2) and (1.3) and thus unifies these two inequalities.

Proposition 1.1. *Let $A \in \mathbb{M}_n^+$ and let $X, Y \in \mathbb{M}_n$ be normal. Then, for all $p \geq 1$,*

$$|A(X + Y)A|^p \prec_{w\log} 2^{p-1} A^p (|X|^p + |Y|^p) A^p.$$

Letting $X = Y = B$ in Proposition 1.1 we have (1.2), more generally,

$$|AXA|^p \prec_{\log} A^p |X|^p A^p \quad (1.4)$$

for all $A \in \mathbb{M}_n^+$ and normal matrices $X \in \mathbb{M}_n$. When X is Hermitian, this was noted by Audenaert [2, Proposition 3]. If A is the identity and $p = 1$, Proposition 1.1 gives (1.3). From (1.4) follows several nice inequalities for the matrix exponential, due to Cohen and al. [7], [8], including the Golden-Thompson trace inequality and the elegant relation

$$|e^Z| \prec_{\log} e^{\operatorname{Re} Z} \quad (1.5)$$

for all matrices $Z \in \mathbb{M}_n$, where $\operatorname{Re} Z = (Z + Z^*)/2$, [7, Theorem 2].

Fixing the other variable in Theorem 1.2 below entails a Hölder inequality due to Kosaki. Several matrix versions of an inequality of Littlewood related to Hölder's inequality will be also obtained.

The two variables in Theorem 1.2 are essential and reflect a construction with the perspective of a convex function. Recall that a norm on \mathbb{M}_n is symmetric whenever $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n$ and all unitary $U, V \in \mathbb{M}_n$. For $X, Y \in \mathbb{M}_n^+$, the condition $X \prec_{w\log} Y$ implies $\|X\| \leq \|Y\|$ for all symmetric norms. We state our log-convexity theorem.

Theorem 1.2. *Let $A, B \in \mathbb{M}_n^+$ and $Z \in \mathbb{M}_n$. Then, for all symmetric norms and $\alpha > 0$, the map*

$$(p, t) \mapsto \left\| |A^{t/p} Z B^{t/p}|^{\alpha p} \right\|$$

is jointly log-convex on $(0, \infty) \times (-\infty, \infty)$.

Here, if $A \in \mathbb{M}_n^+$ is not invertible, we naturally define for $t \geq 0$, $A^{-t} := (A + F)^{-t} E$ where F is the projection onto the nullspace of A and E is the range projection of A .

The next two sections present many hidden consequences of Theorem 1.2, several of them extending (1.2) and/or (1.3), for instance,

$$\left| A \frac{T + T^*}{2} A \right|^p \prec_{w\log} A^p \frac{|T|^p + |T^*|^p}{2} A^p$$

for all $A \in \mathbb{M}_n^+$, $p \geq 1$, and any $T \in \mathbb{M}_n$. The proof of Theorem 1.2 is in Section 4. The last section provides a version of Theorem 1.2 for operators acting on an infinite dimensional Hilbert space.

2 Araki type inequalities

2.1 With positive operators

To obtain new Araki's type inequalities, we fix $t = 1$ in Theorem 1.2 and thus use the following special case.

Corollary 2.1. *Let $A, B \in \mathbb{M}_n^+$ and $Z \in \mathbb{M}_n$. Then, for all symmetric norms and $\alpha > 0$, the map*

$$p \mapsto \left\| |A^{1/p} Z B^{1/p}|^{\alpha p} \right\|$$

is log-convex on $(0, \infty)$.

We may now state a series of corollaries extending Araki's theorem.

Corollary 2.2. *Let $A, B \in \mathbb{M}_n^+$ and $p \geq 1$. Let $Z \in \mathbb{M}_n$ be a contraction. Then, for all symmetric norms and $\alpha > 0$,*

$$\|(AZ^* BZA)^{\alpha p}\| \leq \|(A^p Z^* B^p Z A^p)^\alpha\|.$$

Let I be the identity of \mathbb{M}_n . A matrix Z is contractive, or a contraction, if $Z^* Z \leq I$, equivalently if its operator norm satisfies $\|Z\|_\infty \leq 1$.

Proof. The function $f(p) = \| |B^{1/p} Z A^{1/p}|^{2\alpha p} \|$ is log-convex, hence convex on $(0, \infty)$, and bounded since Z is contractive, $0 \leq f(p) \leq \|B\|_\infty^{2\alpha} \|A\|_\infty^{2\alpha} \|I\|$. Thus $f(p)$ is nonincreasing, so $f(1) \geq f(p)$ for all $p \geq 1$. Replacing B by $B^{p/2}$ and A by A^p completes the proof. \square

Let $\|\cdot\|_{\{k\}}$, $k = 1, \dots, n$, denote the normalized Ky Fan k -norms on \mathbb{M}_n ,

$$\|T\|_{\{k\}} = \frac{1}{k} \sum_{j=1}^k \lambda_j(|T|).$$

Since, for all $A \in \mathbb{M}_n^+$,

$$\lim_{\alpha \rightarrow 0^+} \|A^\alpha\|_{\{k\}}^{1/\alpha} = \left\{ \prod_{j=1}^k \lambda_j(A) \right\}^{1/k}$$

we obtain from Corollary 2.2 applied to the normalized Ky Fan k -norms, with $\alpha \rightarrow 0^+$, a striking weak-log-majorization extending Araki's theorem.

Corollary 2.3. *Let $A, B \in \mathbb{M}_n^+$ and $p \geq 1$. Then, for all contractions $Z \in \mathbb{M}_n$,*

$$(AZ^*BZA)^p \prec_{w\log} A^p Z^* B^p Z A^p.$$

If $Z = I$, we have the determinant equality and thus Araki's log-majorization (1.2). Corollary 2.2 and 2.3 are equivalent. Our proof of these extensions of Araki's theorem follows from the two variables technic of Theorem 1.2. It's worth mentioning that Fumio Hiai also obtained Corollary 2.3 in the beautiful note [11]. Hiai's approach is based on some subtle estimates for the operator geometric mean.

For $X, Y \in \mathbb{M}_n^+$, the notation $X \prec^{w\log} Y$ indicates that the series of n inequalities holds,

$$\prod_{j=1}^k \nu_j(X) \geq \prod_{j=1}^k \nu_j(Y)$$

for $k = 1, \dots, n$, where $\nu_j(\cdot)$ stands for the eigenvalues arranged in increasing order. The following so-called super weak-log-majorization is another extension of Araki's theorem. A matrix Z is expansive when $Z^*Z \geq I$.

Corollary 2.4. *Let $A, B \in \mathbb{M}_n^+$ and $p \geq 1$. Then, for all expansive matrices $Z \in \mathbb{M}_n$,*

$$(AZ^*BZA)^p \prec^{w\log} A^p Z^* B^p Z A^p.$$

Proof. By a limit argument, we may assume invertibility of A and B . Taking inverses, and using that Z^{-1} is contractive, Corollary 2.4 is then equivalent to Corollary 2.3. \square

Corollaries 2.3-2.4 imply a host of trace inequalities. We say that a continuous function $h : [0, \infty) \rightarrow (-\infty, \infty)$ is e-convex, (resp. e-concave), if $h(e^t)$ is convex, (resp. concave) on $(-\infty, \infty)$. For instance, for all $\alpha > 0$, $t \mapsto \log(1 + t^\alpha)$ is e-convex, while $t \mapsto \log(t^\alpha/(t + 1))$ is e-concave. The equivalence between Corollary 2.3 and Corollary 2.5 below is a basic property of majorization discussed in any monograph on this topic such as [4] and [12].

Corollary 2.5. *Let $A, B \in \mathbb{M}_n^+$, $Z \in \mathbb{M}_n$, and $p \geq 1$.*

(a) *If Z is contractive and $f(t)$ is e -convex and nondecreasing, then*

$$\mathrm{Tr} f((AZ^*BZA)^p) \leq \mathrm{Tr} f(A^p Z^* B^p Z A^p).$$

(b) *If Z is expansive and $g(t)$ is e -concave and nondecreasing, then*

$$\mathrm{Tr} g((AZ^*BZA)^p) \geq \mathrm{Tr} g(A^p Z^* B^p Z A^p).$$

We will propose in Section 4 a proof of Theorem 1.2 making use of antisymmetric tensor powers, likewise in the proof of Araki's log-majorization. We will also indicate another, more elementary way, without antisymmetric tensors. The antisymmetric tensor technic goes back to Hermann Weyl, cf. [4], [12]. We use it to derive our next corollary.

Corollary 2.6. *Let $A, B \in \mathbb{M}_n^+$ and $Z \in \mathbb{M}_n$. For each $j = 1, \dots, n$, the function defined on $(0, \infty)$*

$$p \mapsto \lambda_j^{1/p}(A^p Z^* B^p Z A^p)$$

converges as $p \rightarrow \infty$.

Proof. We may assume that Z is contractive. As in the proof of Corollary 2.2 we then see that the function $g(p) = \lambda_1^p(A^{1/p} Z^* B^{1/p} Z A^{1/p})$ is log-convex and bounded, hence nonincreasing on $(0, \infty)$. Therefore $g(p)$ converges as $p \rightarrow 0$ and so $g(1/p)$ converges as $p \rightarrow \infty$. Thus $p \mapsto \lambda_1^{1/p}(A^p Z^* B^p Z A^p)$ converges as $p \rightarrow \infty$. Considering k -th antisymmetric tensor products, $k = 1, \dots, n$, we infer the convergence of

$$p \mapsto \prod_{j=1}^k \lambda_j^{1/p}(A^p Z^* B^p Z A^p) = \lambda_1^{1/p}((\wedge^k A)^p \wedge^k Z^* (\wedge^k B)^p \wedge^k Z (\wedge^k A)^p)$$

and so, the convergence of $p \mapsto \lambda_j^{1/p}(A^p Z^* B^p Z A^p)$ as $p \rightarrow \infty$, for each $j = 1, 2, \dots$. \square

When $Z = I$, Audenaert and Hiai [3] recently gave a remarkable improvement of Corollary 2.6 by showing that $p \mapsto (A^p B^p A^p)^{1/p}$ converges in \mathbb{M}_n as $p \rightarrow \infty$. We do not know whether such a reciprocal Lie-Trotter limit still holds with a third matrix Z as in Corollary 2.6.

It is possible to state Corollary 2.3 in a stronger form involving a positive linear map Φ . Such a map is called sub-unital when $\Phi(I) \leq I$.

Corollary 2.7. *Let $A, B \in \mathbb{M}_n^+$ and $p \geq 1$. Then, for all positive linear, sub-unital map $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_n$,*

$$(A\Phi(B)A)^p \prec_{w\log} A^p \Phi(B^p) A^p.$$

Proof. We may assume (the details are given, for a more general class of maps, in the proof of Corollary 3.7) that

$$\Phi(X) = \sum_{i=1}^m Z_i^* X Z_i$$

where $m = n^2$ and $Z_i \in \mathbb{M}_n$, $i = 1, \dots, m$, satisfy $\sum_{i=1}^m Z_i^* Z_i \leq I$. Corollary 2.7 then follows from Corollary 2.3 applied to the operators $\tilde{A}, \tilde{B}, \tilde{Z} \in \mathbb{M}_{mn}$,

$$\tilde{A} = \begin{pmatrix} A & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B & 0_n & \cdots & 0_n \\ 0_n & B & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & \cdots & B \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} Z_1 & 0_n & \cdots & 0_n \\ Z_2 & 0_n & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ Z_m & 0_n & \cdots & 0_n \end{pmatrix}$$

where 0_n stands for the zero matrix in \mathbb{M}_n . \square

Corollary 2.7 can be applied for the Schur product \circ (i.e., entrywise product) in \mathbb{M}_n .

Corollary 2.8. *Let $A, B, C \in \mathbb{M}_n^+$ and $p \geq 1$. If C has all its diagonal entries less than or equal to one, then*

$$(A(C \circ B)A)^p \prec_{w\log} A^p(C \circ B^p)A^p.$$

Proof. The map $X \mapsto C \circ X$ is a positive linear, sub-unital map on \mathbb{M}_n . \square

Corollary 2.8 with the matrix C whose entries are all equal to one is Araki's log-majorization. With $C = I$, Corollary 2.8 is already an interesting extension of Araki's theorem as we may assume that B is diagonal in (1.2). We warn the reader that the super weak-log-majorization, for $A, B \in \mathbb{M}_n^+$ and $p \geq 1$, $(A(I \circ B)A)^p \prec^{w\log} A^p(I \circ B^p)A^p$ does not hold, in fact, in general, $\det^2 I \circ B < \det I \circ B^2$.

2.2 With normal operators

To obtain Proposition 1.1 we need the following generalization of Corollary 2.7.

Theorem 2.9. *Let $A \in \mathbb{M}_n^+$ and let $N \in \mathbb{M}_m$ be normal. Then, for all positive linear, sub-unital maps $\Phi : \mathbb{M}_m \rightarrow \mathbb{M}_n$, and $p \geq 1$,*

$$|A\Phi(N)A|^p \prec_{w\log} A^p\Phi(|N|^p)A^p.$$

Proof. By completing, if necessary, our matrices A and N with some 0-entries, we may assume that $m = n$ and then, as in the proof of Corollary 2.7, that Φ is a congruence map with a contraction \tilde{Z} , $\Phi(X) = \tilde{Z}X\tilde{Z}^*$. Now, we have with the polar decomposition $N = U|N|$,

$$\begin{aligned} |A\tilde{Z}N\tilde{Z}^*A| &= \left| A\tilde{Z}|N|^{1/2}U|N|^{1/2}\tilde{Z}^*A \right| \\ &\prec_{\log} A\tilde{Z}|N|\tilde{Z}^*A \end{aligned}$$

by using Horn's log-majorization $|XKX^*| \prec_{w\log} XX^*$ for all $X \in \mathbb{M}_n$ and all contractions $K \in \mathbb{M}_n$. Hence, from Corollary 2.3, for all $p \geq 1$,

$$|A\tilde{Z}N\tilde{Z}^*A|^p \prec_{\log} \left| A\tilde{Z}|N|\tilde{Z}^*A \right|^p \prec_{w\log} A^p\tilde{Z}|N|^p\tilde{Z}^*A^p$$

which completes the proof. \square

We are in a position to prove Proposition 1.1 whose m -variables version is given here.

Corollary 2.10. *Let $A \in \mathbb{M}_n^+$ and let $X_1, \dots, X_m \in \mathbb{M}_n$ be normal. Then, for all $p \geq 1$,*

$$\left| A \left(\sum_{k=1}^m X_k \right) A \right|^p \prec_{w\log} m^{p-1} A^p \left(\sum_{k=1}^m |X_k|^p \right) A^p.$$

Proof. Applying Theorem 2.9 to $N = X_1 \oplus \dots \oplus X_m$ and to the unital, positive linear map $\Phi : \mathbb{M}_{mn} \rightarrow \mathbb{M}_n$,

$$\begin{pmatrix} S_{1,1} & \cdots & S_{1,m} \\ \vdots & \ddots & \vdots \\ S_{m,1} & \cdots & S_{m,m} \end{pmatrix} \mapsto \frac{1}{m} \sum_{k=1}^m S_{k,k}$$

yields

$$\left| A \frac{\sum_{k=1}^m X_k}{m} A \right|^p \prec_{w\log} A^p \frac{\sum_{k=1}^m |X_k|^p}{m} A^p$$

which is equivalent to the desired inequality. \square

A special case of Corollary 2.10 deals with the Cartesian decomposition of an arbitrary matrix.

Corollary 2.11. *Let $X, Y \in \mathbb{M}_n$ be Hermitian. Then, for all $p \geq 1$,*

$$|A(X + iY)A|^p \prec_{w\log} 2^{p-1} A^p (|X|^p + |Y|^p) A^p$$

where the constant 2^{p-1} is the best possible.

To check that 2^{p-1} is optimal, take $A = I \in \mathbb{M}_{2n}$ and pick any two-nilpotent matrix,

$$X + iY = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

For a single normal operator, Corollary 2.10 gives (1.4) as we have equality for the determinant. This entails the following remarkable log-majorization for the matrix exponential.

Corollary 2.12. *Let $A, B \in \mathbb{M}_n$. Then,*

$$|e^{A+B}| \prec_{\log} e^{\operatorname{Re} A/2} e^{\operatorname{Re} B} e^{\operatorname{Re} A/2}.$$

Corollary 2.12 contains (1.5) and shows that when A and B are Hermitian we have the famous Thompson log-majorization, [16, Lemma 6],

$$e^{A+B} \prec_{\log} e^{A/2} e^B e^{A/2} \quad (2.1)$$

which entails

$$\|e^{A+B}\| \leq \|e^{A/2} e^B e^{A/2}\|$$

for all symmetric norms. For the operator norm this is Segal's inequality while for the trace norm this is the Golden-Thompson inequality. Taking the logarithms in (2.1), we have a classical majorization between $A+B$ and $\log e^{A/2} e^B e^{A/2}$. Since $t \mapsto |t|$ is convex, we infer, replacing B by $-B$ that

$$\|A - B\| \leq \|\log(e^{A/2} e^{-B} e^{A/2})\|$$

for all symmetric norms. For the Hilbert-Schmidt norm, this is the Exponential Metric Increasing inequality, reflecting the nonpositive curvature of the positive definite cone with its Riemannian structure ([5, Chapter 6]).

Corollary 2.12 follows from (1.4) combined with the Lie Product Formula [4, p. 254] as shown in the next proof. Note that Corollary 2.12 also follows from Cohen's log-majorization (1.5) combined with Thompson's log-majorization (2.1), thus we do not pretend to originality.

Proof. We have a Hermitian matrix C such that, using the Lie Product Formula,

$$e^{A+B} = e^{\operatorname{Re} A + \operatorname{Re} B + iC} = \lim_{n \rightarrow +\infty} \left(e^{(\operatorname{Re} A + \operatorname{Re} B)/2n} e^{iC/n} e^{(\operatorname{Re} A + \operatorname{Re} B)/2n} \right)^n.$$

On the other hand, by (1.4), for all $n \geq 1$,

$$\left| \left(e^{(\operatorname{Re} A + \operatorname{Re} B)/2n} e^{iC/n} e^{(\operatorname{Re} A + \operatorname{Re} B)/2n} \right)^n \right| \prec_{\log} e^{\operatorname{Re} A + \operatorname{Re} B}$$

so that

$$|e^{A+B}| \prec_{\log} e^{\operatorname{Re} A + \operatorname{Re} B}.$$

Using again the Lie Product Formula,

$$e^{\operatorname{Re} A + \operatorname{Re} B} = \lim_{n \rightarrow +\infty} \left(e^{\operatorname{Re} A/2n} e^{\operatorname{Re} B/n} e^{\operatorname{Re} A/2n} \right)^n,$$

combined with (1.4) (or (1.2)) completes the proof. \square

Theorem 2.9 is the main result of Section 2 as all the other results in this section are special cases. One more elegant extension of Araki's inequality follows, involving an arbitrary matrix.

Corollary 2.13. *Let $A \in \mathbb{M}_n^+$ and $p \geq 1$. Then, for any $T \in \mathbb{M}_n$,*

$$\left| A \frac{T + T^*}{2} A \right|^p \prec_{w\log} A^p \frac{|T|^p + |T^*|^p}{2} A^p.$$

Proof. It suffices to apply Theorem 2.9 to

$$N = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

and to the unital, positive linear map $\Phi : \mathbb{M}_{2n} \rightarrow \mathbb{M}_n$,

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix} \mapsto \frac{B + C + D + E}{2}.$$

□

We apply Theorem 2.9 to Schur products in the next two corollaries.

Corollary 2.14. *Let $A \in \mathbb{M}_n^+$ and let $X, Y \in \mathbb{M}_n$ be normal. Then, for all $p \geq 1$,*

$$|A(X \circ Y)A|^p \prec_{w \log} A^p(|X|^p \circ |Y|^p)A^p.$$

Proof. We need to see the Schur product as a positive linear map,

$$X \circ Y = \Phi(X \otimes Y)$$

where $\Phi : \mathbb{M}_n \otimes \mathbb{M}_n \rightarrow \mathbb{M}_n$ merely consists in extracting a principal submatrix. Setting $N = X \otimes Y$ in Theorem 2.9 completes the proof. □

We note that Corollary 2.14 extends (1.4) (with X in diagonal form and $Y = I$) and contains the classical log-majorization for normal operators,

$$|X \circ Y| \prec_{w \log} |X| \circ |Y|.$$

As a last illustration of the scope of Theorem 2.9 we have the following result.

Corollary 2.15. *Let $A \in \mathbb{M}_n^+$ and $p \geq 1$. Then, for any $T \in \mathbb{M}_n$,*

$$|A(T \circ T^*)A|^p \prec_{w \log} A^p(|T|^p \circ |T^*|^p)A^p.$$

Proof. We apply Corollary 2.14 to the pair of Hermitian operators in \mathbb{M}_{2n} ,

$$X = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

with $A \oplus A$ in \mathbb{M}_{2n}^+ . We then obtain

$$\left| \begin{pmatrix} 0 & A(T \circ T^*)A \\ A(T \circ T^*)A & 0 \end{pmatrix} \right|^p \prec_{w \log} \begin{pmatrix} A(|T|^p \circ |T^*|^p)A & 0 \\ 0 & A(|T|^p \circ |T^*|^p)A \end{pmatrix}$$

which is equivalent to the statement of our corollary. □

3 Hölder type inequalities

Now we turn to Hölder's type inequalities. Fixing $p = 1$ in Theorem 1.2, we have the following special case.

Corollary 3.1. *Let $A, B \in \mathbb{M}_n^+$ and $Z \in \mathbb{M}_n$. Then, for all symmetric norms and $\alpha > 0$, the map*

$$t \mapsto \left\| |A^t Z B^t|^\alpha \right\|$$

is log-convex on $(-\infty, \infty)$.

This implies a fundamental fact, the Löwner-Heinz inequality stating the operator monotonicity of t^p , $p \in (0, 1)$.

Corollary 3.2. *Let $A, B \in \mathbb{M}_n^+$. If $A \geq B$, then $A^p \geq B^p$ for all $p \in (0, 1)$.*

Proof. Corollary 3.1 for the operator norm, with $Z = I$, $\alpha = 2$, and the pair $A^{-1/2}, B^{1/2}$ in place of the pair A, B shows that $f(t) = \|A^{-t/2} B^t A^{-t/2}\|_\infty$ is log-convex. Hence for $p \in (0, 1)$, we have $f(p) \leq f(1)^p f(0)^{1-p}$. Since $f(0) = 1$ and by assumption $f(1) \leq 1$, we obtain $f(p) \leq 1$ and so $A^p \geq B^p$. \square

Corollary 3.1 entails a Hölder inequality with a parameter. This inequality was first proved by Kosaki [13, Theorem 3]. Here, we state it without the weight Z .

Corollary 3.3. *Let $X, Y \in \mathbb{M}_n$ and $p, q \geq 1$ such that $p^{-1} + q^{-1} = 1$. Then, for all symmetric norms and $\alpha > 0$,*

$$\left\| |XY|^\alpha \right\| \leq \left\| |X|^{\alpha p} \right\|^{1/p} \left\| |Y|^{q\alpha} \right\|^{1/q}.$$

Proof. Let $A, B \in \mathbb{M}_n^+$ with B invertible. By replacing B with B^{-1} and letting $Z = B$ in Corollary 3.1 show that $t \mapsto \left\| |A^t B^{1-t}|^\alpha \right\|$ is log-convex on $(-\infty, \infty)$. Thus, for $t \in (0, 1)$, $\left\| |A^t B^{1-t}|^\alpha \right\| \leq \|A^\alpha\|^t \|B^\alpha\|^{1-t}$. Then, choose $A = |X|^p$, $B = |Y^*|^q$, $t = 1/p$. \square

More original Hölder's type inequalities are given in the next series of corollaries.

Corollary 3.4. *Let $A \in \mathbb{M}_n^+$ and $Z \in \mathbb{M}_{n,m}$. Then, for all symmetric norms and $\alpha > 0$, the map*

$$(p, t) \mapsto \left\| (Z^* A^{t/p} Z)^\alpha \right\|$$

is jointly log-convex on $(0, \infty) \times (-\infty, \infty)$.

Proof. By completing, if necessary, our matrices with some 0-entries, we may suppose $m = n$ and then apply Theorem 1.2 with $B = I$. \square

Corollary 3.5. Let $a = (a_1, \dots, a_m)$ and $w = (w_1, \dots, w_m)$ be two m -tuples in \mathbb{R}^+ and define, for all $p > 0$, $\|a\|_p := (\sum_{i=1}^m w_i a_i^p)^{1/p}$. Then, for all $p, q > 0$ and $\theta \in (0, 1)$,

$$\|a\|_{\frac{1}{\theta p + (1-\theta)q}} \leq \|a\|_{\frac{\theta}{p}} \|a\|_{\frac{1}{q}}^{1-\theta}.$$

Proof. Fix $t = 1$ and pick $A = \text{diag}(a_1, \dots, a_m)$ and $Z^* = (w_1^{1/2}, \dots, w_m^{1/2})$ in the previous corollary. \square

Corollary 3.5 is the classical log-convexity of $p \rightarrow \|\cdot\|_{1/p}$, or Littlewood's version of Hölder's inequality [9, Theorem 5.5.1]. The next two corollaries, seemingly stronger but actually equivalent to Corollary 3.4, are also generalizations of this inequality.

Corollary 3.6. Let $A_i \in \mathbb{M}_n^+$ and $Z_i \in \mathbb{M}_{n,m}$, $i = 1, \dots, k$. Then, for all symmetric norms and $\alpha > 0$, the map

$$(p, t) \mapsto \left\| \left\{ \sum_{i=1}^k Z_i^* A_i^{t/p} Z_i \right\}^{\alpha p} \right\|$$

is jointly log-convex on $(0, \infty) \times (-\infty, \infty)$.

The unweighted case, $Z_i = I$ for all $i = 1, \dots, k$, is especially interesting. With $t = \alpha = 1$, it is a matrix version of the unweighted Littlewood inequality.

Proof. Apply Corollary 3.4 with $A = A_1 \oplus \dots \oplus A_k$ and $Z^* = (Z_1^*, \dots, Z_k^*)$. \square

Corollary 3.7. Let $A \in \mathbb{M}_m^+$ and let $\Phi : \mathbb{M}_m^+ \rightarrow \mathbb{M}_n^+$ be a positive linear map. Then, for all symmetric norms and $\alpha > 0$, the map

$$(p, t) \mapsto \left\| \{ \Phi(A^{t/p}) \}^{\alpha p} \right\|$$

is jointly log-convex on $(0, \infty) \times (-\infty, \infty)$.

Proof. When restricted to the $*$ -commutative subalgebra spanned by A , the map Φ has the form

$$\Phi(X) = \sum_{i=1}^m \sum_{j=1}^n Z_{i,j}^* X Z_{i,j} \tag{3.1}$$

for some rank 1 or 0 matrices $Z_{i,j} \in \mathbb{M}_{m,n}$, $i = 1, \dots, m$, $j = 1, \dots, n$. So we are in the range of the previous corollary. To check the decomposition (3.1), write the spectral decomposition $A = \sum_{i=1}^m \lambda_i(A) E_i$ with rank one projections $E_i = x_i x_i^*$ for some column vectors $x_i \in \mathbb{M}_{m,1}$ and set $Z_{i,j} = x_i R_{i,j}$ where $R_{i,j} \in \mathbb{M}_{1,n}$ is the j -th row of $\Phi(E_i)^{1/2}$. \square

The above proof shows a classical fact, a positive linear map on a commutative domain is completely positive. Our proof seems shorter than the ones in the literature. We close this section with an application to Schur products.

Corollary 3.8. *Let $A, B \in \mathbb{M}_n^+$. If $p \geq r \geq s \geq q$ and $p + q = r + s$, then, for all symmetric norms and $\alpha > 0$,*

$$\|\{A^r \circ B^s\}^\alpha\| \|\{A^s \circ B^r\}^\alpha\| \leq \|\{A^p \circ B^q\}^\alpha\| \|\{A^q \circ B^p\}^\alpha\|$$

and

$$\|\{A^r \circ B^s\}^\alpha\| + \|\{A^s \circ B^r\}^\alpha\| \leq \|\{A^p \circ B^q\}^\alpha\| + \|\{A^q \circ B^p\}^\alpha\|.$$

Proof. By a limit argument we may assume invertibility of A and B . Let $w := (p+q)/2$. We will show that the maps

$$t \mapsto \|\{A^{w+t} \circ B^{w-t}\}^\alpha\|, \quad t \mapsto \|\{A^{w-t} \circ B^{w+t}\}^\alpha\| \quad (3.2)$$

are log-convex on $(-\infty, \infty)$. This implies that the functions

$$f(t) = \|\{A^{w+t} \circ B^{w-t}\}^\alpha\| \|\{A^{w-t} \circ B^{w+t}\}^\alpha\|$$

and

$$g(t) = \|\{A^{w+t} \circ B^{w-t}\}^\alpha\| + \|\{A^{w-t} \circ B^{w+t}\}^\alpha\|$$

are convex and even, hence nondecreasing on $[0, \infty)$. So we have

$$f((r-s)/2) \leq f((p-q)/2) \text{ and } g((r-s)/2) \leq g((p-q)/2)$$

which prove the corollary.

To check the log-convexity of the maps (3.2), we see the Schur product as a positive linear map acting on a tensor product, $A \circ B = \Psi(A \otimes B)$. By Corollary 3.7, the map

$$t \mapsto \|\{\Phi(Z^t)\}^\alpha\|$$

is log-convex on $(-\infty, \infty)$ for any positive matrix $Z \in \mathbb{M}_n \otimes \mathbb{M}_n$ and any positive linear map $\Phi : \mathbb{M}_n \otimes \mathbb{M}_n \rightarrow \mathbb{M}_n$. Taking $Z = A \otimes B^{-1}$ and

$$\Phi(X) = \Psi(A^{w/2} \otimes B^{w/2} \cdot X \cdot A^{w/2} \otimes B^{w/2})$$

we obtain the log-convexity of the first map $t \mapsto \|\{A^{w+t} \circ B^{w-t}\}^\alpha\|$ in (3.2). The log-convexity of the second one is similar. \square

4 Proof of Theorem 1.2

In the proof of the theorem, we will denote the k -th antisymmetric power $\wedge^k T$ of an operator T simply as T_k . The symbol $\|\cdot\|_\infty$ stands for the usual operator norm while $\rho(\cdot)$ denotes the spectral radius. Given $A \in \mathbb{M}_n^+$ we denote by A^\downarrow the diagonal matrix with the eigenvalues of A in decreasing order down to the diagonal, $A^\downarrow = \text{diag}(\lambda_j(A))$.

Proof. Recall that if $A \in \mathbb{M}_n^+$ is not invertible and $t \geq 0$, we define A^{-t} as the generalized inverse of A^t , i.e., $A^{-t} := (A + F)^{-t}E$ where F is the projection onto the nullspace of A and E is the range projection of A . With this convention, replacing if necessary Z by EZE' where E is the range projection of A and E' that of B , we may and do assume that A and B are invertible.

Let

$$g_k(t) := \prod_{j=1}^k \lambda_j(|A^t Z B^t|) = \|A_k^t Z_k B_k^t\|_\infty$$

Then

$$\begin{aligned} g_k((t+s)/2) &= \|A_k^{(t+s)/2} Z_k B_k^{t+s} Z_k^* A_k^{(t+s)/2}\|_\infty^{1/2} \\ &= \rho^{1/2}(A_k^t Z_k B_k^{t+s} Z_k^* A_k^s) \\ &\leq \|A_k^t Z_k B_k^{t+s} Z_k^* A_k^s\|_\infty^{1/2} \\ &\leq \|A_k^t Z_k B_k^t\|_\infty^{1/2} \|B_k^s Z_k^* A_k^s\|_\infty^{1/2} \\ &= \{g_k(t)g_k(s)\}^{1/2}. \end{aligned}$$

Thus $t \mapsto g_k(t)$ is log-convex on $(-\infty, \infty)$ and so $(p, t) \mapsto g_k^p(t/p)$ is jointly log-convex on $(0, \infty) \times (-\infty, \infty)$. Indeed, its logarithm $p \log g_k(t/p)$ is the perspective of the convex function $\log g_k(t)$, and hence is jointly convex. Therefore

$$g_k^{(p+q)/2} \left(\frac{(t+s)/2}{(p+q)/2} \right) \leq \{g_k^p(t/p)g_k^q(s/q)\}^{1/2} \quad (4.1)$$

for $k = 1, 2, \dots, n$, with equality for $k = n$ as it then involves the determinant. This is equivalent to the log-majorization

$$\left| A_k^{\frac{t+s}{p+q}} Z B_k^{\frac{t+s}{p+q}} \right|^{\frac{p+q}{2}} \prec_{\log} |A^{t/p} Z B^{t/p}|^{\frac{p}{2}} \downarrow |A^{s/q} Z B^{s/q}|^{\frac{q}{2}} \downarrow$$

which is equivalent, for any $\alpha > 0$, to the log-majorization

$$\left| A_k^{\frac{t+s}{p+q}} Z B_k^{\frac{t+s}{p+q}} \right|^{\alpha \frac{p+q}{2}} \prec_{\log} |A^{t/p} Z B^{t/p}|^{\frac{\alpha p}{2}} \downarrow |A^{s/q} Z B^{s/q}|^{\frac{\alpha q}{2}} \downarrow$$

ensuring that

$$\left\| \left| A_k^{\frac{t+s}{p+q}} Z B_k^{\frac{t+s}{p+q}} \right|^{\alpha \frac{p+q}{2}} \right\| \leq \left\| |A^{t/p} Z B^{t/p}|^{\frac{\alpha p}{2}} \downarrow |A^{s/q} Z B^{s/q}|^{\frac{\alpha q}{2}} \downarrow \right\| \quad (4.2)$$

for all symmetric norms. Thanks to the Cauchy-Schwarz inequality for symmetric norms, we then have

$$\left\| \left| A_k^{\frac{t+s}{p+q}} Z B_k^{\frac{t+s}{p+q}} \right|^{\alpha \frac{p+q}{2}} \right\| \leq \left\| |A^{t/p} Z B^{t/p}|^{\alpha p} \right\|^{1/2} \left\| |A^{s/q} Z B^{s/q}|^{\alpha q} \right\|^{1/2} \quad (4.3)$$

which means that

$$(p, t) \mapsto \left\| |A^{t/p} Z B^{t/p}|^{\alpha p} \right\|$$

is jointly log-convex on $(0, \infty) \times (-\infty, \infty)$. \square

Denote by I_k the identity of \mathbb{M}_k and by \det_k the determinant on \mathbb{M}_k . If $n \geq k$, $\Theta(k, n)$ stands for the set of $n \times k$ isometry matrices T , i.e, $T^*T = I_k$. One easily checks the variational formula, for $A \in \mathbb{M}_n$ and $k = 1, \dots, n$,

$$\prod_{j=1}^k \lambda_j(|A|) = \max_{V, W \in \Theta(k, n)} |\det_k V^*AW|.$$

From this formula follow two facts, Horn's inequality,

$$\prod_{j=1}^k \lambda_j(|AB|) \leq \prod_{j=1}^k \lambda_j(|A|)\lambda_j(|B|)$$

for all $A, B \in \mathbb{M}_n$ and $k = 1, \dots, n$, and, making use of Schur's triangularization, the inequality

$$\prod_{j=1}^k \lambda_j(|AB|) \leq \prod_{j=1}^k \lambda_j(|BA|)$$

whenever AB is normal (indeed, by Schur's theorem we may assume that BA is upper triangular with the eigenvalues of BA , hence of AB , down to the diagonal and our variational formula then gives the above log-majorization). This shows that the proof of Theorem 1.2 can be written without the machinery of antisymmetric tensors.

The novelty of this proof consists in using the perspective of a one variable convex function. One more perspective yields the following variation of Corollary 2.1.

Corollary 4.1. *Let $A, B \in \mathbb{M}_m^+$, let $Z \in \mathbb{M}_m$. Then, for all symmetric norms and $\alpha > 0$, the map*

$$p \mapsto \left\| |A^{1/p} Z B^{1/p}|^\alpha \right\|^p$$

is log-convex on $(0, \infty)$.

Proof. By Theorem 1.2 with fixed $p = 1$, the map $t \mapsto \log(\| |A^t Z B^t|^\alpha \|)$ is convex on $(0, \infty)$, thus its perpective

$$(p, t) \mapsto p \log(\| |A^{t/p} Z B^{t/p}|^\alpha \|) = \log(\| |A^{t/p} Z B^{t/p}|^\alpha \|^p)$$

is jointly log-convex on $(0, \infty) \times (0, \infty)$. Now fixing $t = 1$ completes the proof. \square

From this corollary we may derive the next one exactly as Corollary 3.7 follows from Theorem 1.2. This result is another noncommutative version of Littlewood's inequality ([9, Theorem 5.5.1]).

Corollary 4.2. *Let $\Phi : \mathbb{M}_m \rightarrow \mathbb{M}_n$ be a positive linear map and let $A \in \mathbb{M}_m^+$. Then, for all symmetric norms and $\alpha > 0$, the map*

$$p \mapsto \left\| \Phi^\alpha(A^{1/p}) \right\|^p$$

is log-convex on $(0, \infty)$.

5 Hilbert space operators

In this section we give a version of Theorem 1.2 for the algebra \mathbb{B} of bounded linear operators on a separable, infinite dimensional Hilbert space \mathcal{H} . We first include a brief treatment of symmetric norms for operators in \mathbb{B} . Our approach does not require to discuss any underlying ideal, we refer the reader to [15, Chapter 2] for a much more complete discussion.

We may define symmetric norms on \mathbb{B} in a closely related way to the finite dimensional case as follows. Let \mathbb{F} be the set of finite rank operators and \mathbb{F}^+ its positive part.

Definition 5.1. A symmetric norm $\|\cdot\|$ on \mathbb{B} is a functional taking value in $[0, \infty]$ such that:

- (1) $\|\cdot\|$ induces a norm on \mathbb{F} .
- (2) If $\{X_n\}$ is a sequence in \mathbb{F}^+ strongly increasing to X , then $\|X\| = \lim_n \|X_n\|$.
- (3) $\|KZL\| \leq \|Z\|$ for all $Z \in \mathbb{B}$ and all contractions $K, L \in \mathbb{B}$.

The reader familiar to the theory of symmetrically normed ideals may note that our definition of a symmetric norm is equivalent to the usual one. More precisely, restricting $\|\cdot\|$ to the set where it takes finite values, Definition 5.1 yields the classical notion of a symmetric norm defined on its maximal ideal.

Definition 5.1 shows that a symmetric norm on \mathbb{B} induces a symmetric norm on \mathbb{M}_n for each n , say $\|\cdot\|_{\mathbb{M}_n}$. In fact $\|\cdot\|$ can be regarded as a limit of the norms $\|\cdot\|_{\mathbb{M}_n}$, see Lemma 5.6 for a precise statement, so that basic properties of symmetric norms on \mathbb{M}_n can be extended to symmetric norms on \mathbb{B} . For instance the Cauchy-Schwarz inequality also holds for symmetric norms on \mathbb{B} , (with possibly the ∞ value) as well as the Ky Fan principle for $A, B \in \mathbb{B}^+$: If $A \prec_w B$, then $\|A\| \leq \|B\|$ for all symmetric norms. In fact, even for a noncompact operator $A \in \mathbb{B}^+$, the sequence $\{\lambda_j(A)\}_{j=1}^\infty$ and the corresponding diagonal operator $A^\downarrow = \text{diag}(\lambda_j(A))$ are well defined, via the minmax formulae (see [10, Proposition 1.4])

$$\lambda_j(A) = \inf_E \{\|EAE\|_\infty : E \text{ projection with } \text{rank}(I - E) = j - 1\}$$

The Ky Fan principle then still holds for $A, B \in \mathbb{B}^+$ by Lemma 5.6 and the obvious property

$$\lambda_j(A) = \lim_{n \rightarrow \infty} \lambda_j(E_n A E_n)$$

for all sequences of finite rank projections $\{E_n\}_{n=1}^\infty$ strongly converging to the identity. Note also that we still have $\|\wedge^k A\|_\infty = \prod_{j=1}^k \lambda_j(A)$.

Thus we have the same tools as in the matrix case and we will be able to adapt the proof of Theorem 1.2 for \mathbb{B} . The infinite dimensional version of Theorem 1.2 is the following statement.

Theorem 5.2. *Let $A, B \in \mathbb{B}^+$, let $Z \in \mathbb{B}$. Then, for all symmetric norms and $\alpha > 0$, the map*

$$(p, t) \mapsto \left\| |A^{t/p} Z B^{t/p}|^{\alpha p} \right\|$$

is jointly log-convex on $(0, \infty) \times (0, \infty)$. This map takes its finite values in the open quarter-plan

$$\Omega(p_0, t_0) = \{(p, t) \mid p > p_0, t > t_0\}$$

for some $p_0, t_0 \in [0, \infty]$, or on its closure $\overline{\Omega}(p_0, t_0)$.

Note that, contrarily to Theorem 1.2, we confine the variable t to the positive half-line. Indeed, when dealing with a symmetric norm, the operators A and B are often compact, so that, for domain reasons, we cannot consider two unbounded operators such as A^{-1} and B^{-1} .

Proof. Note that $AZB = 0$ if and only if $A^q Z B^q = 0$, for any $q > 0$. In this case, our map is the 0-map, and its logarithm with constant value $-\infty$ can be regarded as convex. Excluding this trivial case, our map takes values in $(0, \infty]$ and it makes sense to consider the log-convexity property. We may reproduce the proof of Theorem 1.2 and obtain (4.1) for all $k = 1, 2, \dots$. This leads to weak-logmajorizations and so to a weak majorization equivalent (Ky Fan's principle in \mathbb{B}) to (4.2), with possibly the ∞ value on the right side or both sides. The Cauchy-Schwarz inequality for symmetric norms in \mathbb{B} yields (4.3) (possibly with the ∞ value). Therefore our map is jointly log-convex. To show that the domain where it takes finite values is $\Omega(p_0, t_0)$ or $\overline{\Omega}(p_0, t_0)$, it suffices to show the following two implications:

Let $0 < t < s$ and $0 < p < q$. If $\left\| |A^{t/p} Z B^{t/p}|^{\alpha p} \right\| < \infty$, then

(i) $\left\| |A^{s/p} Z B^{s/p}|^{\alpha p} \right\| < \infty$, and

(ii) $\left\| |A^{t/q} Z B^{t/q}|^{\alpha q} \right\| < \infty$.

Since $0 < t < s$ ensures that, for some constant $c = c(s, t) > 0$,

$$\lambda_j(|A^{t/p} Z B^{t/p}|) \geq c \lambda_j(|A^{s/p} Z B^{s/p}|)$$

for all $j = 1, 2, \dots$, we obtain (i). To obtain (ii) we may assume that Z is a contraction. Then arguing as in the proof of Corollary 2.2 we see that the finite value map

$$p \mapsto \left\| |A^{t/p} Z B^{t/p}|^{\alpha p} \right\|$$

is nonincreasing for all Ky-Fan norms. Thus this map is also nonincreasing for all symmetric norms. This gives (ii). \square

Exactly as in the matrix case, we can derive the following two corollaries.

Corollary 5.3. *Let $A, B \in \mathbb{B}^+$ and $p \geq 1$. Then, for all contractions $Z \in \mathbb{B}$,*

$$(AZ^* BZA)^p \prec_{w \log} A^p Z^* B^p Z A^p.$$

Corollary 5.4. *Let $A, B \in \mathbb{B}^+$ and let $Z \in \mathbb{B}$ be a contraction. Assume that at least one of these three operators is compact. Then, if $p \geq 1$ and $f(t)$ is e -convex and nondecreasing,*

$$\mathrm{Tr} f((AZ^*BZA)^p) \leq \mathrm{Tr} f(A^pZ^*B^pZA^p).$$

Here, we use the fact that for $X \in \mathbb{K}^+$ and a nondecreasing continuous function $f : [0, \infty) \rightarrow (-\infty, \infty)$, we can define $\mathrm{Tr} f(X)$ as an element in $[-\infty, \infty]$ by

$$\mathrm{Tr} f(X) = \lim_{k \rightarrow \infty} \sum_{j=1}^k f(\lambda_j(X)).$$

Given a symmetric norm $\|\cdot\|$ on \mathbb{B} , the set where $\|\cdot\|$ takes a finite value is an ideal. We call it the maximal ideal of $\|\cdot\|$ or the domain of $\|\cdot\|$. From Theorem 5.2 we immediately infer our last corollary.

Corollary 5.5. *Let $A, B \in \mathbb{B}^+$ and $Z \in \mathbb{B}$. Suppose that $AZB \in \mathbb{J}$, the domain of a symmetric norm. Then, for all $q \in (0, 1)$, we also have $|A^qZB^q|^{1/q} \in \mathbb{J}$.*

Following [15, Chapter 2], we denote by $\mathbb{J}^{(0)}$ the $\|\cdot\|$ -closure of the finite rank operators. In most cases $\mathbb{J} = \mathbb{J}^{(0)}$, however the strict inclusion $\mathbb{J}^{(0)} \subsetneq \mathbb{J}$ may happen. We do not know whether we can replace in the last corollary \mathbb{J} by $\mathbb{J}^{(0)}$.

We close our article with two simple lemmas and show how the Cauchy-Schwarz inequality for the infinite dimensional case follows from the matrix case.

Lemma 5.6. *Let $\|\cdot\|$ be a symmetric norm on \mathbb{B} and let $\{E_n\}_{n=1}^\infty$ be an increasing sequence of finite rank projections in \mathbb{B} , strongly converging to I . Then, for all $X \in \mathbb{B}$, $\|X\| = \lim_n \|E_n X E_n\|$.*

Proof. We first show that $\|E_n X\| \rightarrow \|X\|$ as $n \rightarrow \infty$. Since $\|E_n X\| = \|(X^* E_n X)^{1/2}\|$ and $(X^* E_n X)^{1/2} \nearrow |X|$ by operator monotonicity of $t^{1/2}$, we obtain $\lim_n \|E_n X\| = \|X\|$ by Definition 5.1(2). Similarly, $\lim_k \|E_n X E_k\| = \|E_n X\|$, and so $\lim_n \|E_n X E_{k(n)}\| = \|X\|$, and thus, by Definition 5.1(3), $\lim_p \|E_p X E_p\| = \|X\|$. \square

Lemma 5.7. *Let $\|\cdot\|$ be a symmetric norm on \mathbb{B} and let $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ be two increasing sequences of finite rank projections in \mathbb{B} , strongly converging to I . Then, for all $X \in \mathbb{B}$, $\|X^* X\| = \lim_n \|E_n X^* F_n X E_n\|$.*

Proof. By Definition 5.1(2)-(3), the map $n \mapsto \|E_n X^* F_n X E_n\|$ is nondecreasing. By Definition 5.1(2), for any integer p , its limit is greater than or equal $\|E_p X^* X E_p\|$. By Lemma 5.6, the limit is precisely $\|X^* X\|$. \square

Let $X, Y \in \mathbb{B}$, let $\|\cdot\|$ be a symmetric norm on \mathbb{B} , and let $\{E_n\}_{n=1}^\infty$ be as in the above lemma. Let F_n be the range projection of YE_n . We have by Lemma 5.6

$$\|X^*Y\| = \lim_n \|E_n X^* Y E_n\| = \lim_n \|E_n X^* F_n Y E_n\|.$$

Let \mathcal{H}_n be the sum of the ranges of E_n and F_n . This is a finite dimensional subspace, say $\dim \mathcal{H}_n = d(n)$. Applying the Cauchy-Schwarz inequality for a symmetric norm on $\mathbb{M}_{d(n)}$, we obtain, thanks to Lemma 5.7,

$$\begin{aligned} \|X^*Y\| &= \lim_n \|E_n X^* F_n Y E_n\|_{\mathbb{M}_{d(n)}} \\ &\leq \lim_n \|E_n X^* F_n X E_n\|_{\mathbb{M}_{d(n)}}^{1/2} \|E_n Y^* F_n Y E_n\|_{\mathbb{M}_{d(n)}}^{1/2} \\ &= \|X^* X\|^{1/2} \|Y^* Y\|^{1/2}. \end{aligned}$$

Thus the Cauchy-Schwarz inequality for a symmetric norm on \mathbb{B} follows from the Cauchy-Schwarz inequality for symmetric norms on \mathbb{M}_n . Of course, the two previous lemmas and this discussion are rather trivial, but we wanted to stress on the fact that Theorem 5.2 is essentially of finite dimensional nature. However, it would be also desirable to extend these results in the setting of a semifinite von Neumann algebra.

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Laboratoire de mathématiques, Université de Franche-Comté, 25 000 Besançon, France.
Email: jcbourin@univ-fcomte.fr

Department of mathematics, Kyungpook National University, Daegu 702-701, Korea.
Email: eylee89@knu.ac.kr