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On the Russo-Dye Theorem for positive linear maps

Jean-Christophe Bourin* and Eun-Young Lee†

Abstract. We revisit a classical result, the Russo-Dye Theorem, stating that every positive linear map attains its norm at the identity.

Keywords. Matrix inequalities, Positive linear maps, Unitary orbits, Matrix geometric mean.

2010 mathematics subject classification. 47A30, 15A60.

1 Introduction

Let \mathbb{M}_n denote the space of complex $n \times n$ matrices and let \mathbb{M}_n^+ stand for the positive (semi-definite) cone. A linear map $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ is called positive if $\Phi(\mathbb{M}_n^+) \subset \mathbb{M}_m^+$. A fundamental fact, the Russo-Dye Theorem [7], asserts that every positive linear map attains its norm at the identity. This is discussed in two nice books, [2, pp. 41–44] and, in the operator algebras setting, [6, Corollary 2.9]. If I denotes the identity and $\|\cdot\|_\infty$ the operator norm, the theorem can be stated as:

Theorem 1.1. *If $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ is a positive linear map then, for all contractions $A \in \mathbb{M}_n$,*

$$\|\Phi(A)\|_\infty \leq \|\Phi(I)\|_\infty.$$

This short note presents several refinements of Theorem 1.1. The contractive property of A can be written as an operator inequality, $|A| \leq I$ where $|A| = (A^*A)^{1/2}$ is the right modulus. This suggests that for a general matrix Z the data of its two modulus $|Z|$ and $|Z^*|$ should be useful to obtain more general forms of the Russo-Dye Theorem. For instance, we have the following consequence of our main result.

Proposition 1.2. *Let $Z \in \mathbb{M}_n$, $J \in \mathbb{M}_n^+$, and let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map. If $|Z| \leq J$ and $|Z^*| \leq J$, then there exists a unitary $V \in \mathbb{M}_m$ such that*

$$|\Phi(Z)| \leq \frac{\Phi(J) + V\Phi(J)V^*}{2}.$$

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If $Z = A$ is a contraction and $J = I$, Proposition 1.2 claims that

$$|\Phi(A)| \leq \frac{\Phi(I) + V\Phi(I)V^*}{2} \quad (1.1)$$

and this clearly improves Theorem 1.1. Let us illustrate that with some eigenvalue inequalities for the Schur product with $S \in \mathbb{M}_n^+$. Let s_i^\downarrow be the diagonal entries of S rearranged in decreasing order, and similarly let $\lambda_i^\downarrow(H)$ denote the eigenvalues of the Hermitian matrix H arranged in decreasing order. Theorem 1.1 for the Schur multiplier $\Phi(T) = S \circ T$ says that

$$\lambda_1^\downarrow(|S \circ A|) \leq s_1^\downarrow$$

for all contractions $A \in \mathbb{M}_n$. From (1.1) we infer several other inequalities, for example,

$$\lambda_3^\downarrow(|S \circ A|) \leq s_2^\downarrow. \quad (1.2)$$

We shall prove a considerable improvement of Proposition 1.2. This yields several eigenvalue inequalities which extend and complete the Russo-Dye Theorem. Of course the Russo-Dye Theorem holds for positive linear maps acting on unital C^* -algebras and, at the end of the paper, we state our main result in this setting.

2 Positive linear maps and geometric mean

Let $Z \in \mathbb{M}_n$, $J \in \mathbb{M}_n^+$. We wish to state a version of Proposition 1.2 under a more general assumption than the domination $|Z|, |Z^*| \leq J$. Given two nonnegative functions $f(t)$ and $g(t)$, we use the notation

$$Z \leq_{f,g} J \quad \iff \quad f(|Z|) \leq J \text{ and } g(|Z^*|) \leq J.$$

So, the assumption of Proposition 1.2 can be written $Z \leq_{f,g} J$ with $f(t) = g(t) = t$.

We also sharpen Proposition 1.2 by using the geometric mean instead of the arithmetic mean. We refer to [1] or [2] for a background on the geometric mean $A\#B$ of two matrices $A, B \in \mathbb{M}_n^+$. Since the arithmetic-geometric mean inequality

$$A\#B \leq \frac{A+B}{2}$$

holds, the following theorem extends and improves Proposition 1.2.

Theorem 2.1. *Let $Z \in \mathbb{M}_n$, $J \in \mathbb{M}_n^+$, and let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map. Suppose that $Z \leq_{f,g} J$ for some functions satisfying $f(t)g(t) = t^2$. Then, there exists a unitary $V \in \mathbb{M}_m$ such that*

$$|\Phi(Z)| \leq \Phi(J)\#V\Phi(J)V^*.$$

Proof. We first prove the case when $Z = N$ is normal, $f(t) = g(t) = t$, and $J = |N|$. The theorem then reads as

$$|\Phi(N)| \leq \Phi(|N|) \# V \Phi(|N|) V^*. \quad (2.1)$$

Observe that

$$\begin{pmatrix} |N| & N^* \\ N & |N| \end{pmatrix} \geq 0.$$

We may restrict Φ to the unital C^* -algebra spanned by N , and thus we may assume that Φ is completely positive thanks to Stinespring's lemma [8]. Hence,

$$\begin{pmatrix} \Phi(|N|) & \Phi(N^*) \\ \Phi(N) & \Phi(|N|) \end{pmatrix} \geq 0.$$

Next, applying a unitary congruence

$$\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \Phi(|N|) & \Phi(N^*) \\ \Phi(N) & \Phi(|N|) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V^* \end{pmatrix}$$

where V^* is the unitary factor in the polar decomposition of $\Phi(N)$, we obtain

$$\begin{pmatrix} \Phi(|N|) & |\Phi(N)| \\ |\Phi(N)| & V \Phi(|N|) V^* \end{pmatrix} \geq 0.$$

It then follows from the maximal property of $\#$ that (2.1) holds.

Now, suppose that $Z = A$ is a contraction. Since

$$|A| = \frac{W + W^*}{2}$$

for some unitary W , we infer from the polar decomposition that

$$A = \frac{U_0 + U_1}{2}$$

some unitaries U_0 and U_1 . Applying (2.1) to the normal (unitary) matrix

$$N = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix}$$

and to the positive linear map from $\mathbb{M}_2(\mathbb{M}_n)$ to \mathbb{M}_n ,

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \mapsto \Phi \left(\frac{P + S}{2} \right),$$

we obtain

$$|\Phi(A)| \leq \Phi(I) \# V \Phi(I) V^*. \quad (2.2)$$

This establishes the theorem when Z is a contraction, $f(t) = g(t) = t$, and $J = I$.

We turn to the general case, $f(|Z|) \leq J$ and $g(|Z^*|) \leq J$ with $f(t)g(t) = t^2$. Define a map $\Psi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ by

$$\Psi(X) = \Phi(J^{1/2}XJ^{1/2}).$$

Observe that

$$\Psi(I) = \Phi(J), \quad \text{and} \quad \Psi(Y) = \Phi(Z) \tag{2.3}$$

where $Y = J^{-1/2}ZJ^{-1/2}$, and J^{-1} is the generalized inverse. Thanks to the polar decomposition $Z = |Z^*|U = U|Z|$ and the condition $\sqrt{f(t)g(t)} = t$,

$$Y = J^{-1/2}g^{1/2}(|Z^*|)Uf^{1/2}(|Z|)J^{-1/2}.$$

From the assumption $Z \leq_{f,g} J$ we infer that $J^{-1/2}g^{1/2}(|Z^*|)$ and $f^{1/2}(|Z|)J^{-1/2}$ are contractions, and so Y is a contraction too. Applying (2.2) to Y and Ψ yields

$$|\Psi(Y)| \leq \Psi(I) \# V \Psi(I) V^*$$

and coming back to (2.3) completes the proof. \square

In the course of the proof, we have noted two special cases.

Corollary 2.2. *Let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map and let $N \in \mathbb{M}_n$ be normal. Then, there exists a unitary $V \in \mathbb{M}_m$ such that*

$$|\Phi(N)| \leq \Phi(|N|) \# V \Phi(|N|) V^*.$$

Corollary 2.3. *Let $A \in \mathbb{M}_n$ be a contraction and let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map. Then, for some unitary $V \in \mathbb{M}_m$,*

$$|\Phi(A)| \leq \Phi(I) \# V \Phi(I) V^*.$$

Another extension of the Russo-Dye Theorem is given in the next corollary.

Corollary 2.4. *Let $Z \in \mathbb{M}_n$, $J \in \mathbb{M}_n^+$, and let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map. If $J \geq E$, the range projection of Z , and $J \geq Z^*Z$, then, for some unitary $V \in \mathbb{M}_m$,*

$$|\Phi(Z)| \leq \Phi(J) \# V \Phi(J) V^*.$$

Proof. Apply Theorem 2.1 with $f(t) = t^2$ and $g(t)$ such that $g(0) = 0$ and $g(s) = 1$ for $s > 0$. \square

The next statement extends Corollary 2.2.

Corollary 2.5. *Let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map, let $Z \in \mathbb{M}_n$ be invertible, and ρ the spectral radius of $|Z^*||Z|^{-1}$. Then, there exists a unitary $V \in \mathbb{M}_m$ such that*

$$|\Phi(Z)| \leq \sqrt{\rho} \Phi(|Z|) \# V \Phi(|Z|) V^*.$$

Proof. Note that $|Z^*| \leq \rho|Z|$ and apply Theorem 2.1 for the matrices Z and $J = \sqrt{\rho}|Z|$ with the functions $f(t) = \sqrt{\rho}t$ and $g(t) = (1/\sqrt{\rho})t$. \square

In general the inequality of Corollary 2.5 is sharp as shown with the transpose map $\Phi(X) = X^T$ on \mathbb{M}_2 and

$$Z = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}.$$

Indeed, the spectral radius of $|Z^*||Z|^{-1}$ then equals k , and if e_2 denotes the second vector of the canonical basis of \mathbb{C}^2 , the inequality

$$k = \langle e_2, |Z^T|e_2 \rangle \leq c \langle e_2, |Z|^T \# V |Z|^T V^* e_2 \rangle$$

entails

$$k \leq c \langle e_2, |Z|^T e_2 \rangle \# \langle e_2, V |Z|^T V^* e_2 \rangle \leq c\sqrt{k}$$

and so $c \geq \sqrt{k}$.

For 2-positive maps, Theorem 2.1 can be improved.

Theorem 2.6. *Let $Z \in \mathbb{M}_n$ and let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a 2-positive linear map. Then, for any pair of nonnegative functions $f(t)$ and $g(t)$ satisfying $f(t)g(t) = t^2$, there exists a unitary $V \in \mathbb{M}_m$ such that*

$$|\Phi(Z)| \leq \Phi(f(|Z|)) \# V \Phi(g(|Z^*|)) V^*.$$

Proof. From the polar decomposition $Z = U|Z|$ we have $Z = \sqrt{g(|Z^*|)}U\sqrt{f(|Z|)}$, hence

$$\begin{pmatrix} f(|Z|) & Z^* \\ Z & g(|Z^*|) \end{pmatrix} \geq 0.$$

Since Φ is 2-positive, we then have

$$\begin{pmatrix} \Phi(f(|Z|)) & \Phi(Z^*) \\ \Phi(Z) & \Phi(g(|Z^*|)) \end{pmatrix} \geq 0.$$

Arguing as in the proof of Theorem 2.1 we then infer that

$$|\Phi(Z)| \leq \Phi(f(|Z|)) \# V \Phi(g(|Z^*|)) V^*$$

for some unitary $V \in \mathbb{M}_m$. \square

Corollary 2.7. *Let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a 2-positive linear map, let $Z \in \mathbb{M}_n$ and let $p \in (-\infty, \infty)$. Then, there exists a unitary $V \in \mathbb{M}_m$ such that*

$$|\Phi(Z)| \leq \Phi(|Z|^{1+p}) \# V \Phi(|Z^*|^{1-p}) V^*.$$

Here, if Z is not invertible, the nonpositive powers of $|Z|$ and $|Z^*|$ are understood as generalized inverses, in particular $|Z|^0$ is the support projection of Z and $|Z^*|^0$ the range projection. The case $p = 0$ reads as

$$|\Phi(Z)| \leq \Phi(|Z|) \# V \Phi(|Z^*|) V^*. \quad (2.4)$$

However, in general (2.4) does not hold if the 2-positivity assumption is dropped, as shown by the next example.

Example 2.8. Let $\Phi : \mathbb{M}_2 \rightarrow \mathbb{M}_2$, $\Phi(X) = X + X^T$, and let

$$Z = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

Then, for any pair of unitary matrices $U, V \in \mathbb{M}_2$,

$$25 = \det |\Phi(Z)| > \det \{U \Phi(|Z|) U^* \# V \Phi(|Z^*|) V^*\} = 16.$$

3 Some consequences

From these results follow some eigenvalue inequalities. Given $S, T \in \mathbb{M}_n^+$, recall that the (weak) log-majorisation $S \prec_{w\log} T$ means that a series of n eigenvalue inequalities holds: For $k = 1, \dots, n$,

$$\prod_{j=1}^k \lambda_j^\downarrow(S) \leq \prod_{j=1}^k \lambda_j^\downarrow(T),$$

where $\lambda_j^\downarrow(\cdot)$ still stand for the eigenvalues arranged in nonincreasing order.

Theorem 2.1 and Corollary 2.3 are equivalent, though the most useful statement is Theorem 2.1. For instance, if $Z \in \mathbb{M}_n$ is a shift (a weighted permutation matrix), then $|Z^*|$ and $|Z|$ are diagonal matrices, and comparing with a diagonal J seems natural. A similar remark holds if Z is paranormal. For a general matrix Z , the Kato supremum, [5],

$$|Z| \vee |Z^*| = \lim_{p \rightarrow \infty} \left(\frac{|Z|^p + |Z^*|^p}{2} \right)^{1/p}$$

may be used as J in Theorem 2.1 with $f(t) = g(t) = t$. Some other natural choices for J are $(|Z|^q + |Z^*|^q)^{1/q}$, $1 \leq q < \infty$, still with $f(t) = g(t) = t$.

We list some consequences of Theorem 2.1.

Corollary 3.1. *Let $Z \in \mathbb{M}_n$, $J \in \mathbb{M}_n^+$, and let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map. If $Z \leq_{f,g} J$ for some functions satisfying $f(t)g(t) = t^2$, then,*

$$|\Phi(Z)| \prec_{w\log} \Phi(J)$$

and

$$\lambda_{j+k+1}^\downarrow(|\Phi(Z)|) \leq \sqrt{\lambda_{j+1}^\downarrow(\Phi(J)) \lambda_{k+1}^\downarrow(\Phi(J))}$$

for all $j, k = 0, 1, \dots, n$.

Proof. By a well-known property of the geometric mean, Theorem 2.1 says that

$$|\Phi(Z)| \leq \sqrt{\Phi(J)U} \sqrt{\Phi(J)V^*}$$

for some unitary matrices U and V . From Horn's inequalities we then get the log-majorisation, while the second series of inequalities follows from a standard min-max principle. \square

The Schur product of a matrix with the identity yields the diagonal part of the matrix. From the second statement of Corollary 3.1 we obtain:

Corollary 3.2. *Let $S \in \mathbb{M}_n^+$ and let $A \in \mathbb{M}_n$ be a contraction. If $s_1^\downarrow \geq \dots \geq s_n^\downarrow$ denote the diagonal entries of S arranged in decreasing order, then, for all $j, k = 0, 1, \dots$,*

$$\lambda_{j+k+1}^\downarrow(|S \circ A|) \leq \sqrt{s_{j+1}^\downarrow s_{k+1}^\downarrow}.$$

Remark 3.3. For $j = k = 1$, we recapture (1.2). If $S \in \mathbb{M}_n^+$ is expansive, Corollary 3.2 shows that

$$\lambda_{2j+1}^\downarrow(S \circ S^{-1}) \leq s_{j+1}^\downarrow.$$

If $S \in \mathbb{M}_n^+$ is a contraction, then

$$\lambda_{2j+1}^\downarrow(S \circ S) \leq s_{j+1}^\downarrow.$$

Corollary 3.4. *Let $Z \in \mathbb{M}_n$, $J \in \mathbb{M}_n^+$, and let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map. If $Z \leq_{f,g} J$ for some functions satisfying $f(t)g(t) = t^2$, then*

$$\left\{ \prod_{j=1}^k \lambda_j^\uparrow(|\Phi(Z)|) \right\}^2 \leq \prod_{j=1}^k \lambda_j^\uparrow(\Phi(J)) \lambda_j^\downarrow(\Phi(J))$$

for all $k = 1, \dots, n$.

Proof. Let \mathcal{S} be any subspace of \mathbb{C}^m , and denote by $A_{\mathcal{S}}$ the compression of $A \in \mathbb{M}_m$ onto \mathcal{S} . From Theorem 2.1 and Ando's inequality [1, Theorem 3] we infer

$$|\Phi(Z)|_{\mathcal{S}} \leq (\Phi(J))_{\mathcal{S}} \# (V\Phi(J)V^*)_{\mathcal{S}}.$$

So,

$$\det |\Phi(Z)|_{\mathcal{S}} \leq \left\{ \det (\Phi(J))_{\mathcal{S}} \det (\Phi(J))_{V^*(\mathcal{S})} \right\}^{1/2}.$$

Letting \mathcal{S} be the spectral subspace corresponding to the k smallest eigenvalues of $\Phi(J)$ and using standard min-max principles completes the proof. \square

Every n -by- n complex matrix Z has a decomposition in real and imaginary parts, its so-called Cartesian decomposition, $Z = X + iY$ where X and Y are self-adjoints. In general, the operator inequality

$$|Z| \leq |X| + |Y| \quad (3.1)$$

does not hold. However, a well-known substitute is the (weak) log-majorisation

$$|Z| \prec_{w\log} |X| + |Y|, \quad (3.2)$$

see [3, Corollary 2.11] for a stronger statement. We may use Corollary 2.2 to obtain further extensions of the triangle inequality (3.2).

Corollary 3.5. *Let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ be a positive linear map and let $Z \in \mathbb{M}_n$ with Cartesian decomposition $Z = X + iY$. Then, there exists a unitary $V \in \mathbb{M}_m$ such that*

$$|\Phi(Z)| \leq \Phi(|X| + |Y|) \# V \Phi(|X| + |Y|) V^*.$$

Proof. Consider the normal matrix in \mathbb{M}_{2n}

$$N = \begin{pmatrix} X & 0 \\ 0 & iY \end{pmatrix}$$

and apply Corollary 2.2 to this matrix and the positive linear map $\Psi : \mathbb{M}_{2n} \rightarrow \mathbb{M}_m$ defined as $\Psi = \Phi \circ \Lambda$ where Λ is the partial trace on $\mathbb{M}_{2n} = \mathbb{M}_2(\mathbb{M}_n)$,

$$\Lambda \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = A + D.$$

Since $\Psi(N) = \Phi(Z)$ and $\Psi(|N|) = \Phi(|X| + |Y|)$, the corollary is established. \square

Corollary 3.6. *If $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_m$ is a positive linear map and $Z \in \mathbb{M}_n$ has Cartesian decomposition $Z = X + iY$, then,*

$$|\Phi(Z)| \prec_{w\log} \Phi(|X| + |Y|).$$

Corollary 3.7. *If $Z \in \mathbb{M}_n$ has Cartesian decomposition $Z = X + iY$, then,*

$$\|(|X| + |Y|)^{-1/2} Z (|X| + |Y|)^{-1/2}\|_\infty \leq 1.$$

Proof. This is a direct consequence of Corollary 3.6 applied to the positive linear map $T \mapsto (|X| + |Y|)^{-1/2} T (|X| + |Y|)^{-1/2}$. \square

Remark 3.8. Corollary 3.7 is surprising. We could think that this follows from the polar decomposition $Z = |Z^*|^{1/2} U |Z|^{1/2}$ and the plausible fact that $|Z|^{1/2} (|X| + |Y|)^{-1/2}$ is contractive. But in general this matrix is not contractive, as it would be equivalent to the untrue inequality (3.1).

Corollary 3.9. *If $Z \in \mathbb{M}_n$ has Cartesian decomposition $Z = X + iY$, then,*

$$\rho(Z(|X| + |Y|)^{-1}) \leq 1.$$

Proof. This is a straightforward consequence of Corollary 3.7. □

Remark 3.10. Corollary 3.9 seems surprising too: It may happen that

$$\|Z(|X| + |Y|)^{-1}\|_\infty > 1.$$

Otherwise we would have $|Z|^2 \leq (|X| + |Y|)^2$, entailing the untrue inequality (3.1).

4 Comments

This article is a continuation of [4]. Indeed, the idea of revisiting the Russo-Dye Theorem comes from that paper, where Corollary 2.2 is proved with several related sharp inequalities. We can state a version of Theorem 2.1 for operator algebras. Let \mathcal{A} be a unital C^* -algebra, and let \mathcal{B} be a von Neumann algebra.

Theorem 4.1. *Let $Z \in \mathcal{A}$, $J \in \mathcal{A}^+$, and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a positive linear map. Suppose that $Z \leq_{f,g} J$ for some continuous functions satisfying $f(t)g(t) = t^2$. Then, there exists a partial isometry $V \in \mathcal{B}$ such that*

$$|\Phi(Z)| \leq \Phi(J)\#V\Phi(J)V^*.$$

The proof follows exactly the same steps as in the matrix case. Note that V^* is the partial isometry occurring in the polar decomposition $\Phi(Z) = V^*|\Phi(Z)|$, in general it is not a unitary operator, and it belongs to the von Neumann algebra generated by $\Phi(Z)$. If \mathcal{A} is also a von Neumann algebra, then $f(t)$ and $g(t)$ can be bounded Borel functions.

The assumption of Corollary 2.5 can be replaced by the condition that $|Z^*| \leq \rho|Z|$ for some $\rho > 0$, which makes sense also for noninvertible operators. With $\rho = 1$, Corollary 2.5 then takes the following form for infinite dimensional operators.

Corollary 4.2. *Let $Z \in \mathcal{A}$ be semi-hyponormal and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a positive linear map. Then, there exists a partial isometry $V \in \mathcal{B}$ such that*

$$|\Phi(Z)| \leq \Phi(|Z|)\#V\Phi(|Z|)V^*.$$

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