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Pinchings and positive linear maps

Jean-Christophe Bourin* and Eun-Young Lee†

Abstract. We employ the [pinching theorem](#), ensuring that some operators A admit *any* sequence of contractions as an operator diagonal of A , to deduce/improve two recent theorems of Kennedy-Skoufranis and Loreaux-Weiss for conditional expectations onto a masa in the algebra of operators on a Hilbert space. Similarly, we obtain a proof of a theorem of Akeman and Anderson showing that positive contractions in a continuous masa can be lifted to a projection. We also discuss a few corollaries for sums of two operators in the same unitary orbit.

Keywords: Pinching, essential numerical range, positive linear maps, conditional expectation onto a masa, unitary orbit.

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1 The pinching theorem

We recall two theorems which are fundamental in the next sections to obtain several results about positive linear maps, in particular conditional expectations, and unitary orbits. These theorems were established in [4], we also refer to this article for various definitions and properties of the essential numerical range $W_e(A)$ of an operator A in the algebra $L(\mathcal{H})$ of all (bounded linear) operators on an infinite dimensional, separable (real or complex) Hilbert space \mathcal{H} .

We denote by \mathcal{D} the unit disc of \mathbb{C} . We write $A \simeq B$ to mean that the operators A and B are unitarily equivalent. This relation is extended to operators possibly acting on different Hilbert spaces, typically, A acts on \mathcal{H} and B acts on an infinite dimensional subspace \mathcal{S} of \mathcal{H} , or on the spaces $\mathcal{H} \oplus \mathcal{H}$ or $\oplus^\infty \mathcal{H}$.

Theorem 1.1. *Let $A \in L(\mathcal{H})$ with $W_e(A) \supset \mathcal{D}$ and $\{X_i\}_{i=1}^\infty$ a sequence in $L(\mathcal{H})$ such that $\sup_i \|X_i\| < 1$. Then, a decomposition $\mathcal{H} = \bigoplus_{i=1}^\infty \mathcal{H}_i$ holds with $A_{\mathcal{H}_i} \simeq X_i$ for all i .*

Of course, the direct sum refers to an orthogonal decomposition, and $A_{\mathcal{H}_i}$ stands for the compression of A onto the subspace \mathcal{H}_i .

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Theorem 1.1 tells us that we have a unitary congruence between an operator in $L(\oplus^\infty \mathcal{H})$ and a “pinching” of A ,

$$\bigoplus_{i=1}^{\infty} X_i \simeq \sum_{i=1}^{\infty} E_i A E_i$$

for some sequence of mutually orthogonal infinite dimensional projections $\{E_i\}_{i=1}^{\infty}$ in $L(\mathcal{H})$ summing up to the identity I . Thus $\{X_i\}_{i=1}^{\infty}$ can be regarded as an operator diagonal of A . In particular, if X is an operator on \mathcal{H} with $\|X\| < 1$, then, A is unitarily congruent to an operator on $\mathcal{H} \oplus \mathcal{H}$ of the form,

$$A \simeq \begin{pmatrix} X & * \\ * & * \end{pmatrix}. \quad (1.1)$$

For a sequence of normal operators, Theorem 1.1 admits a variation. Given $\mathcal{A}, \mathcal{B} \subset \mathbb{C}$, the notation $\mathcal{A} \subset_{st} \mathcal{B}$ means that $\mathcal{A} + r\mathcal{D} \subset \mathcal{B}$ for some $r > 0$.

Theorem 1.2. *Let $A \in L(\mathcal{H})$ and let $\{X_i\}_{i=1}^{\infty}$ be a sequence of normal operators in $L(\mathcal{H})$ such that $\cup_{i=1}^{\infty} W(X_i) \subset_{st} W_e(A)$. Then, a decomposition $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ holds with $A_{\mathcal{H}_i} \simeq X_i$ for all i .*

If all the operators are self-adjoint, this is true for the strict inclusion in \mathbb{R} (if $\mathcal{A}, \mathcal{B} \subset \mathbb{R}$, the notation $\mathcal{A} \subset_{st} \mathcal{B}$ then means that $\mathcal{A} + r\mathcal{I} \subset \mathcal{B}$ for some $r > 0$, where $\mathcal{I} = [-1, 1]$). This is actually an easy consequence of Theorem 1.1 or Theorem 1.2.

Corollary 1.3. *Let $A \in L(\mathcal{H})$ be self-adjoint and let $\{X_i\}_{i=1}^{\infty}$ be a sequence of self-adjoint operators in $L(\mathcal{H})$ such that $\cup_{i=1}^{\infty} W(X_i) \subset_{st} W_e(A)$. Then, a decomposition $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ holds with $A_{\mathcal{H}_i} \simeq X_i$ for all i .*

To get Corollary 1.3 from Theorem 1.2, let $W_e(A) = [a, b]$ and write $A = A_1 + A_2 + A_3 + A_4$ where $A_i A_j = 0$ if $i \neq j$, with $a \in \sigma_e(A_1) \cap \sigma_e(A_2)$ and $b \in \sigma_e(A_3) \cap \sigma_e(A_4)$. Apply Theorem 1.2 to the normal operator $\tilde{A} = (1+i)A_1 + (1-i)A_2 + (1+i)A_3 + (1-i)A_4$ as $\cup_{i=1}^{\infty} W(X_i) \subset_{st} \text{conv}\{(1 \pm i)a; (1 \pm i)b\} \subset W_e(\tilde{A})$. We get a decomposition $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ with $\tilde{A}_{\mathcal{H}_i} \simeq X_i$ for all i . Therefore, taking real parts we also have $A_{\mathcal{H}_i} \simeq X_i$.

In Section 3, our concern is the study of generalized diagonals, i.e., conditional expectations onto a masa in $L(\mathcal{H})$, of the unitary orbit of an operator. The pinching theorems are the good tools for this study; we easily obtain and considerably improve two recent theorems, of Kennedy and Skoufranis for normal operators, and Loreaux and Weiss for idempotent operators. For self-adjoint idempotents, i.e., projections, and continuous masas, we obtain a theorem due to Akemann and Anderson. Section 4 deals with an application to the class of unital, positive linear maps which are trace preserving. Section 5 collects a few questions on possible extension of Theorems 1.1 and 1.2 in the setting of von Neumann algebras.

The next section gives applications which only require (1.1). These results mainly focus on sums of two operators in a unitary orbit.

2 Sums in a unitary orbit

We recall a straightforward consequence of (1.1) for the weak convergence, [4, Corollary 2.4].

Corollary 2.1. *Let $A, X \in L(\mathcal{H})$ with $W_e(A) \supset \mathcal{D}$ and $\|X\| \leq 1$. Then there exists a sequence of unitaries $\{U_n\}_{n=1}^\infty$ in $L(\mathcal{H})$ such that*

$$\text{wot} \lim_{n \rightarrow +\infty} U_n A U_n^* = X.$$

Of course, we cannot replace the weak convergence by the strong convergence; for instance if A is invertible and $\|Xh\| < \|A^{-1}\|^{-1}$ for some unit vector h , then X cannot be a strong limit from the unitary orbit of A . However, the next best thing does happen. Moreover, this is even true for the $*$ -strong operator topology.

Corollary 2.2. *Let $A, X \in L(\mathcal{H})$ with $W_e(A) \supset \mathcal{D}$ and $\|X\| \leq 1$. Then there exist two sequences of unitaries $\{U_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ in $L(\mathcal{H})$ such that*

$$* \text{ sot} \lim_{n \rightarrow +\infty} \frac{U_n A U_n^* + V_n A V_n^*}{2} = X.$$

Proof. From (1.1) we have

$$A \simeq \begin{pmatrix} X & -R \\ -S & T \end{pmatrix}.$$

Hence there exist two unitaries $U, V : \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}$ such that

$$\frac{U A U^* + V A V^*}{2} = \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix}. \quad (2.1)$$

Now let $\{e_n\}_{n=1}^\infty$ be a basis of \mathcal{H} and choose any unitary $W_n : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}$ such that $W_n(e_j \oplus 0) = e_j$ for all $j \leq n$. Then

$$X_n := W_n \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} W_n^*$$

strongly converges to X . Indeed, $\{X_n\}$ is bounded in norm and, for all j , $X_n e_j \rightarrow X e_j$. Taking adjoints,

$$X_n^* = W_n \begin{pmatrix} X^* & 0 \\ 0 & T^* \end{pmatrix} W_n^*,$$

we also have $X_n^* \rightarrow X$ strongly. Setting $U_n = W_n U$ and $V_n = W_n V$ and using (2.1) completes the proof. \square

Remark 2.3. Corollary 2.2 does not hold for the convergence in norm. We give an example. Consider the permutation matrix

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and set $A = 2(\oplus^\infty T)$ regarded as an operator in $L(\mathcal{H})$. Then $W_e(A) \supset \mathcal{D}$, however 0 cannot be a norm limit of means of two operators in the unitary orbit of A . Indeed 0 cannot be a norm limit of means of two operators in the unitary orbit of $(A + A^*)/2$ as $(A + A^*)/2 = 2I - 3P$ for some projection P .

Remark 2.4. The converse of Corollary 2.2 holds: if $A \in L(\mathcal{H})$ has the property that any contraction is a strong limit of a mean of two operators in its unitary orbit, then necessarily $W_e(A) \supset \mathcal{D}$. This is checked by arguing as in the proof of Corollary 4.1.

We reserve the word “projection” for self-adjoint idempotent. A strong limit of uniformly bounded idempotent operators is still an idempotent; thus, the next corollary is rather surprising.

Corollary 2.5. *Fix $\alpha > 0$. There exists an idempotent $Q \in L(\mathcal{H})$ such that for every $X \in L(\mathcal{H})$ with $\|X\| \leq \alpha$ we have two sequences of unitaries $\{U_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ in $L(\mathcal{H})$ for which*

$$* \text{ sot } \lim_{n \rightarrow +\infty} U_n Q U_n^* + V_n Q V_n^* = X.$$

Proof. Let $a > 0$, define a two-by-two idempotent matrix

$$M_a = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} \tag{2.2}$$

and set $Q = \oplus^\infty M_a$ regarded as an operator in $L(\mathcal{H})$. Since the numerical range $W(\cdot)$ of

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

is \mathcal{D} , we infer that $W(2\alpha^{-1}M_a) = W_e(2\alpha^{-1}Q) \supset \mathcal{D}$ for a large enough a . The result then follows from Corollary 2.2 with $A = 2\alpha^{-1}Q$ and the contraction $\alpha^{-1}X$. \square

Corollary 2.5 does not hold for the convergence in norm.

Proposition 2.6. *Let $X \in L(\mathcal{H})$ be of the form $\lambda I + K$ for a compact operator K and a scalar $\lambda \notin \{0, 1, 2\}$. Then X is not norm limit of $U_n Q U_n^* + V_n Q V_n^*$ for any sequences of unitaries $\{U_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ and any idempotent Q in $L(\mathcal{H})$.*

Proof. First observe that if $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ are two bounded sequences in $L(\mathcal{H})$ such that $A_n - B_n \rightarrow 0$ in norm, then we also have $A_n^2 - B_n^2 \rightarrow 0$ in norm; indeed

$$A_n^2 - B_n^2 = A_n(A_n - B_n) + (A_n - B_n)B_n.$$

Now, suppose that $\lambda \neq 1$ and that we have the (norm) convergence,

$$U_n Q U_n^* + V_n Q V_n^* \rightarrow \lambda I + K.$$

Then we also have

$$W_n Q W_n^* - (-Q + \lambda I + U_n^* K U_n) \rightarrow 0 \tag{2.3}$$

where $W_n := U_n^* V_n$. Hence, by the previous observation,

$$(W_n Q W_n^*)^2 - (-Q + \lambda I + U_n^* K U_n)^2 \rightarrow 0,$$

that is

$$W_n Q W_n^* - (-Q + \lambda I + U_n^* K U_n)^2 \rightarrow 0. \tag{2.4}$$

Combining (2.3) and (2.4) we get

$$(-Q + \lambda I + U_n^* K U_n) - (-Q + \lambda I + U_n^* K U_n)^2 \rightarrow 0$$

hence

$$(-2 + 2\lambda)Q + (\lambda - \lambda^2)I + K_n \rightarrow 0$$

for some bounded sequence of compact operators K_n . Since $\lambda \neq 1$, we have

$$Q = \frac{\lambda}{2}I + L$$

for some compact operator L . Since Q is idempotent, either $\lambda = 2$ or $\lambda = 0$. \square

The operator X in Proposition 2.6 has the special property that $W_e(X)$ is reduced to a single point. However Proposition 2.6 may also hold when $W_e(X)$ has positive measure.

Corollary 2.7. *Let Q be an idempotent in $L(\mathcal{H})$ and $z \in \mathbb{C} \setminus \{0, 1, 2\}$. Then, there exists $\alpha > 0$ such that the following property holds:*

If $X \in L(\mathcal{H})$ satisfies $\|X - zI\| \leq \alpha$, then X is not norm limit of $U_n Q U_n^ + V_n Q V_n^*$ for any sequences of unitaries $\{U_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ in $L(\mathcal{H})$.*

Proof. By the contrary, zI would be a norm limit of $U_n Q U_n^* + V_n Q V_n^*$ for some unitaries U_n, V_n , contradicting Proposition 2.6. \square

More operators with large numerical and essential numerical ranges are given in the next proposition. An operator X is stable when its real part $(X + X^*)/2$ is negative definite (invertible).

Proposition 2.8. *If $X \in L(\mathcal{H})$ is stable, then X is not norm limit of $U_nQU_n^* + V_nQV_n^*$ for any sequences of unitaries $\{U_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ and any idempotent Q in $L(\mathcal{H})$.*

Proof. We have a decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_{ns}$ in two invariant subspaces of Q such that Q acts on \mathcal{H}_s as a selfadjoint projection P , and Q acts on \mathcal{H}_{ns} as a purely nonselfadjoint idempotent, that is $A_{\mathcal{H}_{ns}}$ is unitarily equivalent to an operator on $\mathcal{F} \oplus \mathcal{F}$ of the form

$$Q_{\mathcal{H}_{ns}} \simeq \begin{pmatrix} I & 0 \\ R & 0 \end{pmatrix} \quad (2.5)$$

where R is a nonsingular (i.e., a zero kernel) positive operator on a Hilbert space \mathcal{F} , so

$$Q \simeq P \oplus \begin{pmatrix} I & 0 \\ R & 0 \end{pmatrix}. \quad (2.6)$$

Let Y be a norm limit of the sum of two sequences in the unitary orbit of Q . If the purely non-selfadjoint part \mathcal{H}_{ns} is vacuous, then Y is positive, hence $Y \neq X$. If \mathcal{H}_{ns} is not vacuous, (2.6) shows that

$$\begin{aligned} Q + Q^* &\simeq 2P \oplus \begin{pmatrix} 2I & R \\ R & 0 \end{pmatrix} \\ &\simeq 2P \oplus \left\{ \begin{pmatrix} I & I \\ I & I \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \right\}. \end{aligned}$$

This implies that $\|(Q + Q^*)_+\| \geq \|(Q + Q^*)_-\|$, therefore $Y + Y^*$ cannot be negative definite, hence $X \neq Y$. \square

It is known [13] that any operator is the sum of five idempotents. We close this section by asking whether Corollary 2.5 admits a substitute for Banach space operators.

Question 2.9. *Let \mathcal{X} be a separable Banach space and $T \in L(\mathcal{X})$, the linear operators on \mathcal{X} . Do there exist two sequences $\{P_n\}_{n=1}^\infty$ and $\{Q_n\}_{n=1}^\infty$ of idempotents in $L(\mathcal{X})$ such that $T = \text{sot} \lim_{n \rightarrow +\infty} (P_n + Q_n)$?*

3 Conditional expectation onto a masa

3.1 Conditional expectation of general operators

Kennedy and Skoufranis have studied the following problem: Let \mathfrak{X} be a maximal abelian $*$ -subalgebra (masa) of a von Neumann algebra \mathfrak{M} , with corresponding expectation $\mathbb{E}_{\mathfrak{X}} : \mathfrak{M} \rightarrow \mathfrak{X}$ (i.e., a unital positive linear map such that $\mathbb{E}_{\mathfrak{X}}(XM) = X\mathbb{E}_{\mathfrak{X}}(M)$ for all $X \in \mathfrak{X}$ and $M \in \mathfrak{M}$). Given a normal operator $A \in \mathfrak{M}$, determine the image by $\mathbb{E}_{\mathfrak{X}}$ of the unitary orbit of A ,

$$\Delta_{\mathfrak{X}}(A) = \{ \mathbb{E}_{\mathfrak{X}}(UAU^*) : U \text{ a unitary in } \mathfrak{M} \}.$$

In several cases, they determined the norm closure of $\Delta_{\mathfrak{X}}(A)$. In particular, [11, Theorem 1.2] can be stated in the following two propositions.

Proposition 3.1. *Let \mathfrak{X} be a masa in $L(\mathcal{H})$, $X \in \mathfrak{X}$, and A a normal operator in $L(\mathcal{H})$. If $\sigma(X) \subset \text{conv}\sigma_e(A)$, then X lies in the norm closure of $\Delta_{\mathfrak{X}}(A)$.*

Proposition 3.2. *Let \mathfrak{X} be a continuous masa in $L(\mathcal{H})$, $X \in \mathfrak{X}$, and A a normal operator in $L(\mathcal{H})$. If X lies in the norm closure of $\Delta_{\mathfrak{X}}(A)$, then $\sigma(X) \subset \text{conv}\sigma_e(A)$.*

Since we deal with normal operators, $\sigma(X) \subset \text{conv}\sigma_e(A)$ means $W(X) \subset W_e(A)$. Proposition 3.2 needs the continuous assumption. It is a rather simple fact; we generalize it in Lemma 3.5: *Conditional expectations reduce essential numerical ranges, $W_e(\mathbb{E}_{\mathfrak{X}}(T)) \subset W_e(T)$ for all $T \in L(\mathcal{H})$.* Thus, the main point of [11, Theorem 1.2] is Proposition 3.1 which says that if $W(X) \subset W_e(A)$ then X can be approximated by operators of the form $\mathbb{E}_{\mathfrak{X}}(UAU^*)$ with unitaries U . With the slightly stronger assumption $W(X) \subset_{st} W_e(A)$, Theorem 1.2 guarantees, via the following corollary, that X is exactly of this form. Furthermore the normality assumption on A is not necessary.

Corollary 3.3. *Let \mathfrak{X} be a masa in $L(\mathcal{H})$, $X \in \mathfrak{X}$ and $A \in L(\mathcal{H})$. If $W(X) \subset_{st} W_e(A)$, then $X = \mathbb{E}_{\mathfrak{X}}(UAU^*)$ for some unitary operator $U \in L(\mathcal{H})$.*

Proof. First, we note a simple fact: Let $\{P_i\}_{i=1}^{\infty}$ be a sequence of orthogonal projections in \mathfrak{X} such that $\sum_{i=1}^{\infty} P_i = I$, and let $Z \in L(\mathcal{H})$ such that $P_i Z P_i \in \mathfrak{X}$ for all i . Then, we have a strong sum

$$\mathbb{E}_{\mathfrak{X}}(Z) = \sum_{i=1}^{\infty} P_i Z P_i.$$

Now, denote by \mathcal{H}_i the range of P_i and assume $\dim \mathcal{H}_i = \infty$ for all i . We have $W(X_{\mathcal{H}_i}) \subset W(X)$, hence

$$\cup_{i=1}^{\infty} W(X_{\mathcal{H}_i}) \subset_{st} W_e(A).$$

We may then apply Theorem 1.2 and get a unitary U on $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ such that

$$A \simeq UAU^* = \begin{pmatrix} X_{\mathcal{H}_1} & * & \cdots & \cdots \\ * & X_{\mathcal{H}_2} & * & \cdots \\ \vdots & * & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Since $0 \oplus \cdots \oplus X_{\mathcal{H}_i} \oplus 0 \cdots \in \mathfrak{X}$ for all i , the previous simple fact shows that

$$\mathbb{E}_{\mathfrak{X}}(UAU^*) = \bigoplus_{i=1}^{\infty} X_{\mathcal{H}_i} = X.$$

□

We remark that Corollary 3.3 also covers the assumption $W(X) \subset W_e(A)$ of Proposition 3.1. Indeed, $W_e(A) \subset_{st} W_e(A + D)$ for some normal operator D with arbitrarily small norm, and we may apply Corollary 3.3 to X and $A + D$.

Kadison's article [9] completely describes the diagonals of a projection, thus, providing in his terminology a carpenter theorem for discrete masas $L(\mathcal{H})$. Our method can be used to obtain a similar statement for continuous masas in $L(\mathcal{H})$ given in the next corollary. This result is due to Akemann and Anderson, see [1, Corollary 6.19]. The case of a masa in a type-II₁ factor is solved in Ravichandran's paper [14].

Corollary 3.4. *Let \mathfrak{X} be a continuous masa in $L(\mathcal{H})$ and let $X \in \mathfrak{X}$ be a positive contraction. Then there exists a projection P in $L(\mathcal{H})$ such that $X = \mathbb{E}_{\mathfrak{X}}(P)$.*

Proof. We may suppose that $0 < \|Xh\| < 1$ for all unit vectors $h \in \mathcal{H}$; otherwise decompose X as $Q + X_0$ for some projection $Q \in \mathfrak{X}$ and consider X_0 in place of X . Then we have a decomposition $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$, where each summand \mathcal{H}_i is the range of some projection in \mathfrak{X} , such that the decomposition $X = \bigoplus_{i=1}^{\infty} X_{\mathcal{H}_i}$ satisfies $W(X_{\mathcal{H}_i}) \subset_{st} [0, 1]$ for all i . We may further decompose each Hilbert space \mathcal{H}_i as $\mathcal{H}_i = \bigoplus_{j=1}^{\infty} \mathcal{H}_{i,j}$ where $\mathcal{H}_{i,j}$ is the range of some nonzero projection in \mathfrak{X} . We have a corresponding decomposition $X = \bigoplus_{i=1}^{\infty} \left(\bigoplus_{j=1}^{\infty} X_{\mathcal{H}_{i,j}} \right)$. Since $\bigcup_{j=1}^{\infty} W(X_{\mathcal{H}_{i,j}}) \subset W(X_{\mathcal{H}_i}) \subset_{st} [0, 1]$, Corollary 1.3 yields some projection $P_i \in L(\mathcal{H}_i)$, $W_e(P) = [0, 1]$, such that

$$P_i = \begin{pmatrix} X_{\mathcal{H}_{i,1}} & * & \cdots & \cdots \\ * & X_{\mathcal{H}_{i,2}} & * & \cdots \\ \vdots & * & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Letting $P = \bigoplus_{i=1}^{\infty} P_i$, we have $X = \mathbb{E}_{\mathfrak{X}}(P)$. □

3.2 A reduction lemma

The following result extends Proposition 3.2, the “easy” part of Kennedy-Skoufranis' theorem [11, Theorem 1.2].

Lemma 3.5. *If \mathfrak{X} is a masa in $L(\mathcal{H})$ and $Z \in L(\mathcal{H})$, then $W(\mathbb{E}_{\mathfrak{X}}(Z)) \subset \overline{W}(Z)$ and $W_e(\mathbb{E}_{\mathfrak{X}}(Z)) \subset W_e(Z)$.*

Proof. (1) Assume Z is normal. We may identify the unital C^* -algebra \mathfrak{A} spanned by Z with $C^0(\sigma(Z))$ via a $*$ -isomorphism $\varphi : C^0(\sigma(Z)) \rightarrow \mathfrak{A}$ with $\varphi(z \mapsto z) = Z$. Let $h \in \mathcal{H}$ be a unit vector. For $f \in C^0(\sigma(Z))$, set

$$\psi(f) = \langle h, \mathbb{E}_{\mathfrak{X}}(\varphi(f))h \rangle.$$

Then ψ is a positive linear functional on $C^0(\sigma(Z))$ and $\psi(1) = 1$. Thus ψ is a Radon measure induced by a probability measure μ ,

$$\psi(f) = \int_{\sigma(Z)} f(z) d\mu(z).$$

We then have $\langle h, \mathbb{E}_{\mathfrak{X}}(Z)h \rangle = \psi(z) \in \text{conv}(\sigma(Z))$. Since $\text{conv}(\sigma(Z)) = \overline{W}(Z)$, we obtain $W(\mathbb{E}_{\mathfrak{X}}(Z)) \subset \overline{W}(Z)$.

(2) Let Z be a general operator in $L(\mathcal{H})$ and define a conditional expectation

$$\mathbb{E}_2 : L(\mathcal{H} \oplus \mathcal{H}) \rightarrow \mathfrak{X} \oplus \mathfrak{X}$$

by

$$\mathbb{E}_2 \left(\begin{pmatrix} A & C \\ D & B \end{pmatrix} \right) = \begin{pmatrix} \mathbb{E}_{\mathfrak{X}}(A) & 0 \\ 0 & \mathbb{E}_{\mathfrak{X}}(B) \end{pmatrix}.$$

From the first part of the proof, we infer

$$W(\mathbb{E}_{\mathfrak{X}}(Z)) \subset W \left(\begin{pmatrix} \mathbb{E}_{\mathfrak{X}}(Z) & 0 \\ 0 & \mathbb{E}_{\mathfrak{X}}(B) \end{pmatrix} \right) \subset \overline{W} \left(\begin{pmatrix} Z & C \\ D & B \end{pmatrix} \right)$$

whenever $\begin{pmatrix} Z & C \\ D & B \end{pmatrix}$ is normal. Since we have, by a simple classical fact [8],

$$\overline{W}(Z) = \bigcap \overline{W} \left(\begin{pmatrix} Z & C \\ D & B \end{pmatrix} \right)$$

where the intersection runs over all B, C, D such that $\begin{pmatrix} Z & C \\ D & B \end{pmatrix}$ is normal, we obtain $W(\mathbb{E}_{\mathfrak{X}}(Z)) \subset \overline{W}(Z)$.

(3) We deal with the essential numerical range inclusion. We can split \mathfrak{X} into its discrete part \mathfrak{D} and continuous part \mathfrak{C} with the corresponding decomposition of the Hilbert space,

$$\mathfrak{X} = \mathfrak{D} \oplus \mathfrak{C}, \quad \mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c.$$

We then have

$$W_e(\mathbb{E}_{\mathfrak{X}}(Z)) = \text{conv} \{W_e(\mathbb{E}_{\mathfrak{D}}(Z_{\mathcal{H}_d})); W_e(\mathbb{E}_{\mathfrak{C}}(Z_{\mathcal{H}_c}))\}. \quad (3.1)$$

We have an obvious inclusion

$$W_e(\mathbb{E}_{\mathfrak{D}}(Z_{\mathcal{H}_d})) \subset W_e(Z_{\mathcal{H}_d}). \quad (3.2)$$

On the other hand, for all compact operators $K \in L(\mathcal{H})$,

$$W_e(\mathbb{E}_{\mathfrak{C}}(Z_{\mathcal{H}_c})) = W_e(\mathbb{E}_{\mathfrak{C}}(Z_{\mathcal{H}_c}) + K_{\mathcal{H}_c}) = W_e(\mathbb{E}_{\mathfrak{C}}(Z_{\mathcal{H}_c} + K_{\mathcal{H}_c})) \subset \overline{W}(Z_{\mathcal{H}_c} + K_{\mathcal{H}_c})$$

by the simple folklore fact that a conditional expectation onto a continuous masa vanishes on compact operators and part (2) of the proof. Thus, when K runs over all compact operators, we obtain

$$W_e(\mathbb{E}_{\mathfrak{D}}(Z_{\mathcal{H}_c})) \subset W_e(Z_{\mathcal{H}_c}). \quad (3.3)$$

Combining (3.1), (3.2) and (3.3) completes the proof. \square

3.3 Conditional expectation of idempotent operators

For discrete masas, unlike continuous masas [10], there is a unique conditional expectation, which merely consists in extracting the diagonal with respect to an orthonormal basis. In a recent article, Loreaux and Weiss give a detailed study of diagonals of idempotents in $L(\mathcal{H})$. They established that a nonzero idempotent Q has a zero diagonal with respect to some orthonormal basis if and only if Q is not a Hilbert-Schmidt perturbation of a projection (i.e., a self-adjoint idempotent). They also showed that any sequence $\{a_n\} \in l^\infty$ such that $|a_n| \leq \alpha$ for all n and, for some a_{n_0} , $a_k = a_{n_0}$ for infinitely many k , one has an idempotent Q such that $\|Q\| \leq 18\alpha + 4$ and Q admits $\{a_n\}$ as a diagonal with respect to some orthonormal basis [12, Proposition 3.4]. Using this, they proved that any sequence in l^∞ is the diagonal of some idempotent operator [12, Theorem 3.6], answering a question of Jasper. This statement is in the range of Theorem 1.1. Further, it is not necessary to confine to diagonals, i.e., discrete masas, and the constant $18\alpha + 4$ can be improved; in the next corollary we have 3 as the best constant when $\alpha = 1$.

Corollary 3.6. *Let \mathfrak{X} be a masa in $L(\mathcal{H})$ and $\alpha > 0$. There exists an idempotent $Q \in L(\mathcal{H})$, such that for all $X \in \mathfrak{X}$ with $\|X\| < \alpha$, we have $X = \mathbb{E}_{\mathfrak{X}}(UQU^*)$ for some unitary operator $U \in L(\mathcal{H})$. If $\alpha = 1$, $\|Q\| = 3$ is the smallest possible norm.*

Proof. As in the proof of Corollary 2.5 we have an idempotent Q such that $W_e(Q) \supset \alpha\mathcal{D}$, hence the first and main part of Corollary 3.6 follows from Corollary 3.3. The remaining parts require a few computations.

To obtain the bound 3 when $\alpha = 1$ we get a closer look at $\oplus^\infty M_a$ with M_a given by (2.2) where a is a positive scalar. We have

$$\begin{aligned} W(M_a) &= \{ \langle h, M_a h \rangle : h \in \mathbb{C}^2, \|h\| = 1 \} \\ &= \{ |h_1|^2 + a\overline{h_2}h_1 : |h_1|^2 + |h_2|^2 = 1 \}, \end{aligned}$$

hence, with $h_1 = re^{i\theta}$, $h_2 = \sqrt{1-r^2}e^{i\alpha}$,

$$W(M_a) = \bigcup_{0 \leq r \leq 1} \left\{ r^2 + ar\sqrt{1-r^2}e^{i(\theta-\alpha)} : \theta, \alpha \in [0, 2\pi] \right\}.$$

Therefore $W(M_a)$ is a union of circles Γ_r with centers r^2 and radii $ar\sqrt{1-r^2}$. To have $\mathcal{D} \subset W(M_a)$ it is necessary and sufficient that $-1 \in \Gamma_r$ for some $r \in [0, 1]$, hence

$$a = \frac{1+r^2}{r\sqrt{1-r^2}}. \quad (3.4)$$

Now we minimize $a = a(r)$ given by (3.4) when $r \in (0, 1)$ and thus obtain the matrix M_{a_*} with smallest norm such that $W(M_{a_*}) \supset \mathcal{D}$. Observe that $a(r) \rightarrow +\infty$ as $r \rightarrow 0$ and as $r \rightarrow 1$, and

$$r^2(1-r^2)^{3/2}a'(r) = 3r^2 - 1.$$

Thus $a(r)$ takes its minimal value a_* when $r = 1/\sqrt{3}$. We have $a_* = 2\sqrt{2}$, hence

$$\|M_{a_*}\| = 3.$$

Now, letting $Q = \bigoplus^{\infty} M_{a_*}$, we have $W_e(Q) = W(M_{a_*})$, so that Q is an idempotent in $L(\mathcal{H})$ such that $W_e(Q) \supset \mathcal{D}$, and thus by Corollary 3.3 any operator X such that $\|X\| < 1$ satisfies $\mathbb{E}_{\mathfrak{X}}(UQU^*) = X$ for some unitary U .

It remains to check that if Q is an idempotent such that Corollary 3.6 holds for any operator X such that $\|X\| < 1$, then $\|Q\| \geq 3$. To this end, we consider the purely nonselfadjoint part $Q_{\mathcal{H}_{ns}}$ of Q in (2.5),

$$Q_{\mathcal{H}_{ns}} \simeq \begin{pmatrix} I & 0 \\ R & 0 \end{pmatrix}.$$

We have $W_e(Q) \supset \mathcal{D}$ if and only if $W_e(Q_{\mathcal{H}_{ns}}) \supset \mathcal{D}$. By Lemma 3.5 this is necessary. We may approximate $W_e(Q_{\mathcal{H}_{ns}})$ with slightly larger essential numerical ranges, by using a positive diagonalizable operator R_{ε} such that $R_{\varepsilon} \geq R \geq R_{\varepsilon} - \varepsilon I$, for which

$$W_e \left(\begin{pmatrix} I & 0 \\ R_{\varepsilon} & 0 \end{pmatrix} \right) = W_e \left(\bigoplus_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ a_n & 0 \end{pmatrix} \right)$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive scalars, the eigenvalues of R_{ε} . By the previous step of the proof, this essential numerical range contains \mathcal{D} if and only if $\overline{\lim} a_n \geq a_*$. If this holds for all $\varepsilon > 0$, then $\|Q\| \geq 3$. \square

4 Unital, trace preserving positive linear maps

Unital positive linear maps $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_n$, the matrix algebra, which preserve the trace play an important role in matrix analysis and its applications. These maps are sometimes called doubly stochastic [3].

We say that $\Phi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ is trace preserving if it preserves the trace ideal \mathcal{T} and $\text{Tr } \Phi(Z) = \text{Tr } Z$ for all $Z \in \mathcal{T}$.

Corollary 4.1. *Let $A \in L(\mathcal{H})$. The following two conditions are equivalent:*

- (i) $W_e(A) \supset \mathcal{D}$.
- (ii) For all $X \in L(\mathcal{H})$ with $\|X\| < 1$, there exists a unital, trace preserving, positive linear map $\Phi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ such that $\Phi(A) = X$.

We may further require in (ii) that Φ is completely positive and sot- and wot-sequentially continuous.

Proof. Assume (i). By Theorem 1.1 we have a unitary $U : \mathcal{H} \rightarrow \oplus^\infty \mathcal{H}$ such that

$$A \simeq UAU^* = \begin{pmatrix} X & * & \cdots & \cdots \\ * & X & * & \cdots \\ \vdots & * & X & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Now consider the map $\Psi : L(\oplus^\infty \mathcal{H}) \rightarrow L(\mathcal{H})$,

$$\begin{pmatrix} Z_{1,1} & Z_{1,2} & \cdots \\ Z_{2,1} & Z_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \mapsto \sum_{i=1}^{\infty} 2^{-i} Z_{i,i}$$

and define $\Phi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ as $\Phi(T) = \Psi(UTU^*)$. Since both Ψ and the unitary congruence with U are sot- and wot-sequentially continuous, and trace preserving, completely positive and unital, so is Φ . Further $\Phi(A) = X$.

Assume (ii) and suppose that $z \notin W_e(A)$ and $|z| < 1$ in order to reach a contradiction. If $z = |z|e^{i\theta}$, replacing A by $e^{-i\theta}A$, we may assume $1 > z \geq 0$. Hence,

$$W_e((A + A^*)/2) \subset (-\infty, z]$$

and there exists a selfadjoint compact operator L such that

$$\frac{A + A^*}{2} \leq zI + L.$$

This implies that $X := \frac{1+z}{2}I$ cannot be in the range of Φ for any unital, trace preserving positive linear map. Indeed, we would have

$$\frac{1+z}{2}I = \frac{X + X^*}{2} = \Phi\left(\frac{A + A^*}{2}\right) \leq zI + \Phi(L)$$

which is not possible as $\Phi(L)$ is compact. \square

In the finite dimensional setting, two Hermitian matrices A and X satisfy the relation $X = \Phi(A)$ for some positive, unital, trace preserving linear map if and only if X is in the convex hull of the unitary orbit of A . In the infinite dimensional setting, if two Hermitian $A, X \in L(\mathcal{H})$ satisfy $W_e(A) \supset [-1, 1]$ and $\|X\| \leq 1$, then X is in the norm closure of the unitary orbit of A . This is easily checked by approximating the operators with diagonal operators. Such an equivalence might not be brought out to the setting of Corollary 4.1.

Question 4.2. *Do there exist $A, X \in L(\mathcal{H})$ such that $W_e(A) \supset \mathcal{D}$, $\|X\| < 1$, and X does not belong to the norm closure of the convex hull of the unitary orbit of A ?*

Here we mention a result of Wu [15, Theorem 6.11]: *If $A \in L(\mathcal{H})$ is not of the form scalar plus compact, then every $X \in L(\mathcal{H})$ is a linear combination of operators in the unitary orbit of A .*

If one deletes the positivity assumption, the most regular class of linear maps on $L(\mathcal{H})$ might be given in the following definition.

Definition 4.3. A linear map $\Psi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ is said ultra-regular if it fulfills two conditions:

- (u1) $\Psi(I) = I$ and Ψ is trace preserving.
- (u2) Whenever a sequence $A_n \rightarrow A$ for either the norm-, strong-, or weak-topology, then we also have $\Psi(A_n) \rightarrow \Psi(A)$ for the same type of convergence.

Any ultra-regular linear map preserves the set of essentially scalar operators (of the form $\lambda I + K$ with $\lambda \in \mathbb{C}$ and a compact operator K). For its complement, we state our last corollary.

Corollary 4.4. *Let $A \in L(\mathcal{H})$ be essentially nonscalar. Then, for all $X \in L(\mathcal{H})$ there exists a ultra-regular linear map $\Psi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ such that $\Psi(A) = X$.*

Proof. An operator is essentially nonscalar precisely when its essential numerical range is not reduced to a single point. So, let $a, b \in W_e(A)$, $a \neq b$. By a lemma of Anderson and Stampfli [2], A is unitarily equivalent to an operator on $\mathcal{H} \oplus \mathcal{H}$ of the form

$$B = \begin{pmatrix} D & * \\ * & * \end{pmatrix}$$

where $D = \bigoplus_{n=1}^{\infty} D_n$, with two by two matrices D_n ,

$$D_n = \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix}$$

such that $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow \infty$. We may assume that, for some $\alpha, \beta > 0$, we have $\alpha > |a_n| + |b_n|$ and $|a_n - b_n| > \beta$. Hence there exist $\gamma > 0$ and two by two invertible matrices T_n such that, for all n , $W(T_n D_n T_n^{-1}) \supset \mathcal{D}$ and $\|T_n\| + \|T_n^{-1}\| \leq \gamma$. So, letting $T = (\bigoplus_{n=1}^{\infty} T_n) \oplus I$, we obtain an invertible operator T on $\mathcal{H} \oplus \mathcal{H}$ such that $W_e(TBT^{-1}) \supset \mathcal{D}$.

Hence we have an invertible operator S on \mathcal{H} such that $W_e(SAS^{-1}) \supset \mathcal{D}$. Therefore we may apply Corollary 4.1 and obtain a wot- and sot-sequentially continuous, unital, trace preserving map Φ such that $\Phi(SAS^{-1}) = X$. Letting $\Psi(\cdot) = \Phi(S \cdot S^{-1})$ completes the proof. \square

We cannot find an alternative proof, not based on the pinching theorem, for Corollaries 4.1 and 4.4.

If we trust in Zorn, there exists a linear map $\Psi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ which satisfies the condition (u1) but not the condition (u2). Indeed, let $\{a_p\}_{p \in \Omega}$ be a basis in the Calkin algebra $\mathfrak{C} = L(\mathcal{H})/K(\mathcal{H})$, indexed on an ordered set Ω , whose first element a_{p_0} is the image of I by the canonical projection $\pi : L(\mathcal{H}) \rightarrow \mathfrak{C}$. Thus, for each operator X , we have a unique decomposition $\pi(X) = \sum_{p \in \Omega} (\pi(X))_p a_p$ with only finitely many nonzero

terms. Further $(\pi(X))_{p_0} = 0$ if X is compact, and $(\pi(I))_{p_0} = 1$. We then define a map $\psi : L(\mathcal{H}) \rightarrow L(\mathcal{H} \oplus \mathcal{H})$ by

$$\psi(X) = \begin{pmatrix} X & 0 \\ 0 & (\pi(X))_{p_0} I \end{pmatrix}.$$

Letting $\Psi(X) = V\psi(X)V^*$ where $V : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}$ is unitary, we obtain a linear map $\Psi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ which satisfies (u1) but not (u2): it is not norm continuous.

Let ω be a Banach limit on l^∞ and define a map $\phi : l^\infty \rightarrow l^\infty$, $\{a_n\} \mapsto \{b_n\}$, where $b_1 = \omega(\{a_n\})$ and $b_n = a_{n-1}$, $n \geq 2$. Letting $\Psi(X) = \phi(\text{diag}(X))$, where $\text{diag}(X)$ is the diagonal of $X \in \mathcal{H}$ in an orthonormal basis, we obtain a linear map Ψ which is norm continuous, satisfies (u1) but not (u2): it is not strongly sequentially continuous.

However, it seems not possible to define explicitly a linear map $\Psi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ satisfying (u1) but not (u2).

5 Pinchings in factors ?

We discuss possible extensions to our results to a von Neumann algebra \mathfrak{R} acting on a separable Hilbert space \mathcal{H} . First, we need to define an essential numerical range $W_e^{\mathfrak{R}}$ for \mathfrak{R} . Let $A \in \mathfrak{R}$. If \mathfrak{R} is type-III, then $W_e^{\mathfrak{R}}(A) := W_e(A)$. If \mathfrak{R} is type-II $_\infty$, then

$$W_e^{\mathfrak{R}}(A) := \bigcap_{K \in \mathcal{T}} \overline{W}(A + K)$$

where \mathcal{T} is the trace ideal in \mathfrak{R} (we may also use its norm closure \mathcal{K} , the ‘‘compact’’ operators in \mathfrak{R} , or any dense sequence in \mathcal{K})

Question 5.1. *In Corollaries 2.1, 2.2 and 2.5, can we replace $L(\mathcal{H})$ by a type-II $_\infty$ or -III factor \mathfrak{R} with $W_e^{\mathfrak{R}}$?*

Question 5.2. *In Corollaries 3.3 and 3.6, can we replace $L(\mathcal{H})$ by a type-II $_\infty$ or -III factor \mathfrak{R} with $W_e^{\mathfrak{R}}$?*

Question 5.3. *In Corollaries 4.1 and 4.4, can we replace $L(\mathcal{H})$ by a type-II $_\infty$ factor \mathfrak{R} with $W_e^{\mathfrak{R}}$?*

Recently, Dragan and Kaftal [7] obtained some decompositions for positive operators in von Neumann factors, which, in the case of $L(\mathcal{H})$ were first investigated in [5]-[6] by using Theorem 1.1. This suggests that our questions dealing with a possible extension to type-II $_\infty$ and -III factors also have an affirmative answer. In fact, it seems plausible that Theorem 1.1 and Theorem 1.2 admit a version for such factors and this would affirmatively answer these questions.

Let \mathfrak{R} be a type-II $_\infty$ or -III factor.

Definition 5.4. A sequence $\{V_i\}_{i=1}^{\infty}$ of isometries in \mathfrak{K} such that $\sum_{i=1}^{\infty} V_i V_i^* = I$ is called an isometric decomposition of \mathfrak{K} .

Conjecture 5.5. Let $A \in \mathfrak{K}$ with $W_e^{\mathfrak{K}}(A) \supset \mathcal{D}$ and $\{X_i\}_{i=1}^{\infty}$ a sequence in \mathfrak{K} such that $\sup_i \|X_i\| < 1$. Then, there exists an isometric decomposition $\{V_i\}_{i=1}^{\infty}$ of \mathfrak{K} such that $V_i^* A V_i = X_i$ for all i .

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