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ℓ^1 -CONTRACTIVE MAPS ON NONCOMMUTATIVE L^p -SPACES

CHRISTIAN LE MERDY AND SAFOURA ZADEH

ABSTRACT. Let $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ be a bounded operator between two noncommutative L^p -spaces, $1 \leq p < \infty$. We say that T is ℓ^1 -bounded (resp. ℓ^1 -contractive) if $T \otimes I_{\ell^1}$ extends to a bounded (resp. contractive) map from $L^p(\mathcal{M}; \ell^1)$ into $L^p(\mathcal{N}; \ell^1)$. We show that Yeadon's factorization theorem for L^p -isometries, $1 \leq p \neq 2 < \infty$, applies to an isometry $T: L^2(\mathcal{M}) \rightarrow L^2(\mathcal{N})$ if and only if T is ℓ^1 -contractive. We also show that a contractive operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is automatically ℓ^1 -contractive if it satisfies one of the following two conditions: either T is 2-positive; or T is separating, that is, for any disjoint $a, b \in L^p(\mathcal{M})$ (i.e. $a^*b = ab^* = 0$), the images $T(a), T(b)$ are disjoint as well.

1. INTRODUCTION

Let \mathcal{M} and \mathcal{N} be two semifinite von Neumann algebras. For any $1 \leq p < \infty$, consider the associated noncommutative L^p -spaces $L^p(\mathcal{M})$ and $L^p(\mathcal{N})$. A remarkable theorem of Yeadon [26] (see Theorem 3.1 below) asserts that if $p \neq 2$ and $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is a linear isometry, then there exist a normal Jordan homomorphism $J: \mathcal{M} \rightarrow \mathcal{N}$, a positive operator B affiliated with \mathcal{N} and a partial isometry $w \in \mathcal{N}$ such that $w^*wB = B$, $J(a)$ commutes with B for all $a \in \mathcal{M}$, and

$$(1) \quad T(a) = wBJ(a),$$

for all $a \in \mathcal{M} \cap L^p(\mathcal{M})$.

This striking factorization property is the noncommutative version of the celebrated description of isometries on classical (=commutative) L^p -spaces due to Banach [1] and Lamperti [13]. We refer to the books [3] and [4] for details on these results, complements and historical background.

The work presented in this paper was originally motivated by the following question, concerning the case $p = 2$: what are the linear isometries $T: L^2(\mathcal{M}) \rightarrow L^2(\mathcal{N})$ which admit a Yeadon type factorization, that is, isometries for which there exist J, B, w as above such that (1) holds true for any $a \in \mathcal{M} \cap L^2(\mathcal{M})$?

This issue leads us to introduce a new property, called ℓ^1 -boundedness, which is defined as follows. Consider the ℓ^1 -valued noncommutative L^p -space $L^p(\mathcal{M}; \ell^1)$ introduced by Junge [7] (see also [20] and [9]). Let $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ be a bounded operator. We say that T is ℓ^1 -bounded if $T \otimes I_{\ell^1}$ extends to a bounded map

$$T \overline{\otimes} I_{\ell^1}: L^p(\mathcal{M}; \ell^1) \longrightarrow L^p(\mathcal{N}; \ell^1).$$

We further say that T is ℓ^1 -contractive if the map $T \overline{\otimes} I_{\ell^1}$ is a contraction. The main result of this paper (Theorem 4.2 below) is that an isometry $T: L^2(\mathcal{M}) \rightarrow L^2(\mathcal{N})$ is ℓ^1 -contractive if and only if it admits a Yeadon type factorization.

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To explain the relevance of Theorem 4.2 we note that ℓ^1 -boundedness is a noncommutative analogue of regularity for maps acting on commutative L^p -spaces. (We refer to [18, Chapter 1] for definitions and background on regular maps.) It follows that Theorem 4.2 is a noncommutative extension of the well-known result stating that a linear isometry between commutative L^2 -spaces is a Lamperti operator if and only if it is contractively regular, if and only if it is a subpositive contraction (see e.g [15]).

The proof of Yeadon's theorem heavily relies on the fact that for $p \neq 2$, any linear isometry $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ has the following property: if $a, b \in L^p(\mathcal{M})$ are disjoint, that is $a^*b = ab^* = 0$, then $T(a)$ and $T(b)$ are disjoint as well. Such maps are called separating in the present paper. We show that a bounded operator $L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is separating if and only if it admits a Yeadon type factorization.

The concept of ℓ^1 -boundedness is interesting in its own sake and this paper aims at studying some of its main features. We show in particular that a contractive operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is automatically ℓ^1 -contractive either if T is separating (see Theorem 3.15) or if T is 2-positive (see Proposition 5.1).

2. NOTION OF ℓ^1 -BOUNDEDNESS AND BACKGROUND

In this section, we provide some background on noncommutative L^p -spaces and on the ℓ^1 -valued spaces $L^p(\mathcal{M}; \ell^1)$. Then we introduce the notions of ℓ^1 -boundedness and ℓ^1 -contractivity and establish some preliminary results.

Let \mathcal{M} be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace $\tau_{\mathcal{M}}$. We briefly recall the noncommutative L^p -spaces $L^p(\mathcal{M})$, $0 < p \leq \infty$, associated with $(\mathcal{M}, \tau_{\mathcal{M}})$ and some of their basic properties. The reader is referred to the survey [16] and references therein for details and further properties.

If $\mathcal{M} \subset B(\mathcal{H})$ acts on some Hilbert space \mathcal{H} , the elements of $L^p(\mathcal{M})$ can be viewed as closed densely defined (possibly unbounded) operators on \mathcal{H} . More precisely, let \mathcal{M}' denote the commutant of \mathcal{M} in $B(\mathcal{H})$. A closed densely defined operator a is said to be affiliated with \mathcal{M} if a commutes with every unitary of \mathcal{M}' . An affiliated operator a is called measurable (with respect to $(\mathcal{M}, \tau_{\mathcal{M}})$) if there is a positive number $\lambda > 0$ such that $\tau_{\mathcal{M}}(\varepsilon_{\lambda}) < \infty$, where $\varepsilon_{\lambda} = \chi_{[\lambda, \infty)}(|a|)$ is the projection associated with the indicator function of $[\lambda, \infty)$ in the Borel functional calculus of $|a|$. Then the set $L^0(\mathcal{M})$ of all measurable operators forms a $*$ -algebra (see e.g. [24, Chapter I] for a proof and also for the definitions of algebraic operations on $L^0(\mathcal{M})$). We proceed with defining $L^p(\mathcal{M})$ as a subspace of $L^0(\mathcal{M})$. First note that for any $a \in L^0(\mathcal{M})$ and any $0 < p < \infty$, the operator $|a|^p = (a^*a)^{\frac{p}{2}}$ belongs to $L^0(\mathcal{M})$. If $L^0(\mathcal{M})^+$ denotes the positive cone of $L^0(\mathcal{M})$, that is the set of all positive operators in $L^0(\mathcal{M})$, the trace $\tau_{\mathcal{M}}$ extends to a positive tracial functional on $L^0(\mathcal{M})^+$, taking values in $[0, \infty]$, also denoted by $\tau_{\mathcal{M}}$. For any $0 < p < \infty$, the noncommutative L^p -space, $L^p(\mathcal{M})$, associated with $(\mathcal{M}, \tau_{\mathcal{M}})$, is

$$L^p(\mathcal{M}) := \{a \in L^0(\mathcal{M}) : \tau_{\mathcal{M}}(|a|^p) < \infty\}.$$

For $a \in L^p(\mathcal{M})$, let $\|a\|_p := \tau_{\mathcal{M}}(|a|^p)^{\frac{1}{p}}$. For $1 \leq p < \infty$, $\|\cdot\|_p$ defines a complete norm, and for $p < 1$, a complete p -norm. We let $L^\infty(\mathcal{M}) := \mathcal{M}$, equipped with its operator norm $\|\cdot\|_\infty$.

For any $0 < p \leq \infty$ and any $a \in L^p(\mathcal{M})$, the adjoint operator a^* belongs to $L^p(\mathcal{M})$ and $\|a^*\|_p = \|a\|_p$. Furthermore, we have that $a^*a \in L^{\frac{p}{2}}(\mathcal{M})$ and $|a| \in L^p(\mathcal{M})$, with

$\|a\|_p = \|a\|_p$. More generally, for any $0 < p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have that $ab \in L^r(\mathcal{M})$ if $a \in L^p(\mathcal{M})$ and $b \in L^q(\mathcal{M})$, with Hölder's inequality

$$(2) \quad \|ab\|_r \leq \|a\|_p \|b\|_q.$$

For any $1 \leq p < \infty$, let $p' := \frac{p}{p-1}$ be the conjugate number of p . Then by (2), ab belongs to $L^1(\mathcal{M})$ for any $a \in L^p(\mathcal{M})$ and $b \in L^{p'}(\mathcal{M})$. Further the duality pairing

$$\langle a, b \rangle = \tau_{\mathcal{M}}(ab), \quad a \in L^p(\mathcal{M}), \quad b \in L^{p'}(\mathcal{M}),$$

yields an isometric isomorphism $L^p(\mathcal{M})^* = L^{p'}(\mathcal{M})$. In particular, we may identify $L^1(\mathcal{M})$ with the (unique) predual \mathcal{M}_* of \mathcal{M} . These duality results will be used without further reference in the paper.

We let $L^p(\mathcal{M})^+ := L^0(\mathcal{M})^+ \cap L^p(\mathcal{M})$ denote the positive cone of $L^p(\mathcal{M})$. A bounded operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ between two noncommutative L^p -spaces is called positive if it maps $L^p(\mathcal{M})^+$ into $L^p(\mathcal{N})^+$.

If $\mathcal{M} = B(\mathcal{H})$, the algebra of all bounded operators on \mathcal{H} , and $\tau_{\mathcal{M}} = tr$, the usual trace on $B(\mathcal{H})$, then the associated noncommutative L^p -space is the Schatten class $S^p(\mathcal{H})$. If $\mathcal{M} = L^\infty(\Omega, \mathcal{F}, \mu)$ is the commutative von Neumann algebra associated with a measure space $(\Omega, \mathcal{F}, \mu)$, then $L^p(\mathcal{M})$ coincides with the classical L^p -space $L^p(\Omega, \mathcal{F}, \mu)$.

Let tr denote the usual trace on $B(\ell^2)$ and consider the von Neumann algebra tensor product $B(\ell^2) \overline{\otimes} \mathcal{M}$, equipped with the normal semifinite faithful trace $tr \overline{\otimes} \tau_{\mathcal{M}}$ (see [23, Chapter V, Proposition 2.14]). Any element of $L^p(B(\ell^2) \overline{\otimes} \mathcal{M})$ can be regarded as an infinite matrix $(b_{ij})_{i,j \geq 1}$, with $b_{ij} \in L^p(\mathcal{M})$. We let $L^p(\mathcal{M}; \ell_c^2)$ denote the subspace of $L^p(B(\ell^2) \overline{\otimes} \mathcal{M})$ consisting of all matrices whose entries off the first column are all zero. We regard this space as a sequence space by saying that a sequence $(b_n)_{n \geq 1}$ of $L^p(\mathcal{M})$ belongs to $L^p(\mathcal{M}; \ell_c^2)$ if the infinite matrix

$$\begin{pmatrix} b_1 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & & \vdots & \\ b_n & 0 & \dots & 0 & \dots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

represents an element of $L^p(B(\ell^2) \overline{\otimes} \mathcal{M})$. Similarly, we define $L^p(\mathcal{M}; \ell_r^2)$ as the subspace of $L^p(B(\ell^2) \overline{\otimes} \mathcal{M})$ consisting of all matrices whose entries off the first row are all zero.

We let E_{ij} , $i, j \geq 1$, denote the usual matrix units of $B(\ell^2)$, and regard $S^p(\ell^2) \otimes L^p(\mathcal{M})$ as a subspace of $L^p(B(\ell^2) \overline{\otimes} \mathcal{M})$ in the usual way. For any finitely supported sequence $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ of $L^p(\mathcal{M})$, we have

$$(3) \quad \left\| \sum_{n=1}^{\infty} E_{n1} \otimes b_n \right\|_{L^p(\mathcal{M}; \ell_c^2)} = \left\| \sum_{n=1}^{\infty} b_n^* b_n \right\|_{\frac{p}{2}}^{\frac{1}{2}}$$

and

$$(4) \quad \left\| \sum_{n=1}^{\infty} E_{1n} \otimes a_n \right\|_{L^p(\mathcal{M}; \ell_r^2)} = \left\| \sum_{n=1}^{\infty} a_n a_n^* \right\|_{\frac{p}{2}}^{\frac{1}{2}}.$$

When $1 \leq p < \infty$, elements of $L^p(\mathcal{M}; \ell_c^2)$ and $L^p(\mathcal{M}; \ell_r^2)$ can be approximated by finitely supported sequences, thanks to the following (easy) result.

Lemma 2.1. *Let $1 \leq p < \infty$ and suppose that $(b_n)_{n \geq 1}$ is a sequence in $L^p(\mathcal{M})$. The following are equivalent:*

- (i) $(b_n)_{n \geq 1}$ belongs to $L^p(\mathcal{M}; \ell_c^2)$.
- (ii) There exists a positive constant K such that for every $N \geq 1$,

$$\left\| \sum_{n=1}^N E_{n1} \otimes b_n \right\|_{L^p(\mathcal{M}; \ell_c^2)} \leq K.$$

- (iii) The series $\sum_{n \geq 1} E_{n1} \otimes b_n$ converges in $L^p(\mathcal{M}; \ell_c^2)$.

Moreover, the same result holds with ℓ_c^2 replaced by ℓ_r^2 and E_{n1} replaced by E_{1n} .

Remark 2.2. By (3) and the Cauchy convergence test, we see that a sequence $(b_n)_{n \geq 1}$ in $L^p(\mathcal{M})$ satisfies the assertion (iii) of Lemma 2.1 if and only if the series $\sum_n b_n^* b_n$ converges in $L^{\frac{p}{2}}(\mathcal{M})$. In this case, the identity (3) holds true for $(b_n)_{n \geq 1}$.

Let $1 \leq p < \infty$. In [7], Junge defined $L^p(\mathcal{M}; \ell^1)$ as the space of all sequences $x = (x_n)_{n \geq 1}$ in $L^p(\mathcal{M})$ for which there exist families $(u_{kn})_{k, n \geq 1}$, $(v_{kn})_{k, n \geq 1} \in L^{2p}(\mathcal{M})$ and a positive constant K such that

$$(5) \quad \left\| \sum_{k, n=1}^N u_{kn} u_{kn}^* \right\|_p \leq K \quad \text{and} \quad \left\| \sum_{k, n=1}^N v_{kn}^* v_{kn} \right\|_p \leq K$$

for any $N \geq 1$, and $\sum_{k=0}^{\infty} u_{kn} v_{kn} = x_n$, for all $n \geq 1$. (The convergence of the series is ensured by (5) and Lemma 2.1.) He showed that this a Banach space when equipped with the norm

$$\|(x_n)_{n \geq 1}\|_{L^p(\mathcal{M}; \ell^1)} = \inf \left\{ \sup_N \left\| \sum_{k, n=1}^N u_{kn} u_{kn}^* \right\|_p^{\frac{1}{2}} \sup_N \left\| \sum_{k, n=1}^N v_{kn}^* v_{kn} \right\|_p^{\frac{1}{2}} \right\},$$

where the infimum is taken over all families $(u_{kn})_{k, n \geq 1}$ and $(v_{kn})_{k, n \geq 1}$ as above.

The following alternative description is well-known to specialists (and implicit in [17, pp. 537-538]). We give a proof for the sake of completeness.

Lemma 2.3. *Suppose that $1 \leq p < \infty$ and that $(x_n)_{n \geq 1}$ is a sequence in $L^p(\mathcal{M})$. Then the following are equivalent:*

- (i) $(x_n)_{n \geq 1}$ belongs to $L^p(\mathcal{M}; \ell^1)$ and $\|(x_n)_{n \geq 1}\|_{L^p(\mathcal{M}; \ell^1)} < 1$.
- (ii) There exist sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ in $L^{2p}(\mathcal{M})$ such that $a_n b_n = x_n$ for all $n \geq 1$, the series $\sum_n a_n a_n^*$ and $\sum_n b_n^* b_n$ converge in $L^p(\mathcal{M})$, and we have

$$\left\| \sum_{n=1}^{\infty} a_n a_n^* \right\|_p < 1 \quad \text{and} \quad \left\| \sum_{n=1}^{\infty} b_n^* b_n \right\|_p < 1.$$

Proof. The assertion “(ii) \Rightarrow (i)” is obvious. Conversely assume (i) and consider u_{kn}, v_{kn} in $L^{2p}(\mathcal{M})$ satisfying (5) for some $K < 1$, and such that

$$x_n = \sum_{k=1}^{\infty} u_{kn} v_{kn}, \quad \text{for all } n \geq 1.$$

We regard $L^p(\mathcal{M})$ as a subspace of $L^p(B(\ell^2) \overline{\otimes} \mathcal{M})$ by identifying any $b \in L^p(\mathcal{M})$ with $E_{11} \otimes b$. We set

$$u_n := (u_{kn})_{k \geq 1} \in L^{2p}(\mathcal{M}; \ell_r^2) \quad \text{and} \quad v_n := (v_{kn})_{k \geq 1} \in L^{2p}(\mathcal{M}; \ell_c^2)$$

for all $n \geq 1$, that we regard as elements of $L^{2p}(B(\ell^2) \overline{\otimes} \mathcal{M})$. Then $x_n = u_n v_n$ for all $n \geq 1$.

By polar decomposition, there exist $\varphi_n \in L^\infty(\mathcal{M}; \ell_r^2)$ and $\psi_n \in L^\infty(\mathcal{M}; \ell_c^2)$ such that

$$\|\varphi_n\|_\infty \leq 1, \quad \|\psi_n\|_\infty \leq 1, \quad u_n = |u_n^*| \varphi_n \quad \text{and} \quad v_n = \psi_n |v_n|.$$

If we let $a_n = |u_n^*|$ and $b_n = \varphi_n \psi_n |v_n|$, then we have

$$x_n = u_n v_n = |u_n^*| \varphi_n \psi_n |v_n| = a_n b_n$$

for all $n \geq 1$. Further a_n, b_n belong to $L^{2p}(\mathcal{M})$. Next we have

$$a_n a_n^* = |u_n^*|^2 = u_n u_n^* = \sum_{k=1}^{\infty} u_{kn} u_{kn}^*,$$

hence for any $N \geq 1$, $\|\sum_{n=1}^N a_n a_n^*\|_p \leq K$. By Lemma 2.1 and Remark 2.2, this implies the convergence of the series of the $a_n a_n^*$ in $L^p(\mathcal{M})$, with $\|\sum_{n=1}^{\infty} a_n a_n^*\|_p \leq K$. Likewise, since $(\varphi_n \psi_n)^* (\varphi_n \psi_n) \leq 1$, we have

$$\begin{aligned} b_n^* b_n &= |v_n| (\varphi_n \psi_n)^* (\varphi_n \psi_n) |v_n| \\ &\leq |v_n|^2 = v_n^* v_n = \sum_{k=1}^{\infty} v_{kn}^* v_{kn}, \end{aligned}$$

from which we deduce that the series of the $b_n^* b_n$ converges in $L^p(\mathcal{M})$, with $\|\sum_{n=1}^{\infty} b_n^* b_n\|_p \leq K$. This proves (ii). \square

When dealing with positive sequences, the study of the $L^p(\mathcal{M}, \ell^1)$ -norm is simple. We learnt the following result from [25].

Lemma 2.4. *Let $1 \leq p < \infty$, let $(x_n)_{n \geq 1}$ be a sequence of $L^p(\mathcal{M})$ and assume that $x_n \geq 0$ for any $n \geq 1$. The following are equivalent.*

- (i) $(x_n)_{n \geq 1}$ belongs to $L^p(\mathcal{M}; \ell^1)$.
- (ii) The series $\sum_n x_n$ converges in $L^p(\mathcal{M})$.

Further in this case, we have

$$(6) \quad \|(x_n)_{n \geq 1}\|_{L^p(\mathcal{M}; \ell^1)} = \left\| \sum_{n=1}^{\infty} x_n \right\|_p.$$

Proof. It follows from (3) and (4) that for any finitely supported families $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ in $L^{2p}(\mathcal{M})$, we have

$$\left\| \sum_{n=1}^{\infty} a_n b_n \right\|_p \leq \left\| \sum_{n=1}^{\infty} a_n a_n^* \right\|_p^{\frac{1}{2}} \left\| \sum_{n=1}^{\infty} b_n^* b_n \right\|_p^{\frac{1}{2}}.$$

The assertion “(i) \Rightarrow (ii)” and the inequality \geq in (6) follow at once (here we do not need any positivity assumption on the x_n).

Assume conversely that the series $\sum_n x_n$ converges in $L^p(\mathcal{M})$ and set $a_n = b_n = x_n^{\frac{1}{2}}$. Then the convergence of $\sum_n a_n a_n^*$ and $\sum_n b_n^* b_n$ are trivial and $x_n = a_n b_n$ for all $n \geq 1$. This implies (i), as well as the inequality \leq in (6). \square

We remark that for any $c = (c_n)_{n \geq 1} \in \ell^1$ and any $x \in L^p(\mathcal{M})$, the sequence $(c_n x)_{n \geq 1}$ belongs to $L^p(\mathcal{M}; \ell^1)$. Further the mapping $x \otimes c \mapsto (c_n x)_{n \geq 1}$ extends to an embedding

$$L^p(\mathcal{M}) \otimes \ell^1 \subset L^p(\mathcal{M}; \ell^1)$$

and with this convention, $L^p(\mathcal{M}) \otimes \ell^1$ is a dense subspace of $L^p(\mathcal{M}; \ell^1)$.

Let $(e_k)_{k \geq 1}$ denote the canonical basis of ℓ^1 . Then we let $L^p(\mathcal{M}; \ell^1_2)$ be the direct sum $L^p(\mathcal{M}) \oplus L^p(\mathcal{M})$ equipped with the norm $\|(x, y)\| = \|x \otimes e_1 + y \otimes e_2\|_{L^p(\mathcal{M}; \ell^1)}$, for any $x, y \in L^p(\mathcal{M})$.

Throughout the paper we will consider two semifinite von Neumann algebras \mathcal{M} and \mathcal{N} equipped with normal semifinite faithful traces $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively, and we will consider various bounded operators $L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, for $1 \leq p < \infty$.

Definition 2.5. We say that a bounded operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is

- (i) ℓ^1 -bounded if $T \otimes I_{\ell^1}$ extends to a bounded map

$$T \overline{\otimes} I_{\ell^1}: L^p(\mathcal{M}; \ell^1) \longrightarrow L^p(\mathcal{N}; \ell^1).$$

In this case, the norm of $T \overline{\otimes} I_{\ell^1}$ is called the ℓ^1 -bounded norm of T and is denoted by $\|T\|_{\ell^1}$;

- (ii) ℓ^1 -contractive if it is ℓ^1 -bounded and $\|T\|_{\ell^1} \leq 1$;
 (iii) ℓ^1_2 -contractive if for every $x, y \in L^p(\mathcal{M})$, we have

$$\|(T(x), T(y))\|_{L^p(\mathcal{N}; \ell^1_2)} \leq \|(x, y)\|_{L^p(\mathcal{M}; \ell^1_2)}.$$

Remark 2.6. In the case $p = 1$, we note that $L^1(\mathcal{M}; \ell^1) \simeq \ell^1(L^1(\mathcal{M}))$ isometrically. This implies that any bounded operator $T: L^1(\mathcal{M}) \rightarrow L^1(\mathcal{N})$ is automatically ℓ^1 -bounded, with $\|T\|_{\ell^1} = \|T\|$.

In the rest of this section, we compare ℓ^1 -boundedness with Pisier's notion of complete regularity. Let us recall that for a hyperfinite von Neumann algebra \mathcal{M} and an operator space E , Pisier [20, Chapter 3] introduced a vector valued noncommutative L^p -space $L^p(\mathcal{M})[E]$. Let $\text{Max}(\ell^1)$ be ℓ^1 equipped with its so-called maximal operator space structure (see e.g. [19, Chapter 3]). It turns out that

$$(7) \quad L^p(\mathcal{M}; \ell^1) \simeq L^p(\mathcal{M})[\text{Max}(\ell^1)] \quad \text{isometrically,}$$

when \mathcal{M} is hyperfinite (see [7, 9]).

Assume that the semifinite von Neumann algebras \mathcal{M}, \mathcal{N} are both hyperfinite. Let $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ be a bounded operator. Following Pisier [21], T is called completely regular if there exists a constant $K \geq 0$ such that for any $n \geq 1$,

$$\|T \otimes I_{M_n}: L^p(\mathcal{M})[M_n] \longrightarrow L^p(\mathcal{N})[M_n]\| \leq K.$$

In this case the least possible K is denoted by $\|T\|_{reg}$ and is called the completely regular norm of T . It is noticed in [21] that if T is completely regular, then for any operator space E , $T \otimes I_E$ (uniquely) extends to a bounded operator $T \overline{\otimes} I_E$ from $L^p(\mathcal{M})[E]$ into $L^p(\mathcal{N})[E]$, with

$$\|T \overline{\otimes} I_E: L^p(\mathcal{M})[E] \longrightarrow L^p(\mathcal{N})[E]\| \leq \|T\|_{reg}.$$

Combining this fact with (7), we obtain that if $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is completely regular, then T is ℓ^1 -bounded, with $\|T\|_{\ell^1} \leq \|T\|_{reg}$.

The next example shows that the converse is wrong.

Example 2.7. We consider the specific case $\mathcal{M} = B(\ell^2)$, and we let $T: S^p(\ell^2) \rightarrow S^p(\ell^2)$ be the transposition map. This map is ℓ^1 -contractive. This is an easy fact, which is a special case of Theorem 3.15 below. Here is a direct argument.

Let $(x_n)_{n \geq 1}$ be in $L^p(\mathcal{M}; \ell^1)$ and let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences belonging to $L^{2p}(\mathcal{M}; \ell_r^2)$ and $L^{2p}(\mathcal{M}; \ell_c^2)$, respectively, such that $x_n = a_n b_n$ for any $n \geq 1$. Then $T(x_n) = {}^t b_n {}^t a_n$ for any $n \geq 1$, $({}^t a_n)_{n \geq 1}$ belongs to $L^{2p}(\mathcal{M}; \ell_c^2)$, $({}^t b_n)_{n \geq 1}$ belongs to $L^{2p}(\mathcal{M}; \ell_r^2)$ and we both have

$$\|({}^t a_n)_n\|_{L^{2p}(\mathcal{M}; \ell_c^2)} = \|(a_n)_n\|_{L^{2p}(\mathcal{M}; \ell_r^2)} \quad \text{and} \quad \|({}^t b_n)_n\|_{L^{2p}(\mathcal{M}; \ell_r^2)} = \|(b_n)_n\|_{L^{2p}(\mathcal{M}; \ell_c^2)}.$$

The result follows at once.

Let us now prove that T is not completely regular. We need a little operator space technology, in particular we use the Haagerup tensor product \otimes_h , the operator spaces R, C and the interpolation spaces $R(\theta) = [C, R]_\theta$, $0 \leq \theta \leq 1$, for which we refer to [19].

Let $(e_k)_{k \geq 1}$ be the canonical basis of ℓ^2 . As it is outlined in [20, Theorem 1.1 and p.20], for any operator space E , the mapping $e_i \otimes x \otimes e_j \mapsto E_{ij} \otimes x$, for $i, j \geq 1$ and $x \in E$, uniquely extends to an isometric isomorphism

$$(8) \quad S^p[E] \simeq R\left(\frac{1}{p}\right) \otimes_h E \otimes_h R\left(1 - \frac{1}{p}\right).$$

Assume that T is completely regular and let $K = \|T\|_{reg}$. Apply (8) with $E = R$. For any $n \geq 1$, we have

$$(T \otimes I_R) \left(\sum_{k=1}^n e_k \otimes e_k \otimes e_1 \right) = \sum_{k=1}^n e_1 \otimes e_k \otimes e_k,$$

hence

$$\left\| \sum_{k=1}^n e_k \otimes e_k \right\|_{R \otimes_h R(1 - \frac{1}{p})} \leq K \left\| \sum_{k=1}^n e_k \otimes e_k \right\|_{R(\frac{1}{p}) \otimes_h R}.$$

It follows from the calculations in [20, Chapter 1] (see also [8]) that

$$R\left(\frac{1}{p}\right) \otimes_h R \simeq S^{2p} \quad \text{and} \quad R \otimes_h R\left(1 - \frac{1}{p}\right) \simeq S^{(2p)'}$$

isometrically, where $(2p)'$ is the conjugate number of $2p$. Let $Q_n \in B(\ell^2)$ be the orthogonal projection onto $\text{Span}\{e_1, \dots, e_n\}$. Then $\sum_{k=1}^n e_k \otimes e_k = Q_n$ in the above identifications. Hence we obtain that

$$\|Q_n\|_{(2p)'} \leq K \|Q_n\|_{2p}.$$

Since $\|Q_n\|_q = n^{\frac{1}{q}}$ for any $1 < q < \infty$, we obtain that $n^{1 - \frac{1}{2p}} \leq K n^{\frac{1}{2p}}$, equivalently, $n \leq K n^{\frac{1}{p}}$, for any $n \geq 1$. This yields a contradiction if $p > 1$. In the case $p = 1$, the fact that T is not completely regular is obtained by applying (8) with $E = C$ instead of $E = R$.

Remark 2.8. Here we consider the commutative case. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For any operator space E , $L^p(L^\infty(\Omega))[E]$ coincides with the Bochner space $L^p(\Omega; E)$. Thus if $(\Omega', \mathcal{F}', \mu')$ is another measure space and $T: L^p(\Omega) \rightarrow L^p(\Omega')$ is any bounded operator, then T is completely regular (in the above sense) if and only if T is regular in the lattice sense (see [18, Chapter 1] for details and background). Moreover in this case, the completely regular norm of T coincides with its regular norm. It follows from [18, Paragraph 1.2] that T is regular if (and only if) $T \otimes I_{\ell^1}$ extends to a bounded map $T \overline{\otimes} I_{\ell^1}$ from $L^p(\Omega; \ell^1)$ into $L^p(\Omega'; \ell^1)$ and in this case, we have $\|T \overline{\otimes} I_{\ell^1}\| = \|T\|_{reg}$. Consequently,

$T: L^p(\Omega) \rightarrow L^p(\Omega')$ is ℓ^1 -bounded if and only if T is regular and in this case, the ℓ^1 -bounded norm of T is equal to its regular norm.

3. DISJOINTNESS AND SEPARATING OPERATORS

In [12], Kan introduced the concept of Lamperti operators on commutative L^p -spaces, which include L^p -isometries, $1 \leq p \neq 2 < \infty$, and positive L^2 -isometries. He then proved a structural theorem for such operators. In this section we provide a noncommutative version of this result, as well as a connection with ℓ^1 -boundedness.

Let us first recall some facts related to Jordan homomorphisms that we require in this section. (We refer to [22], [5] and [11, Exercices 10.5.21-10.5.31] for general information.) A Jordan homomorphism between von Neumann algebras \mathcal{M} and \mathcal{N} is a linear map $J: \mathcal{M} \rightarrow \mathcal{N}$ that satisfies $J(a^2) = J(a)^2$ and $J(a^*) = J(a)^*$, for every $a \in \mathcal{M}$. We say that the Jordan homomorphism $J: \mathcal{M} \rightarrow \mathcal{N}$ is a Jordan monomorphism when J is one-to-one. Any Jordan homomorphism is a positive contraction and any Jordan monomorphism is an isometry.

Let $J: \mathcal{M} \rightarrow \mathcal{N}$ be a Jordan homomorphism and let $\mathcal{D} \subset \mathcal{N}$ be the von Neumann algebra generated by $J(\mathcal{M})$. Let $e := J(1)$. Then e is a projection and e is the unit of \mathcal{D} . According to [22, Theorem 3.3] (see also [5, Corollary 7.4.9.]), there exist projections g and f in the center of \mathcal{D} such that

- (i) $g + f = e$;
- (ii) $a \mapsto J(a)g$ is a $*$ -homomorphism;
- (iii) $a \mapsto J(a)f$ is an anti- $*$ -homomorphism;

Let $\mathcal{N}_1 = g\mathcal{N}g$ and $\mathcal{N}_2 = f\mathcal{N}f$. We let $\pi: \mathcal{M} \rightarrow \mathcal{N}_1$ and $\sigma: \mathcal{M} \rightarrow \mathcal{N}_2$ be defined by $\pi(a) = J(a)g$ and $\sigma(a) = J(a)f$, for all $a \in \mathcal{M}$. Then, $\mathcal{D} = \mathcal{D}_1 \overset{\infty}{\oplus} \mathcal{D}_2$ and $J(a) = \pi(a) + \sigma(a)$, for all $a \in \mathcal{M}$. We will use the suggestive notations

$$(9) \quad J = \begin{pmatrix} \pi & 0 \\ 0 & \sigma \end{pmatrix} \quad \text{and} \quad J(a) = \begin{pmatrix} \pi(a) & 0 \\ 0 & \sigma(a) \end{pmatrix}$$

to refer to such a central decomposition. We note that J is normal (i.e. w^* -continuous) if and only if π and σ are normal.

Assume that $\mathcal{M} \subset B(\mathcal{H})$ acts on some Hilbert space \mathcal{H} and let x be a closed densely defined operator on \mathcal{H} , affiliated with \mathcal{M} . If x is self-adjoint, with polar decomposition $x = w|x|$, we let $s(x)$ denote the projection $w^*w (= ww^*)$, called the support of x .

The following remarkable characterization of L^p -isometries, $1 \leq p \neq 2 < \infty$, is at the root of our investigations.

Theorem 3.1 (Yeadon [26]). *For $1 \leq p < \infty$, $p \neq 2$, a bounded operator*

$$T: L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{N})$$

is an isometry if and only if there exist a normal Jordan monomorphism $J: \mathcal{M} \rightarrow \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a positive operator B affiliated with \mathcal{N} , which verify the following conditions:

- (a) $T(a) = wBJ(a)$ for all $a \in \mathcal{M} \cap L^p(\mathcal{M})$;
- (b) $w^*w = J(1) = s(B)$;
- (c) every spectral projection of B commutes with $J(a)$, for all $a \in \mathcal{M}$;

(d) $\tau_{\mathcal{M}}(a) = \tau_{\mathcal{N}}(B^p J(a))$ for all $a \in \mathcal{M}^+$.

This motivates the introduction of the following concept. Since we would like to consider maps $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ that are not necessarily isometries we drop part (d) of Theorem 3.1 in our definition.

Definition 3.2. We say that a bounded operator $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, $1 \leq p \leq \infty$, has a “**Yeadon type factorization**” if there exist a normal Jordan homomorphism $J : \mathcal{M} \rightarrow \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a positive operator B affiliated with \mathcal{N} , which verify the following conditions:

- (a) $T(a) = wBJ(a)$ for all $a \in \mathcal{M} \cap L^p(\mathcal{M})$;
- (b) $w^*w = J(1) = s(B)$;
- (c) every spectral projection of B commutes with $J(a)$, for all $a \in \mathcal{M}$.

Remark 3.3. The argument in the last paragraph of the proof of [26, Theorem 2] shows that for an operator $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ with a Yeadon type factorization, w , B and J from Definition 3.2 are uniquely determined by T . We call (w, B, J) the Yeadon triple of the operator T .

The crucial property that allowed Yeadon to describe L^p -isometries is that they map disjoint elements to disjoint elements. This property is shared by operators other than isometries and as we show in Proposition 3.11, it characterizes operators with a Yeadon type factorization. We introduce the relevant concepts and supply a few preparatory results.

Definition 3.4. Let a and b be elements in $L^p(\mathcal{M})$, $1 \leq p \leq \infty$. We say a and b are disjoint if $ab^* = a^*b = 0$.

Lemma 3.5. *The elements a and b in $L^p(\mathcal{M})$ are disjoint if and only if $|a|$ and $|b|$ are disjoint and $|a^*|$ and $|b^*|$ are disjoint.*

Proof. Let $a, b \in L^p(\mathcal{M})$. First we note that a and b are disjoint if and only if

$$\text{Im}(b) \subseteq \text{Im}(a)^\perp \quad \text{and} \quad \text{Ker}(a)^\perp \subseteq \text{Ker}(b).$$

This implies that $|a|$ and $|b|$ are disjoint if and only if $\text{Ker}(a)^\perp = \text{Ker}(|a|)^\perp \subseteq \text{Ker}(|b|) = \text{Ker}(b)$. Then $|a^*|$ and $|b^*|$ are disjoint if and only if $\text{Ker}(a^*)^\perp \subseteq \text{Ker}(b^*)$, which is itself equivalent to $\text{Im}(b) \subseteq \text{Im}(a)^\perp$. Therefore, a and b are disjoint if and only if $|a|$ and $|b|$ are disjoint and $|a^*|$ and $|b^*|$ are disjoint. \square

Remark 3.6. Consider the special case $p = 2$ and let a, b be two positive elements in $L^2(\mathcal{M})$. Then a and b are disjoint if (and only if) $\tau_{\mathcal{M}}(ab) = 0$, that is, if and only if a and b are orthogonal in the Hilbertian sense. Indeed assume that $\tau_{\mathcal{M}}(ab) = 0$. Then $0 = \tau_{\mathcal{M}}(ab) = \tau_{\mathcal{M}}((a^{\frac{1}{2}}b^{\frac{1}{2}})(b^{\frac{1}{2}}a^{\frac{1}{2}}))$ and $a^{\frac{1}{2}}b^{\frac{1}{2}}$ is the adjoint of $b^{\frac{1}{2}}a^{\frac{1}{2}}$. Since the trace $\tau_{\mathcal{M}}$ is faithful, this implies that $a^{\frac{1}{2}}b^{\frac{1}{2}} = 0$. Therefore, $ab = a^{\frac{1}{2}}(a^{\frac{1}{2}}b^{\frac{1}{2}})b^{\frac{1}{2}} = 0$. Hence a and b are disjoint.

Definition 3.7. We say that a bounded operator $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, $1 \leq p \leq \infty$, is separating if whenever $a, b \in L^p(\mathcal{M})$ are disjoint, then $T(a)$ and $T(b)$ are disjoint.

Lemma 3.8. *Any Jordan homomorphism $J : \mathcal{M} \rightarrow \mathcal{N}$ is separating.*

Proof. Let $J: \mathcal{M} \rightarrow \mathcal{N}$ be a Jordan homomorphism and consider a decomposition $J = \begin{pmatrix} \pi & 0 \\ 0 & \sigma \end{pmatrix}$ as in (9).

Suppose that a and b are disjoint elements of \mathcal{M} . Then we have

$$\begin{aligned} J(a)^* J(b) &= \begin{pmatrix} \pi(a)^* \pi(b) & 0 \\ 0 & \sigma(a)^* \sigma(b) \end{pmatrix} \\ &= \begin{pmatrix} \pi(a^*b) & 0 \\ 0 & \sigma(ba^*) \end{pmatrix} = 0. \end{aligned}$$

Similarly, we can show that $J(a)J(b)^* = 0$, and therefore J is separating. \square

Lemma 3.9. *Suppose that a bounded operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, $1 \leq p < \infty$, is separating on $\mathcal{M} \cap L^p(\mathcal{M})$, that is, $T(a)$ and $T(b)$ are disjoint for any disjoint a and b in $\mathcal{M} \cap L^p(\mathcal{M})$. Then T is separating.*

Proof. Let $a, b \in L^p(\mathcal{M})$ with $a^*b = ab^* = 0$. We let $a = v|a|$ and $b = w|b|$ be the polar decompositions of a and b , respectively. By Lemma 3.5, we have $|a||b| = 0$.

For any $n \geq 1$, let $p_n := \chi_{[-n, n]}(|a|)$ be the projection associated with the indicator function of $[-n, n]$ in the Borel functional calculus of $|a|$, and similarly let $q_n := \chi_{[-n, n]}(|b|)$. Let $a_n := ap_n$ and $b_n := bq_n$. We have

$$p_n|a| = |a|p_n \rightarrow |a| \quad \text{and} \quad q_n|b| = |b|q_n \rightarrow |b|$$

in $L^p(\mathcal{M})$. This implies that $a_n \rightarrow a$ and $b_n \rightarrow b$ in $L^p(\mathcal{M})$.

Note that for any $n \geq 1$, a_n and b_n belong to $\mathcal{M} \cap L^p(\mathcal{M})$. Further we have

$$a_n^* b_n = p_n a^* b q_n = 0$$

and

$$a_n b_n^* = v|a|p_n q_n |b|w^* = v p_n |a||b| q_n w^* = 0.$$

Thus a_n and b_n are disjoint.

By assumption this implies that $T(a_n)^* T(b_n) = 0$ and $T(a_n) T(b_n)^* = 0$. Passing to the limit, we deduce that $T(a)^* T(b) = 0$ and $T(a) T(b)^* = 0$. \square

From now on, we consider

$$\mathcal{E} := \{e \in \mathcal{M} : e \text{ is a projection and } \tau_{\mathcal{M}}(e) < \infty\} \quad \text{and} \quad \mathcal{A} := \bigcup_{e \in \mathcal{E}} e \mathcal{M} e.$$

For any $x \in \mathcal{M}$, $exe \rightarrow x$ in the w^* -topology of \mathcal{M} , when $e \rightarrow 1$. Further for any $1 \leq p < \infty$ and any $x \in L^p(\mathcal{M})$, $exe \rightarrow x$ in $L^p(\mathcal{M})$. Thus \mathcal{A} is a w^* -dense subspace of \mathcal{M} and a norm dense subspace of $L^p(\mathcal{M})$, for any $1 \leq p < \infty$. Lemma 3.10 below is a w^* -extension result of independent interest, which will be used in the proof of Proposition 3.11.

Given any $w \in \mathcal{M}^*$ and $a, b \in \mathcal{M}$, we let $awb \in \mathcal{M}^*$ be defined by

$$awb(x) := w(bxa), \quad \text{for all } x \in \mathcal{M}.$$

Recall e.g. from [23, pp 126-128] the decomposition

$$(10) \quad \mathcal{M}^* = \mathcal{M}_* \overset{1}{\oplus} \mathcal{M}_s^*,$$

where \mathcal{M}_s^* denotes the space of singular functionals on \mathcal{M} , and \mathcal{M}_* is the predual of \mathcal{M} , which coincides with the space of normal functionals on \mathcal{M} . It is well-known that if $w = w_n + w_s$ is the aforementioned decomposition of w , then aw_nb and aw_sb are the normal part and the singular part of awb , respectively.

Lemma 3.10. *Let Y be a dual Banach space. For any bounded operator $u: \mathcal{A} \rightarrow Y$, the following are equivalent:*

- (i) *For every $e \in \mathcal{E}$, the restriction $u|_{e\mathcal{M}e}: e\mathcal{M}e \rightarrow Y$ is w^* -continuous.*
- (ii) *There exists a w^* -continuous extension $\hat{u}: \mathcal{M} \rightarrow Y$ of u .*

Proof. The implication “(ii) \Rightarrow (i)” is trivial. For the converse, we assume (i).

We first consider the case when $Y = \mathbb{C}$. Suppose that $\alpha \in \mathcal{A}^*$ is such that $\alpha|_{e\mathcal{M}e}$ is w^* -continuous for each $e \in \mathcal{E}$. Using Hahn-Banach, we let $w: \mathcal{M} \rightarrow \mathbb{C}$ be a bounded extension of α to \mathcal{M} and we consider its decomposition $w = w_n + w_s$ according to (10).

For every $e \in \mathcal{E}$, ewe is w^* -continuous. We noticed that $ewe = ew_ne + ew_se$ is the decomposition of ewe . Since ewe and ew_ne are w^* -continuous, the singular part ew_se of ewe must be zero, and consequently, $ewe = ew_ne$, for every $e \in \mathcal{E}$. This implies that the restriction of w_n to \mathcal{A} coincides with α . Thus $\hat{\alpha} = w_n$ is a w^* -continuous extension of α .

For the general case, let $v = u^*|_{Y_*}: Y_* \rightarrow \mathcal{A}^*$ be the restriction of the adjoint of u to the predual of Y . Let

$$\kappa: \mathcal{M}_* \longrightarrow \mathcal{A}^*$$

denote the restriction map taking any $\nu \in \mathcal{M}_*$ to $\nu|_{\mathcal{A}}$. This is an isometry (by Kaplansky’s theorem, say), whose range coincides with the space of all functionals $\mathcal{A} \rightarrow \mathbb{C}$ which admit a w^* -continuous extension to \mathcal{M} .

For each $\eta \in Y_*$, $\eta \circ u|_{e\mathcal{M}e}$ is w^* -continuous, for every $e \in \mathcal{E}$. By our argument for the case $Y = \mathbb{C}$, this implies that $\eta \circ u \in \text{Im}(\kappa)$. This means that v is valued in $\text{Im}(\kappa)$. We can therefore consider $w = \kappa^{-1} \circ v: Y_* \rightarrow \mathcal{M}_*$ and define $\hat{u} = w^*: \mathcal{M} \rightarrow Y$. By construction, \hat{u} is w^* -continuous.

We now claim that \hat{u} is an extension of u . To see this, recall that for any $\eta \in Y_*$, the functional $\kappa^{-1} \circ u^*(\eta)$ is an extension to \mathcal{M} of $u^*(\eta): \mathcal{A} \rightarrow \mathbb{C}$. Consequently,

$$\langle \hat{u}(exe), \eta \rangle = \langle exe, w(\eta) \rangle = \langle exe, \kappa^{-1} \circ u^*(\eta) \rangle = \langle exe, u^*(\eta) \rangle = \langle u(exe), \eta \rangle$$

for any $x \in \mathcal{M}$, any $e \in \mathcal{E}$ and any $\eta \in Y_*$. This proves the claim. \square

Proposition 3.11. *For $1 \leq p < \infty$, a bounded operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is separating if and only if it has a Yeadon type factorization.*

Proof. Assume that T is separating. We adapt Yeadon’s argument from [26], taking into account that our operators are no longer necessarily isometries.

For any $e \in \mathcal{E}$, let $B_e = |T(e)|$ and let $T(e) = w_e B_e$ be the polar decomposition of $T(e)$. We have

$$(11) \quad B_e w_e^* w_e = B_e = w_e^* w_e B_e.$$

Set $J(e) := w_e^* w_e = s(B_e)$. If e and f are in \mathcal{E} and $ef = 0$, then since T is separating we have

$$(12) \quad T(e)^* T(f) = T(e) T(f)^* = 0.$$

Using (12), the argument in the proof of [26, Theorem 2] shows that $J: \mathcal{E} \rightarrow \mathcal{N}$ extends to a linear map $J: \mathcal{A} \rightarrow \mathcal{N}$ such that

$$(13) \quad J(a^2) = J(a)^2, \quad J(a^*) = J(a)^*, \quad \text{and} \quad \|J(a)\|_\infty \leq \|a\|_\infty$$

for all $a \in \mathcal{A}$,

$$(14) \quad T(exe) = w_e B_e J(exe)$$

for all $e \in \mathcal{E}$ and $x \in \mathcal{M}$, and

$$(15) \quad B_f J(e) = J(e) B_f = B_e, \quad \text{for any } e, f \in \mathcal{E}, \text{ with } e \leq f.$$

Note that by (13), the restriction of J to $e\mathcal{M}e$ is a Jordan homomorphism for any $e \in \mathcal{E}$. Consequently,

$$(16) \quad J(exe) = J(e)J(x)J(e)$$

for any $e \in \mathcal{E}$ and any $x \in \mathcal{A}$.

We now show that J admits a normal extension (still denoted by) $J: \mathcal{M} \rightarrow \mathcal{N}$. (Note that Yeadon's argument in the isometric case does not apply to our general case.) According to Lemma 3.10 it suffices to show that the restriction of J to $e\mathcal{M}e$ is normal for any $e \in \mathcal{E}$. To see this, we fix such an e and we let $(ex_i e)_i$ be a bounded net of $e\mathcal{M}e$ converging to exe in the w^* -topology of \mathcal{M} . Then $ex_i e \rightarrow exe$ in the weak topology of $L^p(\mathcal{M})$, hence $w_e^* T(ex_i e) \rightarrow w_e^* T(exe)$ in the weak topology of $L^p(\mathcal{N})$. By (14) and (11), this implies that $\tau_{\mathcal{N}}(AB_e J(ex_i e)) \rightarrow \tau_{\mathcal{N}}(AB_e J(exe))$ for any $A \in L^{p'}(\mathcal{N})$, where p' is the conjugate number of p . Note that by (16), the restriction of J to $e\mathcal{M}e$ is valued in $J(e)\mathcal{N}J(e)$. To deduce from the above convergence property that $J(ex_i e) \rightarrow J(exe)$ in the w^* -topology of \mathcal{N} , it therefore suffices to check that

$$\{AB_e : A \in J(e)L^{p'}(\mathcal{N})J(e)\} \quad \text{is a dense subset of } J(e)L^1(\mathcal{N})J(e).$$

This is indeed the case, since $B_e = |T(e)| \in L^p(\mathcal{N})$ and $s(B_e) = J(e)$.

We note that since J is w^* -continuous and \mathcal{A} is w^* -dense in \mathcal{M} , (16) holds true for any $e \in \mathcal{E}$ and any $x \in \mathcal{M}$.

We now use the increasing net \mathcal{E} and we recall that $e \rightarrow 1$ in the w^* -topology of \mathcal{M} . Since J is normal, $J(1)$ is the w^* -limit of $J(e)$. Then using (11) and (15), the same argument as in [10, Theorem 3.1] can be implemented to obtain extensions B (as supremum of the B_e) and w (as strong limit of the w_e) which satisfy properties (b) and (c) of Definition 3.2. By (15) we further have

$$(17) \quad BJ(e) = J(e)B = B_e \quad \text{and} \quad wJ(e) = w_e,$$

for any $e \in \mathcal{E}$.

We now aim at showing property (a) of Definition 3.2. For any $y \in \mathcal{M}^+ \cap L^1(\mathcal{M})$, using spectral projections, we find a sequence $(e_n)_{n \geq 1}$ in \mathcal{E} such that $e_n y e_n$ is increasing to y when $n \rightarrow \infty$. Since J is normal, this implies that $B^p J(e_n y e_n)$ is increasing to $B^p J(y)$ when $n \rightarrow \infty$. Consequently, using the normality of $\tau_{\mathcal{N}}$, we obtain that

$$(18) \quad \tau_{\mathcal{N}}(B^p J(y)) = \lim_{n \rightarrow \infty} \tau_{\mathcal{N}}(B^p J(e_n y e_n)).$$

Let $x \in \mathcal{M} \cap L^p(\mathcal{M})$. Consider a decomposition $J = \begin{pmatrix} \pi & 0 \\ 0 & \sigma \end{pmatrix}$ as in (9). Then we have

$$(19) \quad J(|x|^p) = \begin{pmatrix} |\pi(x)|^p & 0 \\ 0 & |\sigma(x^*)|^p \end{pmatrix}.$$

Then by a well-known argument (see the proof of the easy implication of [26, Theorem 2]), this implies that

$$(20) \quad \|wBJ(x)\|_p^p = \tau_{\mathcal{N}}(B^p|J(x)|^p) = \tau_{\mathcal{N}}(B^pJ(|x|^p)).$$

Using (18) with $y = |x|^p$, we deduce that for some sequence $(e_n)_{n \geq 1}$ in \mathcal{E} , we have

$$\|wBJ(x)\|_p^p = \lim_{n \rightarrow \infty} \tau_{\mathcal{N}}(B^pJ(e_n|x|^pe_n)).$$

Fix $e \in \mathcal{E}$. Combining (14) and (16), we have $T(exe) = w_e B_e J(x) J(e)$ from which we deduce as in (20) that

$$\|T(exe)\|_p^p = \tau_{\mathcal{N}}(B_e^p|J(x)|^p).$$

Using (16) again, and (17), we have

$$\tau_{\mathcal{N}}(B^pJ(e|x|^pe)) = \tau_{\mathcal{N}}(B^pJ(e)J(|x|^p)J(e)) = \tau_{\mathcal{N}}(B_e^pJ(|x|^p)).$$

Consequently, using (19), we have

$$\begin{aligned} \tau_{\mathcal{N}}(B^pJ(e|x|^pe)) &= \tau_{\mathcal{N}}(B_e^p|\pi(x)|^p) + \tau_{\mathcal{N}}(B_e^p|\sigma(x^*)|^p) \\ &\leq \tau_{\mathcal{N}}(B_e^p|J(x)|^p) + \tau_{\mathcal{N}}(B_e^p|J(x^*)|^p) \\ &\leq \|T(exe)\|_p^p + \|T(ex^*e)\|_p^p \\ &\leq 2\|T\|^p \|x\|_p^p. \end{aligned}$$

Applying this with $e = e_n$ and passing to the limit, we deduce that $wBJ(x) \in L^p(\mathcal{N})$ for any $x \in \mathcal{M} \cap L^p(\mathcal{M})$, with $\|wBJ(x)\|_p \leq 2\|T\| \|x\|_p$. This shows the existence of a (necessarily unique) bounded operator $T': L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ such that $T'(x) = wBJ(x)$ for any $x \in \mathcal{M} \cap L^p(\mathcal{M})$. By (14) and (17), T and T' coincide on \mathcal{A} . Since the latter is dense in $L^p(\mathcal{M})$, we obtain that $T = T'$, hence property (a) of Definition 3.2 is satisfied.

For the converse suppose that $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ has a Yeadon type factorization, with Yeadon triple (w, B, J) , and let us show that T is separating. By Lemma 3.9, it is enough to show that T is separating on $\mathcal{M} \cap L^p(\mathcal{M})$. Let a and b be disjoint elements in $\mathcal{M} \cap L^p(\mathcal{M})$, then

$$\begin{aligned} T(a)^*T(b) &= J(a)^*Bw^*wBJ(b) && \text{by (a) in Definition 3.2} \\ &= J(a)^*B^2J(b) && \text{by (b) in Definition 3.2} \\ &= B^2J(a)^*J(b) && \text{by (c) in Definition 3.2} \\ &= 0 && \text{by Lemma 3.8.} \end{aligned}$$

Similarly we can show that $T(a)T(b)^* = 0$, and hence $T(a)$ and $T(b)$ are disjoint. \square

After a first version of this paper was circulated, we were informed that the ‘only if’ part of Proposition 3.11 was proved independently in [6].

We now give a series of remarks on this statement.

Remark 3.12.

(a) In Proposition 3.11, the proof that a separating map T admits a Yeadon type factorization only uses the separation property on positive elements (even on projections

with finite trace). Hence a bounded operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is separating if and only if for any $a, b \in L^p(\mathcal{M})^+$, $ab = 0$ implies that $T(a)^*T(b) = T(a)T(b)^* = 0$.

(b) Let us say that a separating map $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is 2-separating if the tensor extension $I_{S_2^p} \otimes T: L^p(M_2(\mathcal{M})) \rightarrow L^p(M_2(\mathcal{N}))$ is separating. Combining Proposition 3.11 and the argument in the proof of [10, Proposition 3.2], we obtain that $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is 2-separating if and only if the Jordan homomorphism in the Yeadon factorisation of T is multiplicative. This fact was also observed in [6, Theorem 3.4].

Remark 3.13. Here we discuss the commutative case. Let $(\Omega, \mathcal{F}, \mu)$ and $(\Omega', \mathcal{F}', \mu')$ be two σ -finite measure spaces. Let $\widetilde{\mathcal{F}}$ be the set of classes of \mathcal{F} modulo the null sets (i.e. $E \in \mathcal{F}$ such that $\mu(E) = 0$). We identify any element of \mathcal{F} with its class in $\widetilde{\mathcal{F}}$. We define $\widetilde{\mathcal{F}'}$ similarly.

Recall that a **regular set morphism** (RSM) from $(\Omega, \mathcal{F}, \mu)$ into $(\Omega', \mathcal{F}', \mu')$ is a map $\varphi: \widetilde{\mathcal{F}} \rightarrow \widetilde{\mathcal{F}'}$ satisfying the following two properties:

- (i) For any $E_1, E_2 \in \mathcal{F}$, $E_1 \cap E_2 = \emptyset \Rightarrow \varphi(E_1) \cap \varphi(E_2) = \emptyset$.
- (ii) For any sequence $(E_n)_{n \geq 1}$ of pairwise disjoint sets in \mathcal{F} , $\varphi(\bigcup_{n \geq 1} E_n) = \bigcup_{n \geq 1} \varphi(E_n)$.

Following [13], Kan [12, Theorem 4.1] showed that a bounded operator $T: L^p(\Omega) \rightarrow L^p(\Omega')$ is separating if and only if there exist a measurable function h and a regular set morphism φ from $(\Omega, \mathcal{F}, \mu)$ into $(\Omega', \mathcal{F}', \mu')$ such that for every set of finite measure E , we have

$$T(\chi_E) = h \cdot \chi_{\varphi(E)}.$$

(See [12] and [3, Chapter 3] for more on this factorization property.)

There is a well-known correspondence between RSM and normal $*$ -homomorphisms on L^∞ -spaces. Namely, let $\pi: L^\infty(\Omega, \mathcal{F}, \mu) \rightarrow L^\infty(\Omega', \mathcal{F}', \mu')$ be a normal $*$ -homomorphism. Then for any $E \in \mathcal{F}$, $\pi(\chi_E)$ is a projection, hence an indicator function. We may therefore define $\varphi: \widetilde{\mathcal{F}} \rightarrow \widetilde{\mathcal{F}'}$ by $\pi(\chi_E) = \chi_{\varphi(E)}$. It is easy to check that φ is a RSM. It turns out that any regular set morphism is of this form. Indeed let $\varphi: \widetilde{\mathcal{F}} \rightarrow \widetilde{\mathcal{F}'}$ be a RSM. For any $g \in L^1(\Omega', \mathcal{F}', \mu')$, define $\nu_g: \mathcal{F} \rightarrow \mathbb{C}$ by

$$\nu_g(E) = \int_{\varphi(E)} g(t) d\mu'(t), \quad \text{for any } E \in \mathcal{F}.$$

By (ii) and Lebesgue's theorem, ν_g is a complex measure, whose total variation is less than or equal to $\|g\|_1$. By (i), $\varphi(\emptyset) = \emptyset$ hence ν is absolutely continuous with respect to μ . Hence by the Radon-Nikodym theorem, there exists a necessarily unique $h_g \in L^1(\Omega, \mathcal{F}, \mu)$ such that

$$\nu_g(E) = \int_E h_g(s) d\mu(s), \quad \text{for any } E \in \mathcal{F}.$$

Moreover $\|h_g\|_1 \leq \|g\|_1$. It is clear that the mapping $u: L^1(\Omega', \mathcal{F}', \mu') \rightarrow L^1(\Omega, \mathcal{F}, \mu)$ defined by $u(g) = h_g$ is linear. The above estimate shows that u is contractive. Set

$$\pi = u^*: L^\infty(\Omega, \mathcal{F}, \mu) \longrightarrow L^\infty(\Omega', \mathcal{F}', \mu').$$

By construction, π is w^* -continuous. Further it is easy to check that π is a $*$ -homomorphism and that $\pi(\chi_E) = \chi_{\varphi(E)}$ for any $E \in \mathcal{F}$.

We finally note that all Jordan homomorphisms on L^∞ -spaces are $*$ -homomorphisms. Thereby, through the aforementioned correspondence, Proposition 3.11 reduces to Kan's theorem in the case when \mathcal{M} and \mathcal{N} are commutative.

Remark 3.14.

(a) In general, separating operators may not be one-to-one (contrary to isometries). We observe however that if a bounded operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is separating, with Yeadon triple (w, B, J) , then T is one-to-one if and only if J is a Jordan monomorphism.

Indeed if T is one-to-one, then $J(e) = s(|T(e)|) \neq 0$ for any $e \in \mathcal{E} \setminus \{0\}$. This implies that for any pairwise disjoint non zero e_1, \dots, e_m in \mathcal{E} and any $\alpha_1, \dots, \alpha_m$ in \mathbb{C} , we have

$$\left\| \sum_k \alpha_k J(e_k) \right\| = \max\{|\alpha_k| : k = 1, \dots, m\} = \left\| \sum_k \alpha_k e_k \right\|.$$

Hence the restriction of J to \mathcal{E} is an isometry. This readily implies that J is an isometry, and hence J is one-to-one.

Assume conversely that J is one-to-one and let $x \in L^p(\mathcal{M})$ such that $T(x) = 0$. Let $x = u|x|$ be the polar decomposition of x and for any integer $n \geq 1$, let $p_n = \chi_{[-n, n]}(|x|)$. Then consider $x'_n = xp_n$ and $x''_n = x(1 - p_n)$. We have $x = x'_n + x''_n$ hence $T(x'_n) + T(x''_n) = 0$. Further we have

$$x'_n{}^* x''_n = p_n |x| u^* u |x| (1 - p_n) = p_n |x|^2 (1 - p_n) = 0,$$

whereas $x'_n x''_n{}^* = xp_n(1 - p_n)x^* = 0$. Hence x'_n and x''_n are disjoint. Since T is separating this implies that $T(x'_n)$ and $T(x''_n)$ are disjoint. Since these elements are opposite to each other, this implies that $T(x'_n) = 0$. For any $n \geq 1$, $x'_n \in L^p(\mathcal{M}) \cap M$ hence $T(x'_n) = wBJ(x'_n)$. Since $w^*w = s(B) = J(1)$, this implies that $J(x'_n) = 0$, hence $x'_n = 0$ by our assumption. Since $x'_n \rightarrow x$ in $L^p(\mathcal{M})$, we deduce that $x = 0$. This shows that T is one-to-one.

(b) We also observe that a separating operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ with Yeadon triple (w, B, J) is positive if and only if w is positive (if and only if w is a projection). The verification is left to the reader.

The following theorem shows that separating bounded operators are ℓ^1 -bounded. The converse does not hold true, this can be easily seen on commutative L^p -spaces (see Remark 2.8).

Theorem 3.15. *Suppose that $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is a separating bounded operator, with $1 \leq p < \infty$. Then T is ℓ^1 -bounded and $\|T\|_{\ell^1} = \|T\|$.*

Proof. We apply Proposition 3.11. We let (w, B, J) be the Yeadon triple of the operator T . Next according to (9), we let $J = \begin{pmatrix} \pi & 0 \\ 0 & \sigma \end{pmatrix}$ be a central decomposition of J and we let e, f be the central projections such that $\pi = J(\cdot)e$ and $\sigma = J(\cdot)f$.

Let $T_1: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ and $T_2: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ be defined by

$$T_1(x) = T(x)e \quad \text{and} \quad T_2(x) = T(x)f$$

for any $x \in L^p(\mathcal{M})$. Then, $T = T_1 + T_2$ and for all $u, v \in L^p(\mathcal{M})$, we have

$$(21) \quad \|T_1(u) + T_2(v)\|_p^p = \|T_1(u)\|_p^p + \|T_2(v)\|_p^p.$$

Let $(x_n)_{n \geq 1} \in L^p(\mathcal{M}; \ell^1)$, with $\|(x_n)\|_{L^p(\mathcal{M}; \ell^1)} < 1$. By Lemma 2.3, there exist factorizations $x_n = a_n b_n$, $n \geq 1$, with $a_n, b_n \in L^{2p}(\mathcal{M})$, such that

$$\left\| \sum_{n=1}^{\infty} a_n a_n^* \right\|_p < 1 \quad \text{and} \quad \left\| \sum_{n=1}^{\infty} b_n^* b_n \right\|_p < 1.$$

The identity (20) and its proof show that for any $a \in \mathcal{M} \cap L^{2p}(\mathcal{M})$, we have

$$\|B^{\frac{1}{2}} \pi(a)\|_{2p}^{2p} = \tau_{\mathcal{N}}(B^p \pi(|a|^{2p})) \leq \tau_{\mathcal{N}}(B^p J(|a|^{2p})) = \|T(|a|^2)\|_p^p.$$

Similarly, $\|B^{\frac{1}{2}} \sigma(a)\|_{2p}^{2p} \leq \|T(|a|^2)\|_p^p$. Hence there exist two bounded operators

$$S_1: L^{2p}(\mathcal{M}) \longrightarrow L^{2p}(\mathcal{N}) \quad \text{and} \quad S_2: L^{2p}(\mathcal{M}) \longrightarrow L^{2p}(\mathcal{N})$$

such that

$$S_1(a) = B^{\frac{1}{2}} \pi(a) \quad \text{and} \quad S_2(a) = B^{\frac{1}{2}} \sigma(a),$$

for all a in $\mathcal{M} \cap L^{2p}(\mathcal{M})$. It is clear from above that for any a and b in $L^{2p}(\mathcal{M})$ we have

$$S_1(a)S_1(b) = w^* T_1(ab), \quad S_2(a)S_2(b) = w^* T_2(ba) \quad \text{and} \quad S_1(a)S_2(b) = S_2(b)S_1(a) = 0.$$

We will use this repeatedly in the rest of the proof.

For any $n \geq 1$, we have

$$T(x_n) = T(a_n b_n) = T_1(a_n b_n) + T_2(a_n b_n) = w S_1(a_n) S_1(b_n) + w S_2(b_n) S_2(a_n).$$

Hence $T(x_n) = c_n d_n$, with

$$c_n = w (S_1(a_n) + S_2(b_n)) \quad \text{and} \quad d_n = S_1(b_n) + S_2(a_n).$$

With a similar computation, we obtain

$$\begin{aligned} c_n c_n^* &= w (S_1(a_n) + S_2(b_n)) (S_1(a_n^*) + S_2(b_n^*)) w^* \\ &= (T_1(a_n a_n^*) + T_2(b_n^* b_n)) w^*. \end{aligned}$$

Let $N \geq 1$ and set $u_N = \sum_{n=1}^N a_n a_n^*$ and $v_N = \sum_{n=1}^N b_n^* b_n$. Summing up we obtain

$$\sum_{n=1}^N c_n c_n^* = (T_1(u_N) + T_2(v_N)) w^*$$

Appealing to (21), we deduce that

$$\left\| \sum_{n=1}^N c_n c_n^* \right\|_p^p \leq \|T_1(u_N)\|_p^p + \|T_2(v_N)\|_p^p.$$

Similarly,

$$\left\| \sum_{n=1}^N d_n^* d_n \right\|_p^p \leq \|T_1(v_N)\|_p^p + \|T_2(u_N)\|_p^p.$$

Consequently,

$$\left\| \sum_{n=1}^N c_n c_n^* \right\|_p^p \left\| \sum_{n=1}^N d_n^* d_n \right\|_p^p \leq (\|T_1(u_N)\|_p^p + \|T_2(v_N)\|_p^p) (\|T_1(v_N)\|_p^p + \|T_2(u_N)\|_p^p).$$

Let $\alpha = \|T_1(u_N)\|_p^p$ and $\beta = \|T_1(v_N)\|_p^p$. Since,

$$\|T_1(u_N)\|_p^p + \|T_2(u_N)\|_p^p = \|T(u_N)\|_p^p \leq \|T\|^p \|u_N\|_p^p,$$

by (21), and $\|u_N\|_p < 1$, we have $\|T_2(u)\|_p^p \leq \|T\|^p - \alpha$. Similarly, $\|T_2(v)\|_p^p \leq \|T\|^p - \beta$. Therefore,

$$\begin{aligned} (\|T_1(u_N)\|_p^p + \|T_2(v_N)\|_p^p) (\|T_1(v_N)\|_p^p + \|T_2(u_N)\|_p^p) &\leq (\alpha + (\|T\|^p - \beta))(\beta + (\|T\|^p - \alpha)) \\ &= (\|T\|^p - (\beta - \alpha)) (\|T\|^p + (\beta - \alpha)) \\ &= (\|T\|^{2p} - (\beta - \alpha)^2) \leq \|T\|^{2p}, \end{aligned}$$

and hence

$$\left\| \sum_{n=1}^N c_n c_n^* \right\|_p^{\frac{1}{2}} \left\| \sum_{n=1}^N d_n^* d_n \right\|_p^{\frac{1}{2}} \leq \|T\|.$$

This implies that $(T(x_n))_{n \geq 1}$ belongs to $L^p(\mathcal{N}; \ell^1)$ and that its norm in $L^p(\mathcal{N}; \ell^1)$ is less than or equal to $\|T\|$. This yields the boundedness of $T \otimes I_{\ell^1}$, as well as the equality $\|T\|_{\ell^1} = \|T\|$. \square

4. ISOMETRIES ON L^2 -SPACES WITH A YEADON TYPE FACTORIZATION

As it is outlined in Section 3, the crucial property that allowed Yeadon to describe isometries between noncommutative L^p -spaces, $p \neq 2$, is that they are separating. To show that every isometry is indeed separating he relied on the property that when $1 \leq p \neq 2 < \infty$, the equality condition in Clarkson's inequality, $\|a + b\|_p + \|a - b\|_p = 2(\|a\|_p + \|b\|_p)$, holds true if and only if a and b are disjoint. However, this equality always holds true when $p = 2$ and this is why the study of disjointness on L^2 -spaces requires a different approach. This is the purpose of Lemma 4.1 below and as a consequence, we will characterize isometries between noncommutative L^2 -spaces which admit a Yeadon type factorization.

Lemma 4.1. *Suppose that $a, b \in L^2(\mathcal{M})$. Then, a and b are disjoint if and only if we have*

$$\|(a, b)\|_{L^2(\mathcal{M}; \ell_2^1)} \leq (\|a\|_2^2 + \|b\|_2^2)^{1/2}.$$

Proof. First suppose that for disjoint elements $a, b \in L^2(\mathcal{M})$ the polar decompositions are given by $a = u|a|$ and $b = v|b|$. Define

$$a_1 = u|a|^{1/2}, \quad a_2 = |a|^{1/2}, \quad b_1 = v|b|^{1/2} \quad \text{and} \quad b_2 = |b|^{1/2}.$$

These elements belong to $L^4(\mathcal{M})$ and we have $a = a_1 a_2$ and $b = b_1 b_2$. Further we have

$$a_1 a_1^* + b_1 b_1^* = u|a|u^* + v|b|v^* \quad \text{and} \quad a_2^* a_2 + b_2^* b_2 = |a| + |b|.$$

Consequently,

$$\|(a, b)\|_{L^2(\mathcal{M}; \ell_2^1)}^2 \leq \|u|a|u^* + v|b|v^*\|_2 \| |a| + |b| \|_2.$$

Now, since a and b are disjoint we have that $|a||b| = 0$, by Lemma 3.5, and so

$$\begin{aligned} \| |a| + |b| \|_2^2 &= \| |a| \|_2^2 + \| |b| \|_2^2 + 2\tau_{\mathcal{M}}(|a||b|) \\ &= \| |a| \|_2^2 + \| |b| \|_2^2. \end{aligned}$$

Similarly, since a and b are disjoint, we have $ua^*bv^* = 0$, and so

$$\begin{aligned} \|u|a|u^* + v|b|v^*\|_2^2 &= \|u|a|u^*\|_2^2 + \|v|b|v^*\|_2^2 + 2\tau_{\mathcal{M}}(u|a|u^*v|b|v^*) \\ &= \|u|a|u^*\|_2^2 + \|v|b|v^*\|_2^2 + 2\tau_{\mathcal{M}}(ua^*bv^*) \\ &= \|u|a|u^*\|_2^2 + \|v|b|v^*\|_2^2 \\ &\leq \| |a| \|_2^2 + \| |b| \|_2^2. \end{aligned}$$

This implies that $\|(a, b)\|_{L^2(\mathcal{M}; \ell_2^1)} \leq (\|a\|_2^2 + \|b\|_2^2)^{1/2}$.

Conversely, suppose that $a, b \in L^2(\mathcal{M})$ satisfy $\|(a, b)\|_{L^2(\mathcal{M}; \ell_2^1)} \leq (\|a\|_2^2 + \|b\|_2^2)^{1/2}$. Let $(\varepsilon_k)_k$ be a sequence of positive real numbers with $\lim_k \varepsilon_k = 0$. By Lemma 2.3,

$$\|(a, b)\|_{L^2(\mathcal{M}; \ell^1)} = \inf \left\{ \|uu^* + ww^*\|_2^{\frac{1}{2}} \|v^*v + z^*z\|_2^{\frac{1}{2}} \right\},$$

where the infimum is taken over all factorizations $a = uv$ and $b = wz$, with $u, v, w, z \in L^4(\mathcal{M})$. Thus for any $k \geq 1$, we can find $u_k, v_k, w_k, z_k \in L^4(\mathcal{M})$ such that $a = u_k v_k$, $b = w_k z_k$,

$$(22) \quad \|u_k u_k^* + w_k w_k^*\|_2^{\frac{1}{2}} \leq (1 + \varepsilon_k)^{\frac{1}{8}} (\|a\|_2^2 + \|b\|_2^2)^{\frac{1}{4}},$$

and

$$(23) \quad \|v_k^* v_k + z_k^* z_k\|_2^{\frac{1}{2}} \leq (1 + \varepsilon_k)^{\frac{1}{8}} (\|a\|_2^2 + \|b\|_2^2)^{\frac{1}{4}}.$$

This implies that

$$(24) \quad \|u_k u_k^*\|_2^2 + \|w_k w_k^*\|_2^2 + 2\tau_{\mathcal{M}}(u_k u_k^* w_k w_k^*) \leq (1 + \varepsilon_k)^{\frac{1}{2}} (\|u_k v_k\|_2^2 + \|w_k z_k\|_2^2)$$

and

$$(25) \quad \|v_k^* v_k\|_2^2 + \|z_k^* z_k\|_2^2 + 2\tau_{\mathcal{M}}(v_k^* v_k z_k^* z_k) \leq (1 + \varepsilon_k)^{\frac{1}{2}} (\|u_k v_k\|_2^2 + \|w_k z_k\|_2^2).$$

We claim that

$$(26) \quad (\|u_k v_k\|_2^2 + \|w_k z_k\|_2^2)^2 \leq (\|u_k^* u_k\|_2^2 + \|w_k^* w_k\|_2^2) (\|v_k v_k^*\|_2^2 + \|z_k z_k^*\|_2^2).$$

Indeed

$$\|u_k v_k\|_2^2 = \tau_{\mathcal{M}}((u_k v_k)^*(u_k v_k)) = \tau_{\mathcal{M}}(v_k v_k^* u_k^* u_k) = \langle u_k^* u_k, v_k v_k^* \rangle_{L^2(\mathcal{M})}$$

and similarly, $\|w_k z_k\|_2^2 = \langle w_k^* w_k, z_k z_k^* \rangle_{L^2(\mathcal{M})}$. Hence (26) follows by applying the Cauchy-

Schwarz inequality in the Hilbertian direct sum $L^2(\mathcal{M}) \oplus L^2(\mathcal{M})$.

Multiplying inequalities (24) and (25) and using the fact that $\tau_{\mathcal{M}}(v_k^* v_k z_k^* z_k) \geq 0$, we obtain that

$$(27) \quad (\|u_k u_k^*\|_2^2 + \|w_k w_k^*\|_2^2 + 2\tau_{\mathcal{M}}(u_k u_k^* w_k w_k^*)) (\|v_k^* v_k\|_2^2 + \|z_k^* z_k\|_2^2)$$

is less than or equal to $(1 + \varepsilon_k) (\|u_k v_k\|_2^2 + \|w_k z_k\|_2^2)^2$. Now using (26) we deduce that (27) is less than or equal to

$$(1 + \varepsilon_k) (\|u_k^* u_k\|_2^2 + \|w_k^* w_k\|_2^2) (\|v_k v_k^*\|_2^2 + \|z_k z_k^*\|_2^2).$$

Now we observe that $\|u_k u_k^*\|_2^2 = \|u_k\|_4^4 = \|u_k^* u_k\|_2^2$ and similarly for w_k, v_k and z_k . Hence the above inequality reads

$$(\|u_k\|_4^4 + \|w_k\|_4^4 + 2\tau_{\mathcal{M}}(u_k u_k^* w_k w_k^*)) \leq (1 + \varepsilon_k) (\|u_k\|_4^4 + \|w_k\|_4^4).$$

This yields

$$\tau_{\mathcal{M}}(u_k u_k^* w_k w_k^*) \leq \varepsilon_k \left(\frac{\|u_k\|_4^4 + \|w_k\|_4^4}{2} \right).$$

It follows from (22) that $(\|u_k\|_4)_k$ and $(\|w_k\|_4)_k$ are bounded sequences. Hence we have that $\tau_{\mathcal{M}}(u_k u_k^* w_k w_k^*) \rightarrow 0$ as $k \rightarrow \infty$. Writing

$$\tau_{\mathcal{M}}(u_k u_k^* w_k w_k^*) = \tau_{\mathcal{M}}(u_k^* w_k w_k^* u_k) = \tau_{\mathcal{M}}((u_k^* w_k)(u_k^* w_k)^*) = \|u_k^* w_k\|_2^2,$$

we deduce that $\|u_k^* w_k\|_2 \rightarrow 0$ as $k \rightarrow \infty$.

We have $a^*b = v_k^*u_k^*w_kz_k$, hence

$$\|a^*b\|_1 \leq \|v_k\|_4 \|u_k^*w_k\|_2 \|z_k\|_4.$$

By (23), $(\|v_k\|_4)_k$ and $(\|z_k\|_4)_k$ are bounded sequences, hence the right hand side in the above inequality tends to 0 as $k \rightarrow \infty$. We deduce that $a^*b = 0$.

Finally using $\tau_{\mathcal{M}}(v_k^*v_kz_k^*z_k)$ instead of $\tau_{\mathcal{M}}(u_ku_k^*w_kw_k^*)$, we show as well that $ab^* = 0$ and therefore, a and b are disjoint. \square

Theorem 4.2. *For a linear isometry $T: L^2(\mathcal{M}) \rightarrow L^2(\mathcal{N})$, the following statements are equivalent:*

- (i) T has a Yeadon type factorization.
- (ii) T is ℓ_2^1 -contractive.
- (iii) T is ℓ^1 -contractive.

Proof. In the light of Proposition 3.11 and Theorem 3.15, we only need to establish that if (ii) holds true, then T is separating.

Suppose that T is ℓ_2^1 -contractive. Let $a, b \in L^2(\mathcal{M})$ be disjoint elements. By Lemma 4.1,

$$\|(Ta, Tb)\|_{L^2(\mathcal{N}; \ell_2^1)} \leq \|(a, b)\|_{L^2(\mathcal{M}; \ell_2^1)} \leq (\|a\|_2^2 + \|b\|_2^2)^{\frac{1}{2}}.$$

Since T is an isometry we have $\|T(a)\| = \|a\|$ and $\|T(b)\| = \|b\|$ and hence

$$\|(Ta, Tb)\|_{L^2(\mathcal{N}; \ell_2^1)} \leq (\|Ta\|_2^2 + \|Tb\|_2^2)^{\frac{1}{2}}.$$

By Lemma 4.1 again, this implies that Ta and Tb are disjoint. Hence T is separating. \square

Remark 4.3.

(a) As mentioned in Remark 2.8, when $\mathcal{M} = L^\infty(\Omega)$ and $\mathcal{N} = L^\infty(\Omega')$ are commutative, a bounded operator $T: L^2(\Omega) \rightarrow L^2(\Omega')$ is ℓ^1 -contractive if and only if T is regular, with $\|T\|_{reg} \leq 1$. Hence, Theorem 4.2 implies that in the commutative case, an isometry $T: L^2(\Omega) \rightarrow L^2(\Omega')$ is separating if and only if $\|T\|_{reg} \leq 1$. This result is implicit in [15].

(b) Let $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, $1 \leq p < \infty$, be a separating isometry, with the Yeadon triple (w, B, J) . We show in [14] that when $p = 2$ and \mathcal{M} and \mathcal{N} are hyperfinite, then J is multiplicative (equivalently, J is a $*$ -homomorphism) if and only if T is completely regular with $\|T\|_{reg} = 1$. This is an L^2 -analog of [10, Theorem 3.1] which says that for $1 \leq p \neq 2 < \infty$, J is multiplicative if and only if T is a complete isometry.

In [2], Broise showed that every bijective positive isometry between noncommutative L^2 -spaces associated with semifinite factors admits a Yeadon type factorization. Using Proposition 3.11, one can actually obtain the following more general statement.

Corollary 4.4. *Suppose that $T: L^2(\mathcal{M}) \rightarrow L^2(\mathcal{N})$ is a positive isometry. Then T admits a Yeadon type factorization.*

Proof. Let $a, b \in L^2(\mathcal{M})$ be positive elements, with $ab = 0$. They are orthogonal and isometries preserve orthogonality, hence $T(a)$ and $T(b)$ are orthogonal. Since $T(a)$ and $T(b)$ are positive, Remark 3.6 ensures that $T(a)$ and $T(b)$ are disjoint.

By Remark 3.12 (a) and Proposition 3.11, the above shows that T admits a Yeadon type factorization. \square

5. POSITIVITY AND ℓ^1 -CONTRACTIVITY

For any $n \geq 2$, we let $S_n^p = S^p(\ell_2^n)$ and we let $S_n^p[L^p(\mathcal{M})]$ be the space $S_n^p \otimes L^p(\mathcal{M})$ equipped with the norm and the partial order coming from its identification with the space $L^p(M_n(\mathcal{M}))$, see Section 2.

We say that a bounded operator $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, $1 \leq p < \infty$, is n -positive if

$$I_{S_n^p} \otimes T: S_n^p[L^p(\mathcal{M})] \longrightarrow S_n^p[L^p(\mathcal{N})]$$

is positive. We say that T is completely positive if it is n -positive for all $n \geq 1$.

Proposition 5.1. *Suppose that $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is a 2-positive contraction, then T is ℓ^1 -contractive.*

Proof. Let $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ be a 2-positive contraction and let $(x_n)_{n \geq 1}$ be a sequence in $L^p(\mathcal{M})$ such that $\|(x_n)_{n \geq 1}\|_{L^p(\mathcal{M}; \ell^1)} < 1$. According to Lemma 2.3, we may choose sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ in $L^{2p}(\mathcal{M})$ such that $x_n = a_n b_n$ for any $n \geq 1$,

$$(28) \quad \left\| \sum_{n=1}^{\infty} a_n a_n^* \right\|_p < 1 \quad \text{and} \quad \left\| \sum_{n=1}^{\infty} b_n^* b_n \right\|_p < 1.$$

For any $n \geq 1$, let

$$z_n := \begin{pmatrix} a_n a_n^* & a_n b_n \\ b_n^* a_n^* & b_n^* b_n \end{pmatrix}$$

in $S_2^p[L^p(\mathcal{M})]$. Then $z_n = \begin{pmatrix} a_n & 0 \\ b_n^* & 0 \end{pmatrix} \begin{pmatrix} a_n^* & b_n \\ 0 & 0 \end{pmatrix}$, hence $z_n \geq 0$. Therefore by the 2-positivity of T ,

$$(I_{S_2^p} \otimes T)(z_n) = \begin{pmatrix} T(a_n a_n^*) & T(a_n b_n) \\ T(b_n^* a_n^*) & T(b_n^* b_n) \end{pmatrix} \geq 0.$$

Consider the positive square root

$$\begin{pmatrix} \alpha_n & \beta_n \\ \beta_n^* & \delta_n \end{pmatrix} := ((I_{S_2^p} \otimes T)(z_n))^{\frac{1}{2}}.$$

Then $\alpha_n, \beta_n, \delta_n$ belong to $L^{2p}(\mathcal{N})$, we have $\alpha_n \geq 0, \delta_n \geq 0$, and

$$\begin{aligned} T(a_n a_n^*) &= \alpha_n^2 + \beta_n \beta_n^*, \\ T(b_n^* b_n) &= \beta_n^* \beta_n + \delta_n^2, \\ T(a_n b_n) &= \alpha_n \beta_n + \beta_n \delta_n. \end{aligned}$$

Using the third equation above and Junge's definition of $L^p(\mathcal{N}; \ell^1)$, we get that

$$\|(T(a_n b_n))_{n \geq 1}\|_{L^p(\mathcal{N}; \ell^1)} \leq \left\| \sum_{n=1}^{\infty} \alpha_n^2 + \sum_{n=1}^{\infty} \beta_n \beta_n^* \right\|_p^{\frac{1}{2}} \left\| \sum_{n=1}^{\infty} \beta_n^* \beta_n + \sum_{n=1}^{\infty} \delta_n^2 \right\|_p^{\frac{1}{2}}.$$

(The convergence of the series are justified by the next lines.)

We can now apply the first two equations and (28) to deduce that

$$\begin{aligned} \|(T(x_n))_{n \geq 1}\|_{L^p(\mathcal{N}; \ell^1)} &\leq \left\| \sum_{n=1}^{\infty} T(a_n a_n^*) \right\|_p^{\frac{1}{2}} \left\| \sum_{n=1}^{\infty} T(b_n^* b_n) \right\|_p^{\frac{1}{2}} \\ &\leq \left\| T \left(\sum_{n=1}^{\infty} a_n a_n^* \right) \right\|_p^{\frac{1}{2}} \left\| T \left(\sum_{n=1}^{\infty} b_n^* b_n \right) \right\|_p^{\frac{1}{2}} \\ &\leq \|T\| \left\| \sum_{n=1}^{\infty} a_n a_n^* \right\|_p^{\frac{1}{2}} \left\| \sum_{n=1}^{\infty} b_n^* b_n \right\|_p^{\frac{1}{2}} \\ &< 1. \end{aligned}$$

This shows that T is ℓ^1 -contractive. □

Remark 5.2.

(a) An obvious consequence of Proposition 5.1 is that if T is a completely positive contraction then it is ℓ^1 -contractive.

(b) Let \mathcal{N}^{op} be the opposite von Neumann algebra of \mathcal{N} and let $I_{op}: L^p(\mathcal{N}) \rightarrow L^p(\mathcal{N}^{op})$ denote the identity map. We say that $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is 2-copositive if the operator $I_{op} \circ T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N}^{op})$ is 2-positive. It is easy to check that

$$L^p(\mathcal{N}; \ell^1) = L^p(\mathcal{N}^{op}, \ell^1) \quad \text{isometrically.}$$

Therefore, Proposition 5.1 implies that any contractive 2-copositive map $L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is ℓ^1 -contractive. It therefore follows that if a positive map $L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ can be written as a convex combination of a contractive 2-positive map and a contractive 2-copositive map, then it is ℓ^1 -contractive.

We do not know if any positive contraction is ℓ^1 -contractive, however we show below that positive operators are ℓ^1 -bounded.

Proposition 5.3. *Let $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ be a bounded operator. If T is positive, then T is ℓ^1 -bounded, with $\|T\|_{\ell^1} \leq 4\|T\|$.*

Proof. As in the proof of Proposition 5.1, let $(x_n)_{n \geq 1}$ be a sequence in $L^p(\mathcal{M})$ such that $\|(x_n)_{n \geq 1}\|_{L^p(\mathcal{M}; \ell^1)} < 1$, and let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ in $L^{2p}(\mathcal{M})$ such that $x_n = a_n b_n$ for any $n \geq 1$ and (28) holds.

For any $n \geq 1$, we use the polarization identity,

$$a_n b_n = \frac{1}{4} \sum_{k=0}^3 (-i)^k (a_n^* + i^k b_n)^* (a_n^* + i^k b_n).$$

For $0 \leq k \leq 3$ and $n \geq 1$, let $y_n^k := (a_n^* + i^k b_n)^* (a_n^* + i^k b_n)$. Then $y_n^k \geq 0$ hence $T(y_n^k) \geq 0$. This implies, by Lemma 2.4, that for any k ,

$$\|(T(y_n^k))_n\|_{L^p(\mathcal{N}; \ell^1)} = \left\| \sum_{n=1}^{\infty} T(y_n^k) \right\|_p \leq \|T\| \left\| \sum_{n=1}^{\infty} y_n^k \right\|_p.$$

(The convergence of the series are justified by the next lines.)

Moreover

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} y_n^k \right\|_p &= \left\| \sum_{n=1}^{\infty} E_{n1} \otimes (a_n^* + i^k b_n) \right\|_{L^{2p}(\mathcal{M}; \ell_c^2)}^2 \\ &\leq \left(\left\| \sum_{n=1}^{\infty} E_{n1} \otimes a_n^* \right\|_{L^{2p}(\mathcal{M}; \ell_c^2)} + \left\| \sum_{n=1}^{\infty} E_{n1} \otimes b_n \right\|_{L^{2p}(\mathcal{M}; \ell_c^2)} \right)^2 \\ &\leq 4 \end{aligned}$$

by (3) and (28).

Since

$$T(x_n) = \frac{1}{4} \sum_{k=0}^3 (-i)^k T(y_n^k),$$

we deduce that

$$\|(T(x_n))_n\|_{L^p(\mathcal{N}; \ell^1)} \leq \frac{1}{4} \sum_{k=0}^3 \|(T(y_n^k))_n\|_{L^p(\mathcal{N}; \ell^1)} \leq 4\|T\|.$$

This shows that $\|T\|_{\ell^1} \leq 4\|T\|$. □

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