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BILINEAR OPERATOR MULTIPLIERS INTO THE TRACE CLASS

CHRISTIAN LE MERDY, IVAN G. TODOROV, AND LYUDMILA TUROWSKA

ABSTRACT. Given Hilbert spaces H_1, H_2, H_3 , we consider bilinear maps defined on the cartesian product $S^2(H_2, H_3) \times S^2(H_1, H_2)$ of spaces of Hilbert-Schmidt operators and valued in either the space $B(H_1, H_3)$ of bounded operators, or in the space $S^1(H_1, H_3)$ of trace class operators. We introduce modular properties of such maps with respect to the commutants of von Neumann algebras $M_i \subset B(H_i)$, $i = 1, 2, 3$, as well as an appropriate notion of complete boundedness for such maps. We characterize completely bounded module maps $u: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow B(H_1, H_3)$ by the membership of a natural symbol of u to the von Neumann algebra tensor product $M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$. In the case when M_2 is injective, we characterize completely bounded module maps $u: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow S^1(H_1, H_3)$ by a weak factorization property, which extends to the bilinear setting a famous description of bimodule linear mappings going back to Haagerup, Effros-Kishimoto, Smith and Blecher-Smith. We make crucial use of a theorem of Sinclair-Smith on completely bounded bilinear maps valued in an injective von Neumann algebra, and provide a new proof of it, based on Hilbert C^* -modules.

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1. INTRODUCTION

Factorization properties of completely bounded maps have played a prominent role in the development of operator spaces [3, 7, 17] and in their applications to Hilbertian operator theory, in particular for the study of special classes of operators: Schur multipliers, Fourier multipliers on either commutative or non commutative groups, module maps, decomposable maps, etc. The main purpose of this paper is to establish new such factorization properties for some classes of bilinear maps defined on the cartesian product $S^2(H_2, H_3) \times S^2(H_1, H_2)$ of two spaces of Hilbert-Schmidt operators and valued in their “product space”, namely the space $S^1(H_1, H_3)$ of trace class operators. This line of investigation is motivated by the recent characterization of bounded bilinear Schur multipliers $S^2 \times S^2 \rightarrow S^1$ proved in [5, 6], by various advances on multidimensional operator multipliers, see [13, 12], and by new developments on multiple operator integrals, see e.g. [1] and the references therein.

Let H, K be Hilbert spaces and let M, N be von Neumann algebras acting on H and K , respectively. Let $CB_{(N', M')}(S^1(H, K))$ denote the Banach space of all (N', M') -bimodule completely bounded maps on $S^1(H, K)$, equipped with the completely bounded norm $\|\cdot\|_{cb}$. This space is characterized by the following factorization property.

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Theorem 1.1. *A bounded map $u: S^1(H, K) \rightarrow S^1(H, K)$ belongs to $CB_{(N', M')}(S^1(H, K))$ and $\|u\|_{cb} \leq 1$ if and only if there exist an index set I , a family $(a_i)_{i \in I}$ of elements of M belonging to the row space $R_I^w(M)$ and a family $(b_i)_{i \in I}$ of elements of N belonging to the column space $C_I^w(N)$ such that*

$$u(z) = \sum_{i \in I} b_i z a_i, \quad z \in S^1(H, K),$$

and $\|(a_i)_i\|_{R_I^w} \|(b_i)_i\|_{C_I^w} \leq 1$.

We refer to Section 5 for the precise definitions of the spaces $R_I^w(M)$ and $C_I^w(N)$. The above theorem is a reformulation of [4, Theorem 2.2], a fundamental factorization result going back to [9, 11] (see also [20]). Indeed let $B(K, H)$ (resp. $S^\infty(K, H)$) denote the space of all bounded operators (resp. all compact operators) from K into H . Then by standard operator space duality, the adjoint mapping $u \mapsto u^*$ induces an isometric isomorphism between $CB_{(N', M')}(S^1(H, K))$ and the space $CB_{(M', N')}(S^\infty(K, H), B(K, H))$ of all (M', N') -bimodule completely bounded maps from $S^\infty(K, H)$ into $B(K, H)$. Consequently the description of such maps provided by [4, Theorem 2.2] yields Theorem 1.1. Using the so-called weak* Haagerup tensor product \otimes^{w^*h} introduced in [4], an equivalent formulation of Theorem 1.1 is that we have a natural isometric w^* -homeomorphic identification

$$(1.1) \quad M \otimes^{w^*h} N \simeq CB_{(N', M')}(S^1(H, K)).$$

In this paper we consider three Hilbert spaces H_1, H_2, H_3 as well as von Neumann algebras M_1, M_2, M_3 acting on them. We study bilinear (M'_3, M'_2, M'_1) -module maps

$$u: S^2(H_2, H_3) \times S^2(H_1, H_2) \longrightarrow S^1(H_1, H_3),$$

in the sense that $u(Ty, SxR) = Tu(yS, x)R$ for any $x \in S^2(H_1, H_2)$, $y \in S^2(H_2, H_3)$, $R \in M'_1$, $S \in M'_2$ and $T \in M'_3$. In the case when $H_i = L^2(\Omega_i)$ for some measure spaces Ω_i , $i = 1, 2, 3$, and $M_i = L^\infty(\Omega_i) \subset B(L^2(\Omega_i))$ in the usual way, bilinear (M'_3, M'_2, M'_1) -module maps coincide with the bilinear Schur multipliers discussed in [12, 6].

On the projective tensor product $S^2(H_2, H_3) \widehat{\otimes} S^2(H_1, H_2)$, we introduce a natural operator space structure, denoted by $\Gamma(H_1, H_2, H_3)$, see (3.4). Our main result, Theorem 6.1, is a characterization, in the case when M_2 is injective, of completely bounded (M'_3, M'_2, M'_1) -module maps u as above by a weak factorization property, which extends Theorem 1.1. (see Remark 6.6). This characterization is already new in the non module case (that is, when $M_i = B(H_i)$ for $i = 1, 2, 3$).

The proof of this result has two steps. First we establish an isometric and w^* -homeomorphic identification

$$(1.2) \quad M_2^{op} \overline{\otimes} (M_1 \otimes^{w^*h} M_3) \simeq CB_{(M'_3, M'_2, M'_1)}(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$$

which extends (1.1), see Theorem 4.3. Second we make use of a remarkable factorization result of Sinclair-Smith [19] on completely bounded bilinear maps valued in an injective von Neumann algebra (see Theorem 5.2 for the precise statement), as well as operator space results, to derive Theorem 6.1 from (1.2).

The Sinclair-Smith theorem, which plays a key role in this paper, was proved in [19, Theorem 4.4] using tensor product computations, the Effros-Lance characterization of semidiscrete von Neumann algebras [10] and Connes's fundamental result (completed in [23]) that any injective von Neumann algebra is semidiscrete. In Section 5 below, we give a new, much shorter proof of Theorem 5.2 based on Hilbert C^* -modules.

The paper also contains a thorough study of completely bounded bilinear (M'_3, M'_2, M'_1) -module maps

$$u: S^2(H_2, H_3) \times S^2(H_1, H_2) \longrightarrow B(H_1, H_3).$$

In analogy with (1.2) we show that the space of such maps can be identified with the von Neumann algebra tensor product $M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$, see Corollary 4.2.

2. OPERATOR SPACE AND DUALITY PRELIMINARIES

We start with some general principles and conventions which will be used throughout this paper.

Let E, F and G be Banach spaces. We let $E \otimes F$ denote the algebraic tensor product of E and F . We let $B(E, G)$ denote the Banach space of all bounded operators from E into G . We let $B_2(F \times E, G)$ denote the Banach space of all bounded bilinear operators from $F \times E$ into G .

Let $F \widehat{\otimes} E$ be the projective tensor product of F and E . To any $u \in B_2(F \times E, G)$, one can associate a unique $\tilde{u}: F \otimes E \rightarrow G$ satisfying

$$\tilde{u}(y \otimes x) = u(y, x), \quad x \in E, y \in F.$$

Then \tilde{u} extends to a bounded operator (still denoted by) $\tilde{u}: F \widehat{\otimes} E \rightarrow G$ and we have equality $\|\tilde{u}\| = \|u\|$. Then the mapping $u \mapsto \tilde{u}$ yields an isometric identification

$$(2.1) \quad B_2(F \times E, G) \simeq B(F \widehat{\otimes} E, G).$$

Consider the case $G = \mathbb{C}$. Then (2.1) provides an isometric identification $B_2(F \times E, \mathbb{C}) \simeq (F \widehat{\otimes} E)^*$. Now to any bounded bilinear form $u: F \times E \rightarrow \mathbb{C}$, one can associate two bounded maps

$$u': E \longrightarrow F^* \quad \text{and} \quad u'': F \longrightarrow E^*$$

defined by $\langle u'(x), y \rangle = u(y, x) = \langle u''(y), x \rangle$ for any $x \in E$ and $y \in F$. Moreover the norms of u' and u'' are equal to the norm of u . Hence the mappings $u \mapsto u'$ and $u \mapsto u''$ yield isometric identifications

$$(2.2) \quad (F \widehat{\otimes} E)^* \simeq B(E, F^*) \simeq B(F, E^*).$$

We refer to [8, Chap. 8, Theorem 1 and Corollary 2] for these classical facts.

We assume that the reader is familiar with the basics of Operator Space Theory and completely bounded maps, for which we refer to [7, 17] and [3, Chap. 1]. However we need to review a few important definitions and fundamental results which will be used at length in this paper; the remainder of this section is devoted to this task.

We will make crucial use of the dual operator space E^* of an operator space E as well as of the operator space $CB(E, F)$ of completely bounded maps from E into another operator space F (see e.g. [7, Section 3.2]). Whenever $v: E \rightarrow F$ is a completely bounded map, its completely bounded norm will be denoted by $\|v\|_{cb}$.

Let E, F be operator spaces. We let $F \widehat{\otimes} E$ denote the operator space projective tensor product of F and E (here we adopt the notation from [3, 1.5.11]). We will often use the fact that this tensor product is commutative and associative. The identifications (2.2) have operator space analogs. Namely let $u: F \times E \rightarrow \mathbb{C}$ be a bounded bilinear form. Then \tilde{u} extends to a functional on $F \widehat{\otimes} E$ if and only if $u': E \rightarrow F^*$ is completely bounded, if and only if $u'': F \rightarrow E^*$ is completely bounded. In this case $\|u'\|_{cb} = \|u''\|_{cb} = \|\tilde{u}\|_{(F \widehat{\otimes} E)^*}$. Thus (2.2) restricts to isometric identifications

$$(2.3) \quad (F \widehat{\otimes} E)^* \simeq CB(E, F^*) \simeq CB(F, E^*).$$

It turns out that the latter are actually completely isometric identifications (see e.g. [7, Section 7.1] or [3, (1.51)]).

Let H, K be Hilbert spaces. We let \overline{K} denote the complex conjugate of K . For any $\xi \in K$, the notation $\overline{\xi}$ stands for ξ regarded as an element of \overline{K} . We recall the canonical identification $\overline{K} = K^*$. Thus for any $\xi \in K$ and any $\eta \in H$, $\overline{\xi} \otimes \eta$ may be regarded as the rank one operator $K \rightarrow H$ taking any $\zeta \in K$ to $\langle \zeta, \xi \rangle \eta$. With this convention, the algebraic tensor product $\overline{K} \otimes H$ is identified with the space of all bounded finite rank operators from K into H .

Let $S^1(K, H)$ be the space of trace class operators $v: K \rightarrow H$, equipped with its usual norm $\|v\|_1 = \text{tr}(|v|)$. Then $\overline{K} \otimes H$ is a dense subspace of $S^1(K, H)$ and $\|\cdot\|_1$ coincides with the Banach space projective norm on $\overline{K} \otimes H$. Hence we have an isometric identification

$$(2.4) \quad S^1(K, H) \simeq \overline{K} \widehat{\otimes} H.$$

Let $S^2(K, H)$ be the space of Hilbert-Schmidt operators $v: K \rightarrow H$, equipped with its usual Hilbertian norm $\|v\|_2 = (\text{tr}(v^*v))^{\frac{1}{2}}$. Then $\overline{K} \otimes H$ is a dense subspace of $S^2(K, H)$ and $\|\cdot\|_2$ coincides with the Hilbertian tensor norm on $\overline{K} \otimes H$. Hence we have an isometric identification

$$(2.5) \quad S^2(K, H) \simeq \overline{K} \overset{2}{\otimes} H,$$

where the right hand side denotes the Hilbertian tensor product of \overline{K} and H .

Let $S^\infty(H, K)$ denote the space of all compact operators from H into K , equipped with its usual operator space structure. We recall that through trace duality, we have isometric identifications

$$(2.6) \quad S^\infty(H, K)^* \simeq S^1(K, H) \quad \text{and} \quad S^1(K, H)^* \simeq B(H, K).$$

Throughout we assume that $S^1(K, H)$ is equipped with its canonical operator space structure, so that (2.6) holds completely isometrically (see e.g. [7, Theorem 3.2.3]).

Let E, G be Banach spaces and let $j: E^* \rightarrow G^*$ be a w^* -continuous isometry. Then its range $j(E^*)$ is w^* -closed, hence $j(E^*)$ is a dual space. Further j induces a w^* - w^* -homeomorphism between E^* and $j(E^*)$ (see e.g. [3, A.2.5]). Thus j allows to identify E^* and $j(E^*)$ as dual Banach spaces. In this case, we will say that j induces a w^* -continuous isometric identification between E^* and $j(E^*)$. If E, G are operator spaces and j is a complete isometry, then $j(E^*)$ is a dual operator space and we will call j a w^* -continuous completely isometric identification between E^* and $j(E^*)$.

Let E, F be operator spaces and consider w^* -continuous completely isometric embeddings

$$(2.7) \quad E^* \subset B(H) \quad \text{and} \quad F^* \subset B(K),$$

for some Hilbert spaces H, K (see e.g. [7, Prop. 3.2.4]). The normal spatial tensor product of the dual operator spaces F^* and E^* is defined as the w^* -closure of $F^* \otimes E^*$ into the von Neumann algebra $B(K) \overline{\otimes} B(H)$ and is denoted by

$$F^* \overline{\otimes} E^*.$$

This is a dual operator space. It turns out that its definition does not depend on the specific embeddings (2.7), see e.g. [7, p. 135].

We note for further use that the natural embedding $B(K) \otimes B(H) \subset B(K \overset{2}{\otimes} H)$ extends to a w^* -continuous completely isometric identification

$$(2.8) \quad B(K) \overline{\otimes} B(H) \simeq B(K \overset{2}{\otimes} H).$$

To deal with normal spatial tensor products, it is convenient to use the so-called slice maps. Take any $\lambda \in S^1(K)$ and consider it as a w^* -continuous functional $\lambda: B(K) \rightarrow \mathbb{C}$. Then the operator $\lambda \otimes I_{B(H)}$ extends to a (necessarily unique) w^* -continuous bounded map

$$\ell_\lambda: B(K) \overline{\otimes} B(H) \longrightarrow B(H).$$

Likewise, any $\mu \in S^1(H)$ can be considered as a w^* -continuous functional $\mu: B(H) \rightarrow \mathbb{C}$ and $I_{B(K)} \otimes \mu$ extends to a w^* -continuous bounded map

$$r_\mu: B(K) \overline{\otimes} B(H) \longrightarrow B(K).$$

Then we have the following properties (for which we refer to either [7, Lemma 7.2.2] and its proof, or [3, 1.5.2]).

Lemma 2.1. *Let $z \in B(K) \overline{\otimes} B(H)$. The linear mappings*

$$z': S^1(H) \longrightarrow B(K) \quad \text{and} \quad z'': S^1(K) \longrightarrow B(H)$$

defined by $z'(\mu) = r_\mu(z)$ and $z''(\lambda) = \ell_\lambda(z)$ are completely bounded.

Further the mappings $z \mapsto z'$ and $z \mapsto z''$ are w^ -continuous completely isometric isomorphisms from $B(K) \overline{\otimes} B(H)$ onto $CB(S^1(H), B(K))$ and $CB(S^1(K), B(H))$, respectively.*

According to (2.3), an equivalent formulation of the above lemma is that

$$(2.9) \quad (S^1(K) \widehat{\otimes} S^1(H))^* \simeq B(K) \overline{\otimes} B(H)$$

w^* -continuously and completely isometrically.

Recall (2.7). The space of all $z \in B(K) \overline{\otimes} B(H)$ such that z' is valued in F^* and z'' is valued in E^* is usually called the normal Fubini tensor product of F^* and E^* . This subspace is w^* -continuously completely isometric to $CB(E, F^*)$ (equivalently to $CB(F, E^*)$, by (2.3)). Indeed we may regard $CB(E, F^*)$ as the subspace of $CB(S^1(H), B(K))$ of all $w: S^1(H) \rightarrow B(K)$ such that w is valued in F^* and w vanishes on E^\perp . Then it is not hard to see that z belongs to the normal Fubini tensor product of F^* and E^* if and only if z' belongs to $CB(E, F^*)$. We refer to [7, Theorem 7.2.3] for these facts.

It is easy to check that the normal Fubini tensor product of F^* and E^* contains $F^* \overline{\otimes} E^*$. This yields a w^* -continuous completely isometric embedding

$$F^* \overline{\otimes} E^* \subset CB(E, F^*).$$

However this inclusion may be strict. The next lemma provides a list of cases when the inclusion is an equality. We refer the reader to [7, Sections 7.2 and 11.2] for the proofs.

Whenever M is a von Neumann algebra, we let M_* denote its (unique) predual. We equip it with its natural operator space structure, so that $M = (M_*)^*$ completely isometrically (see e.g. [17, Section 2.5] or [3, Lemma 1.4.6]).

Lemma 2.2.

(a) *For any von Neumann algebras M, N , we have*

$$N \overline{\otimes} M \simeq CB(M_*, N).$$

(b) *For any injective von Neumann algebra M and for any operator space E , we have*

$$M \overline{\otimes} E^* \simeq CB(E, M).$$

(c) *For any Hilbert spaces H, K and for any operator space E , we have*

$$B(H, K) \overline{\otimes} E^* \simeq CB(E, B(H, K)).$$

Let K be a Hilbert space. We let $\{K\}_c$ (resp. $\{K\}_r$) denote the column operator space (resp. the row operator space) over K . We recall that through the canonical identification $K^* = \overline{K}$, we have

$$\{K\}_c^* = \{\overline{K}\}_r \quad \text{and} \quad \{K\}_r^* = \{\overline{K}\}_c \quad \text{completely isometrically.}$$

(See e.g. [7, Section 3.4].)

We let $F \overset{h}{\otimes} E$ denote the Haagerup tensor product of a couple (F, E) of operator spaces. We will use the fact that this is an associative tensor product.

Let $\theta: F \times E \rightarrow \mathbb{C}$ be a bounded bilinear form. Then θ extends to an element of $(F \overset{h}{\otimes} E)^*$ if and only if there exist a Hilbert space \mathcal{H} and two completely bounded maps $\alpha: E \rightarrow \{\mathcal{H}\}_c$ and $\beta: F \rightarrow \{\overline{\mathcal{H}}\}_r$ such that $\theta(y, x) = \langle \alpha(x), \beta(y) \rangle$ for any $x \in E$ and any $y \in F$ (see e.g. [7, Corollary 9.4.2]).

The Haagerup tensor product is projective. This means that if $p: E \rightarrow E_1$ and $q: F \rightarrow F_1$ are complete quotient maps, then $q \otimes p$ extends to a (necessarily unique) complete quotient map $F \overset{h}{\otimes} E \rightarrow F_1 \overset{h}{\otimes} E_1$. Taking the adjoint of the latter, we obtain a w^* -continuous completely isometric embedding

$$(2.10) \quad (F_1 \overset{h}{\otimes} E_1)^* \subset (F \overset{h}{\otimes} E)^*.$$

Lemma 2.3. *Let E, F, E_1, F_1 be operator spaces as above and let $\theta \in (F \overset{h}{\otimes} E)^*$. Let*

$$\theta': E \longrightarrow F^* \quad \text{and} \quad \theta'': F \longrightarrow E^*$$

be the bounded linear maps associated to θ . Then $\theta \in (F_1 \overset{h}{\otimes} E_1)^$ (in the sense given by (2.10)) if and only if θ' is valued in F_1^* and θ'' is valued in E_1^* .*

Proof. If $\theta \in (F_1 \overset{h}{\otimes} E_1)^*$, then $\theta(y, x) = 0$ if either $x \in \text{Ker}(p)$ or $y \in \text{Ker}(q)$. Hence $\langle \theta''(y), x \rangle = 0$ for any $(y, x) \in F \times \text{Ker}(p)$ and $\langle \theta'(x), y \rangle = 0$ for any $(y, x) \in \text{Ker}(q) \times E$. Hence θ'' is valued in $E_1^* = \text{Ker}(p)^\perp$ and θ' is valued in $F_1^* = \text{Ker}(q)^\perp$.

Assume conversely that θ' is valued in F_1^* and that θ'' is valued in E_1^* . Let $\alpha: E \rightarrow \{\mathcal{H}\}_c$ and $\beta: F \rightarrow \{\overline{\mathcal{H}}\}_r$ be completely bounded maps, for some Hilbert space \mathcal{H} , such that $\theta(y, x) = \langle \alpha(x), \beta(y) \rangle$ for any $x \in E$ and any $y \in F$. Changing \mathcal{H} into the closure of the range of α , we may assume that α has dense range. Next changing \mathcal{H} into the closure of the (conjugate of) the range of β , we may actually assume that both α and β have dense range.

The assumption on θ' means that $\langle \alpha(x), \beta(y) \rangle = 0$ for any $x \in E$ and any $y \in \text{Ker}(q)$. Since α has dense range this means that β vanishes on $\text{Ker}(q)$. Likewise the assumption on θ'' means that α vanishes on $\text{Ker}(p)$. We may therefore consider $\alpha_1: E_1 \rightarrow \{\mathcal{H}\}_c$ and $\beta_1: F_1 \rightarrow \{\overline{\mathcal{H}}\}_r$ induced by α and β , that is, $\alpha = \alpha_1 \circ p$ and $\beta = \beta_1 \circ q$. Further α_1 and β_1 are completely bounded, hence the bilinear mapping $(y_1, x_1) \mapsto \langle \alpha_1(x_1), \beta_1(y_1) \rangle$ is an element of $(F_1 \overset{h}{\otimes} E_1)^*$. By construction it identifies with θ in the embedding (2.10), hence θ belongs to $(F_1 \overset{h}{\otimes} E_1)^*$. \square

We will need the so-called weak* Haagerup tensor product of two dual operator spaces [4]. It can be defined by

$$(2.11) \quad F^* \overset{w^*h}{\otimes} E^* = (F \overset{h}{\otimes} E)^*.$$

The reason why this dual space can be considered as a tensor product over the couple (F^*, E^*) is discussed in [3, 1.6.9].

We now recall a few tensor product identities involving the operator space projective tensor product and the Haagerup tensor product.

Proposition 2.4. *Let E be an operator space and let H, K be two Hilbert spaces.*

(a) *We have completely isometric identifications*

$$\{K\}_r \widehat{\otimes} E \simeq \{K\}_r \overset{h}{\otimes} E \quad \text{and} \quad E \widehat{\otimes} \{H\}_c \simeq E \overset{h}{\otimes} \{H\}_c.$$

(b) *We have completely isometric identifications*

$$\{K\}_r \widehat{\otimes} \{H\}_r \simeq \{K \overset{2}{\otimes} H\}_r \quad \text{and} \quad \{K\}_c \widehat{\otimes} \{H\}_c \simeq \{K \overset{2}{\otimes} H\}_c.$$

(c) *The embedding $\overline{K} \otimes H \subset S^1(K, H)$ extends to completely isometric identifications*

$$(2.12) \quad S^1(K, H) \simeq \{\overline{K}\}_r \widehat{\otimes} \{H\}_c$$

and

$$(2.13) \quad S^1(K, H) \widehat{\otimes} E \simeq \{\overline{K}\}_r \widehat{\otimes} E \widehat{\otimes} \{H\}_c.$$

(d) *To any $u: E \rightarrow B(H, K)$, associate $\theta_u: \overline{K} \otimes E \otimes H \rightarrow \mathbb{C}$ by letting $\theta_u(\overline{\xi} \otimes x \otimes \eta) = \langle u(x)\eta, \xi \rangle$, for any $x \in E, \eta \in H, \xi \in K$. Then $u \mapsto \theta_u$ extends to a w^* -continuous completely isometric identification*

$$(\{\overline{K}\}_r \widehat{\otimes} E \widehat{\otimes} \{H\}_c)^* \simeq CB(E, B(H, K)).$$

Proof. We refer to [7, Proposition 9.3.2] for (a) and to [7, Proposition 9.3.5] for (b). Formula (2.12) follows from [7, Proposition 9.3.4] and (a), and formula (2.13) follows by the comutativity of the operator space projective tensor product. Finally (d) is a consequence of (2.13), (2.6) and (2.3). \square

Remark 2.5. Comparing (2.12) with (2.4), we note that at the Banach space level, the operator space projective tensor product of a row and a column Hilbert space coincides with their Banach space projective tensor product.

Remark 2.6. For any $\eta \in H$ and $\xi \in K$, let $T_{\eta,\xi} \in B(H, K)$ be the rank one operator defined by

$$T_{\eta,\xi}(\zeta) = \langle \zeta, \eta \rangle \xi, \quad \zeta \in H.$$

When we consider this operator as an element of $S^\infty(H, K)$ or $B(H, K)$, it is convenient to identify it with $\xi \otimes \bar{\eta} \in K \otimes \bar{H}$, and hence to regard $K \otimes \bar{H}$ as a subspace of $S^\infty(H, K)$. This convention is different from the one used so far when we had to represent rank one (more generally, finite rank) operators as elements of the trace class or of the Hilbert-Schmidt class.

The rationale for this is that the trace duality providing (2.6) extends the natural duality between $K \otimes \bar{H}$ and $\bar{K} \otimes H$. Then the embedding $K \otimes \bar{H} \subset S^\infty(H, K)$ extends to a completely isometric identification

$$(2.14) \quad S^\infty(H, K) \simeq \{K\}_c \otimes^h \{\bar{H}\}_r.$$

(See e.g. [7, Proposition 9.3.4].)

If A is any C^* -algebra, the so-called opposite C^* -algebra A^{op} is the involutive Banach space A equipped with its reversed multiplication $(a, b) \mapsto ba$. Note that as an operator space, A^{op} is not (in general) the same as A , that is, the identity mapping $A \rightarrow A^{op}$ is not a complete isometry. See e.g. [18, Theorem 2.2] for more about this.

In the case when $A = B(H)$, we have the following well-known description (see e.g. [17, Sections 2.9 and 2.10]).

Lemma 2.7. *Let H be a Hilbert space. For any $S \in B(H)$, define*

$$\widehat{S}(\bar{h}) = \overline{S^*(h)}, \quad h \in H.$$

Then $S \mapsto \widehat{S}$ is a $$ -isomorphism from $B(H)^{op}$ onto $B(\bar{H})$.*

In the sequel we will use the operator space M_*^{op} for any von Neumann algebra M . This is both the predual operator space of M^{op} and the opposite operator space of M_* , in the sense of [17, Section 2.10].

3. OPERATOR MULTIPLIERS INTO THE TRACE CLASS

Let H_1, H_2, H_3 be three Hilbert spaces. Using (2.5), we let

$$\Theta: H_1 \otimes^2 \bar{H}_2 \otimes^2 H_3 \longrightarrow S^2(S^2(H_1, H_2), H_3)$$

be the unitary operator obtained by first identifying $H_1 \otimes^2 \bar{H}_2$ with $\overline{S^2(H_1, H_2)}$, and then identifying $\overline{S^2(H_1, H_2)} \otimes^2 H_3$ with $S^2(S^2(H_1, H_2), H_3)$.

For any $\varphi \in B(H_1 \overset{2}{\otimes} \overline{H_2} \overset{2}{\otimes} H_3)$, one may define a bounded bilinear map

$$\tau_\varphi: S^2(H_2, H_3) \times S^2(H_1, H_2) \longrightarrow B(H_1, H_3)$$

by

$$[\tau_\varphi(y, x)](h) = \Theta[\varphi(h \otimes y)](x), \quad x \in S^2(H_1, H_2), y \in S^2(H_2, H_3), h \in H_1.$$

On the right hand side of the above equality, y is regarded as an element of $\overline{H_2} \overset{2}{\otimes} H_3$, and hence $h \otimes y$ is an element of $H_1 \overset{2}{\otimes} \overline{H_2} \overset{2}{\otimes} H_3$. It is clear that

$$\|[\tau_\varphi(y, x)](h)\| \leq \|\varphi\| \|x\|_2 \|y\|_2 \|h\|.$$

Consequently, the above construction defines a contraction

$$(3.1) \quad \tau: B(H_1 \overset{2}{\otimes} \overline{H_2} \overset{2}{\otimes} H_3) \longrightarrow B_2(S^2(H_2, H_3) \times S^2(H_1, H_2), B(H_1, H_3)).$$

The bilinear maps τ_φ were introduced in [12] (however the latter paper focuses on the case when $\|[\tau_\varphi(y, x)](h)\| \leq D\|x\|\|y\|\|h\|$ for some constant $D > 0$). We call τ_φ an operator multiplier and we say that φ is the symbol of τ_φ . We refer to [12] for m -linear versions of such operators for arbitrary $m \geq 2$.

We note that by (2.8) and Lemma 2.7, we have a von Neumann algebra identification

$$(3.2) \quad B(H_1 \overset{2}{\otimes} \overline{H_2} \overset{2}{\otimes} H_3) \simeq B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3).$$

In the sequel we will make no difference between these two von Neumann algebras. In particular, we will consider symbols φ of operator multipliers as elements of $B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$.

One can check (see [12]) that for any $R \in B(H_1)$, $S \in B(H_2)$ and $T \in B(H_3)$, we have

$$(3.3) \quad \tau_{R \otimes S \otimes T}(y, x) = T y S x R, \quad x \in S^2(H_1, H_2), y \in S^2(H_2, H_3).$$

Note that in this identity, S is regarded as an element of $B(H_2)^{op}$ at the left-hand side and as an element of $B(H_2)$ at the right-hand side.

We now define the operator space

$$(3.4) \quad \Gamma(H_1, H_2, H_3) = \{S^2(H_2, H_3)\}_c \widehat{\otimes} \{S^2(H_1, H_2)\}_r.$$

According to Remark 2.5, $\Gamma(H_1, H_2, H_3)$ coincides, at the Banach space level, with the projective tensor product of $S^2(H_2, H_3)$ and $S^2(H_1, H_2)$. Hence

$$(3.5) \quad B_2(S^2(H_2, H_3) \times S^2(H_1, H_2), B(H_1, H_3)) \simeq B(\Gamma(H_1, H_2, H_3), B(H_1, H_3))$$

by (2.1). In the sequel for any $u: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow B(H_1, H_3)$, we let

$$\tilde{u}: \Gamma(H_1, H_2, H_3) \longrightarrow B(H_1, H_3)$$

denote its associated linear map.

The next proposition shows that under the identification (3.5), the range of τ coincides with the space of completely bounded maps from $\Gamma(H_1, H_2, H_3)$ into $B(H_1, H_3)$.

Proposition 3.1. *Let $u: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow B(H_1, H_3)$ be a bounded bilinear map. Then $\tilde{u}: \Gamma(H_1, H_2, H_3) \rightarrow B(H_1, H_3)$ is completely bounded if and only if there exists φ in $B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$ such that $u = \tau_\varphi$. Further τ provides a w^* -continuous completely isometric identification*

$$(3.6) \quad B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3) \simeq CB(\Gamma(H_1, H_2, H_3), B(H_1, H_3)).$$

Proof. For convenience we set

$$\mathcal{H} = H_1 \overset{2}{\otimes} \overline{H_2} \overset{2}{\otimes} H_3.$$

By (2.5) and Proposition 2.4 (b), we have

$$\{S^2(H_1, H_2)\}_r \simeq \{\overline{H_1}\}_r \widehat{\otimes} \{H_2\}_r \quad \text{and} \quad \{S^2(H_2, H_3)\}_c \simeq \{\overline{H_2}\}_c \widehat{\otimes} \{H_3\}_c$$

completely isometrically. Hence applying (3.4), we have

$$(3.7) \quad \Gamma(H_1, H_2, H_3) \simeq \{\overline{H_2}\}_c \widehat{\otimes} \{H_3\}_c \widehat{\otimes} \{\overline{H_1}\}_r \widehat{\otimes} \{H_2\}_r$$

completely isometrically. Using the commutativity of the operator space projective tensor product, we deduce a completely isometric identification

$$\{\overline{H_3}\}_r \widehat{\otimes} \Gamma(H_1, H_2, H_3) \widehat{\otimes} \{H_1\}_c \simeq \{\overline{H_1}\}_r \widehat{\otimes} \{H_2\}_r \widehat{\otimes} \{\overline{H_3}\}_r \widehat{\otimes} \{H_1\}_c \widehat{\otimes} \{\overline{H_2}\}_c \widehat{\otimes} \{H_3\}_c.$$

Using Proposition 2.4 (b) again, we have

$$\{H_1\}_c \widehat{\otimes} \{\overline{H_2}\}_c \widehat{\otimes} \{H_3\}_c \simeq \{\mathcal{H}\}_c \quad \text{and} \quad \{\overline{H_1}\}_r \widehat{\otimes} \{H_2\}_r \widehat{\otimes} \{\overline{H_3}\}_r \simeq \{\overline{\mathcal{H}}\}_r$$

completely isometrically. By Proposition 2.4 (c), this yields a completely isometric identification

$$(3.8) \quad \{\overline{H_3}\}_r \widehat{\otimes} \Gamma(H_1, H_2, H_3) \widehat{\otimes} \{H_1\}_c \simeq S^1(\mathcal{H}).$$

Passing to the duals, using (2.6) and Proposition 2.4 (d), we deduce a w^* -continuous completely isometric identification

$$B(\mathcal{H}) \simeq CB(\Gamma(H_1, H_2, H_3), B(H_1, H_3)).$$

Combining with (3.2), we deduce a w^* -continuous, completely isometric onto mapping

$$(3.9) \quad J: B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3) \longrightarrow CB(\Gamma(H_1, H_2, H_3), B(H_1, H_3)).$$

Now to establish the proposition it suffices to check that

$$(3.10) \quad J(\varphi) = \tilde{\tau}_\varphi$$

for any $\varphi \in B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$.

We claim that it suffices to prove (3.10) in the case when φ belongs to the algebraic tensor product $B(H_1) \otimes B(H_2)^{op} \otimes B(H_3)$. Indeed let $\varphi \in B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$, let $x \in S_2(H_1, H_2)$, $y \in S_2(H_2, H_3)$ and $h \in H_1$. Assume that $(\varphi_t)_t$ is a net of $B(H_1) \otimes B(H_2)^{op} \otimes B(H_3)$ converging to φ in the w^* -topology. Then $\varphi_t(h \otimes y) \rightarrow \varphi(h \otimes y)$ in the weak topology of $H_1 \overset{2}{\otimes} \overline{H_2} \overset{2}{\otimes} H_3$. Hence $\Theta[\varphi_t(h \otimes y)] \rightarrow \Theta[\varphi(h \otimes y)]$ in the weak topology of $S^2(S^2(H_1, H_2), H_3)$, which implies that $\Theta[\varphi_t(h \otimes y)](x) \rightarrow \Theta[\varphi(h \otimes y)](x)$ in the weak topology of H_3 . Equivalently, $[\tau_{\varphi_t}(y, x)](h) \rightarrow [\tau_\varphi(y, x)](h)$ weakly. Since J is w^* -continuous,

we also have, by similar arguments, that $[J(\varphi_t)(y \otimes x)](h) \rightarrow [J(\varphi)(y \otimes x)](h)$ weakly. Hence if $J(\varphi_t) = \tilde{\tau}_{\varphi_t}$ for any t , we have $J(\varphi) = \tilde{\tau}_\varphi$ as well.

Moreover by linearity, it suffices to prove (3.10) when $\varphi = R \otimes S \otimes T$ for some $R \in B(H_1)$, $S \in B(H_2)$ and $T \in B(H_3)$. In view of (3.3), it therefore suffices to show that

$$(3.11) \quad J(R \otimes S \otimes T)(y \otimes x) = TySxR,$$

for any $R \in B(H_1)$, $S \in B(H_2)$, $T \in B(H_3)$, $x \in S^2(H_1, H_2)$ and $y \in S^2(H_2, H_3)$. Since J is linear and w^* -continuous it actually suffices to prove (3.11) when R, S, T, x and y are rank one.

For $i = 1, 2, 3$, let $\xi_i, \eta_i, h_i, k_i \in H_i$ and consider $x = \bar{\xi}_1 \otimes \eta_2$ and $y = \bar{\xi}_2 \otimes \eta_3$, as well as the operators $R = h_1 \otimes \bar{k}_1$, $S = h_2 \otimes \bar{k}_2$ and $T = h_3 \otimes \bar{k}_3$ (see Remark 2.6 for the use of these tensor product notations). Then let

$$\alpha = \bar{\xi}_1 \otimes \eta_2 \otimes \bar{\xi}_3 \otimes \eta_1 \otimes \bar{\xi}_2 \otimes \eta_3 \in (\overline{H_1} \otimes H_2 \otimes \overline{H_3}) \otimes (H_1 \otimes \overline{H_2} \otimes H_3) \subset S^1(\mathcal{H})$$

and let

$$\beta = h_1 \otimes \bar{k}_2 \otimes h_3 \otimes \bar{k}_1 \otimes h_2 \otimes \bar{k}_3 \in (H_1 \otimes \overline{H_2} \otimes H_3) \otimes (\overline{H_1} \otimes H_2 \otimes \overline{H_3}) \subset B(\mathcal{H}).$$

In the identification (3.8), $\bar{\xi}_3 \otimes y \otimes x \otimes \eta_1$ corresponds to α whereas in the identification (3.2), $R \otimes S \otimes T$ corresponds to β . Hence

$$\begin{aligned} \langle [J(R \otimes S \otimes T)(y \otimes x)](\eta_1), \xi_3 \rangle &= \text{tr}(\alpha\beta) \\ &= \langle h_1, \xi_1 \rangle \langle \eta_2, k_2 \rangle \langle h_3, \xi_3 \rangle \langle \eta_1, k_1 \rangle \langle h_2, \xi_2 \rangle \langle \eta_3, k_3 \rangle. \end{aligned}$$

On the other hand,

$$TySxR = \langle h_1, \xi_1 \rangle \langle \eta_2, k_2 \rangle \langle h_2, \xi_2 \rangle \langle \eta_3, k_3 \rangle h_3 \otimes \bar{k}_1,$$

hence

$$(3.12) \quad \langle TySxR(\eta_1), \xi_3 \rangle = \langle h_1, \xi_1 \rangle \langle \eta_2, k_2 \rangle \langle h_3, \xi_3 \rangle \langle \eta_1, k_1 \rangle \langle h_2, \xi_2 \rangle \langle \eta_3, k_3 \rangle.$$

This proves the desired equality. \square

Remark 3.2. Using (2.9) twice we have a w^* -continuous completely isometric identification

$$(3.13) \quad B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3) \simeq (S^1(H_1) \widehat{\otimes} S^1(H_2)^{op} \widehat{\otimes} S^1(H_3))^*.$$

Let $\varphi \in B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$ and let $u = \tau_\varphi$. Let $\xi_1, \eta_1 \in H_1$, $\xi_2, \eta_2 \in H_2$ and $\xi_3, \eta_3 \in H_3$ and regard $\bar{\xi}_i \otimes \eta_i$ as an element of $S^1(H_i)$ for $i = 1, 2, 3$. According to (3.13) we may consider the action of φ on $\bar{\xi}_1 \otimes \eta_1 \otimes \eta_2 \otimes \bar{\xi}_2 \otimes \bar{\xi}_3 \otimes \eta_3$. Then we have

$$\langle \varphi, \bar{\xi}_1 \otimes \eta_1 \otimes \eta_2 \otimes \bar{\xi}_2 \otimes \bar{\xi}_3 \otimes \eta_3 \rangle = \langle [u(\bar{\xi}_2 \otimes \eta_3, \bar{\xi}_1 \otimes \eta_2)](\eta_1), \xi_3 \rangle.$$

Indeed this follows from the arguments in the proof of Proposition 3.1. Details are left to the reader.

Let $\varphi \in B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$. We will say that τ_φ is an S^1 -operator multiplier if it takes values into the trace class $S^1(H_1, H_3)$ and there exists a constant $D \geq 0$ such that

$$\|\tau_\varphi(y, x)\|_1 \leq D\|x\|_2\|y\|_2, \quad x \in S^2(H_1, H_2), \quad y \in S^2(H_2, H_3).$$

Note that, by (3.3), τ_φ is an S^1 -operator multiplier when φ is of the form $R \otimes S \otimes T$. Consequently, τ_φ is an S^1 -operator multiplier whenever φ belongs to the algebraic tensor product $B(H_1) \otimes B(H_2) \otimes B(H_3)$.

In this paper we will be mostly interested in **completely bounded S^1 -operator multipliers**, that is, S^1 -operator multipliers τ_φ such that $\tilde{\tau}_\varphi$ is a completely bounded map from $\Gamma(H_1, H_2, H_3)$ into $S^1(H_1, H_3)$. Note that the canonical inclusion $S^1(H_1, H_3) \subset B(H_1, H_3)$ is a complete contraction, hence

$$CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3)) \subset CB(\Gamma(H_1, H_2, H_3), B(H_1, H_3)) \quad \text{contractively.}$$

It therefore follows from Proposition 3.1 that the space of all completely bounded S^1 -operator multipliers coincides with the space $CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$. The following statement provides a characterization.

Lemma 3.3. *Let $u: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow S^1(H_1, H_3)$ be a bounded bilinear map and let $D > 0$ be a constant. Then $\tilde{u} \in CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$ and $\|\tilde{u}\|_{cb} \leq D$ if and only if for any $n \geq 1$, for any $x_1, \dots, x_n \in S^2(H_1, H_2)$ and for any $y_1, \dots, y_n \in S^2(H_2, H_3)$,*

$$\left\| [u(y_i, x_j)]_{1 \leq i, j \leq n} \right\|_{S^1(\ell_n^2(H_1), \ell_n^2(H_3))} \leq D \left(\sum_{j=1}^n \|x_j\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n \|y_i\|_2^2 \right)^{\frac{1}{2}}.$$

Proof. For any $n \geq 1$, we use the classical notations $R_n = \{\ell_n^2\}_r$, $C_n = \{\ell_n^2\}_c$ and $S_n^1 = S^1(\ell_n^2)$.

Consider u as above and set

$$d_n = \|I_{S_n^1} \otimes \tilde{u}: S_n^1 \widehat{\otimes} \Gamma(H_1, H_2, H_3) \longrightarrow S_n^1 \widehat{\otimes} S^1(H_1, H_3)\|$$

for any $n \geq 1$. By [16, Lemma 1.7], $\tilde{u} \in CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$ if and only if the sequence $(d_n)_{n \geq 1}$ is bounded and in this case, $\|\tilde{u}\|_{cb} = \sup_n d_n$.

By Proposition 2.4 (c),

$$S_n^1 \widehat{\otimes} \Gamma(H_1, H_2, H_3) \simeq R_n \widehat{\otimes} \{S^2(H_1, H_2)\}_r \widehat{\otimes} \{S^2(H_2, H_3)\}_c \widehat{\otimes} C_n$$

completely isometrically. Using Proposition 2.4 (b), this yields

$$S_n^1 \widehat{\otimes} \Gamma(H_1, H_2, H_3) \simeq \{\ell_n^2 \overset{2}{\otimes} S^2(H_1, H_2)\}_r \widehat{\otimes} \{\ell_n^2 \overset{2}{\otimes} S^2(H_2, H_3)\}_c.$$

Applying Remark 2.5, we derive that

$$S_n^1 \widehat{\otimes} \Gamma(H_1, H_2, H_3) \simeq (\ell_n^2 \overset{2}{\otimes} S^2(H_1, H_2)) \widehat{\otimes} (\ell_n^2 \overset{2}{\otimes} S^2(H_2, H_3))$$

isometrically.

Similarly,

$$\begin{aligned} S_n^1 \widehat{\otimes} S^1(H_1, H_3) &\simeq R_n \widehat{\otimes} S^1(H_1, H_3) \widehat{\otimes} C_n \\ &\simeq R_n \widehat{\otimes} \{\overline{H_1}\}_r \widehat{\otimes} \{H_3\}_c \widehat{\otimes} C_n \\ &\simeq \{\ell_n^2 \overset{2}{\otimes} \overline{H_1}\}_r \widehat{\otimes} \{\ell_n^2 \overset{2}{\otimes} H_3\}_c \\ &\simeq S^1(\ell_n^2(H_1), \ell_n^2(H_3)) \end{aligned}$$

isometrically. Hence a thorough look at these identifications shows that

$$d_n = \sup \left\{ \left\| \left[u(y_i, x_j) \right]_{1 \leq i, j \leq n} \right\|_{S^1(\ell_n^2(H_1), \ell_n^2(H_3))} \right\},$$

where the supremum runs over all

$$(x_1, \dots, x_n) \in \ell_n^2 \otimes^2 S^2(H_1, H_2) \quad \text{and} \quad (y_1, \dots, y_n) \in \ell_n^2 \otimes^2 S^2(H_2, H_3)$$

of norms less than or equal to 1. This yields the result. \square

The next result, which should be compared to Proposition 3.1, provides a characterization of completely bounded S^1 -operator multipliers. Before stating it, we note that we have $S^1(H_1) \widehat{\otimes} S^1(H_3) \subset S^1(H_1) \overset{h}{\otimes} S^1(H_3)$ completely contractively (see e.g. [7, Theorem 9.2.1]). Consequently

$$CB(S^1(H_1) \overset{h}{\otimes} S^1(H_3), B(H_2)^{op}) \subset CB(S^1(H_1) \widehat{\otimes} S^1(H_3), B(H_2)^{op})$$

contractively. Applying Lemma 2.2 (c), and using (2.9) and (2.11), we deduce a contractive embedding

$$B(H_2)^{op} \overline{\otimes} (B(H_1) \overset{w^*h}{\otimes} B(H_3)) \subset B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3).$$

Theorem 3.4. *Let $\varphi \in B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$. Then τ_φ is a completely bounded S^1 -operator multiplier if and only if φ belongs to $B(H_2)^{op} \overline{\otimes} (B(H_1) \overset{w^*h}{\otimes} B(H_3))$. Further (3.6) restricts to a w^* -continuous completely isometric identification*

$$(3.14) \quad B(H_2)^{op} \overline{\otimes} (B(H_1) \overset{w^*h}{\otimes} B(H_3)) \simeq CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3)).$$

Proof. The scheme of proof is similar to the one of Proposition 3.1. Recall (3.7) from this proof. On the one hand, using commutativity of the operator space projective tensor product, we deduce a completely isometric identification

$$\Gamma(H_1, H_2, H_3) \simeq \{H_2\}_r \widehat{\otimes} \{\overline{H}_2\}_c \widehat{\otimes} \{\overline{H}_1\}_r \widehat{\otimes} \{H_3\}_c,$$

and then, by Proposition 2.4 (c),

$$(3.15) \quad \Gamma(H_1, H_2, H_3) \simeq S^1(\overline{H}_2) \widehat{\otimes} S^1(H_1, H_3).$$

On the other hand, it follows from (2.13) and Proposition 2.4 (a) that

$$S^1(H_1, H_3) \widehat{\otimes} S^\infty(H_3, H_1) \simeq \{\overline{H}_1\}_r \overset{h}{\otimes} S^\infty(H_3, H_1) \overset{h}{\otimes} \{H_3\}_c.$$

Then using (2.14), we deduce that

$$S^1(H_1, H_3) \widehat{\otimes} S^\infty(H_3, H_1) \simeq (\{\overline{H}_1\}_r \overset{h}{\otimes} \{H_1\}_c) \overset{h}{\otimes} (\{\overline{H}_3\}_r \overset{h}{\otimes} \{H_3\}_c).$$

Applying Proposition 2.4 (a) again together with (2.12), we obtain that

$$S^1(H_1, H_3) \widehat{\otimes} S^\infty(H_3, H_1) \simeq S^1(H_1) \overset{h}{\otimes} S^1(H_3)$$

completely isometrically.

Combining the last identification with (3.15), we find

$$(3.16) \quad \Gamma(H_1, H_2, H_3) \widehat{\otimes} S^\infty(H_3, H_1) \simeq S^1(\overline{H}_2) \widehat{\otimes} (S^1(H_1) \overset{h}{\otimes} S^1(H_3)).$$

We now pass to duals. First by (2.3) and (2.6), we have a w^* -continuous completely isometric identification

$$(\Gamma(H_1, H_2, H_3) \widehat{\otimes} S^\infty(H_3, H_1))^* \simeq CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3)).$$

Second by (2.3) and Lemma 2.7, we have w^* -continuous completely isometric identifications

$$\begin{aligned} (S^1(\overline{H}_2) \widehat{\otimes} (S^1(H_1) \overset{h}{\otimes} S^1(H_3)))^* &\simeq CB(S^1(H_1) \overset{h}{\otimes} S^1(H_3), B(\overline{H}_2)) \\ &\simeq CB(S^1(H_1) \overset{h}{\otimes} S^1(H_3), B(H_2)^{op}). \end{aligned}$$

Equivalently, by Lemma 2.2 (c), we have

$$(S^1(\overline{H}_2) \widehat{\otimes} (S^1(H_1) \overset{h}{\otimes} S^1(H_3)))^* \simeq B(H_2)^{op} \overline{\otimes} (B(H_1) \overset{w^*h}{\otimes} B(H_3)).$$

Thus (3.16) yields a w^* -continuous, completely isometric onto mapping

$$L: B(H_2)^{op} \overline{\otimes} (B(H_1) \overset{w^*h}{\otimes} B(H_3)) \longrightarrow CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3)).$$

Arguing as in the proof of Proposition 3.1, it now suffices to show that for any $R \in B(H_1)$, $S \in B(H_2)$ and $T \in B(H_3)$, $L(S \otimes R \otimes T)$ coincides with $\tilde{\tau}_{R \otimes S \otimes T}$. Next, it suffices to show that

$$(3.17) \quad L(S \otimes R \otimes T)(y \otimes x) = TySxR$$

when R, S, T are rank one and when $x \in S^2(H_1, H_2)$ and $y \in S^2(H_2, H_3)$ are rank one.

We let $\xi_i, \eta_i, h_i, k_i \in H_i$ for $i = 1, 2, 3$ and consider $R = h_1 \otimes \overline{k_1}$, $S = h_2 \otimes \overline{k_2}$, $T = h_3 \otimes \overline{k_3}$, $x = \overline{\xi_1} \otimes \eta_2$ and $y = \overline{\xi_2} \otimes \eta_3$. Then $y \otimes x \in \Gamma(H_1, H_2, H_3)$ corresponds to $(\eta_2 \otimes \overline{\xi_2}) \otimes (\overline{\xi_1} \otimes \eta_3) \in S^1(\overline{H}_2) \otimes S^1(H_1, H_3)$ in the identification (3.15). Hence $y \otimes x \otimes (\eta_1 \otimes \overline{\xi_3})$ regarded as an element of $\Gamma(H_1, H_2, H_3) \otimes S^\infty(H_3, H_1)$ corresponds to

$$(\eta_2 \otimes \overline{\xi_2}) \otimes (\overline{\xi_1} \otimes \eta_1) \otimes (\overline{\xi_3} \otimes \eta_3) \in S^1(\overline{H}_2) \otimes S^1(H_1) \otimes S^1(H_3)$$

in the identification (3.16). Since

$$\widehat{S} \otimes R \otimes T = \overline{k_2} \otimes h_2 \otimes h_1 \otimes \overline{k_1} \otimes h_3 \otimes \overline{k_3} \in B(\overline{H}_2) \otimes B(H_1) \otimes B(H_3),$$

we then have

$$\langle [L(S \otimes R \otimes T)(y \otimes x)](\eta_1), \xi_3 \rangle = \langle \eta_2, k_2 \rangle \langle h_2, \xi_2 \rangle \langle h_1, \xi_1 \rangle \langle \eta_1, k_1 \rangle \langle h_3, \xi_3 \rangle \langle \eta_3, k_3 \rangle.$$

By (3.12), the right hand side of this equality is equal to $\langle TySxR(\eta_1), \xi_3 \rangle$. This proves the identity (3.17), and hence the result. \square

4. MODULE MAPS

As in the previous section, we consider three Hilbert spaces H_1, H_2, H_3 . We further consider von Neumann subalgebras

$$M_1 \subset B(H_1), \quad M_2 \subset B(H_2) \quad \text{and} \quad M_3 \subset B(H_3)$$

acting on these spaces. For $i = 1, 2, 3$, we let $M'_i \subset B(H_i)$ be the commutant of M_i .

Let $u: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow B(H_1, H_3)$ be a bounded bilinear operator. We say that u is an (M'_3, M'_2, M'_1) -module map (or is (M'_3, M'_2, M'_1) -modular) provided that

$$u(Ty, x) = Tu(y, x), \quad u(y, xR) = u(y, x)R \quad \text{and} \quad u(yS, x) = u(y, Sx)$$

for any $x \in S^2(H_1, H_2)$, $y \in S^2(H_2, H_3)$, $R \in M'_1$, $S \in M'_2$ and $T \in M'_3$.

It will be convenient to associate to u the following 4-linear bounded operators. We define

$$(4.1) \quad U_1: \overline{H_2} \times H_2 \times \overline{H_3} \times H_3 \longrightarrow B(H_1)$$

by

$$(4.2) \quad \langle [U_1(\overline{\xi_2}, \eta_2, \overline{\xi_3}, \eta_3)](\eta_1), \xi_1 \rangle = \langle [u(\overline{\xi_2} \otimes \eta_3, \overline{\xi_1} \otimes \eta_2)](\eta_1), \xi_3 \rangle$$

for any $\xi_1, \eta_1 \in H_1$, $\xi_2, \eta_2 \in H_2$ and $\xi_3, \eta_3 \in H_3$. Likewise we define

$$U_2: \overline{H_1} \times H_1 \times \overline{H_3} \times H_3 \rightarrow B(H_2) \quad \text{and} \quad U_3: \overline{H_1} \times H_1 \times \overline{H_2} \times H_2 \rightarrow B(H_3)$$

by

$$\begin{aligned} \langle [U_2(\overline{\xi_1}, \eta_1, \overline{\xi_3}, \eta_3)](\eta_2), \xi_2 \rangle &= \langle [u(\overline{\xi_2} \otimes \eta_3, \overline{\xi_1} \otimes \eta_2)](\eta_1), \xi_3 \rangle \\ \langle [U_3(\overline{\xi_1}, \eta_1, \overline{\xi_2}, \eta_2)](\eta_3), \xi_3 \rangle &= \langle [u(\overline{\xi_2} \otimes \eta_3, \overline{\xi_1} \otimes \eta_2)](\eta_1), \xi_3 \rangle. \end{aligned}$$

Lemma 4.1. *Let $u \in B_2(S^2(H_2, H_3) \times S^2(H_1, H_2), B(H_1, H_3))$. Then u is an (M'_3, M'_2, M'_1) -module map if and only if for any $i = 1, 2, 3$, U_i is valued in M_i .*

Proof. Let $R \in B(H_1)$. For any $\eta_1, \xi_1 \in H_1$, $\eta_2, \xi_2 \in H_2$ and $\eta_3, \xi_3 \in H_3$, we have

$$\langle [u(\overline{\xi_2} \otimes \eta_3, \overline{\xi_1} \otimes \eta_2)]R(\eta_1), \xi_3 \rangle = \langle [U_1(\overline{\xi_2}, \eta_2, \overline{\xi_3}, \eta_3)]R(\eta_1), \xi_1 \rangle.$$

Further $(\overline{\xi_1} \otimes \eta_2)R = \overline{R^*(\xi_1)} \otimes \eta_2$, hence

$$\begin{aligned} \langle [u(\overline{\xi_2} \otimes \eta_3, (\overline{\xi_1} \otimes \eta_2)R)](\eta_1), \xi_3 \rangle &= \langle [U_1(\overline{\xi_2}, \eta_2, \overline{\xi_3}, \eta_3)](\eta_1), R^*(\xi_1) \rangle \\ &= \langle R[U_1(\overline{\xi_2}, \eta_2, \overline{\xi_3}, \eta_3)](\eta_1), \xi_1 \rangle. \end{aligned}$$

Since $\overline{H_1} \otimes H_2$ and $\overline{H_2} \otimes H_3$ are dense in $S^2(H_1, H_2)$ and $S^2(H_2, H_3)$, respectively, we deduce that $u(y, xR) = u(y, x)R$ for any $x \in S^2(H_1, H_2)$ and any $y \in S^2(H_2, H_3)$ if and only if R commutes with $U_1(\overline{\xi_2}, \eta_2, \overline{\xi_3}, \eta_3)$ for any $\xi_2, \eta_2 \in H_2$ and $\xi_3, \eta_3 \in H_3$.

Consequently u is $(\mathbb{C}, \mathbb{C}, M'_1)$ -modular if and only if the range of U_1 commutes with M'_1 . By the Bicommutant Theorem, this means that u is $(\mathbb{C}, \mathbb{C}, M'_1)$ -modular if and only if U_1 is valued in M_1 .

Likewise u is $(\mathbb{C}, M'_2, \mathbb{C})$ -modular (resp. $(M'_3, \mathbb{C}, \mathbb{C})$ -modular) if and only if U_2 is valued in M_2 (resp. U_3 is valued in M_3). This proves the result. \square

Corollary 4.2. *Let $\varphi \in B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$. Then τ_φ is (M'_3, M'_2, M'_1) -modular if and only if $\varphi \in M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$. This provides (as a restriction of (3.6)) a w^* -continuous completely isometric identification*

$$M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3 \simeq CB_{(M'_3, M'_2, M'_1)}(\Gamma(H_1, H_2, H_3), B(H_1, H_3)),$$

where the right-hand side denotes the subspace of $CB(\Gamma(H_1, H_2, H_3), B^1(H_1, H_3))$ of all completely bounded maps \tilde{u} such that u is an (M'_3, M'_2, M'_1) -module map.

Proof. Consider the duality relation

$$B(H_2)^{op} \overline{\otimes} B(H_3) = (S^1(H_2)^{op} \widehat{\otimes} S^1(H_3))^*$$

provided by (2.9). We claim that in the space $S^1(H_2)^{op} \widehat{\otimes} S^1(H_3)$, we have equality

$$(4.3) \quad (M_2^{op} \overline{\otimes} M_3)_\perp = \overline{(M_{2\perp}^{op} \otimes S^1(H_3) + S^1(H_2)^{op} \otimes M_{3\perp})}.$$

Indeed let $z \in B(H_2)^{op} \overline{\otimes} B(H_3)$ and let $z': S^1(H_3) \rightarrow B(H_2)^{op}$ and $z'': S^1(H_2)^{op} \rightarrow B(H_3)$ be associated with z (see Lemma 2.1). Then $z \in (M_{2\perp}^{op} \otimes S^1(H_3))^\perp$ if and only if z' is valued in M_2^{op} , whereas $z \in (S^1(H_2)^{op} \otimes M_{3\perp})^\perp$ if and only if z'' is valued in M_3 . Consequently, z belongs to the orthogonal of $M_{2\perp}^{op} \otimes S^1(H_3) + S^1(H_2)^{op} \otimes M_{3\perp}$ if and only if z' is valued in M_2^{op} and z'' is valued in M_3 . In turn this is equivalent to $z' \in CB(M_{3*}, M_2^{op})$. Applying Lemma 2.2 (a), we deduce that the orthogonal of $M_{2\perp}^{op} \otimes S^1(H_3) + S^1(H_2)^{op} \otimes M_{3\perp}$ is equal to $M_2^{op} \overline{\otimes} M_3$. The claim (4.3) follows at once.

Let $\varphi \in B(H_1) \overline{\otimes} B(H_2)^{op} \overline{\otimes} B(H_3)$. Using Lemma 2.1, we may associate 3 completely bounded operators

$$\begin{aligned} \varphi^1: S^1(H_2)^{op} \widehat{\otimes} S^1(H_3) &\longrightarrow B(H_1), \\ \varphi^2: S^1(H_1) \widehat{\otimes} S^1(H_3) &\longrightarrow B(H_2)^{op}, \\ \varphi^3: S^1(H_1) \widehat{\otimes} S^1(H_2)^{op} &\longrightarrow B(H_3) \end{aligned}$$

to φ . According to Lemma 2.2 (a), φ belongs to $M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$ if and only if φ^1 is valued in M_1 and φ^1 vanishes on $(M_2^{op} \overline{\otimes} M_3)_\perp$. By (4.3), φ^1 vanishes on $(M_2^{op} \overline{\otimes} M_3)_\perp$ if and only if it both vanishes on $M_{2\perp}^{op} \otimes S^1(H_3)$ and $S^1(H_2)^{op} \otimes M_{3\perp}$. A quick look at the definitions of $\varphi^1, \varphi^2, \varphi^3$ reveals that φ^1 vanishes on $M_{2\perp}^{op} \otimes S^1(H_3)$ if and only if φ^2 is valued in M_2^{op} and that φ^1 vanishes on $S^1(H_2)^{op} \otimes M_{3\perp}$ if and only if φ^3 is valued in M_3 . Altogether we obtain that φ belongs to $M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$ if and only if φ^1 is valued in M_1 , φ^2 is valued in M_2^{op} and φ^3 is valued in M_3 .

Let $u = \tau_\varphi$. It follows from Remark 3.2 that for any $\eta_2, \xi_2 \in H_2$ and $\eta_3, \xi_3 \in H_3$, we have

$$\varphi^1(\eta_2 \otimes \overline{\xi_2} \otimes \overline{\xi_3} \otimes \eta_3) = U_1(\overline{\xi_2}, \eta_2, \overline{\xi_3}, \eta_3),$$

where U_1 is defined by (4.1) and (4.2). Thus φ^1 is valued in M_1 if and only if U_1 is valued in M_1 . Likewise φ^2 is valued in M_2^{op} if and only if U_2 is valued in M_2 and φ^3 is valued in M_3 if and only if U_3 is valued in M_3 . By Lemma 4.1 we deduce that u is (M'_1, M'_2, M'_3) -modular if and only if $\varphi \in M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$. \square

We now turn to the study of modular completely bounded S^1 -multipliers. We let

$$CB_{(M'_3, M'_2, M'_1)}(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$$

denote the subspace of $CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$ of all completely bounded maps \tilde{u} such that u is an (M'_3, M'_2, M'_1) -module map.

According to (2.10) and (2.11), $M_1 \overset{w^*h}{\otimes} M_3$ can be regarded as a w^* -closed subspace of the dual operator space $B(H_1) \overset{w^*h}{\otimes} B(H_3)$. Consequently, $M_2^{op} \overline{\otimes} (M_1 \overset{w^*h}{\otimes} M_3)$ can be regarded as a w^* -closed subspace of the dual operator space $B(H_2)^{op} \overline{\otimes} (B(H_1) \overset{w^*h}{\otimes} B(H_3))$. The next statement is a continuation of Theorem 3.4.

Theorem 4.3. *Assume that M_2 is injective.*

- (a) *Let $\varphi \in B(H_2)^{op} \overline{\otimes} (B(H_1) \overset{w^*h}{\otimes} B(H_3))$. Then φ belongs to $M_2^{op} \overline{\otimes} (M_1 \overset{w^*h}{\otimes} M_3)$ if and only if τ_φ is (M'_3, M'_2, M'_1) -modular.*
- (b) *The identification (3.14) restricts to*

$$(4.4) \quad M_2^{op} \overline{\otimes} (M_1 \overset{w^*h}{\otimes} M_3) \simeq CB_{(M'_3, M'_2, M'_1)}(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3)).$$

Proof. Clearly (b) is a consequence of (a) so we only treat this first item.

Let $\varphi \in B(H_2)^{op} \overline{\otimes} (B(H_1) \overset{w^*h}{\otimes} B(H_3))$. Let

$$\sigma: S^1(H_1) \overset{h}{\otimes} S^1(H_3) \longrightarrow B(H_2)^{op}$$

be corresponding to φ in the identification provided by Lemma 2.2 (c). Then let

$$\rho: S^1(H_2)^{op} \longrightarrow B(H_1) \overset{w^*h}{\otimes} B(H_3)$$

be the restriction of the adjoint of σ to $S^1(H_2)^{op}$.

We assumed that M_2 is injective. It therefore follows from Lemma 2.2 (b) that $\varphi \in M_2^{op} \overline{\otimes} (M_1 \overset{w^*h}{\otimes} M_3)$ if and only if

$$(4.5) \quad \sigma(S^1(H_1) \overset{h}{\otimes} S^1(H_3)) \subset M_2^{op}$$

and

$$(4.6) \quad \rho(S^1(H_2)^{op}) \subset M_1 \overset{w^*h}{\otimes} M_3.$$

Let $u = \tau_\varphi$. We will now show that u is an (M'_3, M'_2, M'_1) -module map if and only if (4.5) and (4.6) hold true.

First we observe that for any $\xi_1, \eta_1 \in H_1$ and $\xi_3, \eta_3 \in H_3$,

$$\sigma((\overline{\xi_1} \otimes \eta_1) \otimes (\overline{\xi_3} \otimes \eta_3)) = U_2(\overline{\xi_1}, \eta_1, \overline{\xi_3}, \eta_3).$$

Indeed, this follows from Remark 3.2 and the definition of U_2 . Since $\overline{H_1} \otimes H_1$ and $\overline{H_3} \otimes H_3$ are dense in $S^1(H_1)$ and $S^1(H_3)$, respectively, we deduce that (4.5) holds true if and only if U_2 is valued in M_2 .

For any $v \in S^1(H_2)^{op}$, we may regard $\rho(v)$ as an element of $(S^1(H_1) \overset{h}{\otimes} S^1(H_3))^*$. Then following the notation in Lemma 2.3, we let

$$[\rho(v)]': S^1(H_3) \longrightarrow B(H_1) \quad \text{and} \quad [\rho(v)]'': S^1(H_1) \longrightarrow B(H_3)$$

be the bounded linear maps associated to $\rho(v)$.

For any $\xi_2, \eta_2 \in H_2$ and $\xi_3, \eta_3 \in H_3$, we have

$$[\rho(\eta_2 \otimes \overline{\xi_2})]'(\overline{\xi_3} \otimes \eta_3) = U_1(\overline{\xi_2}, \eta_2, \overline{\xi_3}, \eta_3).$$

Indeed this follows again from Remark 3.2. Since $H_2 \otimes \overline{H_2}$ and $\overline{H_3} \otimes H_3$ are dense in $S^1(H_2)^{op}$ and $S^1(H_3)$, respectively, we deduce that $[\rho(v)]'$ maps $S^1(H_3)$ into M_1 for any $v \in S^1(H_2)^{op}$ if and only if U_1 is valued in M_1 . Likewise, $[\rho(v)]''$ maps $S^1(H_1)$ into M_3 for any $v \in S^1(H_2)^{op}$ if and only if U_3 is valued in M_3 . Applying Lemma 2.3, we deduce (4.6) holds true if and only if U_1 is valued in M_1 and U_3 is valued in M_3 .

Altogether we have that (4.5) and (4.6) both hold true if and only if for any $i = 1, 2, 3$, U_i is valued in M_i . According to Lemma 4.1, this is equivalent to $u = \tau_\varphi$ being (M'_3, M'_2, M'_1) -modular. \square

5. THE SINCLAIR-SMITH FACTORIZATION THEOREM

Let I be an index set, and consider the Hilbertian operator spaces

$$C_I = \{\ell_I^2\}_c \quad \text{and} \quad R_I = \{\ell_I^2\}_r.$$

For any operator space G , we set

$$C_I^w(G^*) = C_I \overline{\otimes} G^* \quad \text{and} \quad R_I^w(G^*) = R_I \overline{\otimes} G^*.$$

This notation is taken from [3, 1.2.26–1.2.29], to which we refer for more information.

We recall that $C_I^w(G^*)$ can be equivalently defined as the space of all families $(x_i)_{i \in I}$ of elements of G^* such that the sums $\sum_{i \in J} x_i^* x_i$, for finite $J \subset I$, are uniformly bounded. Likewise, $R_I^w(G^*)$ is equal to the space of all families $(y_i)_{i \in I}$ of elements of G^* such that the sums $\sum_{i \in J} y_i y_i^*$, for finite $J \subset I$, are uniformly bounded.

Assume that $G^* = M$ is a von Neumann algebra, and consider $(x_i)_{i \in I} \in C_I^w(M)$ and $(y_i)_{i \in I} \in R_I^w(M)$. Then the family $(y_i x_i)_{i \in I}$ is summable in the w^* -topology of M and we let

$$(5.1) \quad \sum_{i \in I} y_i x_i \in M$$

denote its sum.

We note the obvious fact that for any $x_i \in M, i \in I$, $(x_i)_{i \in I}$ belongs to $R_I^w(M)$ if and only if $(x_i^*)_{i \in I}$ belongs to $C_I^w(M)$. In this case we set

$$[(x_i)_{i \in I}]^* = (x_i^*)_{i \in I}.$$

Lemma 5.1. *Let E, G be operator spaces and let I be an index set. For any $\alpha = (\alpha_i)_{i \in I} \in C_I^w(CB(E, G^*))$, the (well-defined) operator $\hat{\alpha}: E \rightarrow C_I^w(G^*)$, $\hat{\alpha}(x) = (\alpha_i(x))_{i \in I}$, is completely bounded and the mapping $\alpha \mapsto \hat{\alpha}$ induces a w^* -continuous completely isometric identification*

$$C_I^w(CB(E, G^*)) \simeq CB(E, C_I^w(G^*)).$$

Likewise we have

$$R_I^w(CB(E, G^*)) \simeq CB(E, R_I^w(G^*)).$$

Proof. According to Lemma 2.2 (c) and (2.3), $C_I^w(Z^*) \simeq (R_I \widehat{\otimes} Z)^*$ for any operator space Z . Applying this identification, first with $Z = E \widehat{\otimes} G$ and then with $Z = G$, we obtain that

$$\begin{aligned} C_I^w(CB(E, G^*)) &\simeq C_I^w((E \widehat{\otimes} G)^*) \quad \text{by (2.3)} \\ &\simeq (R_I \widehat{\otimes} E \widehat{\otimes} G)^* \\ &\simeq CB(E, (R_I \widehat{\otimes} G)^*) \quad \text{by (2.3)} \\ &\simeq CB(E, C_I^w(G^*)). \end{aligned}$$

A straightforward verification reveals that this identification is implemented by $\widehat{\alpha}$. This yields the first part of the lemma. The proof of the second part is identical. \square

We can now state the Sinclair-Smith factorization theorem, which will be use in the next section.

Theorem 5.2. ([19]) *Let E, F be operator spaces, let M be an injective von Neumann algebra and let $w: F \overset{h}{\otimes} E \rightarrow M$ be a completely bounded map. Then there exist an index set I and two families*

$$\alpha = (\alpha_i)_{i \in I} \in C_I^w(CB(E, M)) \quad \text{and} \quad \beta = (\beta_i)_{i \in I} \in R_I^w(CB(F, M))$$

such that $\|\alpha\|_{cb}\|\beta\|_{cb} = \|w\|_{cb}$ and

$$w(y \otimes x) = \sum_{i \in I} \beta_i(y) \alpha_i(x), \quad x \in E, y \in F.$$

In the rest of this section, we give a new (shorter) proof of Theorem 5.2 based on Hilbert C^* -modules.

In the following we give the necessary background on Hilbert C^* -modules. Let M be a C^* -algebra. Recall that a pre-Hilbert M -module is a right M -module \mathcal{X} equipped with a map $\langle \cdot, \cdot \rangle: \mathcal{X} \times \mathcal{X} \rightarrow M$ (called an M -valued inner product) satisfying the following properties:

- $\langle s, s \rangle \geq 0$ for every $s \in \mathcal{X}$;
- $\langle s, s \rangle = 0$ if and only if $s = 0$;
- $\langle s, t \rangle = \langle t, s \rangle^*$ for every $s, t \in \mathcal{X}$;
- $\langle s, t_1 m_1 + t_2 m_2 \rangle = \langle s, t_1 \rangle m_1 + \langle s, t_2 \rangle m_2$ for every $s, t_1, t_2 \in \mathcal{X}$ and $m_1, m_2 \in M$.

In this setting, the map $\|\cdot\|: \mathcal{X} \rightarrow \mathbb{R}^+$, defined by

$$\|s\| = \|\langle s, s \rangle\|^{1/2}, \quad s \in \mathcal{X},$$

is a norm on \mathcal{X} . A pre-Hilbert M -module which is complete with respect to its norm is said to be a Hilbert M -module.

By [2] (see also [3, 8.2.1]), a Hilbert M -module \mathcal{X} has a canonical operator space structure obtained by letting, for any $n \geq 1$,

$$\|(s_{ij})_{i,j}\| = \left\| \left(\sum_{k=1}^n \langle s_{ki}, s_{kj} \rangle \right)_{i,j} \right\|_{M_n(M)}^{1/2}, \quad (s_{ij})_{i,j} \in M_n(\mathcal{X}).$$

A morphism between two Hilbert M -modules \mathcal{X}_1 and \mathcal{X}_2 is a bounded M -module map $u: \mathcal{X}_1 \rightarrow \mathcal{X}_2$. A unitary isomorphism $u: \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is an isomorphism preserving the M -valued inner products. Any such map is a complete isometry (see e.g. [3, Proposition 8.2.2]).

Assume now that M is a von Neumann algebra. As a basic example, we recall that whenever $p \in M$ is a projection, then the subspace pM of M is a Hilbert M -module, when equipped with multiplication on the right as the M -module action, and with the M -valued inner product $\langle x, y \rangle = x^*y$, for $x, y \in pM$.

We recall the construction of the ultraweak direct sum Hilbert M -module. Let I be an index set and let $\{\mathcal{X}_i : i \in I\}$ be a collection of Hilbert M -modules indexed by I . We let $\langle \cdot, \cdot \rangle_i$ denote the M -valued inner product of \mathcal{X}_i , for any $i \in I$. Let \mathcal{X} be the set of all families $s = (s_i)_{i \in I}$, with $s_i \in \mathcal{X}_i$, such that the sums $\sum_{i \in J} \langle s_i, s_i \rangle_i$, for finite $J \subset I$, are uniformly bounded. Since $\langle s_i, s_i \rangle_i \geq 0$ for each $i \in I$, the family $(\langle s_i, s_i \rangle_i)_{i \in I}$ is then summable in the w^* -topology of M . Using polarization identity, it is easy to deduce that for any $s = (s_i)_{i \in I}$ and any $t = (t_i)_{i \in I}$ in \mathcal{X} , the family $(\langle s_i, t_i \rangle_i)_{i \in I}$ is summable in the w^* -topology of M . Then one defines

$$\langle s, t \rangle = \sum_{i \in I} \langle s_i, t_i \rangle_i.$$

It turns out that \mathcal{X} is a right M -module for the action $(s_i)_{i \in I} \cdot m = (s_i m)_{i \in I}$, and that equipped with $\langle \cdot, \cdot \rangle$, \mathcal{X} is a Hilbert M -module. The latter is called the ultraweak direct sum of $\{\mathcal{X}_i : i \in I\}$ and it is denoted by

$$\mathcal{X} = \oplus_{i \in I} \mathcal{X}_i.$$

See e.g. [3, 8.5.26] for more on this construction.

Let I be an index set, consider $C_I^w(M)$ as a right M -module in the obvious way. For any $(s_i)_{i \in I}$ and $(t_i)_{i \in I}$ in $C_I^w(M)$ set

$$\langle (s_i)_{i \in I}, (t_i)_{i \in I} \rangle = \sum_{i \in I} s_i^* t_i,$$

where this sum is defined by (5.1). This is an M -valued inner product, which makes $C_I^w(M)$ a Hilbert M -module. Moreover the canonical operator space structure of $C_I^w(M)$ as a Hilbert M -module coincides with the one given by writing $C_I^w(M) = C_I \widehat{\otimes} M$, see [3, 8.2.3]. Further we clearly have

$$C_I^w(M) \simeq \oplus_{i \in I} M \quad \text{as Hilbert } M\text{-modules.}$$

Proof of Theorem 5.2. Assume that $M \subset B(K)$ for some Hilbert space K . Let $w: F \overset{h}{\otimes} E \rightarrow M$ be a completely bounded map. By the Christensen-Sinclair factorization theorem (see e.g. [7, Theorem 9.4.4]), there exist a Hilbert space \mathcal{H} and two completely bounded maps

$$a: E \rightarrow B(K, \mathcal{H}) \quad \text{and} \quad b: F \rightarrow B(\mathcal{H}, K)$$

such that $\|a\|_{cb}\|b\|_{cb} = \|w\|_{cb}$ and $w(y \otimes x) = b(y)a(x)$ for any $x \in E$ and any $y \in F$.

Since M is injective, there exists a unital completely positive projection

$$\Psi: B(K) \longrightarrow M.$$

As Ψ is valued in M , we then have

$$(5.2) \quad w(y \otimes x) = \Psi(b(y)a(x)), \quad x \in E, y \in F.$$

We introduce

$$C = \{T \in B(K, \mathcal{H}) : \Psi(T^*T) = 0\}.$$

For any $k \in K$, $(T, S) \mapsto \langle \Psi(T^*S)k, k \rangle$ is a nonnegative sesquilinear form on $B(K, \mathcal{H})$, which vanishes on $\{(T, T) : T \in C\}$. This implies (by the Cauchy-Schwarz inequality) that $\langle \Psi(T^*S)k, k \rangle = 0$ for any $T \in C$ and any $S \in B(K, \mathcal{H})$. Consequently,

$$C = \{T \in B(K, \mathcal{H}) : \Psi(T^*S) = 0 \text{ for any } S \in B(K, \mathcal{H})\}.$$

In particular C is a subspace of $B(K, \mathcal{H})$. Moreover Ψ is an M -bimodule map by [22], hence

$$\Psi((Tm)^*(Tm)) = \Psi(m^*T^*Tm) = m^*\Psi(T^*T)m, \quad m \in M, T \in B(K, \mathcal{H}).$$

Consequently, C is invariant under right multiplication by elements of M .

Let $N = B(K, \mathcal{H})/C$ and let $q: B(K, \mathcal{H}) \rightarrow N$ be the quotient map. The M -invariance of C allows to define a right M -module action on N by

$$q(T) \cdot m = q(Tm), \quad m \in M, T \in B(K, \mathcal{H}).$$

For any $S, T \in B(K, \mathcal{H})$, set

$$\langle q(T), q(S) \rangle_N = \Psi(T^*S).$$

Then $\langle \cdot, \cdot \rangle_N$ is a well-defined, M -valued inner product on N , and hence N is a pre-Hilbert M -module. For convenience, we keep the notation N to denote its completion, which is a Hilbert M -module. The factorization property (5.2) can now be rephrased as

$$(5.3) \quad w(y \otimes x) = \langle q(b(y)^*), q(a(x)) \rangle_N, \quad x \in E, y \in F.$$

Recall from Paschke's fundamental paper [14] that the dual of N (in the Hilbert M -module sense) is the space

$$N' = \{\phi: N \rightarrow M : \phi \text{ is a bounded } M\text{-module map}\}.$$

Equip N' with the linear structure obtained with usual addition of maps and scalar multiplication given by $(\lambda \cdot \phi)(t) = \bar{\lambda}\phi(t)$ for any $\phi \in N'$, $\lambda \in \mathbb{C}$, and $t \in N$. Then N' is a right M -module for the action given by

$$(\phi \cdot m)(t) = m^*\phi(t), \quad \phi \in N', m \in M, t \in N.$$

Let $\kappa: N \rightarrow N'$ be defined by $\kappa(s): t \in N \mapsto \langle s, t \rangle \in M$. Then κ is a linear map. By [14, Theorem 3.2], there exists an M -valued inner product $\langle \cdot, \cdot \rangle_{N'}$ on N' such that

$$(5.4) \quad \langle \kappa(s), \kappa(t) \rangle_{N'} = \langle s, t \rangle_N, \quad s, t \in N,$$

and such that N' is selfdual (see [14, Section 3] for the definition). Then by [14, Theorem 3.12], N' is unitarily isomorphic to an ultraweak direct sum $\bigoplus_{i \in I} p_i M$, where $(p_i)_{i \in I}$ is a family of non-zero projections in M . Summarizing, we then have

$$(5.5) \quad N \xrightarrow{\kappa} N' \simeq \bigoplus_i p_i M \subset \bigoplus_i M \simeq C_I^w(M).$$

Note that by (5.4), κ is a complete isometry.

We claim that the quotient map $q: B(K, \mathcal{H}) \rightarrow N$ is completely contractive, when N is equipped with its Hilbert M -module operator space structure. Indeed, Ψ is completely contractive hence, for any $(S_{ij})_{i,j} \in M_n(B(K, \mathcal{H}))$, we have

$$\begin{aligned} \|(q(S_{ij}))_{i,j}\|_{M_n(N)}^2 &= \left\| \left(\sum_{k=1}^n \Psi(S_{ki}^* S_{kj}) \right)_{i,j} \right\|_{M_n(M)} \\ &\leq \left\| \left(\sum_{k=1}^n S_{ki}^* S_{kj} \right)_{i,j} \right\|_{M_n(B(K))} \\ &= \|(S_{ij})_{i,j}^* (S_{ij})_{i,j}\|_{M_n(B(K))} \\ &= \|(S_{ij})_{i,j}\|_{M_n(B(K, \mathcal{H}))}^2. \end{aligned}$$

Using (5.5), we define $\alpha: E \rightarrow C_I^w(M)$ by $\alpha(x) = \kappa(q(a(x)))$. It follows from above that α is completely bounded, with $\|\alpha\|_{cb} \leq \|a\|_{cb}$. Likewise we define $\beta: F \rightarrow R_I^w(M)$ by $\beta(y) = [\kappa(q(b(y)^*))]^*$. Then β is completely bounded, with $\|\beta\|_{cb} \leq \|b\|_{cb}$. Consequently, $\|\alpha\|_{cb} \|\beta\|_{cb} \leq \|w\|_{cb}$.

In accordance with Lemma 5.1, let $(\alpha_i)_{i \in I} \in C_I^w(CB(E, M))$ and $(\beta_i)_{i \in I} \in R_I^w(CB(F, M))$ be corresponding to α and β , respectively. Then by (5.3) and (5.5), we have

$$w(y \otimes x) = \langle \beta(y)^*, \alpha(x) \rangle_{N'} = \langle (\beta_i(y)^*)_{i \in I}, (\alpha_i(x))_{i \in I} \rangle_{C_I^w(M)} = \sum_{i \in I} \beta_i(y) \alpha_i(x)$$

for any $x \in E$ and $y \in F$.

Once this identity is established, the inequality $\|w\|_{cb} \leq \|\alpha\|_{cb} \|\beta\|_{cb}$ is a classical fact. \square

6. FACTORIZATION OF MODULAR OPERATORS

Consider H_1, H_2, H_3 and M_1, M_2, M_3 , $M_i \subset B(H_i)$, as in Sections 3 and 4.

Using the Hilbert space identification $S^2(H_1, H_2) \simeq \overline{H_1} \overset{2}{\otimes} H_2$, Lemma 2.7 and (2.8), we have von Neumann algebra identifications

$$B(H_1) \overline{\otimes} B(H_2)^{op} \simeq B(H_1) \overline{\otimes} B(\overline{H_2}) \simeq B(H_1 \overset{2}{\otimes} \overline{H_2}) \simeq B(S^2(H_1, H_2))^{op}$$

and hence a von Neumann algebra embedding

$$\tau^1: M_1 \overline{\otimes} M_2^{op} \hookrightarrow B(S^2(H_1, H_2))^{op}.$$

Unraveling the above identifications, we see that

$$(6.1) \quad [\tau^1(R \otimes S)](x) = SxR, \quad x \in S^2(H_1, H_2), R \in M_1, S \in M_2.$$

Further this property determines τ^1 . Likewise we may consider

$$\tau^3: M_2^{op} \overline{\otimes} M_3 \hookrightarrow B(S^2(H_2, H_3)),$$

the (necessarily unique) von Neumann algebra embedding satisfying

$$(6.2) \quad [\tau^3(S \otimes T)](y) = TyS, \quad y \in S^2(H_2, H_3), T \in M_3, S \in M_2.$$

For convenience, for any $a \in M_1 \overline{\otimes} M_2^{op}$ and any $b \in M_2^{op} \overline{\otimes} M_3$, we write τ_a^1 instead of $\tau^1(a)$ and τ_b^3 instead of $\tau^3(b)$.

The main objective of this section is to prove the following description of modular completely bounded S^1 -multipliers.

Theorem 6.1. *Assume that M_2 is injective.*

(a) *Let I be an index set and let*

$$A = (a_i)_{i \in I} \in R_I^w(M_1 \overline{\otimes} M_2^{op}) \quad \text{and} \quad B = (b_i)_{i \in I} \in C_I^w(M_2^{op} \overline{\otimes} M_3).$$

For any $x \in S^2(H_1, H_2)$ and any $y \in S^2(H_2, H_3)$,

$$\sum_{i \in I} \|\tau_{b_i}^3(y) \tau_{a_i}^1(x)\|_1 < \infty.$$

Let $u_{A,B}: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow S^1(H_1, H_3)$ be the resulting mapping defined by

$$u_{A,B}(y, x) = \sum_{i \in I} \tau_{b_i}^3(y) \tau_{a_i}^1(x), \quad x \in S^2(H_1, H_2), y \in S^2(H_2, H_3).$$

Then $\tilde{u}_{A,B} \in CB_{(M'_3, M'_2, M'_1)}(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$ and

$$(6.3) \quad \|\tilde{u}_{A,B}\|_{cb} \leq \|A\|_{R_I^w} \|B\|_{C_I^w}.$$

(b) *Conversely, let $u: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow S^1(H_1, H_3)$ be a bounded bilinear map and assume that \tilde{u} belongs to $CB_{(M'_3, M'_2, M'_1)}(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$. Then there exist an index set I and two families*

$$A = (a_i)_{i \in I} \in R_I^w(M_1 \overline{\otimes} M_2^{op}) \quad \text{and} \quad B = (b_i)_{i \in I} \in C_I^w(M_2^{op} \overline{\otimes} M_3)$$

such that $u = u_{A,B}$ and $\|A\|_{R_I^w} \|B\|_{C_I^w} = \|u\|_{cb}$.

We will establish two intermediate lemmas before proceeding to the proof. We recall the mapping τ from (3.1). In the sequel we use the notation 1 for the unit of either $B(H_1)$ or $B(H_3)$. Thus for any $a \in M_1 \overline{\otimes} M_2^{op}$, we may consider $a \otimes 1 \in M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$. Likewise, for any $b \in M_2^{op} \overline{\otimes} M_3$, we may consider $1 \otimes b \in M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$. The following is a generalization of [6, Lemma 20].

Lemma 6.2. *For any $a \in M_1 \overline{\otimes} M_2^{op}$, for any $b \in M_2^{op} \overline{\otimes} M_3$, and for any $x \in S^2(H_1, H_2)$ and $y \in S^2(H_2, H_3)$, we have*

$$(6.4) \quad \tau_{(a \otimes 1)(1 \otimes b)}(y, x) = \tau_b^3(y) \tau_a^1(x).$$

Proof. We fix $x \in S^2(H_1, H_2)$, $y \in S^2(H_2, H_3)$, $\eta_1 \in H_1$ and $\xi_3 \in H_3$.

Let $R \in M_1, S, S' \in M_2^{op}, T \in M_3$. Then $(R \otimes S \otimes 1)(1 \otimes S' \otimes T) = R \otimes S'S \otimes T$. Hence by (3.3), (6.1) and (6.2), we have

$$\tau_{(R \otimes S \otimes 1)(1 \otimes S' \otimes T)}(y, x) = TyS'SxR = \tau_{S' \otimes T}^3(y)\tau_{R \otimes S}^1(x).$$

Hence the result holds true when a and b are elementary tensors. By linearity, this implies (6.4) in the case when a and b belong to the algebraic tensor products $M_1 \otimes M_2^{op}$ and $M_2^{op} \otimes M_3$, respectively.

We now use a limit process. Let $a \in M_1 \overline{\otimes} M_2^{op}$ and $b \in M_2^{op} \overline{\otimes} M_3$ be arbitrary. Let $(a_s)_s$ be a net in $M_1 \otimes M_2^{op}$ converging to a in the w^* -topology of $M_1 \overline{\otimes} M_2^{op}$ and let $(b_t)_t$ be a net in $M_2^{op} \otimes M_3$ converging to b in the w^* -topology of $M_2^{op} \overline{\otimes} M_3$. For any s, t , we have

$$(6.5) \quad \tau_{(a_s \otimes 1)(1 \otimes b_t)}(y, x) = \tau_{b_t}^3(y)\tau_{a_s}^1(x)$$

by the preceding paragraph.

On the one hand, since the product is separately w^* -continuous on von Neumann algebras,

$$(6.6) \quad (a \otimes 1)(1 \otimes b) = w^* \text{-} \lim_s \lim_t (a_s \otimes 1)(1 \otimes b_t)$$

in $M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$. Since τ is w^* -continuous, this implies that

$$\langle [\tau_{(a \otimes 1)(1 \otimes b)}(y, x)](\eta_1), \xi_3 \rangle = \lim_s \lim_t \langle [\tau_{(a_s \otimes 1)(1 \otimes b_t)}(y, x)](\eta_1), \xi_3 \rangle.$$

On the other hand, by the w^* -continuity of τ^1 and τ^3 , $\tau_{a_s}^1 \rightarrow \tau_a^1$ in the w^* -topology of $B(S^2(H_1, H_2))$ and $\tau_{b_t}^3 \rightarrow \tau_b^3$ in the w^* -topology of $B(S^2(H_2, H_3))$. Consequently, $\tau_{a_s}^1(x) \rightarrow \tau_a^1(x)$ in the weak topology of $S^2(H_1, H_2)$ whereas $\tau_{b_t}^3(y) \rightarrow \tau_b^3(y)$ in the weak topology of $S^2(H_2, H_3)$. This readily implies that

$$\langle [\tau_b^3(y)\tau_a^1(x)](\eta_1), \xi_3 \rangle = \lim_s \lim_t \langle [\tau_{b_t}^3(y)\tau_{a_s}^1(x)](\eta_1), \xi_3 \rangle.$$

Combining these two limit results with (6.5), we deduce the formula (6.4). \square

It follows from Lemma 2.2 (a) that we have w^* -continuous and completely isometric identifications

$$(6.7) \quad M_1 \overline{\otimes} M_2^{op} \simeq CB(M_{1*}, M_2^{op}) \quad \text{and} \quad M_2^{op} \overline{\otimes} M_3 \simeq CB(M_{3*}, M_2^{op}).$$

Likewise, $M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3 \simeq CB((M_1 \overline{\otimes} M_3)_*, M_2^{op})$ hence by [7, Theorem 7.2.4], we have a w^* -continuous and completely isometric identification

$$(6.8) \quad M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3 \simeq CB(M_{1*} \widehat{\otimes} M_{3*}, M_2^{op}).$$

Lemma 6.3. *Assume that M_2 is injective. Let $a \in M_1 \overline{\otimes} M_2^{op}$ and $b \in M_2^{op} \overline{\otimes} M_3$. Let $\alpha \in CB(M_{1*}, M_2^{op})$ and $\beta \in CB(M_{3*}, M_2^{op})$ be corresponding to a and b , respectively, through the identifications (6.7). Let*

$$\sigma_{a,b}: M_{1*} \widehat{\otimes} M_{3*} \longrightarrow M_2^{op}$$

be the completely bounded map corresponding to $(a \otimes 1)(1 \otimes b)$ through the identification (6.8). Then we have

$$(6.9) \quad \sigma_{a,b}(v_1 \otimes v_3) = \alpha(v_1)\beta(v_3)$$

for any $v_1 \in M_{1}$ and any $v_3 \in M_{3*}$.*

Proof. We fix $v_1 \in M_{1*}$ and $v_3 \in M_{3*}$.

Let $R \in M_1$, $S, S' \in M_2^{op}$ and $T \in M_3$, and assume first that $a = R \otimes S$ and $b = S' \otimes T$. Then $\alpha(v_1) = \langle R, v_1 \rangle_{M_1, M_{1*}} S$ and $\beta(v_3) = \langle T, v_3 \rangle_{M_3, M_{3*}} S'$. Hence

$$\alpha(v_1)\beta(v_3) = \langle R, v_1 \rangle_{M_1, M_{1*}} \langle T, v_3 \rangle_{M_3, M_{3*}} S' S.$$

Since $(a \otimes 1)(1 \otimes b) = R \otimes S' S \otimes T$, $\sigma_{a,b}(v_1 \otimes v_3)$ is also equal to $\langle R, v_1 \rangle_{M_1, M_{1*}} \langle T, v_3 \rangle_{M_3, M_{3*}} S' S$. This proves the result in this special case. By linearity, we deduce that (6.9) holds true when a and b belong to the algebraic tensor products $M_1 \otimes M_2^{op}$ and $M_2^{op} \otimes M_3$.

As in the proof of the preceding lemma, we deduce the general case by a limit process. Let $a \in M_1 \overline{\otimes} M_2^{op}$ and $b \in M_2^{op} \overline{\otimes} M_3$ be arbitrary. Let $(a_s)_s$ be a net in $M_1 \otimes M_2^{op}$ converging to a in the w^* -topology of $M_1 \overline{\otimes} M_2^{op}$ and let $(b_t)_t$ be a net in $M_2^{op} \otimes M_3$ converging to b in the w^* -topology of $M_2^{op} \overline{\otimes} M_3$. Then for any s, t , let $\alpha_s \in CB(M_{1*}, M_2^{op})$ and $\beta_t \in CB(M_{3*}, M_2^{op})$ be corresponding to a_s and b_t , respectively. By the preceding paragraph,

$$\sigma_{a_s, b_t}(v_1 \otimes v_3) = \alpha_s(v_1)\beta_t(v_3)$$

for any s, t .

Since the identifications (6.7) are w^* -continuous, $\alpha_s(v_1) \rightarrow \alpha(v_1)$ and $\beta_t(v_3) \rightarrow \beta(v_3)$ in the w^* -topology of M_2^{op} . Since the product is separately w^* -continuous on von Neumann algebras, this implies that

$$\alpha(v_1)\beta(v_3) = w^* \text{-} \lim_s \lim_t \alpha_s(v_1)\beta_t(v_3).$$

Next since the identification (6.8) is w^* -continuous, it follows from (6.6) that

$$\sigma_{a,b}(v_1 \otimes v_3) = w^* \text{-} \lim_s \lim_t \sigma_{a_s, b_t}(v_1 \otimes v_3).$$

The identity (6.9) follows at once. □

Note that if M_2 is injective, then by Lemma 2.2 (b) the identification (6.8) restricts to an identification between $M_2^{op} \overline{\otimes} (M_1 \overset{w^*h}{\otimes} M_3)$ and $CB(M_{1*} \overset{h}{\otimes} M_{3*}, M_2^{op})$. Combining with (4.4), we deduce a w^* -continuous and completely isometric identification

$$(6.10) \quad CB_{(M'_3, M'_2, M'_1)}(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3)) \simeq CB(M_{1*} \overset{h}{\otimes} M_{3*}, M_2^{op}).$$

This will be used in the proof below.

Proof of Theorem 6.1.

(a): Consider $x \in S^2(H_1, H_2)$ and $y \in S^2(H_2, H_3)$. We have

$$\begin{aligned} \sum_{i \in I} \|\tau_{b_i}^3(y)\tau_{a_i}^1(x)\|_1 &\leq \sum_{i \in I} \|\tau_{b_i}^3(y)\|_2 \|\tau_{a_i}^1(x)\|_2 \\ &\leq \left(\sum_{i \in I} \|\tau_{b_i}^3(y)\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I} \|\tau_{a_i}^1(x)\|_2^2 \right)^{\frac{1}{2}}, \end{aligned}$$

by the Cauchy-Schwarz inequality.

Let $J \subset I$ be a finite subset. Since τ^3 is a $*$ -homomorphism, we have

$$\begin{aligned} \sum_{i \in J} \|\tau_{b_i}^3(y)\|_2^2 &= \sum_{i \in J} \langle \tau_{b_i}^{3*} \tau_{b_i}^3(y), y \rangle_{S^2} \\ &= \left\langle \tau^3 \left(\sum_{i \in J} b_i^* b_i \right) (y), y \right\rangle_{S^2} \\ &\leq \left\| \sum_{i \in J} b_i^* b_i \right\| \|y\|_2^2 \\ &\leq \|B\|_{C_I^w}^2 \|y\|_2^2. \end{aligned}$$

Since J is arbitrary, this implies that

$$(6.11) \quad \sum_{i \in I} \|\tau_{b_i}^3(y)\|_2^2 \leq \|B\|_{C_I^w}^2 \|y\|_2^2.$$

Likewise,

$$(6.12) \quad \sum_{i \in I} \|\tau_{a_i}^1(x)\|_2^2 \leq \|A\|_{R_I^w}^2 \|x\|_2^2.$$

This implies

$$\sum_{i \in I} \|\tau_{b_i}^3(y) \tau_{a_i}^1(x)\|_1 \leq \|A\|_{R_I^w} \|B\|_{C_I^w} \|x\|_2 \|y\|_2,$$

which allows the definition of $u_{A,B}$.

Let $n \geq 1$ be an integer, let $x_1, \dots, x_n \in S^2(H_1, H_2)$ and let $y_1, \dots, y_n \in S^2(H_2, H_3)$. In the space $S^1(\ell_n^2(H_1), \ell_n^2(H_3))$, we have the equality

$$[u_{A,B}(y_k, x_l)]_{1 \leq k, l \leq n} = \sum_{i \in I} [\tau_{b_i}^3(y_k) \tau_{a_i}^1(x_l)]_{1 \leq k, l \leq n}.$$

Further for any $i \in I$, we have

$$\|[\tau_{b_i}^3(y_k) \tau_{a_i}^1(x_l)]_{1 \leq k, l \leq n}\|_{S^1(\ell_n^2(H_1), \ell_n^2(H_3))} \leq \left(\sum_{k=1}^n \|\tau_{b_i}^3(y_k)\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{l=1}^n \|\tau_{a_i}^1(x_l)\|_2^2 \right)^{\frac{1}{2}}.$$

Consequently, using Cauchy-Schwarz,

$$\begin{aligned} \| [u_{A,B}(y_k, x_l)]_{1 \leq k, l \leq n} \|_{S^1(\ell_n^2(H_1), \ell_n^2(H_3))} &\leq \sum_{i \in I} \left(\sum_{k=1}^n \|\tau_{b_i}^3(y_k)\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{l=1}^n \|\tau_{a_i}^1(x_l)\|_2^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i \in I} \sum_{k=1}^n \|\tau_{b_i}^3(y_k)\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I} \sum_{l=1}^n \|\tau_{a_i}^1(x_l)\|_2^2 \right)^{\frac{1}{2}}. \end{aligned}$$

It therefore follows from (6.11) and (6.12) that

$$\| [u_{A,B}(y_k, x_l)]_{1 \leq k, l \leq n} \|_{S^1(\ell_n^2(H_1), \ell_n^2(H_3))} \leq \|A\|_{R_I^w} \|B\|_{C_I^w} \left(\sum_{k=1}^n \|y_k\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{l=1}^n \|x_l\|_2^2 \right)^{\frac{1}{2}}.$$

According to Lemma 3.3, this shows that $\tilde{u}_{A,B}$ is completely bounded and that (6.3) holds.

Again let $x \in S^2(H_1, H_2)$ and $y \in S^2(H_2, H_3)$. Using a simple approximation process, one can check that for any $R \in M'_1$, $S \in M'_2$ and $T \in M'_3$, we have

$$\tau_a^1(xR) = \tau_a^1(x)R, \quad \tau_a^1(Sx) = S\tau_a^1(x), \quad \tau_b^3(yS) = \tau_b^3(y)S \quad \text{and} \quad \tau_b^3(Ty) = T\tau_b^3(y)$$

whenever $a \in M_1 \overline{\otimes} M_2^{op}$ and $b \in M_2^{op} \overline{\otimes} M_3$. This implies that $(y, x) \mapsto \tau_b^3(y)\tau_a^1(x)$ is an (M'_3, M'_2, M'_1) -module map for any $a \in M_1 \overline{\otimes} M_2^{op}$ and $b \in M_2^{op} \overline{\otimes} M_3$. This readily implies that $u_{A,B}$ is an (M'_3, M'_2, M'_1) -module map.

(b): Assume that $\tilde{u} \in CB_{(M'_3, M'_2, M'_1)}(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$. Let

$$\sigma: M_{1*} \overset{h}{\otimes} M_{3*} \longrightarrow M_2^{op}$$

be the completely bounded map corresponding to \tilde{u} through the identification (6.10). Since M_2 is injective, we may apply Theorem 5.2 to σ . We obtain the existence of an index set I and two families $(\alpha_i)_{i \in I} \in R_I^w(CB(M_{1*}, M_2^{op}))$ and $(\beta_i)_{i \in I} \in C_I^w(CB(M_{3*}, M_2^{op}))$ such that

$$\sigma(v_1 \otimes v_3) = \sum_{i \in I} \alpha_i(v_1)\beta_i(v_3), \quad v_1 \in M_{1*}, \quad v_3 \in M_{3*}.$$

For any $i \in I$, we let $a_i \in M_1 \overline{\otimes} M_2^{op}$ and $b_i \in M_2^{op} \overline{\otimes} M_3$ be corresponding to α_i and β_i , respectively, through the identifications (6.7). Then we set $A = (a_i)_{i \in I}$ and $B = (b_i)_{i \in I}$. By Theorem 5.2, we may assume that $\|A\|_{R_I^w} \|B\|_{C_I^w} = \|u\|_{cb}$.

For any finite subset $J \subset I$, we may define

$$u_J: S^2(H_2, H_3) \times S^2(H_1, H_2) \rightarrow S_1(H_1, H_3) \quad \text{and} \quad \sigma_J: M_{1*} \overset{h}{\otimes} M_{3*} \rightarrow M_2^{op}$$

by

$$u_J(y, x) = \sum_{i \in J} \tau_{b_i}^3(y)\tau_{a_i}^1(x), \quad x \in S^2(H_1, H_2), \quad y \in S^2(H_2, H_3),$$

and

$$\sigma_J(v_1 \otimes v_3) = \sum_{i \in J} \alpha_i(v_1)\beta_i(v_3), \quad v_1 \in M_{1*}, \quad v_3 \in M_{3*}.$$

It follows from Lemmas 6.2 and 6.3 that for any i , the mapping $(v_1 \otimes v_3) \rightarrow \alpha_i(v_1)\beta_i(v_3)$ corresponds to the mapping $y \otimes x \mapsto \tau_{b_i}^3(y)\tau_{a_i}^1(x)$ through the identification (6.10). By linearity we deduce that σ_J corresponds to \tilde{u}_J through (6.10).

We observe that by the easy (and well-known) converse to Theorem 5.2, we have

$$\|\sigma_J\|_{cb} \leq \|(\alpha_i)_{i \in J}\|_{R_J^w(CB(M_{1*}, M_2^{op}))} \|(\beta_i)_{i \in J}\|_{C_J^w(CB(M_{3*}, M_2^{op}))}.$$

This implies the following uniform boundedness,

$$(6.13) \quad \forall J \subset I \text{ finite}, \quad \|\sigma_J\|_{cb} \leq \|A\|_{R_I^w} \|B\|_{C_I^w}.$$

In the sequel we consider the set of finite subsets of I as directed by inclusion. We observe that for any $v_1 \in M_{1*}$ and $v_3 \in M_{3*}$, $\sigma_J(v_1 \otimes v_3) \rightarrow \sigma(v_1 \otimes v_3)$ in the w^* -topology of M_2^{op} . Using the uniform boundedness (6.13), this implies that $\sigma_J \rightarrow \sigma$ in the point- w^* -topology of $CB(M_{1*} \overset{h}{\otimes} M_{3*}, M_2^{op})$. Applying (6.13) again, we deduce that $\sigma_J \rightarrow \sigma$ in the w^* -topology

of $CB(M_{1*} \overset{h}{\otimes} M_{3*}, M_2^{op})$. Since the identification (6.10) is a w^* -continuous one, this implies that $\tilde{u}_J \rightarrow \tilde{u}$ is the w^* -topology of $CB(\Gamma(H_1, H_2, H_3), S^1(H_1, H_3))$.

Let $x \in S^2(H_1, H_2)$ and $y \in S^2(H_2, H_3)$. The above implies that $u_J(y, x) \rightarrow u(y, x)$ in the w^* -topology of $S^1(H_1, H_3)$. However by part (a) of the theorem,

$$u_J(y, x) \longrightarrow \sum_{i \in I} \tau_{b_i}^3(y) \tau_{a_i}^1(x)$$

in the norm topology of $S^1(H_1, H_3)$. This shows that $u(y, x)$ is equal to this sum, and proves the result. \square

Remark 6.4. It is clear from its proof that part (a) of Theorem 6.1 is true without assuming that M_2 is injective.

The injectivity assumption in Theorem 5.2 is necessary, see [19, Theorem 5.3], however we do not know if it is necessary in part (b) of Theorem 6.1.

The next corollary follows from the above proof.

Corollary 6.5. *Assume that M_2 is injective and let $\varphi \in M_1 \overline{\otimes} M_2^{op} \overline{\otimes} M_3$. Then τ_φ is a completely bounded S^1 -multiplier if and only if there exist an index set I and families*

$$(a_i)_{i \in I} \in R_I^w(M_1 \overline{\otimes} M_2^{op}) \quad \text{and} \quad (b_i)_{i \in I} \in C_I^w(M_2^{op} \overline{\otimes} M_3)$$

such that

$$\varphi = \sum_{i \in I} (a_i \otimes 1)(1 \otimes b_i),$$

where the convergence is taken in the w^* -topology. Further

$$\|\tau_\varphi\|_{cb} = \inf \left\{ \left\| (a_i)_{i \in I} \right\|_{R_I^w} \left\| (b_i)_{i \in I} \right\|_{C_I^w} \right\},$$

where the infimum runs over all possible families $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ providing such a factorization of φ .

Remark 6.6.

(a) Assume that $H_2 = \mathbb{C}$ is trivial. Then

$$\Gamma(H_1, \mathbb{C}, H_3) = \{H_3\}_c \widehat{\otimes} \{\overline{H_1}\}_r \simeq S^1(H_1, H_3),$$

by (2.12). Hence $CB(\Gamma(H_1, \mathbb{C}, H_3), S^1(H_1, H_3)) \simeq CB(S^1(H_1, H_3))$ and in this identification, $CB_{(M'_3, \mathbb{C}, M'_1)}(\Gamma(H_1, \mathbb{C}, H_3), S^1(H_1, H_3))$ coincides with $CB_{(M'_3, M'_1)}(S^1(H_1, H_3))$, the space of all (M'_3, M'_1) -bimodule completely bounded maps from $S^1(H_1, H_3)$ into itself. Further $\tau^1: M_1 \hookrightarrow B(\overline{H_1})^{op} \simeq B(H_1)$ and $\tau^3: M_3 \hookrightarrow B(H_3)$ coincide with the canonical embeddings. Hence in this case, Theorem 6.1 reduces to Theorem 1.1 (see also (1.1)).

(b) A tensor product reformulation of Corollary 6.5 is that the bilinear mapping $(a, b) \mapsto (a \otimes 1)(1 \otimes b)$ extends to a complete quotient map

$$(M_1 \overline{\otimes} M_2^{op}) \overset{w^*h}{\otimes} (M_2^{op} \overline{\otimes} M_3) \longrightarrow M_2^{op} \overline{\otimes} (M_1 \overset{w^*h}{\otimes} M_3).$$

We conclude this paper by considering the special case of Schur multipliers. Our presentation follows [6]. We let (Ω_1, μ_1) , (Ω_2, μ_2) and (Ω_3, μ_3) be three separable measure spaces. (The separability assumption is not essential but avoids technical measurability issues.) Recall the classical fact that to any $f \in L^2(\Omega_1 \times \Omega_2)$, one may associate an operator $x_f \in S^2(L^2(\Omega_1), L^2(\Omega_2))$ given by

$$x_f(\eta) = \int_{\Omega_1} f(t_1, \cdot) \eta(t_1) d\mu_1(t_1), \quad \eta \in L^2(\Omega_1),$$

and the mapping $f \mapsto x_f$ is a unitary which yields a Hilbert space identification

$$L^2(\Omega_1 \times \Omega_2) \simeq S^2(L^2(\Omega_1), L^2(\Omega_2)).$$

Of course the same holds with the pairs (Ω_2, Ω_3) and (Ω_1, Ω_3) . For any $g \in L^2(\Omega_2 \times \Omega_3)$ (resp. $h \in L^2(\Omega_1 \times \Omega_3)$) we let $y_g \in S^2(L^2(\Omega_2), L^2(\Omega_3))$ (resp. $z_h \in S^2(L^2(\Omega_1), L^2(\Omega_3))$) be the corresponding Hilbert-Schmidt operator.

To any $\varphi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$, one may associate a bounded bilinear map

$$\Lambda_\varphi: S^2(L^2(\Omega_2), L^2(\Omega_3)) \times S^2(L^2(\Omega_1), L^2(\Omega_2)) \longrightarrow S^2(L^2(\Omega_1), L^2(\Omega_3))$$

given for any $f \in L^2(\Omega_1 \times \Omega_2)$ and $g \in L^2(\Omega_2 \times \Omega_3)$ by

$$\Lambda_\varphi(y_g, x_f) = z_h$$

where, for almost every $(t_1, t_3) \in \Omega_1 \times \Omega_3$,

$$h(t_1, t_3) = \int_{\Omega_2} \varphi(t_1, t_2, t_3) f(t_1, t_2) g(t_2, t_3) d\mu_2(t_2).$$

We refer to [12, Theorem 3.1] or [6, Subsection 3.2] for the proof, and also for the fact that

$$\|\Lambda_\varphi: S^2 \times S^2 \longrightarrow S^2\| = \|\varphi\|_\infty.$$

Bilinear maps of this form will be called **bilinear Schur multipliers** in the sequel. Since

$$S^2(L^2(\Omega_1), L^2(\Omega_3)) \subset B(L^2(\Omega_1), L^2(\Omega_3))$$

contractively, we may regard any bilinear Schur multiplier as valued in $B(L^2(\Omega_1), L^2(\Omega_3))$. Then it follows from the proof of [6, Corollary 10] that

$$(6.14) \quad \|\Lambda_\varphi: S^2 \times S^2 \longrightarrow B(L^2(\Omega_1), L^2(\Omega_3))\| = \|\varphi\|_\infty.$$

For any $i = 1, 2, 3$, let us regard

$$(6.15) \quad L^\infty(\Omega_i) \subset B(L^2(\Omega_i))$$

as a von Neumann algebra in the usual way, that is, any $r \in L^\infty(\Omega_i)$ is identified with the multiplication operator $f \mapsto rf$, $f \in L^2(\Omega_i)$. In the sequel we use the notions considered so far in the case when $H_i = L^2(\Omega_i)$ and $M_i = L^\infty(\Omega_i)$. We note that

$$L^\infty(\Omega_i)' = L^\infty(\Omega_i) \quad \text{and} \quad L^\infty(\Omega_i)^{op} = L^\infty(\Omega_i).$$

Using the classical von Neumann algebra identification

$$L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3) = L^\infty(\Omega_1) \overline{\otimes} L^\infty(\Omega_2) \overline{\otimes} L^\infty(\Omega_3),$$

we may apply the construction from Sections 3 and 4 to any $\varphi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$ and consider the operator multiplier

$$\tau_\varphi: S^2(L^2(\Omega_2), L^2(\Omega_3)) \times S^2(L^2(\Omega_1), L^2(\Omega_2)) \longrightarrow B(L^2(\Omega_1), L^2(\Omega_3)).$$

It turns out that

$$(6.16) \quad \tau_\varphi = \Lambda_\varphi.$$

The easy verification is left to the reader.

The next proposition should be compared with [12, Theorem 3.1]. In the latter result, the authors established a similar characterization of bilinear module maps, but under the assumption that they take values in $S^2(L^2(\Omega_1), L^2(\Omega_3))$.

Proposition 6.7. *For any*

$$u \in B_2(S^2(L^2(\Omega_2), L^2(\Omega_3)) \times S^2(L^2(\Omega_1), L^2(\Omega_2)), B(L^2(\Omega_1), L^2(\Omega_3))),$$

the following are equivalent.

- (i) *u is a bilinear Schur multiplier.*
- (ii) *u is an $(L^\infty(\Omega_3), L^\infty(\Omega_2), L^\infty(\Omega_1))$ -module map.*

Proof. The implication “(i) \Rightarrow (ii)” follows from (6.16) and Corollary 4.2. (It is also possible to write a direct proof.)

To prove the converse, assume that u is $(L^\infty(\Omega_3), L^\infty(\Omega_2), L^\infty(\Omega_1))$ -modular. We let

$$U: S^1(L^2(\Omega_1)) \times S^1(L^2(\Omega_2)) \times S^1(L^2(\Omega_3)) \longrightarrow \mathbb{C}$$

be the unique trilinear form satisfying

$$U(\overline{\xi_1} \otimes \eta_1, \overline{\xi_2} \otimes \eta_2, \overline{\xi_3} \otimes \eta_3) = \langle [u(\overline{\xi_2} \otimes \eta_3, \overline{\xi_1} \otimes \eta_2)](\eta_1), \xi_3 \rangle$$

for any $\xi_1, \eta_1 \in L^2(\Omega_1)$, $\xi_2, \eta_2 \in L^2(\Omega_2)$ and $\xi_3, \eta_3 \in L^2(\Omega_3)$.

Then for $i = 1, 2, 3$, let

$$q_i: S^1(L^2(\Omega_i)) \longrightarrow L^1(\Omega_i)$$

be the unique bounded operator satisfying $q_i(\overline{\xi_i} \otimes \eta_i) = \overline{\xi_i} \eta_i$ for any $\xi_i, \eta_i \in L^2(\Omega_i)$. This is a quotient map, whose adjoint coincides with the embedding (6.15).

Recall the operators U_1, U_2, U_3 defined at the beginning of Section 4. By Lemma 4.1, U_i is valued in $L^\infty(\Omega_i)$ for any $i = 1, 2, 3$. This implies that U vanishes on the union of $\text{Ker}(q_1) \times S^1(L^2(\Omega_2)) \times S^1(L^2(\Omega_3))$, $S^1(L^2(\Omega_1)) \times \text{Ker}(q_2) \times S^1(L^2(\Omega_3))$ and $S^1(L^2(\Omega_1)) \times S^1(L^2(\Omega_2)) \times \text{Ker}(q_3)$. Consequently, there exists a trilinear form

$$\widehat{u}: L^1(\Omega_1) \times L^1(\Omega_2) \times L^1(\Omega_3) \longrightarrow \mathbb{C}$$

factorizing U in the sense that

$$U(v_1, v_2, v_3) = \widehat{u}(q_1(v_1), q_2(v_2), q_3(v_3)), \quad v_i \in S^1(L^2(\Omega_i)).$$

Since $L^1(\Omega_1) \widehat{\otimes} L^1(\Omega_2) \widehat{\otimes} L^1(\Omega_3) = L^1(\Omega_1 \times \Omega_2 \times \Omega_3)$ (see e.g. [8, Chap. VIII, Example 10]), there exists $\varphi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$ such that

$$\widehat{u}(\phi_1, \phi_2, \phi_3) = \int_{\Omega_1 \times \Omega_2 \times \Omega_3} \varphi(t_1, t_2, t_3) \phi_1(t_1) \phi_2(t_2) \phi_3(t_3) d\mu_1(t_1) d\mu_2(t_2) d\mu_3(t_3)$$

for any $\phi_i \in L^1(\Omega_i)$. A thorough look at the definitions of U and Λ_φ then reveals that $u = \Lambda_\varphi$. \square

Combining (6.14), (6.16) and Proposition 3.1, we obtain that any bilinear Schur multiplier u induces a completely bounded

$$\tilde{u}: \Gamma(L^2(\Omega_1), L^2(\Omega_2), L^2(\Omega_3)) \longrightarrow B(L^2(\Omega_1), L^2(\Omega_3))$$

and that $\|\tilde{u}\|_{cb} = \|\tilde{u}\| (= \|u\|)$.

The next result, which essentially follows from [6], shows that similarly, S^1 -Schur multipliers are automatically completely bounded and that their norm and completely bounded norm coincide.

Theorem 6.8. *Let $\varphi \in L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$.*

- (a) *Λ_φ is an S^1 -operator multiplier if and only if there exist a separable Hilbert space H and two functions*

$$a \in L^\infty(\Omega_1 \times \Omega_2; H) \quad \text{and} \quad b \in L^\infty(\Omega_2 \times \Omega_3; H)$$

such that

$$(6.17) \quad \varphi(t_1, t_2, t_3) = \langle a(t_1, t_2), b(t_2, t_3) \rangle$$

for a.e. $(t_1, t_2, t_3) \in \Omega_1 \times \Omega_2 \times \Omega_3$.

In this case,

$$\|\Lambda_\varphi: S^2 \times S^2 \rightarrow S^1\| = \inf\{\|a\|_\infty \|b\|_\infty\},$$

where the infimum runs over all pairs (a, b) verifying the above factorization property.

- (b) *If Λ_φ is an S^1 -operator multiplier, then*

$$\widetilde{\Lambda}_\varphi: \Gamma(L^2(\Omega_1), L^2(\Omega_2), L^2(\Omega_3)) \longrightarrow S^1(L^2(\Omega_1), L^2(\Omega_3))$$

is completely bounded, with $\|\widetilde{\Lambda}_\varphi\|_{cb} = \|\widetilde{\Lambda}_\varphi\|$.

Proof. Part (a) is given by [6, Theorem 22].

Assume that Λ_φ is an S^1 -operator multiplier. Let

$$\mathcal{S}_{3,1} \subset B(S^\infty(L^2(\Omega_3), L^2(\Omega_1)), B(L^2(\Omega_3), L^2(\Omega_1)))$$

be the space of all measurable Schur multipliers from $L^2(\Omega_3)$ into $L^2(\Omega_1)$, in the sense of [6, Subsection 2.4]. Then using the notation from the latter paper (to which we refer for more explanations), part (a) implies that $\varphi \in L^\infty_\sigma(\Omega_2; \mathcal{S}_{3,1})$. Indeed this follows from Peller's description of measurable Schur multipliers given by [15, Theorem 1] (see also [21, Theorem 3.3], [6, Theorem 23] and [11]). Measurable Schur multipliers are $(L^\infty(\Omega_1), L^\infty(\Omega_3))$ -bimodule maps hence by [20, Theorem 2.1], any element of $\mathcal{S}_{3,1}$ is a completely bounded map, whose completely bounded norm coincides with its usual norm. Thus we have

$$\mathcal{S}_{3,1} \subset CB(S^\infty(L^2(\Omega_3), L^2(\Omega_1)), B(L^2(\Omega_3), L^2(\Omega_1))) \quad \text{isometrically.}$$

We deduce that

$$\varphi \in L^\infty_\sigma(\Omega_2; CB(S^\infty(L^2(\Omega_3), L^2(\Omega_1)), B(L^2(\Omega_3), L^2(\Omega_1)))).$$

Recall that by [4, Theorem 2.2] (see also Theorem 4.2 in the latter paper), we have a w^* -continuous isometric identification

$$CB(S^\infty(L^2(\Omega_3), L^2(\Omega_1)), B(L^2(\Omega_3), L^2(\Omega_1))) \simeq B(L^2(\Omega_1)) \overset{w^*h}{\otimes} B(L^2(\Omega_3)).$$

Hence we obtain that φ belongs to $L^\infty(\Omega_2; B(L^2(\Omega_1)) \overset{w^*h}{\otimes} B(L^2(\Omega_3)))$. Equivalently, φ belongs to $L^\infty(\Omega_2) \overline{\otimes} (B(L^2(\Omega_1)) \overset{w^*h}{\otimes} B(L^2(\Omega_3)))$. Moreover the norm of $\Lambda_\varphi: S^2 \times S^2 \rightarrow S^1$ is equal to the norm of φ in the latter space.

Now applying Theorem 3.4, we deduce that $\Lambda_\varphi: S^2 \times S^2 \rightarrow S^1$ is completely bounded, with $\|\widetilde{\Lambda_\varphi}\|_{cb} = \|\widetilde{\Lambda_\varphi}\|$. \square

Remark 6.9. In Theorem 6.8 above, (a) can be deduced from (b) as follows. Assume that Λ_φ is a completely bounded S^1 -operator multiplier, with completely bounded norm < 1 . By Proposition 6.7 and (6.16), $\Lambda_\varphi = \tau_\varphi$ is $(L^\infty(\Omega_3), L^\infty(\Omega_2), L^\infty(\Omega_1))$ -modular. Further $L^\infty(\Omega_2)$ is injective. Hence by Corollary 6.5, there exist an index set I , a family $(a_i)_{i \in I}$ in $L^\infty(\Omega_1 \times \Omega_2)$ and a family $(b_i)_{i \in I}$ in $L^\infty(\Omega_2 \times \Omega_3)$ such that

$$\sum_{i \in I} |a_i|^2 < 1 \quad \text{and} \quad \sum_{i \in I} |b_i|^2 < 1$$

almost everywhere on $\Omega_1 \times \Omega_2$ and on $\Omega_2 \times \Omega_3$, respectively, and $\varphi = \sum_{i \in I} (a_i \otimes 1)(1 \otimes b_i)$ in the w^* -topology of $L^\infty(\Omega_1 \times \Omega_2 \times \Omega_3)$. Since we assumed that the three measure spaces (Ω_j, μ_j) are separable, it follows from the proof of Corollary 6.5 that I can be chosen to be a countable set. Then we have

$$(6.18) \quad \varphi(t_1, t_2, t_3) = \sum_{i \in I} a_i(t_1, t_2) b_i(t_2, t_3)$$

for a.e. $(t_1, t_2, t_3) \in \Omega_1 \times \Omega_2 \times \Omega_3$. Further we may define $a \in L^\infty(\Omega_1 \times \Omega_2; \ell_I^2)$ and $b \in L^\infty(\Omega_2 \times \Omega_3; \ell_I^2)$ by $a(t_1, t_2) = (a_i(t_1, t_2))_{i \in I}$ and $b(t_2, t_3) = (b_i(t_2, t_3))_{i \in I}$, respectively. Then we both have $\|a\|_\infty \leq 1$ and $\|b\|_\infty \leq 1$, and the identity (6.18) yields (6.17), with $H = \ell_I^2$.

Note however we do not know any direct proof of Theorem 6.8 (b), not using some of the arguments from [6].

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