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FUNCTIONAL CALCULUS FOR A BOUNDED C_0 -SEMIGROUP ON HILBERT SPACE

LORIS ARNOLD AND CHRISTIAN LE MERDY

ABSTRACT. We introduce a new Banach algebra $\mathcal{A}(\mathbb{C}_+)$ of bounded analytic functions on $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ which is an analytic version of the Figa-Talamanca-Herz algebras on \mathbb{R} . Then we prove that the negative generator A of any bounded C_0 -semigroup on Hilbert space H admits a bounded (natural) functional calculus $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$. We prove that this is an improvement of the bounded functional calculus $\mathcal{B}(\mathbb{C}_+) \rightarrow B(H)$ recently devised by Batty-Gomilko-Tomilov on a certain Besov algebra $\mathcal{B}(\mathbb{C}_+)$ of analytic functions on \mathbb{C}_+ , by showing that $\mathcal{B}(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+)$ and $\mathcal{B}(\mathbb{C}_+) \neq \mathcal{A}(\mathbb{C}_+)$. In the Banach space setting, we give similar results for negative generators of γ -bounded C_0 -semigroups. The study of $\mathcal{A}(\mathbb{C}_+)$ requires to deal with Fourier multipliers on the Hardy space $H^1(\mathbb{R}) \subset L^1(\mathbb{R})$ and we establish new results on this topic.

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Key words: Functional calculus, Semigroups, Hardy spaces, Fourier multipliers, Besov spaces, γ -boundedness.

1. INTRODUCTION

Let H be a Hilbert space and let $-A$ be the infinitesimal generator of a bounded C_0 -semigroup $(T_t)_{t \geq 0}$ on H . To any $b \in L^1(\mathbb{R}_+)$, one may associate the operator $\Gamma(A, b) \in B(H)$ defined by

$$[\Gamma(A, b)](x) = \int_0^\infty b(t)T_t(x) dt, \quad x \in H.$$

The mapping $b \mapsto \Gamma(A, b)$ is the so-called Hille-Phillips functional calculus ([15], see also [12, Section 3.3]) and we obviously have

$$\|\Gamma(A, b)\| \leq C\|b\|_1, \quad b \in L^1(\mathbb{R}_+),$$

where $C = \sup_{t \geq 0} \|T_t\|$. This holds true as well for any bounded C_0 -semigroup on Banach space. However we focus here on semigroups acting on Hilbert space.

If $(T_t)_{t \geq 0}$ is a contractive semigroup (i.e. $\|T_t\| \leq 1$ for all $t \geq 0$) on H , then we have the much stronger estimate $\|\Gamma(A, b)\| \leq \|\widehat{b}\|_\infty$ for any $b \in L^1(\mathbb{R}_+)$, where \widehat{b} denotes the Fourier transform of b . This is a semigroup version of von Neumann's inequality, see [12, Section 7.1.3] for a proof. Hence more generally, if $(T_t)_{t \geq 0}$ is similar to a contractive semigroup, then

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there exists a constant $C \geq 1$ such that

$$(1.1) \quad \|\Gamma(A, b)\| \leq C \|\widehat{b}\|_\infty, \quad b \in L^1(\mathbb{R}_+).$$

However not all negative generators of bounded C_0 -semigroups satisfy such an estimate. Indeed if A is sectorial of type $< \frac{\pi}{2}$, it follows from [12, Section 3.3] that A satisfies an estimate of the form (1.1) exactly when A has a bounded H^∞ -functional calculus, see Subsection 4.4 for more on this.

The motivation for this paper is the search of sharp estimates of $\|\Gamma(A, b)\|$, and of the norms of other functions of A , valid for all negative generators of bounded C_0 -semigroups. A major breakthrough was achieved by Haase [13, Corollary 5.5] who proved an estimate

$$\|\Gamma(A, b)\| \leq C \|L_b\|_{\mathcal{B}},$$

where $\|\cdot\|_{\mathcal{B}}$ denotes the norm with respect to a suitable Besov algebra $\mathcal{B}(\mathbb{C}_+)$ of analytic functions, and

$$L_b: \mathbb{C}_+ = \{\operatorname{Re}(\cdot) > 0\} \rightarrow \mathbb{C}, \quad L_b(z) = \int_0^\infty b(t)e^{-tz},$$

is the Laplace transform of b . It is worth mentioning the related works by White [35] and Vitse [34]. More recently, Batty-Gomilko-Tomilov [4] (see also [5]) extended Haase's result by providing an explicit construction of a bounded functional calculus $\mathcal{B}(\mathbb{C}_+) \rightarrow B(H)$ associated with A , extending the Hille-Phillips functional calculus.

In this paper we introduce the space $\mathcal{A}(\mathbb{R}) \subset H^\infty(\mathbb{R})$ defined by

$$\mathcal{A}(\mathbb{R}) = \left\{ F = \sum_{k=1}^{\infty} f_k \star h_k : (f_k)_{k \in \mathbb{N}} \subset BUC(\mathbb{R}), (h_k)_{k \in \mathbb{N}} \subset H^1(\mathbb{R}), \sum_{k=1}^{\infty} \|f_k\|_\infty \|h_k\|_1 < \infty \right\},$$

equipped with the norm $\|F\|_{\mathcal{A}} = \inf\{\sum_{k=1}^{\infty} \|f_k\|_\infty \|h_k\|_1\}$, where the infimum runs over all sequences $(f_k)_{k \in \mathbb{N}} \subset BUC(\mathbb{R})$ and $(h_k)_{k \in \mathbb{N}} \subset H^1(\mathbb{R})$ such that $\sum_{k=1}^{\infty} \|f_k\|_\infty \|h_k\|_1 < \infty$ and $F = \sum_{k=1}^{\infty} f_k \star h_k$. The definition of this space is inspired by Peller's paper [26], where a discrete analogue of $\mathcal{A}(\mathbb{R})$ was introduced to study functions of power bounded operators on Hilbert space. Also, $\mathcal{A}(\mathbb{R})$ can be regarded as an analytic version of the the Figa-Talamanca-Herz algebras $A_p(\mathbb{R})$, $1 < p < \infty$, for which we refer e.g. to [7, Chapter 3].

We prove in Section 3 that $\mathcal{A}(\mathbb{R})$ is indeed a Banach algebra for pointwise multiplication. Next in Section 4 we introduce the natural half-plane version $\mathcal{A}(\mathbb{C}_+) \subset H^\infty(\mathbb{C}_+)$ of $\mathcal{A}(\mathbb{R})$ and we show (Corollary 4.7) that whenever A is the negative generator of a bounded C_0 -semigroup $(T_t)_{t \geq 0}$ on Hilbert space, there is a unique bounded homomorphism $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$, such that

$$(1.2) \quad \rho_A(L_b) = \Gamma(A, b), \quad b \in L^1(\mathbb{R}_+).$$

In particular we show that

$$\|\Gamma(A, b)\| \leq \left(\sup_{t \geq 0} \|T_t\| \right)^2 \|L_b\|_{\mathcal{A}}, \quad b \in L^1(\mathbb{R}_+).$$

This improves Haase's estimate mentioned above. Our work also improves [4, Theorem 4.4] in the Hilbert space case. Indeed we show in Section 5 that the Besov algebra considered in [12, 4] is included in $\mathcal{A}(\mathbb{C}_+)$, with an estimate $\|\cdot\|_{\mathcal{A}} \lesssim \|\cdot\|_{\mathcal{B}}$, and we also show that the converse is wrong.

In general, our main result (Corollary 4.7) does hold true on non Hilbertian Banach spaces. In Section 6, following ideas from [1, 2, 23], we give a Banach space version of Corollary 4.7, using the notion of γ -boundedness. Namely we show that if A is the negative generator of a bounded C_0 -semigroup $(T_t)_{t \geq 0}$ on a Banach space X , then the set $\{T_t : t \geq 0\} \subset B(X)$ is γ -bounded if and only if there exists a γ -bounded homomorphism $\rho_A : \mathcal{A}(\mathbb{C}_+) \rightarrow B(X)$ satisfying (1.2). This should be regarded as a semigroup version of [23, Theorem 4.4], where a characterization of γ -bounded continuous representations of amenable groups was established.

Our results make crucial use of Fourier multipliers on the Hardy space $H^1(\mathbb{R})$. Section 2 is devoted to this topic. In particular we establish the following result of independent interest: if a bounded operator $T : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ commutes with translations, then there exists a bounded continuous function $m : \mathbb{R}_+^* \rightarrow \mathbb{C}$ such that $\|m\|_\infty \leq \|T\|$ and for any $h \in H^1(\mathbb{R})$, $\widehat{T(h)} = m\widehat{h}$.

Notation and convention. We will use the following open half-planes of \mathbb{C} ,

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}, \quad P_+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}, \quad P_- := \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}.$$

Also for any real $\alpha \in \mathbb{R}$, we set

$$\mathcal{H}_\alpha = \{z \in \mathbb{C} : \operatorname{Re}(z) > \alpha\}.$$

In particular, $\mathcal{H}_0 = \mathbb{C}_+$.

For any $s \in \mathbb{R}$, we let $\tau_s : L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \rightarrow L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ denote the translation operator defined by

$$\tau_s f(t) = f(t - s), \quad t \in \mathbb{R},$$

for any $f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$.

The Fourier transform of any $f \in L^1(\mathbb{R})$ is defined by

$$\widehat{f}(u) = \int_{-\infty}^{\infty} f(t) e^{-itu} dt, \quad u \in \mathbb{R}.$$

Sometimes we write $\mathcal{F}(f)$ instead of \widehat{f} . We will also let $\mathcal{F}(f)$ or \widehat{f} denote the Fourier transform of any $f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$. Wherever it makes sense, we will use \mathcal{F}^{-1} to denote the inverse Fourier transform.

We will use several times the following elementary result (which follows from Fubini's Theorem and the Fourier inversion Theorem).

Lemma 1.1. *Let $f_1, f_2 \in L^1(\mathbb{R})$ such that either \widehat{f}_1 or \widehat{f}_2 belongs to $L^1(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}_1(u) \widehat{f}_2(-u) du.$$

The norm on $L^p(\mathbb{R})$ will be denoted by $\|\cdot\|_p$. We let $C_0(\mathbb{R})$ (resp. $BUC(\mathbb{R})$, resp. $C_b(\mathbb{R})$) denote the Banach algebra of continuous functions on \mathbb{R} which vanish at infinity (resp. of bounded and uniformly continuous functions on \mathbb{R} , resp. of bounded continuous functions on \mathbb{R}), equipped with the sup-norm $\|\cdot\|_\infty$. We set

$$(1.3) \quad C_{00}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : \widehat{f} \in L^1(\mathbb{R})\},$$

this is a dense subspace of $C_0(\mathbb{R})$. Further we let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space on \mathbb{R} and we let $M(\mathbb{R})$ denote the Banach algebra of all bounded Borel measures on \mathbb{R} .

We will use the identification $M(\mathbb{R}) \simeq C_0(\mathbb{R})^*$ (Riesz's Theorem) provided by the duality pairing

$$(1.4) \quad \langle \mu, f \rangle = \int_{\mathbb{R}} f(-t) d\mu(t), \quad \mu \in M(\mathbb{R}), f \in C_0(\mathbb{R}).$$

For any non empty open set $\mathcal{O} \subset \mathbb{C}$, we let $H^\infty(\mathcal{O})$ denote the Banach algebra of all bounded analytic functions on \mathcal{O} , equipped with the sup-norm $\|\cdot\|_\infty$.

Let X, Y be (complex) Banach spaces. We let $B(X, Y)$ denote the Banach space of all bounded operators $X \rightarrow Y$. We simply write $B(X)$ instead of $B(X, X)$, when $Y = X$. We let I_X denote the identity operator on X .

The domain of an operator A on some Banach space X is denoted by $\text{Dom}(A)$. Its kernel and range are denoted by $\text{Ker}(A)$ and $\text{Ran}(A)$, respectively. If $z \in \mathbb{C}$ belongs to the resolvent set of A , we let $R(z, A) = (zI_X - A)^{-1}$ denote the corresponding resolvent operator.

2. FOURIER MULTIPLIERS ON $H^1(\mathbb{R})$

We denote by $H^1(\mathbb{R})$ the classical Hardy space, defined as the closed subspace of $L^1(\mathbb{R})$ of all functions h such that $\widehat{h}(u) = 0$ for any $u \leq 0$. For any $1 < p < \infty$, we denote by $H^p(\mathbb{R})$ the closure of $H^1(\mathbb{R}) \cap L^p(\mathbb{R})$ in $L^p(\mathbb{R})$. Also we let $H^\infty(\mathbb{R})$ denote the w^* -closure of $H^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ in $L^\infty(\mathbb{R})$. We recall (see e.g [9], [16] or [19]) that for any $1 \leq p \leq \infty$, $H^p(\mathbb{R})$ coincides with the subspace of all functions $f \in L^p(\mathbb{R})$ whose Poisson integral $\mathcal{P}[f]: P_+ \rightarrow \mathbb{C}$ is analytic.

It is well-known that $H^2(\mathbb{R})$ is the subspace of all functions in $L^2(\mathbb{R})$ whose Fourier transform vanishes almost everywhere on \mathbb{R}_- . This can be expressed by the identification

$$(2.1) \quad \mathcal{F}(H^2(\mathbb{R})) = L^2(\mathbb{R}_+).$$

Let $m \in L^\infty(\mathbb{R}_+)$. Using (2.1), we may associate

$$\begin{aligned} T_m : H^2(\mathbb{R}) &\rightarrow H^2(\mathbb{R}) \\ h &\mapsto \mathcal{F}^{-1}(m\widehat{h}), \end{aligned}$$

and we have $\|T_m\| = \|m\|_\infty$. The function m is called the symbol of T_m .

Let $1 \leq p < \infty$. Assume that T_m is bounded with respect to the $H^p(\mathbb{R})$ -norm, that is, there exists a constant $C > 0$ such that

$$(2.2) \quad \|\mathcal{F}^{-1}(m\widehat{h})\|_p \leq C \|h\|_p, \quad h \in H^p(\mathbb{R}) \cap H^2(\mathbb{R}).$$

Then since $H^p(\mathbb{R}) \cap H^2(\mathbb{R})$ is dense in $H^p(\mathbb{R})$, T_m uniquely extends to a bounded operator on $H^p(\mathbb{R})$ whose norm is the least possible constant C satisfying (2.2). In this case we keep the same notation $T_m: H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$ for this extension. Operators of this form are called bounded Fourier multipliers on $H^p(\mathbb{R})$. They form a subspace of $B(H^p(\mathbb{R}))$, that we denote by $\mathcal{M}(H^p(\mathbb{R}))$. It is plain that $\mathcal{M}(H^2(\mathbb{R})) \simeq L^\infty(\mathbb{R}_+)$ isometrically. In the sequel we will be mostly interested by $\mathcal{M}(H^1(\mathbb{R}))$.

The above definitions parallel the classical definitions of bounded Fourier multipliers on $L^p(\mathbb{R})$, that we will use without any further reference.

Example 2.1. Let $s \in \mathbb{R}$. For all $h \in L^1(\mathbb{R})$, one has $\widehat{\tau_s h}(u) = e^{-isu} \widehat{h}(u)$ for any $u \in \mathbb{R}$. Hence τ_s maps $H^1(\mathbb{R})$ into itself. Further τ_s is a bounded Fourier multiplier on $H^1(\mathbb{R})$, with symbol $m(u) = e^{-isu}$.

In the sequel we say that a bounded operator $T: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ commutes with translations if $T\tau_s = \tau_s T$ for each $s \in \mathbb{R}$. Classical properties of the Fourier transform easily imply that any bounded Fourier multiplier on $H^1(\mathbb{R})$ commutes with translations. The next result implies that the converse is true. This is the $H^1(\mathbb{R})$ -version of the well-known characterization of $L^p(\mathbb{R})$ -Fourier multipliers as the operators $L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ commuting with translations (see e.g. [32, Theorem 3.16]). The $H^1(\mathbb{R})$ -case requires different arguments.

Theorem 2.2. *Let $T \in B(H^1(\mathbb{R}))$ and assume that T commutes with translations. Then there exists a bounded continuous function $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$ such that $\widehat{Th} = m\widehat{h}$ for any $h \in H^1(\mathbb{R})$ (and hence $T = T_m$). In this case, we have*

$$(2.3) \quad \|m\|_\infty \leq \|T\|.$$

Proof. We use Bochner spaces and Bochner integrals, for which we refer to [8]. Let $T \in B(H^1(\mathbb{R}))$ and assume that T commutes with translations.

Let $h, g \in H^1(\mathbb{R})$. The identification $L^1(\mathbb{R}^2) = L^1(\mathbb{R}; L^1(\mathbb{R}))$ and the fact that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(t-s)| |g(s)| dt ds = \|h\|_1 \|g\|_1 < \infty$$

imply that $s \mapsto g(s)\tau_s h$ in an almost everywhere defined function belonging to the Bochner space $L^1(\mathbb{R}; L^1(\mathbb{R}))$. Since $\tau_s h$ belongs to $H^1(\mathbb{R})$ for any $s \in \mathbb{R}$, the latter is actually an element of $L^1(\mathbb{R}; H^1(\mathbb{R}))$. Further its integral (which is an element of $H^1(\mathbb{R})$) is equal to the convolution of h and g , that is,

$$(2.4) \quad h \star g = \int_{-\infty}^{\infty} \tau_s h g(s) ds.$$

It follows, using the assumption, that

$$\begin{aligned} T(h \star g) &= \int_{-\infty}^{\infty} T(\tau_s h) g(s) ds \\ &= \int_{-\infty}^{\infty} \tau_s (Th) g(s) ds \\ &= Th \star g. \end{aligned}$$

(The last equality comes from (2.4) replacing h by Th .) Likewise $T(h \star g) = h \star Tg$, whence $Th \star g = h \star Tg$. Applying the Fourier transform to the latter equality, one obtains

$$(2.5) \quad \widehat{Th} \cdot \widehat{g} = \widehat{h} \cdot \widehat{Tg}.$$

Now let $f \in \mathcal{S}(\mathbb{R})$ satisfying $f = 0$ on \mathbb{R}_- and $f > 0$ on \mathbb{R}_+^* . The function $g = \mathcal{F}^{-1}(f)$ belongs to $H^1(\mathbb{R})$ and we may define $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$ by

$$(2.6) \quad m(u) = \frac{\widehat{Tg}(u)}{\widehat{g}(u)}, \quad u > 0.$$

Obviously m is continuous. Furthermore it follows from (2.5) that for any $h \in H^1(\mathbb{R})$, we have $\widehat{T}h = m\widehat{h}$ on \mathbb{R}_+^* . It therefore suffices to show that m is bounded and that (2.3) holds true.

We adapt an argument from [22]. Let us first prove the inequality

$$(2.7) \quad |m(1)| \leq \|T\|.$$

To obtain this, we let $\gamma_1, \gamma_2 \in C_{00}(\mathbb{R})$ (see (1.3)) such that $\|\gamma_1\|_1 = 1$, $\text{Supp}(\widehat{\gamma}_1) \subset [-1, 1]$ and $\|\gamma_2\|_\infty = 1$. For each $0 < \varepsilon \leq \frac{1}{2}$, we define $h_\varepsilon, g_\varepsilon: \mathbb{R} \rightarrow \mathbb{C}$ by

$$h_\varepsilon(t) = \varepsilon e^{it} \gamma_1(\varepsilon t) \quad \text{and} \quad g_\varepsilon(t) = e^{-it} \gamma_2(\varepsilon t), \quad t \in \mathbb{R}.$$

Then, $h_\varepsilon \in H^1(\mathbb{R})$, $g_\varepsilon \in C_{00}(\mathbb{R})$, $\|h_\varepsilon\|_1 = 1$, $\|g_\varepsilon\|_\infty = 1$ and we both have

$$\widehat{h}_\varepsilon(u) = \widehat{\gamma}_1\left(\frac{u-1}{\varepsilon}\right) \quad \text{and} \quad \widehat{g}_\varepsilon(u) = \frac{1}{\varepsilon} \widehat{\gamma}_2\left(\frac{u+1}{\varepsilon}\right)$$

for any $u \in \mathbb{R}$. Moreover using Lemma 1.1, we have

$$\int_{-\infty}^{\infty} [T(h_\varepsilon)](t) g_\varepsilon(t) dt = \frac{1}{2\pi} \int_0^{\infty} \widehat{T(h_\varepsilon)}(u) \widehat{g}_\varepsilon(-u) du.$$

We infer that

$$\begin{aligned} \int_{-\infty}^{\infty} [T(h_\varepsilon)](t) g_\varepsilon(t) dt &= \frac{1}{2\pi} \int_0^{\infty} m(u) \widehat{h}_\varepsilon(u) \widehat{g}_\varepsilon(-u) du \\ &= \frac{1}{2\pi} \int_{1-\varepsilon}^{1+\varepsilon} \frac{m(u)}{\varepsilon} \widehat{\gamma}_1\left(\frac{u-1}{\varepsilon}\right) \widehat{\gamma}_2\left(\frac{-u+1}{\varepsilon}\right) du. \end{aligned}$$

Therefore,

$$\frac{1}{\varepsilon} \left| \int_{1-\varepsilon}^{1+\varepsilon} m(u) \widehat{\gamma}_1\left(\frac{u-1}{\varepsilon}\right) \widehat{\gamma}_2\left(\frac{-u+1}{\varepsilon}\right) du \right| \leq 2\pi \|T\|.$$

By the change of variable $u \mapsto \frac{u-1}{\varepsilon}$, this reads

$$\left| \int_{-1}^1 m(1 + \varepsilon u) \widehat{\gamma}_1(u) \widehat{\gamma}_2(-u) du \right| \leq 2\pi \|T\|.$$

Since m is continuous on \mathbb{R}_+^* and $\varepsilon \in (0, \frac{1}{2})$, the function $u \mapsto m(1 + \varepsilon u)$ is bounded on $(-1, 1)$, independently of ε . Moreover $\widehat{\gamma}_1$ is bounded on $(-1, 1)$ and $\widehat{\gamma}_2$ is integrable. Hence by Lebesgue's dominated convergence Theorem,

$$\int_{-1}^1 m(1 + \varepsilon u) \widehat{\gamma}_1(u) \widehat{\gamma}_2(-u) du \xrightarrow{\varepsilon \rightarrow 0} m(1) \int_{-1}^1 \widehat{\gamma}_1(u) \widehat{\gamma}_2(-u) du.$$

Since $\int_0^{\infty} \widehat{\gamma}_1(u) \widehat{\gamma}_2(-u) du = 2\pi \int_{-\infty}^{\infty} \gamma_1(t) \gamma_2(t) dt$, by Lemma 1.1, we deduce the inequality

$$|m(1)| \left| \int_{-\infty}^{\infty} \gamma_1(t) \gamma_2(t) dt \right| \leq \|T\|.$$

It turns out that the supremum of $\left\{ \left| \int_{-\infty}^{\infty} \gamma_1(t) \gamma_2(t) dt \right| \right\}$, for γ_1 and γ_2 defined as above, is equal to 1. Hence (2.7) holds true.

Now consider an arbitrary $a > 0$. Define $r_a: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ by

$$[r_a h](t) = \frac{1}{a} h\left(\frac{t}{a}\right), \quad h \in H^1(\mathbb{R}), \quad t \in \mathbb{R}.$$

Then r_a is an isometry satisfying $\widehat{r_a h}(u) = \widehat{h}(au)$ for any $h \in H^1(\mathbb{R})$ and $u \in \mathbb{R}$. Set

$$T_{(a)} = r_a T r_{\frac{1}{a}}: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R}).$$

Then $\tau_s T_{(a)} = T_{(a)} \tau_s$ for any $s \in \mathbb{R}$ and the function $\mathbb{R}_+^* \rightarrow \mathbb{C}$ associated with $T_{(a)}$ by the formula (2.6) is $u \mapsto m(au)$. Further $\|T_{(a)}\| = \|T\|$. Hence applying (2.7) to $T_{(a)}$ we deduce that

$$|m(a)| \leq \|T\|.$$

This proves the boundness of m and (2.3). \square

Remark 2.3. Let $1 < p < \infty$.

(1) Let $S: L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ be a bounded Fourier multiplier. Then S maps $H^p(\mathbb{R})$ into itself and the restriction $S|_{H^p}: H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$ is a bounded Fourier multiplier.

Let $Q: L^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$ be the Riesz projection and let $J: H^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ be the canonical embedding. Then conversely, for any bounded Fourier multiplier $T: H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$, $S = JTQ$ is a bounded Fourier multiplier on $L^p(\mathbb{R})$, whose restriction to $H^p(\mathbb{R})$ coincides with T . Thus $\mathcal{M}(H^p(\mathbb{R}))$ can be simply regarded as a subspace of $\mathcal{M}(L^p(\mathbb{R}))$, the space of bounded Fourier multipliers on $L^p(\mathbb{R})$.

It is also easy to check that a bounded operator $H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$ belongs to $\mathcal{M}(H^p(\mathbb{R}))$ if and only if it commutes with translations, using the similar result on $L^p(\mathbb{R})$.

(2) Let $p' = \frac{p}{p-1}$ be the conjugate number of p . Using Q again, we see that given any $m \in L^\infty(\mathbb{R}_+)$, the operator $T_m: H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$ extends to a bounded Fourier multiplier on $H^p(\mathbb{R})$ if and only if it extends to a bounded Fourier multiplier on $H^{p'}(\mathbb{R})$. Thus,

$$(2.8) \quad \mathcal{M}(H^p(\mathbb{R})) \simeq \mathcal{M}(H^{p'}(\mathbb{R}))$$

isomorphically.

Recall that the bounded Fourier multipliers on $L^1(\mathbb{R})$ are the operators of the form $h \mapsto \mu \star h$, with $\mu \in M(\mathbb{R})$, and that the norm of the latter operator is equal to $\|\mu\|_{M(\mathbb{R})}$ (see e.g. [32, Theorem 3.19]).

For any $\mu \in M(\mathbb{R})$, let $R_\mu: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ be the restriction of $h \mapsto \mu \star h$ to $H^1(\mathbb{R})$. This is a bounded Fourier multiplier whose symbol is equal to the restriction of $\widehat{\mu}$ to \mathbb{R}_+ . We set

$$(2.9) \quad \mathcal{R} = \{R_\mu : \mu \in M(\mathbb{R})\} \subset \mathcal{M}(H^1(\mathbb{R})).$$

In contrast with the result in Remark 2.3 (1), we will see in Proposition 2.4 below that $\mathcal{R} \neq \mathcal{M}(H^1(\mathbb{R}))$.

Following [28], we say that an operator $T \in B(H^1(\mathbb{R}))$ is regular if there exists a constant $C \geq 0$ such that

$$\left\| \sup_k |T(h_k)| \right\|_1 \leq C \left\| \sup_k |h_k| \right\|_1$$

for any finite family $(h_k)_k$ of $H^1(\mathbb{R})$.

For any Banach space X , let $H^1(\mathbb{R}; X)$ denote the closure of $H^1(\mathbb{R}) \otimes X$ in the Bochner space $L^1(\mathbb{R}; X)$. As is well-known (see [30, Chapter 1]), $T \in B(H^1(\mathbb{R}))$ is regular if and only if $T \otimes I_X: H^1(\mathbb{R}) \otimes X \rightarrow H^1(\mathbb{R}) \otimes X$ extends to a bounded operator from $H^1(\mathbb{R}; X)$ into itself, for any Banach space X .

Proposition 2.4.

- (1) Let $T \in \mathcal{M}(H^1(\mathbb{R}))$. Then $T \in \mathcal{R}$ if and only if T is regular.
- (2) $\mathcal{R} \neq \mathcal{M}(H^1(\mathbb{R}))$.

Proof. Let $\mu \in M(\mathbb{R})$. For any $F = \sum_j h_j \otimes x_j$ (finite sum) in $H^1(\mathbb{R}) \otimes X$, we have $(R_\mu \otimes I_X)(F) = \mu \star F$. Hence arguing as in the scalar case, we find that $\|(R_\mu \otimes I_X)(F)\|_1 \leq \|\mu\|_{M(\mathbb{R})} \|F\|_1$. This shows that any element of \mathcal{R} is regular.

Conversely, let $T \in \mathcal{M}(H^1(\mathbb{R}))$ and assume that T is regular. By [24] (see also [28, Theorem 3]), T admits a bounded extension $S: L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$. Using an entirely classical averaging argument, we are going to show that T actually admits a bounded extension on $L^1(\mathbb{R})$ which commutes with translations.

Let us regard $L^1(\mathbb{R}) \subset M(\mathbb{R})$ in the usual way. Owing to the fact that \mathbb{R} is amenable, let $\nu: L^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ be a state such that $\nu \circ \tau_s = \nu$ for any $s \in \mathbb{R}$. Define $d: L^1(\mathbb{R}) \times C_0(\mathbb{R}) \rightarrow \mathbb{C}$ by

$$d(h, f) = \nu\left(s \mapsto \langle \tau_{-s} S \tau_s(h), f \rangle\right), \quad h \in L^1(\mathbb{R}), f \in C_0(\mathbb{R}),$$

where the duality pairing is given by (1.4). Then d is a bounded bilinear map which satisfies

$$(2.10) \quad d(\tau_s h, f) = d(h, \tau_s f), \quad h \in L^1(\mathbb{R}), f \in C_0(\mathbb{R}), s \in \mathbb{R}.$$

Let $S_\nu: L^1(\mathbb{R}) \rightarrow M(\mathbb{R})$ be induced by d , that is, $\langle S_\nu(h), f \rangle = d(h, f)$ for any $h \in L^1(\mathbb{R})$ and any $f \in C_0(\mathbb{R})$. Let $q: M(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ be the projection which takes any element of $M(\mathbb{R})$ to (the density of) its absolutely continuous part. Then $\tau_s q = q \tau_s$ for any $s \in \mathbb{R}$. Hence (2.10) implies that $T_\nu := q S_\nu: L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ satisfies

$$(2.11) \quad \tau_s T_\nu = T_\nu \tau_s, \quad s \in \mathbb{R}.$$

Since T commutes with translations, $\langle \tau_{-s} S \tau_s(h), f \rangle = \langle S(h), f \rangle$ for any $h \in L^1(\mathbb{R})$, $f \in C_0(\mathbb{R})$ and $s \in \mathbb{R}$. Further $\nu(1) = 1$. Consequently, $T_\nu h = T h$ for any $h \in H^1(\mathbb{R})$, that is, T_ν is an extension of T .

By (2.11), there exists $\mu \in M(\mathbb{R})$ such that $T_\nu = \mu \star \cdot$. This implies that $T = R_\mu$, which shows (1).

Let us now prove (2). Fix $s \in \mathbb{R}^*$ and define $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$ by $m(u) = u^{is}$. By [22, Lemma 4.1 and Theorem 1.2], m is the symbol of a bounded Fourier multiplier $T_m: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ and if X is a Banach space without the so-called AUMD property, then $T_m \otimes I_X$ does not extend to a bounded operator on $H^1(\mathbb{R}; X)$. Using (1), this implies that $T_m \notin \mathcal{R}$. \square

The following lemma will play a crucial role.

Lemma 2.5. *For any $1 \leq p < \infty$, we have $\mathcal{M}(H^1(\mathbb{R})) \subset \mathcal{M}(H^p(\mathbb{R}))$.*

Proof. By definition we have $\mathcal{M}(H^1(\mathbb{R})) \subset \mathcal{M}(H^2(\mathbb{R}))$. By (2.8) we may assume that $p \in (1, 2)$. Let $\theta = 2(1 - \frac{1}{p})$. Then in the complex interpolation method, we have

$$[H^1(\mathbb{R}), H^2(\mathbb{R})]_\theta \simeq H^p(\mathbb{R}),$$

by [27, Theorem 4.3]. The result follows at once. \square

3. ALGEBRAS \mathcal{A}_0 AND \mathcal{A}

We introduce and study new algebras of functions which will be used in Section 4 to establish a functional calculus for negative generators of bounded C_0 -semigroups on Hilbert space. The next definitions are inspired by [26], see also the “Notes and Remarks on Chapter 6” in [29].

3.1. Definitions and properties.

Definition 3.1. We let $\mathcal{A}_0(\mathbb{R})$ (resp. $\mathcal{A}(\mathbb{R})$) be the set of all functions $F: \mathbb{R} \rightarrow \mathbb{C}$ such that there exist two sequences $(f_k)_{k \in \mathbb{N}}$ in $C_0(\mathbb{R})$ (resp. $BUC(\mathbb{R})$) and $(h_k)_{k \in \mathbb{N}}$ in $H^1(\mathbb{R})$ satisfying

$$(3.1) \quad \sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 < \infty$$

and

$$(3.2) \quad \forall s \in \mathbb{R}, \quad F(s) = \sum_{k=1}^{\infty} f_k \star h_k(s).$$

Note that for any $f \in C_0(\mathbb{R})$ (resp. $BUC(\mathbb{R})$) and any $h \in H^1(\mathbb{R})$, $f \star h$ belongs to $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$ (resp. $BUC(\mathbb{R}) \cap H^\infty(\mathbb{R})$). Further for any $(f_k)_{k \in \mathbb{N}}$ and $(h_k)_{k \in \mathbb{N}}$ as in Definition 3.1, we have $\|f_k \star h_k\|_{\infty} \leq \|f_k\|_{\infty} \|h_k\|_1$, and hence $\sum_{k=1}^{\infty} \|f_k \star h_k\|_{\infty} < \infty$, by (3.1). This insures the convergence of the series in (3.2) and implies that

$$\mathcal{A}_0(\mathbb{R}) \subset C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}) \quad \text{and} \quad \mathcal{A}(\mathbb{R}) \subset BUC(\mathbb{R}) \cap H^\infty(\mathbb{R}).$$

Definition 3.2. For all $F \in \mathcal{A}_0(\mathbb{R})$ (resp. $F \in \mathcal{A}(\mathbb{R})$), we set

$$\|F\|_{\mathcal{A}_0} = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 \right\} \quad \left(\text{resp. } \|F\|_{\mathcal{A}} = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 \right\} \right),$$

where the infimum runs over all sequences $(f_k)_{k \in \mathbb{N}}$ in $C_0(\mathbb{R})$ (resp. $BUC(\mathbb{R})$) and $(h_k)_{k \in \mathbb{N}}$ in $H^1(\mathbb{R})$ satisfying (3.1) and (3.2).

It is clear that

$$\|F\|_{\infty} \leq \|F\|_{\mathcal{A}}, \quad F \in \mathcal{A}(\mathbb{R}).$$

To show that $\|\cdot\|_{\mathcal{A}_0}$ and $\|\cdot\|_{\mathcal{A}}$ are complete norms, we make a connection with projective tensor products, which will be useful along this section.

If X and Y are any Banach spaces, the projective norm of $\zeta \in X \otimes Y$ is defined by

$$\|\zeta\|_{\wedge} = \inf \left\{ \sum_k \|x_k\| \|y_k\| \right\},$$

where the infimum runs over all finite families $(x_k)_k$ in X and $(y_k)_k$ in Y satisfying $\zeta = \sum_k x_k \otimes y_k$. The completion of $(X \otimes Y, \|\cdot\|_{\wedge})$, denoted by $X \widehat{\otimes} Y$, is called the projective tensor product of X and Y .

Let Z be a third Banach space. To any $\ell \in B_2(X \times Y, Z)$, the space of bounded bilinear maps from $X \times Y$ into Z , one can associate a linear map $\overset{\circ}{\ell}: X \otimes Y \rightarrow Z$ by the formula

$$\overset{\circ}{\ell}(x \otimes y) = \ell(x, y), \quad x \in X, y \in Y.$$

Then $\overset{\circ}{\ell}$ extends to a bounded operator (still denoted by) $\overset{\circ}{\ell}: X \widehat{\otimes} Y \rightarrow Z$, with $\|\overset{\circ}{\ell}\| = \|\ell\|$. Further the mapping $\ell \mapsto \overset{\circ}{\ell}$ yields an isometric identification

$$(3.3) \quad B_2(X \times Y, Z) \simeq B(X \widehat{\otimes} Y, Z).$$

Let us apply the above property in the case when $Z = \mathbb{C}$. Using the standard identification $B_2(X \times Y, \mathbb{C}) = B(X, Y^*)$, we obtain an isometric identification

$$(3.4) \quad (X \widehat{\otimes} Y)^* \simeq B(X, Y^*).$$

We refer to [8, Chapter 8, Theorem 1 & Corollary 2] for these classical facts.

Consider the bilinear map $\sigma: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ defined by

$$(3.5) \quad \sigma(f, h) = f \star h, \quad f \in C_0(\mathbb{R}), h \in H^1(\mathbb{R}).$$

Applying (3.3), let

$$\overset{\circ}{\sigma}: C_0(\mathbb{R}) \widehat{\otimes} H^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$$

be associated with σ . Then $\mathcal{A}_0(\mathbb{R}) = \text{Ran}(\overset{\circ}{\sigma})$. Through the resulting algebraic isomorphism between $\mathcal{A}_0(\mathbb{R})$ and $(C_0(\mathbb{R}) \widehat{\otimes} H^1(\mathbb{R})) / \text{Ker}(\overset{\circ}{\sigma})$, $\|\cdot\|_{\mathcal{A}_0}$ corresponds to the quotient norm on the latter space. Thus $(\mathcal{A}_0(\mathbb{R}), \|\cdot\|_{\mathcal{A}_0})$ is a Banach space and $\overset{\circ}{\sigma}$ induces an isometric identification

$$(3.6) \quad \mathcal{A}_0(\mathbb{R}) \simeq \frac{C_0(\mathbb{R}) \widehat{\otimes} H^1(\mathbb{R})}{\text{Ker}(\overset{\circ}{\sigma})}.$$

Similarly, $\|\cdot\|_{\mathcal{A}}$ is a norm on $\mathcal{A}(\mathbb{R})$ and $(\mathcal{A}(\mathbb{R}), \|\cdot\|_{\mathcal{A}})$ is a Banach space.

Remark 3.3. It is clear from Definition 3.2 that $\mathcal{A}_0(\mathbb{R}) \subset \mathcal{A}(\mathbb{R})$ and that for any $F \in \mathcal{A}_0(\mathbb{R})$, we have

$$(3.7) \quad \|F\|_{\mathcal{A}} \leq \|F\|_{\mathcal{A}_0}.$$

3.2. \mathcal{A} and \mathcal{A}_0 are Banach algebras.

Proposition 3.4. *The spaces $\mathcal{A}_0(\mathbb{R})$ and $\mathcal{A}(\mathbb{R})$ are Banach algebras for the pointwise multiplication. Furthermore, $\mathcal{A}_0(\mathbb{R})$ is an ideal of $\mathcal{A}(\mathbb{R})$ and for any $F \in \mathcal{A}(\mathbb{R})$ and $G \in \mathcal{A}_0(\mathbb{R})$, we have*

$$(3.8) \quad \|FG\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}} \|G\|_{\mathcal{A}_0}.$$

Proof. Let $f_1, f_2 \in C_0(\mathbb{R})$ and $h_1, h_2 \in H^1(\mathbb{R})$. We note that

$$(3.9) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h_1(t)| |h_2(t+s)| dt ds = \|h_1\|_1 \|h_2\|_1.$$

We define, for $s \in \mathbb{R}$, $\varphi_s: \mathbb{R} \rightarrow \mathbb{C}$ and $\psi_s: \mathbb{R} \rightarrow \mathbb{C}$ by

$$\varphi_s(t) = f_1(t)f_2(t-s) \quad \text{and} \quad \psi_s(t) = h_1(t)h_2(t+s).$$

Since f_2 is uniformly continuous, the function $s \mapsto \varphi_s$ is continuous from \mathbb{R} into $C_0(\mathbb{R})$. Thus $s \mapsto \varphi_s$ belongs to the Bochner space $L^\infty(\mathbb{R}; C_0(\mathbb{R}))$. Using (3.9) and arguing as at the beginning of the proof of Theorem 2.2, we see that $s \mapsto \psi_s$ belongs to $L^1(\mathbb{R}; H^1(\mathbb{R}))$. It follows that $s \mapsto \varphi_s \star \psi_s$ is defined almost everywhere and belongs to $L^1(\mathbb{R}; \mathcal{A}_0(\mathbb{R}))$. Moreover,

$$(3.10) \quad \int_{-\infty}^{\infty} \|\varphi_s \star \psi_s\|_{\mathcal{A}_0} ds \leq \|f_1\|_\infty \|f_2\|_\infty \|h_1\|_1 \|h_2\|_1.$$

Indeed, using (3.9) and Fubini's Theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} \|\varphi_s \star \psi_s\|_{\mathcal{A}_0} ds &\leq \int_{-\infty}^{\infty} \|\varphi_s\|_\infty \|\psi_s\|_1 ds \\ &\leq \|f_1\|_\infty \|f_2\|_\infty \int_{-\infty}^{\infty} \|\psi_s\|_1 ds, \end{aligned}$$

which is equal to the right-hand side of (3.10).

The integral of $s \mapsto \varphi_s \star \psi_s$ is an element of $\mathcal{A}_0(\mathbb{R})$. We claim that we actually have

$$(3.11) \quad \int_{-\infty}^{\infty} \varphi_s \star \psi_s ds = (f_1 \star h_1)(f_2 \star h_2).$$

Indeed, using again (3.9) and Fubini's Theorem, we have for all $u \in \mathbb{R}$,

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_s \star \psi_s(u) ds &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t) f_2(t-s) h_1(u-t) h_2(u-t+s) dt ds \\ &= \int_{-\infty}^{\infty} f_1(t) h_1(u-t) \int_{-\infty}^{\infty} f_2(t-s) h_2(u-(t-s)) ds dt \\ &= (f_1 \star h_1)(u) (f_2 \star h_2)(u). \end{aligned}$$

Combining (3.11) and (3.10), we obtain that $(f_1 \star h_1)(f_2 \star h_2) \in \mathcal{A}_0(\mathbb{R})$, with

$$\|(f_1 \star h_1)(f_2 \star h_2)\|_{\mathcal{A}_0} \leq \|f_1\|_\infty \|f_2\|_\infty \|h_1\|_1 \|h_2\|_1.$$

Now let $F, G \in \mathcal{A}_0(\mathbb{R})$ and let $\varepsilon > 0$. Consider sequences $(f_k^1)_{k \in \mathbb{N}}$, $(f_k^2)_{k \in \mathbb{N}}$ in $C_0(\mathbb{R})$ and $(h_k^1)_{k \in \mathbb{N}}$, $(h_k^2)_{k \in \mathbb{N}}$ in $H^1(\mathbb{R})$ such that

$$F = \sum_{k=1}^{\infty} f_k^1 \star h_k^1 \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k^1\|_\infty \|h_k^1\|_1 \leq \|F\|_{\mathcal{A}_0} + \varepsilon,$$

as well as

$$G = \sum_{k=1}^{\infty} f_k^2 \star h_k^2 \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k^2\|_\infty \|h_k^2\|_1 \leq \|G\|_{\mathcal{A}_0} + \varepsilon.$$

Then, using summation in $C_0(\mathbb{R})$, we have

$$FG = \sum_{k,l=1}^{\infty} (f_k^1 \star h_k^1)(f_l^2 \star h_l^2).$$

Further $(f_k^1 \star h_k^1)(f_l^2 \star h_l^2) \in \mathcal{A}_0(\mathbb{R})$ for any $k, l \geq 1$, and

$$\begin{aligned} \sum_{k,l=1}^{\infty} \|(f_k^1 \star h_k^1)(f_l^2 \star h_l^2)\|_{\mathcal{A}_0} &\leq \sum_{k,l=1}^{\infty} \|f_k^1\|_{\infty} \|f_l^2\|_{\infty} \|h_k^1\|_1 \|h_l^2\|_1 \\ &= \left(\sum_{k=1}^{\infty} \|f_k^1\|_{\infty} \|h_k^1\|_1 \right) \left(\sum_{l=1}^{\infty} \|f_l^2\|_{\infty} \|h_l^2\|_1 \right) \\ &\leq (\|F\|_{\mathcal{A}_0} + \varepsilon) \leq (\|G\|_{\mathcal{A}_0} + \varepsilon). \end{aligned}$$

Since $\mathcal{A}_0(\mathbb{R})$ is complete, this shows that $FG \in \mathcal{A}_0(\mathbb{R})$. Since $\varepsilon > 0$ is arbitrary, we obtain that $\|FG\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}_0} \|G\|_{\mathcal{A}_0}$. This shows that $\mathcal{A}_0(\mathbb{R})$ is a Banach algebra.

Analogously, $\mathcal{A}(\mathbb{R})$ is a Banach algebra. Moreover if $f_1 \in BUC(\mathbb{R})$ and $f_2 \in C_0(\mathbb{R})$, then for each $s \in \mathbb{R}$, $\varphi_s: t \mapsto f_1(t)f_2(t-s)$ belongs to $C_0(\mathbb{R})$. Hence the computations above show that $\mathcal{A}_0(\mathbb{R})$ is an ideal of $\mathcal{A}(\mathbb{R})$, as well as (3.8). \square

3.3. Duality results and consequences. The main aim of this subsection is to identify $\mathcal{A}_0(\mathbb{R})^*$ with a subspace of $\mathcal{M}(H^1(\mathbb{R}))$, the space of bounded Fourier multilpliers on $H^1(\mathbb{R})$. This requires the use of duality tools.

Recall the identification $M(\mathbb{R}) \simeq C_0(\mathbb{R})^*$ provided by (1.4) and regard $H^1(\mathbb{R}) \subset L^1(\mathbb{R}) \subset M(\mathbb{R})$ in the usual way. By [9, Chapter II, Theorem 3.8], we have

$$(3.12) \quad (C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}))^\perp = H^1(\mathbb{R}).$$

Set $Z_0 := C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$ for convenience, then the above result yields an isometric identification

$$(3.13) \quad \left(\frac{C_0(\mathbb{R})}{Z_0} \right)^* \simeq H^1(\mathbb{R}).$$

In the sequel, we let $\dot{f} \in \frac{C_0(\mathbb{R})}{Z_0}$ denote the class of any $f \in C_0(\mathbb{R})$.

We note that for any $f \in C_0(\mathbb{R})$, $h \in H^1(\mathbb{R})$ and $s \in \mathbb{R}$, we have

$$(3.14) \quad (f \star h)(s) = \langle \tau_{-s}h, f \rangle.$$

Thus $f \star h = 0$ for any $f \in Z_0$ and $h \in H^1(\mathbb{R})$. The bilinear map $\sigma: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ defined by (3.5) therefore induces a bilinear map $\delta: \frac{C_0(\mathbb{R})}{Z_0} \times H^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ given by

$$\delta(\dot{f}, h) = f \star h, \quad f \in C_0(\mathbb{R}), \quad h \in H^1(\mathbb{R}).$$

Let

$$(3.15) \quad \overset{\circ}{\delta}: \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$$

be the bounded map induced by δ . Then the argument leading to (3.6) shows as well that $\mathcal{A}_0(\mathbb{R})$ is equal to the range of $\overset{\circ}{\delta}$ and that

$$(3.16) \quad \mathcal{A}_0(\mathbb{R}) \simeq \left(\frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R}) \right) / \text{Ker}(\overset{\circ}{\delta}).$$

Since $H^1(\mathbb{R})$ is a dual space, by (3.13), $B(H^1(\mathbb{R}))$ is a dual space as well. Indeed applying (3.4), we have an isometric identification

$$(3.17) \quad B(H^1(\mathbb{R})) \simeq \left(\frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R}) \right)^*.$$

If we unravel the identifications leading to (3.17), we obtain that the latter is given by

$$(3.18) \quad \langle T, \dot{f} \otimes h \rangle = \int_{-\infty}^{\infty} (Th)(t) f(-t) dt, \quad T \in B(H^1(\mathbb{R})), f \in C_0(\mathbb{R}), h \in H^1(\mathbb{R}).$$

Lemma 3.5. *The space $\mathcal{M}(H^1(\mathbb{R}))$ is w^* -closed in $B(H^1(\mathbb{R}))$.*

Proof. According to Theorem 2.2, an operator $T \in B(H^1(\mathbb{R}))$ belongs to $\mathcal{M}(H^1(\mathbb{R}))$ if and only if $\tau_s T = T \tau_s$ for any $s \in \mathbb{R}$. Hence it suffices to show that the maps $T \mapsto \tau_s T$ and $T \mapsto T \tau_s$ are w^* -continuous on $B(H^1(\mathbb{R}))$, for any $s \in \mathbb{R}$.

Fix such an s and let $(T_i)_i$ be a bounded net of $B(H^1(\mathbb{R}))$ converging to some $T \in B(H^1(\mathbb{R}))$ in the w^* -topology. It readily follows from (3.18) that for any $f \in C_0(\mathbb{R})$ and $h \in H^1(\mathbb{R})$, we have

$$\langle \tau_s T, \dot{f} \otimes h \rangle = \langle T, \overbrace{\tau_s \dot{f}} \otimes h \rangle \quad \text{and} \quad \langle \tau_s T_i, \dot{f} \otimes h \rangle = \langle T_i, \overbrace{\tau_s \dot{f}} \otimes h \rangle,$$

for all i . This implies that $\langle \tau_s T_i, \dot{f} \otimes h \rangle \rightarrow \langle \tau_s T, \dot{f} \otimes h \rangle$ when $i \rightarrow \infty$. By linearity, this implies that for any $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R})$, we have $\langle \tau_s T_i, \Phi \rangle \rightarrow \langle \tau_s T, \Phi \rangle$ when $i \rightarrow \infty$. Since $(T_i)_i$ is bounded, this implies that $\tau_s T_i \rightarrow \tau_s T$ in the w^* -topology. This shows that $T \mapsto \tau_s T$ is w^* -continuous. The proof that $T \mapsto T \tau_s$ is w^* -continuous is similar. \square

We introduce

$$\mathcal{PM} = \overline{\text{Span}}^{w^*} \{ \tau_s : s \in \mathbb{R} \} \subset B(H^1(\mathbb{R})).$$

A direct consequence of Lemma 3.5 is that

$$(3.19) \quad \mathcal{PM} \subset \mathcal{M}(H^1(\mathbb{R})).$$

Lemma 3.6. *Recall the mapping $\overset{\circ}{\delta}$ from (3.15). Then $[\text{Ker}(\overset{\circ}{\delta})]^\perp = \mathcal{PM}$.*

Proof. Let $s \in \mathbb{R}$ and let $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R})$. Choose two sequences $(f_k)_{k \in \mathbb{N}}$ in $C_0(\mathbb{R})$ and $(h_k)_{k \in \mathbb{N}}$ in $H^1(\mathbb{R})$ such that

$$\sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 < \infty \quad \text{and} \quad \Phi = \sum_{k=1}^{\infty} \dot{f}_k \otimes h_k.$$

Then by (3.14),

$$[\overset{\circ}{\delta}(\Phi)](s) = \sum_{k=1}^{\infty} (f_k \star h_k)(s) = \sum_{k=1}^{\infty} \langle \tau_{-s} h_k, \dot{f}_k \rangle = \langle \tau_{-s}, \Phi \rangle.$$

This shows that

$$\text{Span} \{ \tau_s : s \in \mathbb{R} \}_\perp = \text{Ker}(\overset{\circ}{\delta}).$$

The result follows at once. \square

By standard duality and (3.16), the dual space $\mathcal{A}_0(\mathbb{R})^*$ may be identified with $[\text{Ker}(\overset{\circ}{\delta})]^\perp$. Applying Lemma 3.6 and (3.18), we therefore obtain the following.

Theorem 3.7.

(1) For any $T \in \mathcal{PM}$, there exists a unique $\eta_T \in \mathcal{A}_0(\mathbb{R})^*$ such that

$$\langle \eta_T, f \star h \rangle = \int_{-\infty}^{\infty} (Th)(t) f(-t) dt$$

for any $f \in C_0(\mathbb{R})$ and any $h \in H^1(\mathbb{R})$.

(2) The mapping $T \mapsto \eta_T$ induces a w^* -homeomorphic and isometric identification

$$\mathcal{A}_0(\mathbb{R})^* \simeq \mathcal{PM}.$$

Remark 3.8.

(1) Recall \mathcal{R} from (2.9). It turns out that

$$\mathcal{PM} = \overline{\mathcal{R}}^{w^*}.$$

Indeed for any $s \in \mathbb{R}$, $\tau_s: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$ is equal to the convolution by the Dirac mass at s , which yields \subset . To show the converse inclusion, we observe that for any $\mu \in M(\mathbb{R})$ and any $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R})$, we have

$$(3.20) \quad \langle \mu, \overset{\circ}{\delta}(\Phi) \rangle = \langle R_\mu, \Phi \rangle.$$

Here we use the identification $M(\mathbb{R}) \simeq C_0(\mathbb{R})^*$ given by (1.4) on the left-hand side and we use (3.17) on the right-hand side. To prove this identity, let $f \in C_0(\mathbb{R})$ and $h \in H^1(\mathbb{R})$. Then

$$\begin{aligned} \langle \mu, \overset{\circ}{\delta}(f \otimes h) \rangle &= \langle \mu, f \star h \rangle \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} f(-s) h(s-t) ds d\mu(t) \\ &= \langle \mu \star h, f \rangle. \end{aligned}$$

This shows (3.20) when $\Phi = f \otimes h$. By linearity, this implies (3.20) for $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \otimes H^1(\mathbb{R})$. Then by density, we deduce (3.20) for all Φ .

It clearly follows from (3.20) that $\mathcal{R} \subset [\text{Ker}(\overset{\circ}{\delta})]^\perp$. By Lemma 3.6, this yields \supset .

(2) The Banach algebra $\mathcal{A}_0(\mathbb{R})$ can be naturally regarded as an analytic version of the Figa-Talamanca-Herz algebras $A_p(\mathbb{R})$, $1 < p < \infty$ (see e.g. [7, Chapter 3]). A remarkable property of these algebras is that the dual of $A_p(\mathbb{R})$ can be identified with the space $\mathcal{M}(L^p(\mathbb{R}))$ of bounded Fourier multipliers on $L^p(\mathbb{R})$. By analogy, one may expect that the dual of $\mathcal{A}_0(\mathbb{R})$ identifies with $\mathcal{M}(H^1(\mathbb{R}))$. However at the time of this writing we do not know if the inclusion (3.19) is an equality.

We now give a few consequences of the above duality results.

Proposition 3.9. *For any $b \in L^1(\mathbb{R}_+)$, the function $\widehat{b}(-\cdot): \mathbb{R} \rightarrow \mathbb{C}$ belongs to $\mathcal{A}_0(\mathbb{R})$ and*

$$\|\widehat{b}(-\cdot)\|_{\mathcal{A}_0} \leq \|b\|_1.$$

Moreover the mapping $L^1(\mathbb{R}_+) \rightarrow \mathcal{A}_0(\mathbb{R})$ taking b to $\widehat{b}(-\cdot)$ is a Banach algebra homomorphism with dense range.

Proof. We will use the space $C_{00}(\mathbb{R})$ defined by (1.3). Let $C_{00} \star H^1 \subset \mathcal{A}_0(\mathbb{R})$ be the linear span of the functions $f \star h$, for $f \in C_{00}(\mathbb{R})$ and $h \in H^1(\mathbb{R})$. By definition of $C_{00}(\mathbb{R})$, the Fourier transform maps $C_{00} \star H^1$ into $L^1(\mathbb{R}_+)$. Then we consider

$$\mathcal{C}_{0,1} = \{\widehat{F} : F \in C_{00} \star H^1\} \subset L^1(\mathbb{R}_+).$$

Let $b \in C^\infty(\mathbb{R}_+^*)$ with compact support. Let $c \in C^\infty(\mathbb{R}_+^*)$ with compact support such that $c \equiv 1$ on the support of b , so that $b = bc$. Then $\mathcal{F}^{-1}(b) \in C_{00}(\mathbb{R})$, $\mathcal{F}^{-1}(c) \in H^1(\mathbb{R})$ and the Fourier transform of $\mathcal{F}^{-1}(b) \star \mathcal{F}^{-1}(c)$ is equal to b . Thus $b \in \mathcal{C}_{0,1}$. Consequently, $\mathcal{C}_{0,1}$ is dense in $L^1(\mathbb{R}_+)$.

Let $b \in \mathcal{C}_{0,1}$ and let $F = \widehat{b}(-\cdot)$, so that

$$\widehat{F} = (2\pi)b.$$

Take finite families $(f_k)_k$ in $C_{00}(\mathbb{R})$ and $(h_k)_k$ in $H^1(\mathbb{R})$ such that $F = \sum_k f_k \star h_k$. Pick $\eta \in \mathcal{A}_0(\mathbb{R})^*$ such that $\|\eta\| = 1$ and $\|F\|_{\mathcal{A}_0} = \langle \eta, F \rangle$. By Theorem 3.7, there exists $T \in \mathcal{M}(H^1(\mathbb{R}))$ such that $\|T\|_{B(H^1)} = 1$ and for any k ,

$$\langle \eta, f_k \star h_k \rangle = \int_{-\infty}^{\infty} (Th_k)(u) f_k(-u) du.$$

By Theorem 2.2, the symbol m of the multiplier T satisfies $\|m\|_\infty \leq 1$. Moreover by Lemma 1.1, we have

$$\begin{aligned} \int_{-\infty}^{\infty} (Th_k)(u) f_k(-u) du &= \frac{1}{2\pi} \int_0^\infty \widehat{Th_k}(t) \widehat{f_k}(t) dt \\ &= \frac{1}{2\pi} \int_0^\infty m(t) \widehat{h_k}(t) \widehat{f_k}(t) dt, \end{aligned}$$

for all k . Summing over k , we deduce that

$$\langle \eta, F \rangle = \frac{1}{2\pi} \sum_k \int_0^\infty m(t) \widehat{h_k}(t) \widehat{f_k}(t) dt = \frac{1}{2\pi} \int_0^\infty m(t) \widehat{F}(t) dt,$$

and hence

$$\langle \eta, F \rangle = \int_0^\infty m(t) b(t) dt.$$

We deduce that

$$\|F\|_{\mathcal{A}_0} \leq \|b\|_1.$$

Since $\mathcal{C}_{0,1}$ is dense in $L^1(\mathbb{R}_+)$ and $\mathcal{A}_0(\mathbb{R})$ is complete, this estimate implies that $\widehat{b}(-\cdot): \mathbb{R} \rightarrow \mathbb{C}$ belongs to $\mathcal{A}_0(\mathbb{R})$ for any $b \in L^1(\mathbb{R}_+)$, with $\|\widehat{b}(-\cdot)\|_{\mathcal{A}_0} \leq \|b\|_1$.

It is plain that $b \mapsto \widehat{b}(-\cdot)$ is a Banach algebra homomorphism. Its range contains $C_{00} \star H^1$ and the latter is dense in $\mathcal{A}_0(\mathbb{R})$, because $C_{00}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$. \square

Remark 3.10. Let $\eta \in \mathcal{A}_0(\mathbb{R})^*$, let $T \in \mathcal{M}(H^1(\mathbb{R}))$ be the multiplier associated with η , according to Theorem 3.7, and let $m \in C_b(\mathbb{R}_+^*)$ be the symbol of T . Then it follows from the previous result and its proof that for any $b \in L^1(\mathbb{R}_+)$,

$$\langle \eta, \widehat{b}(-\cdot) \rangle = \int_0^\infty m(t)b(t) dt.$$

Remark 3.11. For any $\lambda \in P_-$, we let

$$b_\lambda(t) = ie^{-i\lambda t}, \quad t > 0.$$

Then $b_\lambda \in L^1(\mathbb{R}_+)$ and we have

$$\widehat{b}_\lambda(-u) = \frac{1}{\lambda - u}, \quad u \in \mathbb{R}.$$

Applying Proposition 3.9, we obtain that $(\lambda - \cdot)^{-1}$ belongs to $\mathcal{A}_0(\mathbb{R})$ for any $\lambda \in P_-$. Since $\mathcal{A}_0(\mathbb{R})$ is a Banach algebra, this implies that any rational function $F: \mathbb{R} \rightarrow \mathbb{C}$ with degree $\deg(F) \leq -1$ and poles in P_- belongs to $\mathcal{A}_0(\mathbb{R})$.

We can now strengthen Remark 3.3 as follows.

Proposition 3.12. *For any $F \in \mathcal{A}_0(\mathbb{R})$, we have*

$$\|F\|_{\mathcal{A}_0} = \|F\|_{\mathcal{A}}.$$

Proof. We will use Proposition 3.9 again. For any $N \in \mathbb{N}$, let $G_N: \mathbb{R} \rightarrow \mathbb{C}$ be defined by

$$G_N(u) = \frac{N}{N - iu}, \quad u \in \mathbb{R}.$$

Then $G_N = \widehat{Ne^{-N\cdot}}(-\cdot)$ belongs to $\mathcal{A}_0(\mathbb{R})$. We note that the sequence $(Ne^{-N\cdot})_{N \in \mathbb{N}}$ is a contractive approximate unit of $L^1(\mathbb{R}_+)$. This implies that $(G_N)_{N \in \mathbb{N}}$ is a contractive approximate unit of $\mathcal{A}_0(\mathbb{R})$.

Let $F \in \mathcal{A}_0(\mathbb{R})$. By Proposition 3.4, we have

$$(3.21) \quad \|FG_N\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}} \|G_N\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}}.$$

Moreover $FG_N \rightarrow F$ in $\mathcal{A}_0(\mathbb{R})$ hence $\|FG_N\|_{\mathcal{A}_0} \rightarrow \|F\|_{\mathcal{A}_0}$ when $N \rightarrow \infty$. We deduce that

$$\|F\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}}.$$

Combining with (3.7), we obtain $\|F\|_{\mathcal{A}_0} = \|F\|_{\mathcal{A}}$. □

Remark 3.13. According to [9, Chapter II, Theorem 3.8], $H^1(\mathbb{R}) \subset M(\mathbb{R})$ is the orthogonal space of the functions $u \mapsto \frac{1}{\lambda - u}$, for λ running in P_- . Hence we deduce from Remark 3.11 that $\mathcal{A}_0(\mathbb{R})^\perp$ contains $H^1(\mathbb{R})$. Combining with (3.12) we deduce that we actually have $\mathcal{A}_0(\mathbb{R})^\perp = H^1(\mathbb{R}) = (C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}))^\perp$. Consequently, $\mathcal{A}_0(\mathbb{R})$ is dense in $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$.

Using Proposition 3.9, or repeating the above argument, we also obtain that the space $\{\widehat{b}(-\cdot) : b \in L^1(\mathbb{R}_+)\}$ is dense in $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$.

4. FUNCTIONAL CALCULUS ON \mathcal{A}_0 AND \mathcal{A}

The goal of this section is to construct a bounded functional calculus for the negative generator of a bounded C_0 -semigroup on Hilbert space, defined on a suitable half-plane version of $\mathcal{A}(\mathbb{R})$.

This half-plane version $\mathcal{A}(\mathbb{C}_+)$ (and its companion $\mathcal{A}_0(\mathbb{C}_+)$) are defined as follows. Let $F \in H^\infty(\mathbb{R})$. We may consider its Poisson integral $\mathcal{P}[F]: P_+ \rightarrow \mathbb{C}$ and the latter is a bounded holomorphic function (see e.g. [9, Sect. I.3]). Then we define

$$\tilde{F}: \mathbb{C}_+ \longrightarrow \mathbb{C}$$

by setting

$$\tilde{F}(z) = \mathcal{P}[F](iz), \quad z \in \mathbb{C}_+.$$

We set

$$\mathcal{A}_0(\mathbb{C}_+) = \{\tilde{F} : F \in \mathcal{A}_0(\mathbb{R})\} \quad \text{and} \quad \mathcal{A}(\mathbb{C}_+) = \{\tilde{F} : F \in \mathcal{A}(\mathbb{R})\}.$$

We equip these spaces with the norms induced by $\mathcal{A}_0(\mathbb{R})$ and $\mathcal{A}(\mathbb{R})$, respectively. That is, we set $\|\tilde{F}\|_{\mathcal{A}_0} = \|F\|_{\mathcal{A}_0}$ (resp. $\|\tilde{F}\|_{\mathcal{A}} = \|F\|_{\mathcal{A}}$) for any $F \in \mathcal{A}_0(\mathbb{R})$ (resp. $F \in \mathcal{A}(\mathbb{R})$). By construction, we have

$$(4.1) \quad \mathcal{A}_0(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+) \quad \text{and} \quad \mathcal{A}(\mathbb{C}_+) \subset H^\infty(\mathbb{C}_+).$$

It clearly follows from Section 3 that $\mathcal{A}(\mathbb{C}_+)$ is a Banach algebra for pointwise multiplication, that the second inclusion in (4.1) is contractive, that $\mathcal{A}_0(\mathbb{C}_+)$ is an ideal of $\mathcal{A}(\mathbb{C}_+)$ and that the first inclusion in (4.1) is an isometry.

Let $b \in L^1(\mathbb{R}_+)$ and consider $F = \widehat{b}(-\cdot): \mathbb{R} \rightarrow \mathbb{C}$. Then \tilde{F} coincides with the Laplace transform $L_b: \mathbb{C}_+ \rightarrow \mathbb{C}$ of b defined by

$$(4.2) \quad L_b(z) = \int_0^\infty e^{-tz} b(t) dt, \quad z \in \mathbb{C}_+.$$

As a consequence of Proposition 3.9, we therefore obtain the following.

Lemma 4.1. *For any $b \in L^1(\mathbb{R}_+)$, L_b belongs to $\mathcal{A}_0(\mathbb{C}_+)$ and $\|L_b\|_{\mathcal{A}_0} \leq \|b\|_1$. Moreover the mapping $b \mapsto L_b$ is a Banach algebra homomorphism from $L^1(\mathbb{R}_+)$ into $\mathcal{A}_0(\mathbb{C}_+)$, and the space $\{L_b : b \in L^1(\mathbb{R}_+)\}$ is dense in $\mathcal{A}_0(\mathbb{C}_+)$.*

4.1. Half-plane holomorphic functional calculus. We need some background on the half-plane holomorphic functional calculus introduced by Batty-Haase-Mubeen in [6], to which we refer for details.

Let X be an arbitrary Banach space. Let $\omega \in \mathbb{R}$ and let A be a closed and densely defined operator on X such that the spectrum of A is included in the closed half-plane $\overline{\mathcal{H}_\omega}$ and

$$(4.3) \quad \forall \alpha < \omega, \quad \sup\{\|R(z, A)\| : \operatorname{Re}(z) \leq \alpha\} < \infty.$$

Consider the auxiliary algebra

$$\mathcal{E}(\mathcal{H}_\alpha) := \{\varphi \in H^\infty(\mathcal{H}_\alpha) : \exists s > 0, |\varphi(z)| = O(|z|^{-(1+s)}) \text{ as } |z| \rightarrow \infty\},$$

for any $\alpha < \omega$. For any $\varphi \in \mathcal{E}(\mathcal{H}_\alpha)$ and for any $\beta \in (\alpha, \omega)$, the assumption (4.3) insures that the integral

$$(4.4) \quad \varphi(A) := \frac{-1}{2\pi} \int_{-\infty}^{\infty} \varphi(\beta + is) R(\beta + is, A) ds$$

is absolutely convergent in $B(X)$. Further its value is independent of the choice of β (this is due to Cauchy's Theorem for vector-valued holomorphic functions) and the mapping $\varphi \mapsto \varphi(A)$ in an algebra homomorphism from $\mathcal{E}(\mathcal{H}_\alpha)$ into $B(X)$. This definition is compatible with the usual rational functional calculus; indeed for any $\mu \in \mathbb{C}$ with $\operatorname{Re}(\mu) < \alpha$ and any integer $m \geq 2$, the function

$$e_{\mu,m}: z \mapsto (\mu - z)^{-m},$$

which belongs to $\mathcal{E}(\mathcal{H}_\alpha)$, satisfies $e_{\mu,m}(A) = R(\mu, A)^m$.

Let $\varphi \in H^\infty(\mathcal{H}_\alpha)$. We can define a closed, densely defined, operator $\varphi(A)$ by regularisation, as follows (see [6] and [12] for more on such constructions). Take $\mu \in \mathbb{C}$ with $\operatorname{Re}(\mu) < \alpha$, and set $e = e_{\mu,2}$. Then $e\varphi \in \mathcal{E}(\mathcal{H}_\alpha)$ and $e(A) = R(\mu, A)^2$ is injective. Then $\varphi(A)$ is defined by

$$(4.5) \quad \varphi(A) = e(A)^{-1}(e\varphi)(A),$$

with domain $\operatorname{Dom}(\varphi(A))$ equal to the space of all $x \in X$ such that $[(e\varphi)(A)](x)$ belongs to the range of $e(A)$ ($= \operatorname{Dom}(A^2)$). It turns out that this definition does not depend on the choice of μ .

The half-plane holomorphic functional calculus $\varphi \mapsto \varphi(A)$ satisfies the following ‘‘Convergence Lemma’’, provided by [6, Theorem 3.1].

Lemma 4.2. *Assume that A satisfies (4.3) for some $\omega \in \mathbb{R}$ and let $\alpha < \omega$. Let $(\varphi_i)_i$ be a net of $H^\infty(\mathcal{H}_\alpha)$ such that $\varphi_i(A) \in B(X)$ for any i and let $\varphi \in H^\infty(\mathcal{H}_\alpha)$ such that $\varphi_i \rightarrow \varphi$ pointwise on R_α , when $i \rightarrow \infty$. If*

$$\sup_i \|\varphi_i\|_{H^\infty(R_\alpha)} < \infty \quad \text{and} \quad \sup_i \|\varphi_i(A)\|_{B(X)} < \infty,$$

then $\varphi(A) \in B(X)$ and for any $x \in X$, $[\varphi_i(A)](x) \rightarrow [\varphi(A)](x)$ when $i \rightarrow \infty$.

Let $(T_t)_{t \geq 0}$ be a bounded C_0 -semigroup on X and let $-A$ denote its infinitesimal generator. For any $b \in L^1(\mathbb{R}_+)$, we set

$$\Gamma(A, b) := \int_0^\infty b(t) T_t dt,$$

where the latter integral is defined in the strong sense. The mapping $b \mapsto \Gamma(A, b)$ is the so-called Hille-Phillips functional calculus. We refer to [12, Section 3.3] for information and background. We recall that this functional calculus is a Banach algebra homomorphism

$$L^1(\mathbb{R}_+) \longrightarrow B(X).$$

We now provide a compatibility result between the half-plane holomorphic functional calculus and the Hille-Phillips functional calculus. Note that any A as above satisfies (4.3) for $\omega = 0$. Thus for any $\varepsilon > 0$, the operator $A + \varepsilon$ satisfies (4.3) for $\omega = \varepsilon$. For any $b \in L^1(\mathbb{R}_+)$, this allows to define $L_b(A + \varepsilon)$, where L_b is the Laplace transform defined by (4.2).

Lemma 4.3. *Let $b \in L^1(\mathbb{R}_+)$ and let $-A$ be the generator of a bounded C_0 -semigroup on X . Then for any $\varepsilon > 0$, we have*

$$L_b(A + \varepsilon) = \Gamma(A + \varepsilon, b).$$

Proof. Let $\varepsilon > 0$, let $\beta \in (0, \varepsilon)$, and let $b \in L^1(\mathbb{R}_+)$.

First suppose that $L_b \in \mathcal{E}(\mathcal{H}_0)$. Then by the Laplace formula,

$$L_b(A + \varepsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} L_b(\beta + is) \left(\int_0^{\infty} e^{ist} e^{\beta t} e^{-\varepsilon t} T_t dt \right) ds.$$

Since $L_b \in \mathcal{E}(\mathcal{H}_0)$, the function $s \mapsto L_b(\beta + is)$ belongs to $L^1(\mathbb{R})$. Hence by Fubini's Theorem,

$$L_b(A + \varepsilon) = \frac{1}{2\pi} \int_0^{\infty} e^{\beta t} \left(\int_{-\infty}^{\infty} L_b(\beta + is) e^{ist} ds \right) e^{-\varepsilon t} T_t dt.$$

By the Fourier inversion Theorem, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} L_b(\delta + is) e^{ist} ds = e^{-\beta t} b(t)$$

for each $t \in \mathbb{R}$. We deduce that

$$L_b(A + \varepsilon) = \int_0^{\infty} b(t) e^{-\varepsilon t} T_t dt = \Gamma(A + \varepsilon, b).$$

For the general case, let us consider $e \in \mathcal{E}(\mathcal{H}_0)$ defined by $e(z) = (1 + z)^{-2}$. We note that e is the Laplace transform of the function $c \in L^1(\mathbb{R}_+)$ defined by $c(t) = te^{-t}$. The product eL_b , which belongs to $\mathcal{E}(\mathcal{H}_0)$, is therefore the Laplace transform of $b \star c$. Hence by the first part of this proof,

$$(eL_b)(A + \varepsilon) = \Gamma(A + \varepsilon, b \star c).$$

The multiplicativity of the Hille-Phillips functional calculus yields $\Gamma(A + \varepsilon, b \star c) = \Gamma(A + \varepsilon, c)\Gamma(A + \varepsilon, b)$. Further $e(A + \varepsilon) = \Gamma(A + \varepsilon, c)$ by the Laplace formula. Thus we have

$$e(A + \varepsilon)\Gamma(A + \varepsilon, b) = (eL_b)(A + \varepsilon).$$

Applying (4.5), we obtain that $L_b(A + \varepsilon) = \Gamma(A + \varepsilon, b)$ as wanted. \square

4.2. Functional calculus on $\mathcal{A}_0(\mathbb{C}_+)$. Throughout this subsection, we fix a Hilbert space H , we let $(T_t)_{t \geq 0}$ be a bounded C_0 -semigroup on H and we let $-A$ denote its infinitesimal generator. We set

$$C := \sup\{\|T_t\| : t \geq 0\}.$$

For any $f \in C_{00}(\mathbb{R})$ and any $h \in H^1(\mathbb{R})$, the function

$$b = (2\pi)^{-1} \widehat{f\widehat{h}}$$

belongs to $L^1(\mathbb{R}_+)$ and we have $\widehat{b(-\cdot)} = f \star h$. Consequently,

$$(f \star h)^\sim = L_b.$$

Further we have the following key estimate, which is inspired by [29, Proposition 4.16].

Lemma 4.4. *For any $f \in C_{00}(\mathbb{R})$ and any $h \in H^1(\mathbb{R})$,*

$$(4.6) \quad \|\Gamma(A, (2\pi)^{-1} \widehat{f\widehat{h}})\| \leq C^2 \|f\|_\infty \|h\|_1.$$

Proof. We fix $f \in C_{00}(\mathbb{R})$. Let $w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ and let $h = wv$. By definition,

$$\Gamma(A, (2\pi)^{-1}f\widehat{h}) = \frac{1}{2\pi} \int_0^\infty \widehat{f}(t)\widehat{wv}(t)T_t dt.$$

By assumption, \widehat{w} and \widehat{v} belong to $L^1(\mathbb{R}_+)$ hence $\widehat{wv} = (2\pi)^{-1}\widehat{w} \star \widehat{v}$ belongs to $L^1(\mathbb{R}_+)$. Further f belongs to $L^1(\mathbb{R})$. We can therefore apply Fubini's Theorem and we obtain that

$$\frac{1}{2\pi} \int_0^\infty \widehat{f}(t)\widehat{wv}(t)T_t dt = \frac{1}{2\pi} \int_{-\infty}^\infty f(s) \left(\int_0^\infty \widehat{wv}(t)e^{-its}T_t dt \right) ds.$$

Note that for any $s \in \mathbb{R}$,

$$\int_0^\infty \widehat{wv}(t)e^{-its}T_t dt = \Gamma(A + is, \widehat{wv}).$$

According to the multiplicativity of the Hille-Phillips functional calculus, this implies that

$$\int_0^\infty \widehat{wv}(t)e^{-its}T_t dt = \frac{1}{2\pi} \left(\int_0^\infty \widehat{w}(r)e^{-irs}T_r dr \right) \left(\int_0^\infty \widehat{v}(t)e^{-its}T_t dt \right).$$

Let $W, V: \mathbb{R} \rightarrow B(H)$ be defined by

$$W(s) = \int_0^\infty \widehat{w}(r)e^{-irs}T_r dr \quad \text{and} \quad V(s) = \int_0^\infty \widehat{v}(t)e^{-its}T_t dt, \quad s \in \mathbb{R}.$$

It follows from above that for any $x, x^* \in H$, we have

$$(4.7) \quad \langle \Gamma(A, (2\pi)^{-1}f\widehat{h})x, x^* \rangle = \frac{1}{4\pi^2} \int_{-\infty}^\infty f(s) \langle W(s)x, V(s)^*x^* \rangle ds.$$

Applying the Cauchy-Schwarz inequality, we deduce

$$|\langle \Gamma(A, (2\pi)^{-1}f\widehat{h})x, x^* \rangle| \leq \frac{1}{4\pi^2} \|f\|_\infty \left(\int_{-\infty}^\infty \|W(s)x\|^2 ds \right)^{\frac{1}{2}} \left(\int_{-\infty}^\infty \|V(s)^*x^*\|^2 ds \right)^{\frac{1}{2}}.$$

According to the Fourier-Plancherel equality on $L^2(\mathbb{R}; H)$, we have

$$\int_{-\infty}^\infty \|W(s)x\|^2 ds = 2\pi \int_{-\infty}^\infty |\widehat{w}(r)|^2 \|T_r x\|^2 dr.$$

This implies

$$\begin{aligned} \int_{-\infty}^\infty \|W(s)x\|^2 ds &\leq 2\pi C^2 \int_{-\infty}^\infty |\widehat{w}(r)|^2 \|x\|^2 dr \\ &= 4\pi^2 C^2 \|w\|_2^2 \|x\|^2. \end{aligned}$$

Similarly, we have

$$\int_{-\infty}^\infty \|V(s)^*x^*\|^2 ds \leq 4\pi^2 C^2 \|v\|_2^2 \|x^*\|^2.$$

Hence,

$$|\langle \Gamma(A, (2\pi)^{-1}f\widehat{h})x, x^* \rangle| \leq C^2 \|f\|_\infty \|w\|_2 \|v\|_2 \|x\| \|x^*\|.$$

Since this is true for any x, x^* , we have proved that

$$\|\Gamma(A, (2\pi)^{-1}f\widehat{h})\| \leq C^2 \|f\|_\infty \|w\|_2 \|v\|_2.$$

Now let h be an arbitrary element of $H^1(\mathbb{R})$. As is well-known (see e.g. [9, Exercise 1, p.84]), there exist $w, v \in H^2(\mathbb{R})$ such that $h = wv$ and $\|w\|_2^2 = \|v\|_2^2 = \|h\|_1$.

Since $\mathcal{F}(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$, it follows from (2.1) that $\mathcal{F}(H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})) = L^2(\mathbb{R}_+) \cap \mathcal{S}(\mathbb{R})$. Since $L^2(\mathbb{R}_+) \cap \mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R}_+)$, we readily deduce that $H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ is dense in $H^2(\mathbb{R})$.

Thus, there exist sequences $(w_k)_{k \in \mathbb{N}}, (v_k)_{k \in \mathbb{N}}$ of $H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ such that $w_k \rightarrow w$ and $v_k \rightarrow v$ in $H^2(\mathbb{R})$, when $k \rightarrow \infty$. This implies that

$$\|w_k\|_2 \|v_k\|_2 \longrightarrow \|h\|_1,$$

when $k \rightarrow \infty$. Moreover, $w_k v_k \rightarrow wv = h$ in $H^1(\mathbb{R})$, when $k \rightarrow \infty$. Consequently,

$$\|\Gamma(A, (2\pi)^{-1} \widehat{f w_k v_k}) - \Gamma(A, (2\pi)^{-1} \widehat{f h})\| \longrightarrow 0$$

when $k \rightarrow \infty$. Indeed,

$$\begin{aligned} \|\Gamma(A, (2\pi)^{-1} \widehat{f w_k v_k}) - \Gamma(A, (2\pi)^{-1} \widehat{f h})\| &= \left\| \int_0^\infty \widehat{f}(t) (\widehat{w_k v_k} - \widehat{wv})(t) T_t dt \right\| \\ &\leq C \|\widehat{f}\|_1 \|(\widehat{w_k v_k} - \widehat{wv})\|_\infty \\ &\leq C \|\widehat{f}\|_1 \|w_k v_k - wv\|_1. \end{aligned}$$

For all $k \in \mathbb{N}$, we have

$$\|\Gamma(A, (2\pi)^{-1} \widehat{f w_k v_k})\| \leq C^2 \|f\|_\infty \|u_k\|_2 \|v_k\|_2,$$

by the first part of the proof. Passing to the limit, we obtain (4.6). \square

We now arrive at the main result of this subsection.

Theorem 4.5. *There exists a unique bounded homomorphism $\rho_{0,A}: \mathcal{A}_0(\mathbb{C}_+) \rightarrow B(H)$ such that*

$$(4.8) \quad \rho_{0,A}(L_b) = \int_0^\infty b(t) T_t dt$$

for any $b \in L^1(\mathbb{R}_+)$. Moreover $\|\rho_{0,A}\| \leq C^2$.

Proof. By Lemma 4.4 and the density of $C_{00}(\mathbb{R})$ in $C_0(\mathbb{R})$, there exists a unique bounded bilinear map

$$u_A: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \longrightarrow B(H)$$

such that $u_A(f, h) = \Gamma(A, (2\pi)^{-1} \widehat{f h})$ for each $(f, h) \in C_{00}(\mathbb{R}) \times H^1(\mathbb{R})$. Moreover $\|u_A\| \leq C^2$.

For each $\varepsilon > 0$, $(A + \varepsilon)$ is the negative generator of the semigroup $(e^{-\varepsilon t} T_t)_{t \geq 0}$. Therefore, in the same manner as above, one can define $u_{A+\varepsilon}: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow B(H)$ and we have the uniform estimate

$$(4.9) \quad \forall \varepsilon > 0, \quad \|u_{A+\varepsilon}: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \longrightarrow B(H)\| \leq C^2.$$

We claim that for each $\varepsilon > 0$, we have

$$(4.10) \quad u_{A+\varepsilon}(f, h) = (f \star h)^\sim(A + \varepsilon), \quad f \in C_0(\mathbb{R}), h \in H^1(\mathbb{R}).$$

(We recall that the operator on the right-hand side is defined by the half-plane holomorphic functional calculus. In particular the above formula shows that $(f \star h)^\sim(A + \varepsilon)$ is bounded.)

Recall that if $f \in C_{00}(\mathbb{R})$, then $b = (2\pi)^{-1}\widehat{f\widehat{h}} \in L^1(\mathbb{R}_+)$ and $(f \star h)^\sim = L_b$. Hence (4.10) is given by Lemma 4.3 in this case. In the general case, let $(f_n)_{n \in \mathbb{N}}$ be a sequence of $C_{00}(\mathbb{R})$ such that $f_n \rightarrow f$ in $C_0(\mathbb{R})$, when $n \rightarrow \infty$. Then $u_{A+\varepsilon}(f_n, h) \rightarrow u_{A+\varepsilon}(f, h)$, hence $(f_n \star h)^\sim(A + \varepsilon) \rightarrow u_{A+\varepsilon}(f, h)$. Moreover $(f_n \star h)^\sim \rightarrow (f \star h)^\sim$ in $H^\infty(\mathbb{C}_+)$. Therefore by the Convergence Lemma 4.2, $(f \star h)^\sim(A + \varepsilon)$ is bounded and (4.10) holds true.

Next we show that in $B(H)$, we have

$$(4.11) \quad u_{A+\varepsilon}(f, h) \xrightarrow{\varepsilon \rightarrow 0} u_A(f, h), \quad f \in C_0(\mathbb{R}), h \in H^1(\mathbb{R}).$$

In the case when $f \in C_{00}(\mathbb{R})$,

$$u_{A+\varepsilon}(f, h) = \frac{1}{2\pi} \int_0^\infty \widehat{f}(t)\widehat{h}(t)e^{-\varepsilon t}T_t dt$$

for any $\varepsilon \geq 0$. Hence

$$\|u_A(f, h) - u_{A+\varepsilon}(f, h)\| \leq \frac{C}{2\pi} \int_0^\infty |\widehat{f}(t)\widehat{h}(t)|(1 - e^{-\varepsilon t}) dt.$$

This integral goes to 0 when $\varepsilon \rightarrow 0$, by Lebesgue's dominated convergence Theorem. This yields the result in this case. The general case follows from the density of $C_{00}(\mathbb{R})$ in $C_0(\mathbb{R})$ and the uniform estimate (4.9).

We now construct $\rho_{0,A}$. Let $F \in \mathcal{A}_0(\mathbb{R})$ and consider two sequences $(f_k)_{k \in \mathbb{N}}$ of $C_0(\mathbb{R})$ and $(h_k)_{k \in \mathbb{N}}$ of $H^1(\mathbb{R})$ satisfying (3.1) and (3.2). We let

$$F_N = \sum_{k=1}^N f_k \star h_k, \quad N \geq 1.$$

For any fixed $\varepsilon > 0$, it follows from (4.10) that for any $N \geq 1$,

$$\widetilde{F}_N(A + \varepsilon) = \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k).$$

We both have that $\widetilde{F}_N \rightarrow \widetilde{F}$ in $H^\infty(\mathbb{C}_+)$ and that $\sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) \rightarrow \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k)$ in $B(H)$. Appealing again to Lemma 4.2, we deduce that $\widetilde{F}(A + \varepsilon) \in B(H)$ and that

$$(4.12) \quad \widetilde{F}(A + \varepsilon) = \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k).$$

We observe that

$$(4.13) \quad \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) \xrightarrow{\varepsilon \rightarrow 0} \sum_{k=1}^\infty u_A(f_k, h_k)$$

in $B(H)$. To check this, let $a > 0$ and choose $N \geq 1$ such that $\sum_{k=N+1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 \leq a$. We have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^{\infty} u_A(f_k, h_k) \right\| &\leq \left\| \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^N u_A(f_k, h_k) \right\| \\ &\quad + \sum_{k=N+1}^{\infty} \|u_{A+\varepsilon}(f_k, h_k)\| + \sum_{k=N+1}^{\infty} \|u_A(f_k, h_k)\|. \end{aligned}$$

Using the uniform estimate (4.9), this implies that

$$\left\| \sum_{k=1}^{\infty} u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^{\infty} u_A(f_k, h_k) \right\| \leq \left\| \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^N u_A(f_k, h_k) \right\| + 2Ca.$$

Applying (4.11), we deduce that

$$\left\| \sum_{k=1}^{\infty} u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^{\infty} u_A(f_k, h_k) \right\| \leq 3Ca$$

for $\varepsilon > 0$ small enough, which shows the result.

Combining (4.12) and (4.13) we obtain that $\tilde{F}(A + \varepsilon)$ has a limit in $B(H)$, when $\varepsilon \rightarrow 0$. We set

$$\rho_{0,A}(\tilde{F}) := \lim_{\varepsilon \rightarrow 0} \tilde{F}(A + \varepsilon).$$

It is plain that $\rho_{0,A} : \mathcal{A}_0(\mathbb{C}_+) \rightarrow B(H)$ is a linear map. It follows from the construction that

$$\|\rho_{0,A}(\tilde{F})\|_{\mathcal{A}_0} \leq C^2 \|\tilde{F}\|_{\mathcal{A}_0}$$

for any $F \in \mathcal{A}_0(\mathbb{R})$, hence $\rho_{0,A}$ is bounded with $\|\rho_{0,A}\| \leq C^2$.

Let $b \in L^1(\mathbb{R}_+)$. By the compatibility Lemma 4.3, we have

$$L_b(A + \varepsilon) = \int_0^{\infty} \hat{b}(t) e^{-\varepsilon t} T_t dt$$

for any $\varepsilon > 0$. Passing to the limit and using Lebesgue's dominated convergence Theorem, we obtain (4.8).

It follows from the density of $\{L_b : b \in L^1(\mathbb{R}_+)\}$ in $\mathcal{A}_0(\mathbb{C}_+)$, given by Lemma 4.1, that $\rho_{0,A}$ is unique. Moreover the multiplicativity of the Hille-Phillips functional calculus insures that $\rho_{0,A}$ is a Banach algebra homomorphism. \square

Remark 4.6. Let $F \in \mathcal{A}_0(\mathbb{R})$ and let $(f_k)_{k \in \mathbb{N}}$ and $(h_k)_{k \in \mathbb{N}}$ be sequences of $C_0(\mathbb{R})$ and $H^1(\mathbb{R})$, respectively, satisfying (3.1) and (3.2). It follows from the proof of Theorem 4.5 that

$$(4.14) \quad \rho_{0,A}(\tilde{F}) = \sum_{k=1}^{\infty} u_A(f_k, h_k).$$

This equality shows that the right-hand side of (4.14) does not depend on the choice of $(f_k)_{k \in \mathbb{N}}$ and $(h_k)_{k \in \mathbb{N}}$. The reason why we did not take (4.14) as a definition of $\rho_{0,A}$ is precisely that we did not know a priori that $\sum_{k=1}^{\infty} u_A(f_k, h_k)$ was independent of the representation of F .

4.3. **Functional calculus on $\mathcal{A}(\mathbb{C}_+)$.** We keep the notation from the previous subsection. We can extend Theorem 4.5 as follows.

Corollary 4.7. *There exists a unique bounded homomorphism $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$ extending $\rho_{0,A}$. Moreover $\|\rho_A\| \leq C^2$.*

Proof. We follow an idea from [4], using regularization. Consider the sequence $(G_N)_{N \in \mathbb{N}}$ defined in the proof of Proposition 3.12. Then

$$\widetilde{G}_N(z) = \frac{N}{N+z}, \quad z \in \mathbb{C}_+, N \geq 1.$$

For any $\varphi \in \mathcal{A}(\mathbb{C}_+)$, we let S_φ be the operator defined by

$$S_\varphi = (1+A)\rho_{0,A}(\varphi\widetilde{G}_1),$$

with domain $\text{Dom}(S_\varphi) = \{x \in H : [\rho_{0,A}(\varphi\widetilde{G}_1)](x) \in \text{Dom}(A)\}$. In this definition, we use the fact that $\varphi\widetilde{G}_1$ belongs to $\mathcal{A}_0(\mathbb{C}_+)$, which follows from Proposition 3.4. It is clear that S_φ is closed. Further $\text{Dom}(A) \subset \text{Dom}(S_\varphi)$, hence S_φ is densely defined. More precisely, if $x \in \text{Dom}(A)$, then $x = \rho_{A,0}(\widetilde{G}_1)(1+A)(x)$ hence

$$[\rho_{0,A}(\varphi\widetilde{G}_1)](x) = \rho_{0,A}(\varphi\widetilde{G}_1^2)(1+A)(x) = (1+A)^{-1}\rho_{0,A}(\varphi\widetilde{G}_1)(1+A)(x)$$

belongs to $\text{Dom}(A)$ and we have

$$(4.15) \quad S_\varphi(x) = \rho_{0,A}(\varphi\widetilde{G}_1)(1+A)(x).$$

Since $\rho_{0,A}$ is multiplicative, we have $\rho_{0,A}(\varphi\widetilde{G}_N\widetilde{G}_1) = \rho_{0,A}(\varphi\widetilde{G}_N)(1+A)^{-1}$ for any $N \geq 1$. Moreover as noticed in the proof of Proposition 3.12, $(G_N)_{N \in \mathbb{N}}$ is an approximate unit of $\mathcal{A}_0(\mathbb{R})$, hence $\varphi\widetilde{G}_N\widetilde{G}_1 \rightarrow \varphi\widetilde{G}_1$ in $\mathcal{A}_0(\mathbb{C}_+)$, when $N \rightarrow \infty$. We deduce, using (4.15), that for any $x \in \text{Dom}(A)$,

$$S_\varphi(x) = \lim_N \rho_{0,A}(\varphi\widetilde{G}_N\widetilde{G}_1)(1+A)(x) = \lim_N \rho_{0,A}(\varphi\widetilde{G}_N)(x).$$

For any $N \geq 1$,

$$\|\rho_{0,A}(\varphi\widetilde{G}_N)\| \leq C^2\|\varphi\widetilde{G}_N\|_{\mathcal{A}_0} \leq C^2\|\varphi\|_{\mathcal{A}},$$

by (3.8). Consequently, $\|S_\varphi(x)\| \leq C^2\|\varphi\|_{\mathcal{A}}\|x\|$ for any $x \in \text{Dom}(A)$. This shows that $\text{Dom}(S_\varphi) = H$ and $S_\varphi \in B(H)$.

We now define $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$ by $\rho_A(\varphi) = S_\varphi$. It is clear from above that ρ_A is linear and bounded, with $\|\rho_A\| \leq C^2$. It extends $\rho_{0,A}$ because if $F \in \mathcal{A}_0(\mathbb{C}_+)$, then we have $\rho_{0,A}(\varphi\widetilde{G}_1) = \rho_{0,A}(\widetilde{G}_1)\rho_{0,A}(\varphi) = (1+A)^{-1}\rho_{0,A}(\varphi)$, hence $S_\varphi = \rho_{0,A}(\varphi)$. Using similar arguments, one easily obtains that ρ_A is multiplicative, as well as the uniqueness property. \square

4.4. Operators with a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus. The goal of this subsection is to explain the connections between our main results (Theorem 4.5, Corollary 4.7) and H^∞ -functional calculus.

We will assume that the reader is familiar with sectorial operators and their H^∞ -functional calculus, for which we refer to [12] or [17, Chapter 10]. Using usual notation, for any $\theta \in (0, \pi)$ we let $\Sigma_\theta = \{z \in \mathbb{C}^* : |\text{Arg}(z)| < \theta\}$ and

$$H_0^\infty(\Sigma_\theta) = \{\varphi \in H^\infty(\Sigma_\theta) : \exists s > 0, |\varphi(z)| \lesssim \min\{|z|^s, |z|^{-s}\} \text{ on } \Sigma_\theta\}.$$

Let $(T_t)_{t \geq 0}$ be a bounded C_0 -semigroup on some Banach space X , with generator $-A$. Recall that A is a sectorial operator of type $\frac{\pi}{2}$.

The following lemma is probably known to specialists, we include a proof for the sake of completeness. In part (i), the operator $\varphi(A)$ is defined by (4.4) whereas in part (ii), the operator $\varphi(A)$ is defined by [12, (2.5)]. It is worth noting that if $\varphi \in \mathcal{E}(\mathcal{H}_\alpha) \cap H_0^\infty(\Sigma_\theta)$, then these two definitions coincide.

Lemma 4.8. *The following assertions are equivalent.*

(i) *There exists a constant $C > 0$ such that for any $\alpha < 0$ and for any $\varphi \in \mathcal{E}(\mathcal{H}_\alpha)$,*

$$(4.16) \quad \|\varphi(A)\| \leq C \|\varphi\|_{H^\infty(\mathbb{C}_+)}.$$

(ii) *There exists a constant $C > 0$ such that for any $\theta \in (\frac{\pi}{2}, \pi)$ and for any $\varphi \in H_0^\infty(\Sigma_\theta)$,*

$$\|\varphi(A)\| \leq C \|\varphi\|_{H^\infty(\mathbb{C}_+)}.$$

(iii) *There exists a constant $C > 0$ such that for any $b \in L^1(\mathbb{R}_+)$,*

$$(4.17) \quad \left\| \int_0^\infty b(t) T_t dt \right\| \leq C \|\widehat{b}\|_\infty.$$

Proof. Assume (i). By the approximation argument at the beginning of [6, Section 5], (4.16) holds as well for any $\varphi \in H^\infty(\mathcal{H}_\alpha)$. Let $b \in L^1(\mathbb{R}_+)$. The function $L_b(\cdot + \varepsilon)$ belongs to $H^\infty(\mathcal{H}_{-\varepsilon})$ for any $\varepsilon > 0$, hence we have

$$\|L_b(A + \varepsilon)\| \leq C \|L_b(\cdot + \varepsilon)\|_{H^\infty(\mathbb{C}_+)} \leq C \|L_b\|_{H^\infty(\mathbb{C}_+)} = C \|\widehat{b}\|_\infty.$$

Applying Lemma 4.3 and letting $\varepsilon \rightarrow 0$, we obtain (4.17), which proves (iii).

The fact that (iii) implies (ii) follows from [12, Lemma 3.3.1 & Proposition 3.3.2], see also [21, Lemma 2.12].

Assume (ii) and let us prove (i). For any $\varepsilon \in (0, 1)$, consider the rational function q_ε defined by

$$q_\varepsilon(z) = \frac{\varepsilon + z}{1 + \varepsilon z}, \quad z \neq \frac{-1}{\varepsilon}.$$

We may and do assume that $\alpha \in (-1, 0)$ when proving (i). Fix some $\varepsilon \in (0, 1)$. It is easy to check (left to the reader) that q_ε maps \mathcal{H}_α into itself. Moreover there exists $\theta \in (\frac{\pi}{2}, \pi)$ such that q_ε maps Σ_θ into \mathbb{C}_+ .

Let $\varphi \in \mathcal{E}(\mathcal{H}_\alpha)$, then

$$\varphi_\varepsilon := \varphi \circ q_\varepsilon : \mathcal{H}_\alpha \cup \Sigma_\theta \longrightarrow \mathbb{C}$$

is a well-defined bounded holomorphic function. Moreover we have

$$(4.18) \quad \|\varphi_\varepsilon\|_{H^\infty(\mathcal{H}_\alpha)} \leq \|\varphi\|_{H^\infty(\mathcal{H}_\alpha)}.$$

By [12, Lemma 2.2.3], φ_ε belongs to $H_0^\infty(\Sigma_\theta) \oplus \text{Span}\{1, (1+\cdot)^{-1}\}$. Further the definition of $\varphi_\varepsilon(A)$ provided by the functional calculus of sectorial operators coincides with the definition of $\varphi_\varepsilon(A)$ provided by the half-plane functional calculus. Hence for some constant $C' > 0$ not depending on ε , we have

$$\|\varphi_\varepsilon(A)\| \leq C' \|\varphi_\varepsilon\|_{H^\infty(\mathbb{C}_+)} \leq C' \|\varphi\|_{H^\infty(\mathbb{C}_+)},$$

by (ii). Since $\varphi_\varepsilon \rightarrow \varphi$ pointwise on \mathcal{H}_α , it now follows from (4.18) and the Convergence Lemma 4.2 that $\|\varphi(A)\| \leq C' \|\varphi\|_{H^\infty(\mathbb{C}_+)}$, which proves (i). \square

We say that A admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus if one of (equivalently, all of) the properties of Lemma 4.8 hold true. If A is sectorial of type $< \frac{\pi}{2}$, the latter is equivalent to A having a bounded H^∞ -functional calculus of angle $\frac{\pi}{2}$ in the usual sense. The main feature of the “bounded $H^\infty(\mathbb{C}_+)$ -functional calculus” property considered here is that it may apply to the case when the sectorial type of A is not $< \frac{\pi}{2}$.

We now come back to the specific case when $X = H$ is a Hilbert space. Here are a few known facts in this setting:

- (f1) If $(T_t)_{t \geq 0}$ is a contractive semigroup (that is, $\|T_t\| \leq 1$ for all $t \geq 0$), then A admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus. See [12, Section 7.1.3] for a proof and more on this theme.
- (f2) We say that $(T_t)_{t \geq 0}$ is similar to a contractive semigroup if there exists an invertible operator $S \in B(H)$ such that $(ST_tS^{-1})_{t \geq 0}$ is a contractive semigroup. A straightforward application of the previous result is that in this case, A admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus.
- (f3) If A is sectorial of type $< \frac{\pi}{2}$, then A admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus (if and) only if $(T_t)_{t \geq 0}$ is similar to a contractive semigroup. This goes back to [20, Section 4].
- (f4) There exist sectorial operators of type $< \frac{\pi}{2}$ which do not admit a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus, by [25, 3] (see also [12, Section 7.3.4]).
- (f5) There exists a bounded C_0 -semigroup $(T_t)_{t \geq 0}$ such that A admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus but $(T_t)_{t \geq 0}$ is not similar to a contractive semigroup. This follows from [20, Proposition 4.8] and its proof.

We now establish analogues of Theorem 4.5 and Corollary 4.7 in the case when A admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus. Just as we did at the beginning of Section 4, we set

$$\mathcal{C}_0(\mathbb{C}_+) = \{\tilde{F} : F \in C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})\} \quad \text{and} \quad \mathcal{C}(\mathbb{C}_+) = \{\tilde{F} : F \in C_b(\mathbb{R}) \cap H^\infty(\mathbb{R})\}.$$

Since $\{\widehat{b}(\cdot) : b \in L^1(\mathbb{R}_+)\}$ is dense in $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$, by Remark 3.13, the following is straightforward.

Proposition 4.9. *Assume that A admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus. Then there exists a unique bounded homomorphism $\nu_{0,A} : \mathcal{C}_0(\mathbb{C}_+) \rightarrow B(H)$ such that*

$$(4.19) \quad \nu_{0,A}(L_b) = \int_0^\infty b(t)T_t dt$$

for any $b \in L^1(\mathbb{R}_+)$.

Now arguing as in the proof of Corollary 4.7, we deduce the following.

Corollary 4.10. *Assume that A admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus. Then there exists a unique bounded homomorphism $\nu_A: \mathcal{C}(\mathbb{C}_+) \rightarrow B(H)$ such that (4.19) holds true for any $b \in L^1(\mathbb{R}_+)$.*

Of course when the above corollary applies, ν_A is an extension of the mapping ρ_A from Corollary 4.7. Thus our main results (Theorem 4.5, Corollary 4.7) should be regarded as a way to obtain a “good” functional calculus for negative generators of bounded C_0 -semigroups which do not admit a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus.

Remark 4.11. We can easily deduce from the above results a new proof of Proposition 2.4 (2), as follows. Recall the identifications $C_0(\mathbb{R})^* = M(\mathbb{R})$ and $\mathcal{A}_0(\mathbb{R})^* \subset \mathcal{M}(H^1(\mathbb{R}))$ given by (1.4) and Subsection 3.3. The identity (3.20) shows that through these identifications, the mapping $R: M(\mathbb{R}) \rightarrow \mathcal{M}(H^1(\mathbb{R}))$ taking any $\mu \in M(\mathbb{R})$ to R_μ is the adjoint of the canonical embedding $j: \mathcal{A}_0(\mathbb{R}) \rightarrow C_0(\mathbb{R})$. Assume, by contradiction, that $\mathcal{R} = \mathcal{M}(H^1(\mathbb{R}))$. Then R is onto, hence j induces an isomorphism from $\mathcal{A}_0(\mathbb{R})$ onto its range. This implies that $\mathcal{A}_0(\mathbb{R}) = C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$ and that $\|\cdot\|_{\mathcal{A}_0}$ is equivalent to $\|\cdot\|_\infty$ on $\mathcal{A}_0(\mathbb{R})$. By Theorem 4.5, this implies that the negative generator of any bounded C_0 -semigroup on Hilbert space admits a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus. This is false, as noted in (f4), hence we have $\mathcal{R} \neq \mathcal{M}(H^1(\mathbb{R}))$.

5. COMPARISON WITH THE BESOV FUNCTIONAL CALCULUS

In this section we compare the functional calculus constructed in Section 4 (Theorem 4.5 and Corollary 4.7) with the Besov functional calculus from [13, Subsection 5.5] and [4]. We start with some background on the analytic homogeneous Besov space used in the latter paper. We refer to [4] for further details.

Let $\psi \in \mathcal{S}(\mathbb{R})$ such that $\text{Supp}(\psi) \subset [\frac{1}{2}, 2]$, $\psi(t) \geq 0$ for all $t \in \mathbb{R}$, and $\psi(t) + \psi(\frac{t}{2}) = 1$ for all $t \in [1, 2]$. For any $k \in \mathbb{Z}$, we let $\psi_k \in \mathcal{S}(\mathbb{R})$ be defined by $\psi_k(t) = \psi(2^{-k}t)$, $t \in \mathbb{R}$. A key property of the sequence $(\psi_k)_{k \in \mathbb{Z}}$ is that for any $k_0 \in \mathbb{Z}$, we have

$$(5.1) \quad \forall t \in [2^{k_0}, 2^{k_0+1}) : \quad \psi_{k_0}(t) + \psi_{k_0+1}(t) = 1 \quad \text{and} \quad \psi_k(t) = 0 \text{ if } k \notin \{k_0, k_0 + 1\}.$$

Next define $\phi_k = \mathcal{F}^{-1}(\psi_k)$. It is plain that for any $k \in \mathbb{Z}$,

$$(5.2) \quad \phi_k \in H^1(\mathbb{R}) \quad \text{and} \quad \|\phi_k\|_1 = \|\phi_0\|_1.$$

It follows that for any $F \in BUC(\mathbb{R})$ and any $k \in \mathbb{Z}$, $F \star \phi_k$ belongs to $BUC(\mathbb{R}) \cap H^\infty(\mathbb{R})$. We define a Besov space $\mathcal{B}(\mathbb{R})$ by

$$\mathcal{B}(\mathbb{R}) = \left\{ F \in BUC(\mathbb{R}) : \sum_{k \in \mathbb{Z}} \|F \star \phi_k\|_\infty < \infty \text{ and } F = \sum_{k \in \mathbb{Z}} F \star \phi_k \right\}.$$

This is a Banach space for the norm

$$\|F\|_{\mathcal{B}} = \sum_{k \in \mathbb{Z}} \|F \star \phi_k\|_\infty.$$

Next we set $\mathcal{B}_0(\mathbb{R}) = \mathcal{B}(\mathbb{R}) \cap C_0(\mathbb{R})$, equipped with the norm of $\mathcal{B}(\mathbb{R})$. Then $\mathcal{B}_0(\mathbb{R})$ is a closed subspace of $\mathcal{B}(\mathbb{R})$ and we clearly have

$$\mathcal{B}_0(\mathbb{R}) \subset C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}) \quad \text{and} \quad \mathcal{B}(\mathbb{R}) \subset BUC(\mathbb{R}) \cap H^\infty(\mathbb{R}).$$

We warn the reader that our notation $\mathcal{B}/\mathcal{B}_0$ differs from the one introduced in [4]. We make this choice to match with the notation in Section 3.

We wish to underline that the above definitions of $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}_0(\mathbb{R})$ do not depend on the choice of the function ψ . More precisely if $\psi^{(1)}, \psi^{(2)}$ are two functions as above and if we let $\mathcal{B}^{\psi^{(1)}}(\mathbb{R})$ and $\mathcal{B}^{\psi^{(2)}}(\mathbb{R})$ denote the associated spaces, then $\mathcal{B}^{\psi^{(1)}}(\mathbb{R})$ and $\mathcal{B}^{\psi^{(2)}}(\mathbb{R})$ coincide as vector spaces and the norms $\|\cdot\|_{\mathcal{B}^{\psi^{(1)}}}$ and $\|\cdot\|_{\mathcal{B}^{\psi^{(2)}}}$ are equivalent. We refer to [4] and the references therein for these properties.

Similarly to the beginning of Section 4, we introduce half-plane versions of $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}_0(\mathbb{R})$, by setting

$$\mathcal{B}_0(\mathbb{C}_+) = \{\tilde{F} : F \in \mathcal{B}_0(\mathbb{R})\} \quad \text{and} \quad \mathcal{B}(\mathbb{C}_+) = \{\tilde{F} : F \in \mathcal{B}(\mathbb{R})\}.$$

In [4], Batty-Gomilko-Tomilov provided an equivalent definition of $\mathcal{B}(\mathbb{C}_+)$, from which it follows that

$$(5.3) \quad \{L_b : b \in L^1(\mathbb{R}_+)\} \subset \mathcal{B}_0(\mathbb{C}_+),$$

see [4, Subsection 2.4 & Proposition 6.2].

Moreover they established the following remarkable functional calculus result.

Theorem 5.1. ([4, Theorem 4.4], [5, Theorem 6.1]) *Let X be a Banach space, let $(T_t)_{t \geq 0}$ be a bounded C_0 -semigroup and let $-A$ denote its generator. The following are equivalent.*

(i) *There exists a constant $K > 0$ such that*

$$\int_{-\infty}^{\infty} |\langle R(\beta + it, A)^2(x), x^* \rangle| dt \leq \frac{-K}{\beta} \|x\| \|x^*\|$$

for any $\beta < 0$, any $x \in X$ and any $x^ \in X^*$.*

(ii) *There exists a bounded homomorphism $\gamma_A : \mathcal{B}(\mathbb{C}_+) \rightarrow B(X)$ such that*

$$\gamma_A(L_b) = \int_0^{\infty} b(t) T_t dt, \quad b \in L^1(\mathbb{R}_+).$$

In this case, γ_A is unique.

Condition (i) in Theorem 5.1 goes back at least to [10] and [31]. In fact, condition (i) can be defined for any closed and densely defined operator A satisfying (4.3) for $\omega = 0$. Then it follows from [10, 31] that (i) actually implies that $-A$ generates a bounded C_0 -semigroup on X . (See also [6, Theorem 6.4]). Conversely, if $X = H$ is a Hilbert space, it is proved in [10, 31] that if $-A$ generates a bounded C_0 -semigroup, then A satisfies (i). (The assumption that $X = H$ is a Hilbert space is crucial here, see the beginning of Section 6 for more on this.)

Thus if $(T_t)_{t \geq 0}$ is a bounded C_0 -semigroup with generator $-A$ on Hilbert space, then the property (ii) in Theorem 5.1 holds true. It is therefore natural to compare Corollary 4.7 with that property. This is the aim of the rest of this section.

Proposition 5.2. *We both have*

$$\mathcal{B}(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+) \quad \text{and} \quad \mathcal{B}_0(\mathbb{C}_+) \subset \mathcal{A}_0(\mathbb{C}_+).$$

Moreover there exists a constant $K > 0$ such that $\|\varphi\|_{\mathcal{A}} \leq K\|\varphi\|_{\mathcal{B}}$ for any $\varphi \in \mathcal{B}(\mathbb{C}_+)$.

Proof. It follows from (5.1) that

$$(5.4) \quad \psi_k = \psi_k(\psi_{k-1} + \psi_k + \psi_{k+1}), \quad k \in \mathbb{Z}.$$

Consequently,

$$\phi_k = \phi_k \star (\phi_{k-1} + \phi_k + \phi_{k+1}), \quad k \in \mathbb{Z}.$$

Let $F \in \mathcal{B}(\mathbb{R})$. Applying the above identity, we have

$$F = \sum_{k \in \mathbb{Z}} F \star \phi_k \star (\phi_{k-1} + \phi_k + \phi_{k+1}).$$

Appealing to (5.2), we observe that $F \star \phi_k \in BUC(\mathbb{R})$ and $\phi_{k-1} + \phi_k + \phi_{k+1} \in H^1(\mathbb{R})$ for each $k \in \mathbb{Z}$, and that

$$\sum_{k \in \mathbb{Z}} \|F \star \phi_k\|_{\infty} \|\phi_{k-1} + \phi_k + \phi_{k+1}\|_1 \leq 3\|\phi_0\|_1 \|F\|_{\mathcal{B}}.$$

This shows that $F \in \mathcal{A}(\mathbb{R})$, with

$$\|F\|_{\mathcal{A}} \leq 3\|\phi_0\|_1 \|F\|_{\mathcal{B}}.$$

This yields $\mathcal{B}(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+)$. The above argument also shows that $\mathcal{B}_0(\mathbb{C}_+) \subset \mathcal{A}_0(\mathbb{C}_+)$. \square

Let H be a Hilbert space and let A be the negative generator of a bounded C_0 -semigroup on H . We already noticed that A satisfies property (ii) in Theorem 5.1. According to Proposition 5.2, the functional calculus $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$ from Corollary 4.7 extends the functional calculus $\gamma_A: \mathcal{B}(\mathbb{C}_+) \rightarrow B(H)$.

It turns out that the extension from γ_A to ρ_A is an actual improvement, because of the following result.

Theorem 5.3. *We both have*

$$\mathcal{B}(\mathbb{C}_+) \neq \mathcal{A}(\mathbb{C}_+) \quad \text{and} \quad \mathcal{B}_0(\mathbb{C}_+) \neq \mathcal{A}_0(\mathbb{C}_+).$$

We need some preparation before coming to the proof. We use an idea from [33, Paragraph 2.6.4]. First for the definition of the Besov space $\mathcal{B}(\mathbb{R})$, we make the additional assumption that $\psi(t) = 1$ for any $t \in [\frac{3}{4}, 1]$. This is allowed by the aforementioned fact that the definition of $\mathcal{B}(\mathbb{R})$ does not depend on ψ . This implies that $\text{Supp}(\psi) \subset [\frac{1}{2}, \frac{3}{2}]$. Second we fix a non-zero function $f_0 \in \mathcal{S}(\mathbb{R})$ such that $\text{Supp}(f_0) \subset [\frac{3}{4}, 1]$. Next for any integer $n \geq 0$, we set $N_n = 2^n - 1$ and $f_n = \tau_{N_n} f_0 = f_0(\cdot - N_n)$. By construction, $\text{Supp}(\psi_k) \subset [2^{k-1}, \frac{3}{2}2^k]$ for all $k \in \mathbb{Z}$ and $\text{Supp}(f_n) \subset [2^n - \frac{1}{4}, 2^n]$ for all $n \geq 0$. We derive that

$$(5.5) \quad \forall k \geq 0: \quad f_k \psi_k = f_k \quad \text{and} \quad f_n \psi_k = 0 \text{ if } n \neq k,$$

as well as

$$(5.6) \quad \forall n, n' \geq 0: \quad f_n f_{n'} = 0 \text{ if } n \neq n'.$$

Lemma 5.4. *There exists a bounded continuous function $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$ such that*

$$(5.7) \quad \sup_{k \in \mathbb{Z}} \|\widehat{m\psi}_k\|_1 < \infty$$

and the mapping $T_m: H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$ does not belong to $\mathcal{M}(H^1(\mathbb{R}))$.

Proof. Using the definitions preceding the lemma, we set

$$m(t) = \sum_{n=0}^{\infty} e^{iN_n t} f_n(t), \quad t > 0.$$

At most one term is non zero in this sum, hence this is well-defined and $m \in C_b(\mathbb{R}_+^*)$. Let $k \geq 0$. According to (5.5), we have $m\psi_k = e^{iN_k} \cdot f_k$ hence

$$\|\widehat{m\psi}_k\|_1 = \|\widehat{f}_k(\cdot - N_k)\|_1 = \|\widehat{f}_k\|_1 = \|\widehat{f}_0\|_1.$$

Since $m\psi_k = 0$ if $k < 0$, this shows (5.7).

Define

$$g_N = \mathcal{F}^{-1} \left(\sum_{n=0}^N e^{-iN_n \cdot} f_n \right)$$

for all $N \geq 0$. Then $g_N \in \mathcal{S}(\mathbb{R}) \cap H^1(\mathbb{R})$ hence $g_N \in H^p(\mathbb{R})$ for any $1 \leq p \leq \infty$. Let us estimate its L^p -norm. On the one hand, we have

$$\|g_N\|_1 \leq \sum_{n=0}^N \|\mathcal{F}^{-1}(e^{-iN_n \cdot} f_n)\|_1 = \sum_{n=0}^N \|\mathcal{F}^{-1}(f_n)(\cdot - N_n)\|_1 = \sum_{n=0}^N \|\mathcal{F}^{-1}(f_n)\|_1,$$

hence

$$\|g_N\|_1 \leq (N+1) \|\mathcal{F}^{-1}(f_0)\|_1.$$

On the other hand, for any $t \in \mathbb{R}$, we have

$$g_N(t) = \sum_{n=0}^N \mathcal{F}^{-1}(f_n)(t - N_n),$$

hence

$$|g_N(t)| \leq \sum_{n=0}^N |g_0(t - N_n)|.$$

Since $g_0 = \mathcal{F}^{-1}(f_0) \in \mathcal{S}(\mathbb{R})$ we infer that

$$\sup_{N \geq 0} \|g_N\|_{\infty} < \infty.$$

For any $1 < p < \infty$, we have $\|g_N\|_p \leq \|g_N\|_1^{\frac{1}{p}} \|g_N\|_{\infty}^{1-\frac{1}{p}}$, hence the above estimates imply the existence of a constant $K > 0$ such that

$$(5.8) \quad \|g_N\|_p \leq K N^{\frac{1}{p}}, \quad N \geq 1.$$

By (5.6), we have

$$m\widehat{g}_N = \left(\sum_{n=0}^{\infty} e^{iNn} \cdot f_n \right) \left(\sum_{n=0}^N e^{-iNn} \cdot f_n \right) = \sum_{n=0}^N f_n^2.$$

For any $n \geq 0$, $\text{Supp}(f_n^2) \subset [2^{n-1}, 2^n]$ hence by [11, Theorem 5.1.5.], we have an estimate

$$\|\mathcal{F}^{-1}(m\widehat{g}_N)\|_p \approx \left\| \left(\sum_{n=0}^N |\mathcal{F}^{-1}(f_n^2)|^2 \right)^{\frac{1}{2}} \right\|_p.$$

Further, $f_n^2 = f_0^2(\cdot - N_n)$ hence $|\mathcal{F}^{-1}(f_n^2)| = |\mathcal{F}^{-1}(f_0^2)|$ for any $n \geq 0$. Consequently,

$$\left(\sum_{n=0}^N |\mathcal{F}^{-1}(f_n^2)|^2 \right)^{\frac{1}{2}} = (N+1)^{\frac{1}{2}} |\mathcal{F}^{-1}(f_0^2)|.$$

Thus we have

$$\|\mathcal{F}^{-1}(m\widehat{g}_N)\|_p \approx N^{\frac{1}{2}}.$$

Comparing with (5.8) we deduce that if $2 < p < \infty$, then $T_m: H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$ is not a bounded Fourier multiplier on $H^p(\mathbb{R})$. By Lemma 2.5, we deduce that $T_m \notin \mathcal{M}(H^1(\mathbb{R}))$. \square

Proof of Theorem 5.3. If $\mathcal{A}(\mathbb{C}_+)$ were equal to $\mathcal{B}(\mathbb{C}_+)$, we would have $\mathcal{A}_0(\mathbb{C}_+) = \mathcal{B}_0(\mathbb{C}_+)$ which in turn is equivalent to $\mathcal{A}_0(\mathbb{R}) = \mathcal{B}_0(\mathbb{R})$. So it suffices to show that this equality fails. Let us assume, by contradiction, that $\mathcal{A}_0(\mathbb{R}) = \mathcal{B}_0(\mathbb{R})$.

Let m be given by Lemma 5.4. Let $b \in L^1(\mathbb{R}_+)$. For any $k \in \mathbb{Z}$, we have

$$\psi_k b = \frac{1}{2\pi} \mathcal{F}(\phi_k \star \widehat{b}(-\cdot)).$$

Hence using (5.1), (5.4) and Lemma 1.1, we have

$$\begin{aligned} \int_{-\infty}^{\infty} m(t)b(t) dt &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} m(t)\psi_k(t)b(t) dt \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} (\psi_{k-1}(t) + \psi_k(t) + \psi_{k+1}(t))m(t)\psi_k(t)b(t) dt \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} [\mathcal{F}((\psi_{k-1} + \psi_k + \psi_{k+1})m)](u) [\mathcal{F}(\psi_k b)](-u) du \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} [\mathcal{F}((\psi_{k-1} + \psi_k + \psi_{k+1})m)](u) [\phi_k \star \widehat{b}(-\cdot)](u) du \end{aligned}$$

Therefore,

$$\left| \int_{-\infty}^{\infty} m(t)b(t) dt \right| \leq \sum_{k \in \mathbb{Z}} \|\mathcal{F}((\psi_{k-1} + \psi_k + \psi_{k+1})m)\|_1 \|\phi_k \star \widehat{b}(-\cdot)\|_{\infty}.$$

Applying (5.7), we deduce the existence of a constant $K > 0$ such that

$$\left| \int_{-\infty}^{\infty} m(t)b(t) dt \right| \leq K \sum_{k \in \mathbb{Z}} \|\phi_k \star \widehat{b}(-\cdot)\|_{\infty} = K \|\widehat{b}(-\cdot)\|_{\mathcal{B}}.$$

Therefore there exists $\eta \in \mathcal{B}_0(\mathbb{R})^*$ such that

$$\langle \eta, \widehat{b}(-\cdot) \rangle = \int_{-\infty}^{\infty} m(t)b(t) dt, \quad b \in L^1(\mathbb{R}_+).$$

By assumption, $\eta \in \mathcal{A}_0(\mathbb{R})^*$. Applying Theorem 3.7, let $T \in \mathcal{M}(H^1(\mathbb{R}))$ be associated to η and let $m_0 \in C_b(\mathbb{R}_+^*)$ be the symbol of T . Then by Remark 3.10, we have

$$\langle \eta, \widehat{b}(-\cdot) \rangle = \int_{-\infty}^{\infty} m_0(t)b(t) dt, \quad b \in L^1(\mathbb{R}_+).$$

We deduce that $m_0 = m$, and this contradicts the fact that $T_m \notin \mathcal{M}(H^1(\mathbb{R}))$. \square

Remark 5.5. Let $\mathcal{D} \subset H^1(\mathbb{R})$ be the space of all $h \in H^1(\mathbb{R})$ such that $\text{Supp}(\widehat{h})$ is a compact subset of \mathbb{R}_+^* . It is well-known that \mathcal{D} is dense in $H^1(\mathbb{R})$. To check this, take any $h \in H^1(\mathbb{R})$ and recall that there exist $v, w \in H^2(\mathbb{R})$ such that $h = wv$. Let $(d_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ be sequences of $C_b(\mathbb{R}_+^*)$ with compact supports such that $d_n \rightarrow \widehat{w}$ and $c_n \rightarrow \widehat{v}$ in $L^2(\mathbb{R}_+)$. Then $\mathcal{F}^{-1}(d_n) \rightarrow w$ and $\mathcal{F}^{-1}(c_n) \rightarrow v$ in $H^2(\mathbb{R})$, hence $\mathcal{F}^{-1}(d_n)\mathcal{F}^{-1}(c_n) \rightarrow h$ in $H^1(\mathbb{R})$. Now it is easy to see that $\mathcal{F}^{-1}(d_n)\mathcal{F}^{-1}(c_n)$ belongs to \mathcal{D} for any $n \in \mathbb{N}$.

Let $BUC \star \mathcal{D} \subset \mathcal{A}(\mathbb{R})$ be the linear span of the functions $f \star h$, for $f \in BUC(\mathbb{R})$ and $h \in \mathcal{D}$. It follows from above that this is a dense subspace of $\mathcal{A}(\mathbb{R})$.

Let $\mathcal{G} \subset H^\infty(\mathbb{R})$ be the space of all $F \in H^\infty(\mathbb{R})$ such that $\text{Supp}(\widehat{F})$ is a compact subset of \mathbb{R}_+^* . Then we have

$$BUC \star \mathcal{D} \subset \mathcal{G} \subset \mathcal{B}(\mathbb{R}).$$

The first inequality is obvious and the second one is given by [4, Proposition 6.2 & Lemma 2.4]. Note that [4, Proposition 2.9] also asserts that \mathcal{G} is dense in $\mathcal{B}(\mathbb{R})$.

It follows that $\mathcal{B}(\mathbb{R})$ is dense in $\mathcal{A}(\mathbb{R})$, or equivalently that $\mathcal{B}(\mathbb{C}_+)$ is dense in $\mathcal{A}(\mathbb{C}_+)$.

Also $\mathcal{B}_0(\mathbb{C}_+)$ is dense in $\mathcal{A}_0(\mathbb{C}_+)$, by (5.3).

Remark 5.6.

(1) Let A be as in Subsections 4.2 and 4.3. Let $\mu \in M(\mathbb{R})$, with $\mu(\{0\}) = 0$. According to [4, Subsection 2.2 & Proposition 6.2], its Laplace transform $L_\mu: \mathbb{C}_+ \rightarrow \mathbb{C}$ belongs to $\mathcal{B}(\mathbb{C}_+)$. Hence L_μ belongs to $\mathcal{A}(\mathbb{C}_+)$, by Proposition 5.2. The argument in the proof of Corollary 4.7 shows that $\rho_A(L_\mu)$ is the strong limit of $\rho_{0,A}(L_\mu \widehat{G}_N)$, when $N \rightarrow \infty$. Define $c_N(t) = Ne^{-Nt}$ for any $t > 0$ and recall that $\widehat{G}_N = L_{c_N}$. Then $L_\mu \widehat{G}_N = L_{\mu \star c_N}$ for any $N \geq 1$. Further $\mu \star c_N \rightarrow \mu$ narrowly, when $N \rightarrow \infty$. It therefore follows from (4.8) that

$$[\rho_A(L_\mu)](x) = \int_{\mathbb{R}} T_t(x) d\mu(t), \quad x \in H.$$

(2) It follows from [4, Subsection 2.2] that $\lim_{y \rightarrow \infty} \varphi(y) = 0$ for any $\varphi \in \mathcal{B}(\mathbb{C}_+)$. We noticed in Remark 5.5 that $\mathcal{B}(\mathbb{C}_+)$ is dense in $\mathcal{A}(\mathbb{C}_+)$. Since $\|\cdot\|_{H^\infty(\mathbb{C}_+)} \leq \|\cdot\|_{\mathcal{A}(\mathbb{C}_+)}$, this implies that any element of $\mathcal{A}(\mathbb{C}_+)$ is the uniform limit of a sequence of $\mathcal{B}(\mathbb{C}_+)$. Consequently, $\lim_{y \rightarrow \infty} \varphi(y) = 0$ for any $\varphi \in \mathcal{A}(\mathbb{C}_+)$.

Thus the algebra $\mathcal{A}(\mathbb{C}_+)$ (equivalently, the algebra $\mathcal{A}(\mathbb{R})$) does not contain any non zero constant function and hence is not unital.

6. γ -BOUNDED SEMIGROUPS ON BANACH SPACES

In general, Theorem 4.5 and Corollary 4.7 do not hold true if H is replaced by an arbitrary Banach space. Indeed as a consequence of [10, Proposition], the translation semigroup $(T_t)_{t \geq 0}$ on $L^p(\mathbb{R})$, for $1 \leq p \neq 2 < \infty$, does not satisfy condition (i) in Theorem 5.1. Hence by the latter theorem and Proposition 5.2, the mapping

$$L_b \mapsto \int_0^\infty b(t)T_t dt, \quad b \in L^1(\mathbb{R}_+),$$

is not bounded with respect to the $\mathcal{A}_0(\mathbb{C}_+)$ -norm.

In this section we will however establish Banach space versions of Theorem 4.5 and Corollary 4.7 on Banach spaces, involving γ -boundedness. We start with some background and basic facts on this topic and refer to [17, Chapter 9] for details and more information.

Let X be a Banach space. Let $(\gamma_n)_{n \geq 1}$ be a sequence of independent complex valued standard Gaussian variables on some probability space Σ and let $G_0 \subset L^2(\Sigma)$ be the linear span of the γ_n . We denote by $G(X)$ the closure of

$$G_0 \otimes X = \left\{ \sum_{k=1}^N \gamma_k \otimes x_k : x_k \in X, N \in \mathbb{N} \right\}$$

in the Bochner space $L^2(\Sigma; X)$, equipped with the induced norm. Next we let $G'(X^*)$ denote the closure of $G_0 \otimes X^*$ in the dual space $G(X)^*$.

A bounded set $\mathcal{T} \subset B(X)$ is called γ -bounded if there exists a constant $C \geq 0$ such that for all finite sequences $(S_k)_{k=1}^N \subset \mathcal{T}$ and $(x_k)_{k=1}^N \subset X$, we have:

$$(6.1) \quad \left\| \sum_{k=1}^N \gamma_k \otimes S_k(x_k) \right\|_{G(X)} \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}.$$

The least admissible constant C in the above inequality is called the γ -bound of \mathcal{T} and is denoted by $\gamma(\mathcal{T})$.

Let Z be any Banach space and let $\text{Ball}(Z)$ denote its closed unit ball. A bounded operator $\rho: Z \rightarrow B(X)$ is called γ -bounded if the set $\rho(\text{Ball}(Z)) \subset B(X)$ is γ -bounded. In this case we set $\gamma(\rho) = \gamma(\rho(\text{Ball}(Z)))$.

We now turn to the definition of γ -spaces, which goes back to the paper [18] (which began to circulate 20 years ago). Let H be a Hilbert space. A bounded operator $T: H \rightarrow X$ is called γ -summing if

$$\|T\|_\gamma := \sup \left\{ \left\| \sum_{k=1}^N \gamma_k \otimes T(e_k) \right\|_{G(X)} \right\} < \infty,$$

where the supremum is taken over all finite orthonormal systems $(e_k)_{k=1}^N$ in H . We let $\gamma_\infty(H; X)$ denote the space of all γ -summing operators and we endow it with the norm $\|\cdot\|_\gamma$. Then $\gamma_\infty(H; X)$ is a Banach space. Any finite rank bounded operator is γ -summing and we let $\gamma(H; X)$ denote their closure in $\gamma_\infty(H; X)$. In the sequel, finite rank bounded operator are represented by the algebraic tensor product $H^* \otimes X$ in the usual way.

Following [18, Section 5], we let $\gamma'_+(H^*; X^*)$ be the space of all bounded operators $S: H^* \rightarrow X^*$ such that

$$\|S\|_{\gamma'} := \sup\{|\operatorname{tr}(T^*S)| \mid T: H \rightarrow X, \operatorname{rk}(T) < \infty, \|T\|_{\gamma} \leq 1\} < \infty.$$

Then $\|\cdot\|_{\gamma'}$ is a norm on $\gamma'_+(H^*; X^*)$ and according to [18, Proposition 5.1], we have

$$(6.2) \quad \gamma'_+(H^*; X^*) = \gamma(H; X)^*$$

isometrically, through the duality pairing

$$(S, T) \mapsto \operatorname{tr}(T^*S), \quad T \in \gamma(H; X), S \in \gamma'_+(H^*; X^*).$$

We will focus on the case when H is an L^2 -space. Let (Ω, μ) be a σ -finite measure space. We identify $L^2(\Omega)^*$ and $L^2(\Omega)$ in the usual way. A function $\xi: \Omega \rightarrow X$ is called weakly- L^2 if for each $x^* \in X^*$, the function $\langle x^*, \xi(\cdot) \rangle$ belongs to $L^2(\Omega)$. Then the operator $x^* \mapsto \langle x^*, \xi(\cdot) \rangle$ from X^* into $L^2(\Omega)$ is bounded. If ξ is both measurable and weakly- L^2 , then its adjoint takes values in X and we let $\mathbb{I}_{\xi}: L^2(\Omega) \rightarrow X$ denote the resulting operator. More explicitly,

$$\langle x^*, \mathbb{I}_{\xi}(g) \rangle = \int_{\Omega} g(t) \langle x^*, \xi(t) \rangle d\mu(t), \quad g \in L^2(\Omega), x^* \in X^*.$$

We let $\gamma(\Omega; X)$ be the space of all measurable and weakly- L^2 functions $\xi: \Omega \rightarrow X$ such that \mathbb{I}_{ξ} belongs to $\gamma(L^2(\Omega); X)$, and we write $\|\xi\|_{\gamma} = \|\mathbb{I}_{\xi}\|_{\gamma}$ for any such function.

Likewise a function $\zeta: \Omega \rightarrow X^*$ is called weakly*- L^2 if for each $x \in X$, the function $\langle \zeta(\cdot), x \rangle$ belongs to $L^2(\Omega)$. In this case, the operator $x \mapsto \langle \zeta(\cdot), x \rangle$ from X into $L^2(\Omega)$ is bounded and we let $\mathbb{I}_{\zeta}: L^2(\Omega) \rightarrow X^*$ denote its adjoint. We let $\gamma'_+(\Omega; X^*)$ be the space of all weakly*- L^2 functions $\zeta: \Omega \rightarrow X^*$ such that \mathbb{I}_{ζ} belongs to $\gamma'_+(L^2(\Omega); X)$, and we write $\|\zeta\|_{\gamma'} = \|\mathbb{I}_{\zeta}\|_{\gamma'}$ for any such function.

Note that our space $\gamma'_+(\Omega; X^*)$ is a priori bigger than the one from [18, Definition 4.5], where only measurable functions $\Omega \rightarrow X^*$ are considered.

Lemma 6.1. *For any $\xi \in \gamma(\Omega; X)$ and any $\zeta \in \gamma'_+(\Omega; X^*)$, the function $t \mapsto \langle \zeta(t), \xi(t) \rangle$ belongs to $L^1(\Omega)$ and in the duality (6.2), we have*

$$\langle \mathbb{I}_{\zeta}, \mathbb{I}_{\xi} \rangle = \int_{\Omega} \langle \zeta(t), \xi(t) \rangle d\mu(t).$$

Moreover

$$\int_{\Omega} |\langle \zeta(t), \xi(t) \rangle| d\mu(t) \leq \|\xi\|_{\gamma} \|\zeta\|_{\gamma'}.$$

If we consider measurable functions $\zeta: \Omega \rightarrow X^*$ only, the above statement is provided by [18, Corollary 5.5]. The fact that it holds as well in the more general setting of the present paper follows from the proof of [17, Theorem 9.2.14].

The main result of this section is the following.

Theorem 6.2. *Let $(T_t)_{t \geq 0}$ be a bounded C_0 -semigroup on X and let A be its negative generator. The following assertions are equivalent.*

- (i) *The semigroup $(T_t)_{t \geq 0}$ is γ -bounded, that is, the set $\mathcal{T}_A = \{T_t : t \geq 0\}$ is γ -bounded;*
- (ii) *There exists a γ -bounded homomorphism $\rho_{0,A}: \mathcal{A}_0(\mathbb{C}_+) \rightarrow B(X)$ such that (4.8) holds true for any $b \in L^1(\mathbb{R}_+)$.*

In this case, $\rho_{0,A}$ is unique and $\gamma(\mathcal{T}_A) \leq \gamma(\rho_{0,A}) \leq \gamma(\mathcal{T}_A)^2$.

Further there exists a unique bounded homomorphism $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(X)$ extending $\rho_{0,A}$, this homomorphism is γ -bounded and $\gamma(\rho_A) = \gamma(\rho_{0,A})$.

A thorough look at the proofs of Theorem 4.5 and Corollary 4.7 reveals that in Subsections 4.2 and 4.3, the Hilbertian structure was used only in Lemma 4.4. So without any surprise the main point in proving Theorem 6.2 is the following γ -bounded version of Lemma 4.4.

Lemma 6.3. *Let $(T_t)_{t \geq 0}$ be a γ -bounded C_0 -semigroup on X and let A be its negative generator. Let $C = \gamma(\mathcal{T}_A)$. Then the set*

$$(6.3) \quad \left\{ \Gamma(A, (2\pi)^{-1} \widehat{f \widehat{wv}}) : f \in C_{00}(\mathbb{R}), w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}), \{\|f\|_\infty, \|w\|_2, \|v\|_2\} \leq 1 \right\}$$

is γ -bounded, with γ -bound $\leq C^2$.

Proof. Let $N \in \mathbb{N}$ and let $f_1, \dots, f_N \in C_{00}(\mathbb{R})$, $w_1, \dots, w_N, v_1, \dots, v_N \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ such that $\|f_k\|_\infty \leq 1$, $\|w_k\|_2 \leq 1$ and $\|v_k\|_2 \leq 1$ for any $k = 1, \dots, N$. We set

$$S_k = \Gamma(A, (2\pi)^{-1} \widehat{f_k \widehat{w_k v_k}}), \quad k = 1, \dots, N.$$

Let $x_1, \dots, x_N \in X$ and $x_1^*, \dots, x_N^* \in X^*$. Following the notation in the proof of Lemma 4.4, we define, for any $k = 1, \dots, N$, two strongly continuous functions $W_k, V_k: \mathbb{R} \rightarrow B(X)$ by

$$W_k(s) = \int_0^\infty \widehat{w_k}(r) e^{-irs} T_r dr \quad \text{and} \quad V_k(s) = \int_0^\infty \widehat{v_k}(t) e^{-its} T_t dt, \quad s \in \mathbb{R}.$$

According to (4.7),

$$\sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle = \frac{1}{4\pi^2} \sum_{k=1}^N \int_{-\infty}^\infty f_k(s) \langle W_k(s)x_k, V_k(s)^* x_k^* \rangle ds,$$

hence

$$\left| \sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle \right| \leq \frac{1}{4\pi^2} \sum_{k=1}^N \int_{-\infty}^\infty |\langle W_k(s)x_k, V_k(s)^* x_k^* \rangle| ds.$$

We let $\mathbb{N}_N = \{1, \dots, N\}$ for convenience. We will use γ -spaces on either \mathbb{R} or $\mathbb{N}_N \times \mathbb{R}$. For any $k = 1, \dots, N$, the function

$$\alpha_k := W_k(\cdot)x_k: \mathbb{R} \longrightarrow X$$

is measurable and weakly- L^2 . Likewise,

$$\beta_k := V_k(\cdot)^* x_k^*: \mathbb{R} \longrightarrow X^*$$

is weakly*- L^2 . If we are able to show that $\alpha_k \in \gamma(\mathbb{R}; X)$ and $\beta_k \in \gamma'_+(\mathbb{R}; X^*)$ for any $k = 1, \dots, N$, then Lemma 6.1 insures that

$$(6.4) \quad \left| \sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle \right| \leq \frac{1}{4\pi^2} \left\| (k, s) \mapsto W_k(s)x_k \right\|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} \left\| (k, s) \mapsto V_k^*(s)x_k^* \right\|_{\gamma'(\mathbb{N}_N \times \mathbb{R}; X^*)}.$$

Our aim is now to check that $\alpha_k \in \gamma(\mathbb{R}; X)$ and $\beta_k \in \gamma'_+(\mathbb{R}; X^*)$ for any k and to estimate the right-hand side of (6.4).

By assumption, $\mathcal{T}_A = \{T_t : t \geq 0\}$ is γ -bounded. According to the Multiplier Theorem stated as [14, Theorem 6.1], there exists a bounded operator

$$M: \gamma(L^2(\mathbb{R}); X) \longrightarrow \gamma(L^2(\mathbb{R}); X)$$

with norm $\leq C = \gamma(\mathcal{T}_A)$, mapping $\gamma(\mathbb{R}; X)$ into itself, and such that for any $\xi \in \gamma(\mathbb{R}; X)$, $[M(\xi)](t) = T_t(\xi(t))$ if $t \geq 0$, and $[M(\xi)](t) = 0$ if $t < 0$. Further by the Extension Theorem stated as [17, Theorem 9.6.1], $\mathcal{F} \otimes I_X: L^2(\mathbb{R}) \otimes X \rightarrow L^2(\mathbb{R}) \otimes X$ admits a (necessarily unique) bounded extension

$$\Psi: \gamma(L^2(\mathbb{R}); X) \longrightarrow \gamma(L^2(\mathbb{R}); X),$$

with norm $\leq \sqrt{2\pi}$. According to [2, Lemma 2.19], $\mathbb{I}_{\alpha_k} = (\Psi \circ M)(\widehat{w}_k \otimes x_k)$ for any $k = 1, \dots, N$. This shows that $\alpha_k \in \gamma(\mathbb{R}; X)$. Let $(e_k)_{k=1}^N$ be the canonical basis of ℓ_N^2 . It follows from above that

$$\begin{aligned} \|(k, s) \mapsto W_k(s)x_k\|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} &= \left\| \sum_{k=1}^N e_k \otimes (\Psi \circ M)(\widehat{w}_k \otimes x_k) \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)} \\ &\leq \sqrt{2\pi} C \left\| \sum_{k=1}^N e_k \otimes \widehat{w}_k \otimes x_k \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)}. \end{aligned}$$

The finite sequence $(e_k \otimes \widehat{w}_k)_{k=1}^N$ is an orthogonal family of $L^2(\mathbb{N}_N \times \mathbb{R})$. Consequently,

$$\begin{aligned} \left\| \sum_{k=1}^N e_k \otimes \widehat{w}_k \otimes x_k \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)} &= \left\| \sum_{k=1}^N \|\widehat{w}_k\|_2 \gamma_k \otimes x_k \right\|_{G(X)} \\ &\leq \max_k \|\widehat{w}_k\|_2 \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}. \end{aligned}$$

Since $\|\widehat{w}_k\|_2 = \sqrt{2\pi} \|w_k\|_2 \leq \sqrt{2\pi}$ for any $k = 1, \dots, N$, we finally obtain that

$$\|(k, s) \mapsto W_k(s)x_k\|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} \leq 2\pi C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}.$$

We now analyse the β_k . Fix k and consider $g \in L^2(\mathbb{R})$ and $x \in X$. Using Lemma 6.1, we have

$$\begin{aligned} \langle \mathbb{I}_{\beta_k}(g), x \rangle &= \int_{-\infty}^{\infty} g(s) \langle x_k^*, V_k(s)x \rangle ds \\ &= \int_{-\infty}^{\infty} g(s) \mathcal{F}(\widehat{v}_k \langle x_k^*, T_t(x) \rangle)(s) ds \\ &= \int_0^{\infty} \widehat{g}(t) \widehat{v}_k(t) \langle x_k^*, T_t(x) \rangle dt \\ &= \langle \widehat{v}_k \otimes x_k^*, M(\widehat{g} \otimes x) \rangle \\ &= \langle \widehat{v}_k \otimes x_k^*, (M \circ \Psi)(g \otimes x) \rangle \\ &= \langle (\Psi^* \circ M^*)(\widehat{v}_k \otimes x_k^*), g \otimes x \rangle. \end{aligned}$$

This shows that $\beta_k \in \gamma'_+(\mathbb{R}; X^*)$, with $\mathbb{I}_{\beta_k} = (\Psi^* \circ M^*)(\widehat{v}_k \otimes x_k^*)$. Now arguing as in the $W_k(\cdot)x_k$ case, we obtain that

$$\|(k, s) \mapsto V_k^*(s)x_k^*\|_{\gamma'(\mathbb{N}_N \times \mathbb{R}; X^*)} \leq 2\pi C \left\| \sum_{k=1}^N \gamma_k \otimes x_k^* \right\|_{G'(X)}.$$

We now implement these estimates in (6.4) to obtain that

$$\left| \sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle \right| \leq C^2 \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes x_k^* \right\|_{G'(X)}.$$

By the very definition of $G'(X)$, this means that

$$\left\| \sum_{k=1}^N \gamma_k \otimes S_k(x_k) \right\|_{G(X)} \leq C^2 \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)},$$

which completes the proof. \square

Proof of Theorem 6.2. Assume (i). By Lemma 6.3, any element in the set (6.3) has norm $\leq C$. Hence the proof of Theorem 4.5 shows the existence of a unique bounded homomorphism $\rho_{0,A}: \mathcal{A}_0(\mathbb{C}_+) \rightarrow B(X)$ such that (4.8) holds true for any $b \in L^1(\mathbb{R}_+)$.

To prove γ -boundedness of $\rho_{0,A}$, we introduce the set

$$\mathcal{L} = \{(f \star wv)^\sim : f \in C_{00}(\mathbb{R}), w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}), \{\|f\|_\infty, \|w\|_2, \|v\|_2\} \leq 1\} \subset \mathcal{A}_0(\mathbb{C}_+).$$

Recall (see the proof of Lemma 4.4) that any $h \in H^1(\mathbb{R})$ can be written as a product $h = wv$, with $w, v \in H^2(\mathbb{R})$ and $\|w\|_2^2 = \|v\|_2^2 = \|h\|_1$, and that $H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ is dense in $H^2(\mathbb{R})$. Going back to Definition 3.2, we derive that

$$\text{Ball}(\mathcal{A}_0(\mathbb{C}_+)) = \overline{\text{Conv}}\{\mathcal{L}\}.$$

This implies that

$$\rho_{0,A}(\text{Ball}(\mathcal{A}_0(\mathbb{C}_+))) \subset \overline{\text{Conv}}\{\rho_{0,A}(\mathcal{L})\}.$$

Since $\rho_{0,A}((f \star wv)^\sim) = \Gamma(A, (2\pi)^{-1} \widehat{f} \widehat{w} \widehat{v})$ for any $f \in C_{00}(\mathbb{R})$ and any $w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$, Lemma 6.3 says that $\rho_{0,A}(\mathcal{L})$ is γ -bounded, with γ -bound $\leq C^2$. Owing to the fact that γ -boundedness and γ -bounds are preserved by convex hulls (see e.g. [17, Proposition 8.1.21]) and uniform limits, we infer that $\rho_{0,A}$ is γ -bounded, with $\gamma(\rho_{0,A}) \leq C^2$. This proves (ii).

Conversely assume (ii). The proof of Corollary 4.7 shows the existence of a unique bounded homomorphism $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(X)$ extending $\rho_{0,A}$ as well as the fact that $\rho_A(\text{Ball}(\mathcal{A}(\mathbb{C}_+)))$ belongs to the strong closure of $\rho_{0,A}(\text{Ball}(\mathcal{A}_0(\mathbb{C}_+)))$. Since γ -boundedness and γ -bounds are preserved by strong limits, we obtain that ρ_A is γ -bounded, with $\gamma(\rho_A) = \gamma(\rho_{0,A})$.

Finally the argument in Remark 5.6 (1) shows that for any $t > 0$,

$$T_t \in \rho_A(\text{Ball}(\mathcal{A}(\mathbb{C}_+))).$$

This implies (i), with $\gamma(\mathcal{T}_A) \leq \gamma(\rho_A)$. \square

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