Discrete approach of nonlinear electromagnetism and Lorentz acceleration

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Abstract

An original physical model of nonlinear electromagnetism is derived from the fundamental principles of discrete mechanics. This model, based on the conservation of acceleration and energy, expresses that acceleration is the sum of two orthogonal contributions, one to divergence-free and another to curl-free, as a formal Hodge-Helmholtz decomposition. The two scalar and vector potentials of the acceleration represent the energies of the longitudinal and transverse waves. The equation of electromagnetism obtained is presented as an alternative to Maxwell's equations. Among the important differences between the two formulations, the Lorentz force naturally appears as one of the inertial terms of acceleration. As examples of applications, we consider some cases of verifications serving to validate the alternative formulation. These test cases make it possible to find the behaviors predicted by the classical field theory. The mimetic approach is the ideal framework to discretize the equation of motion from discrete primary operators and their derived operators.

Keywords

Discrete Mechanics; Helmholtz-Hodge Decomposition; Maxwell's equations; Retarded potentials; Lorentz's force; Mimetic Discretizations

1 Introduction

In the general case, a progressive electromagnetic wave is formed by an electric field and a magnetic field varying in time. The variations in each of these two fields lead to the existence of the other. This dependence, conditioned by the celerity of the wave c, is described by partial derivatives in time, making it possible to couple the equations of Maxwell. Whatever the form of the equations of electromagnetism, relativistic or otherwise, none is based on a single variable allowing the construction of the different fields. The equations of J.C. Maxwell [1] were the result of a fine analysis of the laws of Gauss, Ampère and Faraday which led to the synthesis of the laws of modern electromagnetism [2]. The evolutions of these laws with the contribution of special relativity, including the invariance of the equations of electromagnetism under a Lorentz transformation, will not be treated here.

Maxwell's equations are valid in rest frame only but not valid for a moving domain, as noted by A. Einstein [3]. This limit of the classical Maxwell theory also originates from the fact that the force of a moving particle cannot be derived directly from the field equations, but should be postulated *a posteriori*. In any frame of reference, the covariant expression becomes $\mathbf{F} = q\ (\mathbf{E} + \mathbf{V} \times \mathbf{B})$ where q is the charge of the particle, $\mathbf{E}$ the local electric field, $\mathbf{V}$ the velocity of the material particle and $\mathbf{B}$ the magnetic induction. Starting from this limitation of the classical Maxwell formalism, a different approach has been proposed [4], [5], [6]. The most commonly used corresponds to the introduction of the potentials $(\phi, \mathbf{A})$, respectively of the electric scalar potential and the magnetic vector potential.

However, the nonlinear Lorentz term $\mathbf{V} \times \nabla \times \mathbf{A}$ in electrodynamics, or the Lamb vector $\mathbf{V} \times \nabla \times \mathbf{V}$ in fluid dynamics [7], can never be written in the form of a curl in a continuous medium. This loss of symmetry with respect to the first inertia term $\nabla \left(\|\mathbf{V}\|^2/2 \right)$ inhibits any Hodge-Helmholtz decomposition of the acceleration. This limitation is of primary importance for a certain number of fields of physics. In particular, the divergence of the nonlinear term of Lorentz or the vector of Lamb is not zero, whereas $\nabla \cdot \nabla \times \boldsymbol{\psi}$ is always zero regardless of the vector function $\boldsymbol{\psi}$. Discrete mechanics [8] removes this limitation by introducing simple concepts of differential geometry.

In electromagnetics the number of variables is important, and not all of them are independent. Similarly, the number of units with which they are expressed sometimes overlap. The number of fundamental SI electromagnetic units, lengths, masses, times and currents is increased if temperature is a variable used. The perspective of possible unification within the framework of field theory seems compromised by this profusion of variables and units. Only analogies between the quantities of different physical domains make it possible to highlight similarities between the corresponding variables, but they are only formal and partial.

The contribution developed here concerns the field of electromagnetism for which the laws established for centuries have been validated in the classical framework, but also in that of special relativity. The objective is to find the physical behaviors and the emblematic results of electromagnetism using the concepts of discrete mechanics [9], [8]. To this end, it is imperative to reduce the number of variables, physical parameters and fundamental units that describe them. The laws of classical mechanics in fluids and solids have been replaced by a single equation of motion based on the conservation of acceleration associated with its two scalar and vector potentials. The number of physical parameters has been reduced to longitudinal and transverse celerities and the number of fundamental units to two, a length and a time. These same concepts are applied to electromagnetism and the results obtained confronted with those of the literature.

In fact, discrete mechanics abandons the notion of continuum and with it, analysis, derivation at a point, integration, as well as any global reference linked to an inertial reference frame. These radical changes make it possible to reconstruct a formalism on the basis of essential principles, notably the principle of equivalence and that of Galilean relativity. The current laws are not called into question because they perfectly restore the observations in all the fields of physics, in particular in electromagnetism. However, it is not impossible to find a formalism that allows us to do this differently.

2 Discrete formulation

2.1 Physical modelization

The discrete mechanics developed in recent years [9] is based on the abandonment of the notion of continuous medium, replacing it by a discrete geometric topology formed of interconnected edges where the components of the vectors and the scalars are located. The equation derived on this basis translates the conservation of acceleration considered as an absolute quantity. Numerous examples in fluid and solid mechanics show that this makes it possible to restore the solutions of the Navier-Stokes and Navier-Lamé equations; reference [8] presents its main properties.

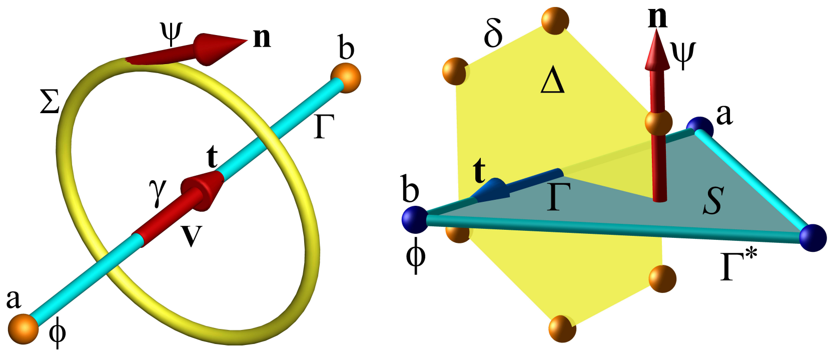
Whatever the field of physics considered, i.e. fluid mechanics, solids, electromagnetism, heat transfer, etc., it turns out that wave propagation plays a major role in the description and modeling of associated effects. Waves can be classified into two categories, directional longitudinal waves by nature and transverse waves attached to surfaces endowed with particular properties, for example polarization. While these waves can be dissociated in certain cases from stationary phenomena, they are in general very overlapping, insofar as the existence of one requires the presence of the others. As they are described by orthogonal fields, the exchanges between them become possible only if the phenomena are unsteady. Thus acceleration, longitudinal waves and transverse waves are the essential elements of a discrete field theory.

The geometric topology created in discrete mechanics corresponds to the stencil in figure (1 a). A rectilinear oriented edge Γ of length d and of unit vector $\mathbf{t}$ is limited by two points a and b. It is on this edge that the derivation of the equation of motion is carried out; a material medium or a particle on this oriented edge can only have an acceleration (in the generalized sense) in the direction of the unit vector $\mathbf{t}$; this is considered constant on the elementary edge Γ. The principle of conservation of acceleration leads to:

$$\int_\Gamma \boldsymbol{\gamma} \cdot \mathbf{t} \, dl = - \int_\Gamma \nabla \phi \cdot \mathbf{t} \, dl + \int_\Gamma \nabla \times \boldsymbol{\psi} \cdot \mathbf{t} \, dl \tag{1}$$

where ϕ is the scalar potential of $\boldsymbol{\gamma}$ defined on points a and b and $\boldsymbol{\psi}$ is the vector potential of the acceleration carried by the unit vector $\mathbf{n}$.

The acceleration on Γ is the sum of two contributions: the first is due to a difference of potential ϕ at the ends of the edge and the second is generated by the circulation of the vector $\boldsymbol{\psi}$ along the closed contour Σ, whose flow is oriented according to $\mathbf{t}$. Thus the addition of scalars of the equation (1) translates the conservation of the acceleration considered as a scalar on the oriented edge Γ.



(a) (b)

Figure 1. *(a) Discrete geometric topology from the direct local reference frame is associated with the edge Γ of unit vector $\mathbf{t}$ whose ends a and b are distant by a length d. The dual contour Σ is orthogonal to the edge Γ. (b) A primitive planar facet $\mathcal{S}$ defined by a contour Γ^* is oriented according to the normal $\mathbf{n}$ such that $\mathbf{n} \cdot \mathbf{t} = 0$; the dual surface Δ connecting the centroids of the cells defined by the dual contour Σ is also flat.*

In this first part, acceleration is to be considered generically: it can be a mechanical acceleration, an electric current, a heat flow, etc., depending on the area of physics concerned. Likewise for the scalar potential, pressure, electrical potential, temperature, etc., and for the electromagnetic potential, curl of velocity or magnetic field.

The energy per unit of mass, of dimension $[e] = m^2 \, s^{-2}$, or rather its difference between points a and a, can be calculated directly from the acceleration under the form:

$$\int_a^b \boldsymbol{\gamma} \cdot \mathbf{t} \, dl = e_b - e_a \tag{2}$$

The energy e defined to a constant represents the directional compression effects and the polarizable orthogonal shear-rotation effects. The other effects, such as gravitation, capillarity, etc., are projected on the Γ edge and thus contribute to modifying the proper acceleration of the medium (vacuum, particle, material medium).

This acceleration, considered as an absolute quantity and qualified as "own", is equal to the sum of the external actions planned on this same edge. This postulate is the cornerstone of discrete mechanics, which abandons the notion of volume balance of continuum mechanics.

The other physical quantities follow from acceleration, as the velocity component, which is not an absolute quantity, is calculated according to the principle of Galilean relativity with $\mathbf{V} = \mathbf{V}^o + \boldsymbol{\gamma}\, dt$; the velocity at time t^o is necessary to calculate its value at time $t^o + dt$; this is the principle of causality where the future can only be predicted if we know the past.

The discrete geometric topology built from the edge Γ is then continued ine figure (1 b), linking three segments Γ defining the primal topology, a contour Γ^* and a planar facet $\mathcal{S}$ of oriented unit normal $\mathbf{n}$. The dual planar surface Δ joins the barycentres of the cells formed by the set of facets $\mathcal{S}$ having the edge Γ in common. The unit vectors are thus orthogonal by construction $\mathbf{t} \cdot \mathbf{n} = 0$; a third unit vector $\mathbf{m}$ orthogonal to the other two makes it possible to construct a local reference frame $(\mathbf{m}, \mathbf{n}, \mathbf{t})$ for each of the primal facets. Acceleration and velocity, or rather their components, are defined on each edge where they are considered to be constant. Like the scalar potential ϕ or the mass, the scalar quantities are attached to the vertices a or b of the primal geometry, and the same goes for the divergence of the velocity $\nabla \cdot \mathbf{V}$. The primal curl $\nabla \times \mathbf{V}$, calculated as the circulation of the velocity vector along the contour Γ^*, is carried by the unit normal $\mathbf{n}$. The gradient operator $\nabla \phi$ is simply defined on the edge Γ as a difference in potential at the ends, and the dual curl $\nabla \times \boldsymbol{\psi}$ projects the results of the circulation along the contour of the facet Δ on the edge Γ.

As we can see in figure (1b), the edge Γ contributes to the creation of the contour Γ^* whose circulation of velocities, on all the corresponding segments, produces the primal curl $\nabla \times \mathbf{V}$ which defines the vector potential $\boldsymbol{\psi} = -\nu \nabla \times \mathbf{V}$ corresponding to the induced acceleration, where ν is a viscosity. In addition, the gradient of the scalar potential $\phi = -r \nabla \cdot \mathbf{V}$ between a and b defines the direct acceleration, where r is a coefficient of incompressibility which will be specified. Thus the velocity $\mathbf{V}$ is the intermediate variable connecting the acceleration to its potentials.

The tessellation of the physical domain is carried out from this element to form geometric topologies of dimension two or three, based on polygons or polyhedra with any number of facets. Whatever the geometry, the properties $\nabla_h \cdot \nabla_h \boldsymbol{\psi} = 0$ and $\nabla_h \times \nabla_h \phi = 0$, where the index h designates the discrete operators, is checked exactly. In addition, the decomposition of acceleration into a divergence-free and a curl-free is orthogonal on the domain [8]. The absence of an inertial reference frame is compensated for by the local frame of reference of all the elementary geometric topologies interconnected by the vertices. The propagation of information throughout the field occurs through cause and effect with a celerity c.

2.2 Acceleration-energy formulation

The equation of motion is derived in one dimension of space on each edge Γ, where the acceleration $\boldsymbol{\gamma}$ or the other vectors can be considered as scalars assigned to this oriented edge. Contrary to the continuous medium where the notion of conservation requires that of volume, we can attribute to acceleration the property of conservation on this edge. The conservation law of the acceleration $\boldsymbol{\gamma} = \boldsymbol{h}$ of the particle or the material medium is written in the form of a Hodge-Helmholtz decomposition:

$$\boldsymbol{\gamma} = -\nabla \phi + \nabla \times \boldsymbol{\psi} \tag{3}$$

where $\boldsymbol{\gamma}$ is the own acceleration of a particle or material medium and where the right-hand side $\boldsymbol{h} = -\nabla \phi + \nabla \times \boldsymbol{\psi}$ represents the sum of the accelerations imposed by the external environment.

The modeling of each of the phenomena in each of the fields of physics [9] leads to a unique system of equations:

$$\left\{ \begin{array}{l} \boldsymbol{\gamma} = -\nabla \left(\phi^o - dt\, c_l^2\, \nabla \cdot \mathbf{V} \right) + \nabla \times \left(\boldsymbol{\psi}^o - dt\, c_t^2\, \nabla \times \mathbf{V} \right) + \boldsymbol{g}_s \\ \\ (1 - \alpha_l)\ \phi^o - c_l^2\, dt\, \nabla \cdot \mathbf{V} \longmapsto \phi^o \\ \\ (1 - \alpha_t)\ \boldsymbol{\psi}^o - c_t^2\, dt\, \nabla \times \mathbf{V} \longmapsto \boldsymbol{\psi}^o \end{array} \right. \tag{4}$$

where c_l and c_t are the longitudinal and transverse celerities of the media concerned, fluid, solid or vacuum; the velocity $\mathbf{V}$ is a function of the acceleration and the velocity at the previous instant, $\mathbf{V} = \mathbf{V}^o + \boldsymbol{\gamma}\, dt$. The $\longmapsto$ symbol corresponds to an explicit upgrade of the quantity concerned after solving the vector equation of the system (4). The factors α_l and α_t, varying between 0 and 1, correspond respectively to the attenuation of the longitudinal and transverse waves. The source term $\boldsymbol{h}_s$, for example gravitational, capillary effects, etc., will be written in the same way, in the form of a Hodge-Helmholtz decomposition [10]. All the variables and the physical parameters of this system are expressed by the two fundamental units, those of length and time.

The system of equations (4) constitutes a true model with continuous memory. The energies per unit of mass of compression ϕ^o and of shear-rotation $\boldsymbol{\psi}^o$ are upgraded at each step in time and can be described by the integrals:

$$\phi^o = -\int_0^t c_l^2\, \nabla \cdot \mathbf{V}\, d\tau; \qquad \boldsymbol{\psi}^o = -\int_0^t c_t^2\, \nabla \times \mathbf{V}\, d\tau \tag{5}$$

The two main terms of the second member of (4) form two oscillators where each of them is a Lagrangian with potential energy stored in ϕ^o (or $\boldsymbol{\psi}^o$) and a velocity-dependent disturbance. These terms? described by a gradient and a dual-curl? are orthogonal and cannot exchange energy directly; the variations of each of them will modify the proper acceleration $\boldsymbol{\gamma}$ which transfers this acceleration to the other oscillator. In a one-space dimension, for example the longitudinal direction, only the first oscillator will describe the waves of celerity c_l.

It is also necessary to note the invariance of the second member of the vector equation (4) with respect to a translation at constant velocity (Galilean transformation), but also for a rotational motion of rigid body; indeed, the operator $\nabla \cdot \mathbf{V}$ filters these motions and the dual-curl operator of $\nabla \times \mathbf{V}$ is also reduced to zero. For these two rigid motions the material derivative of the velocity is zero. According to Noether's theorem [11] these invariances reflect the fact that the laws of physics for conducting an experiment remain unchanged and that there is no absolute reference.

It is important to distinguish two physical cases corresponding to different orders of magnitude of elapsed time dt:

- the case where the observations are made with orders of magnitude of dt compatible with the physics of the phenomena $dt \approx 1/c_l^2$ for the longitudinal effects or $dt \approx 1/c_t^2$ for transverse effects. For example, for light waves, we must then adopt values of $dt < 10^{-16}\, s$ to capture the physical reality of the phenomenon;

- the case where $dt >> 1/c_l^2$ or $dt >> 1/c_t^2$ for which the wave phenomena are no longer represented with the correct time scale in order to obtain a quasi-stationary solution. The products $dt\, c_l^2$ or $dt\, c_t^2$ must then be replaced by coefficients homogeneous with a diffusivity noted ν_l or ν_t.

For example, in fluid mechanics the condition of incompressibility or in electromagnetism the Coulomb gauge, $\nabla \cdot \mathbf{V} = 0$, is obtained by considering the condition $dt >> 1/c_l^2$. The incompressibility constraint of flow or wave is ensured for arbitrarily large values of elapsed time, whatever the medium and the motion considered.

An equivalent condition for transverse effects, i.e. $1/c_l^2 << dt \rightarrow \infty$, leads to the consideration of a curl-free motion. For non-zero transverse effects it is the product $\nu_t \nabla \times \mathbf{V}$ which will represent the notion of viscosity or magnetic field via an axial vector. As ν_t is constant on the facet $\mathcal{S}$ in figure (1), the product $\nu_t \nabla \times \mathbf{V}$ is indeed with divergence-free.

Thus the celerities c_l and c_t are with the coefficients of attenuation of the waves α_l and α_t, the only physical parameters to know in order to carry out predictive simulations in different domains of physics. The celerities and the attenuation coefficients are attached to the frequencies of the waves (tide, acoustics, light, etc.) and to material media (fluids, solids, vacuum). In some cases the celerities are equal, $c_l = c_t$, but this is not generally the case. These parameters can also depend on other variables but in all cases these functions must be known.

The initial instant must correspond to a state of mechanical equilibrium, i.e. the vector equation of the system (4) must be verified identically. In practice it is possible to choose the relative rest state where the motions are associated with a translation or a rotation of rigid body. The proper acceleration is the material derivative $d\mathbf{V}/dt$ which makes it possible to express the vector equation in terms of velocity and thus implicitly all the terms. The nonlinear inertia terms contained in the material derivative will themselves be linearized.

The autonomous nature of this formulation should be emphasized here; the solution is obtained without any constitutive law or additional conservation law. In particular, the conservation of the mass is not necessary to obtain a solution, as is the case for the Navier-Stokes equation to which it is added. This formulation in $(\boldsymbol{\gamma}, \phi^o, \boldsymbol{\psi}^o)$ makes it possible to evaluate *a posteriori* the velocity, the energy, etc. It presents itself as a law of conservation of energy per unit of mass, where ϕ^o is the compression energy and $\boldsymbol{\psi}^o$ the shear-rotation energy. The energy-mass duality of the theory of relativity is also verified in discrete mechanics, where the conservation of acceleration and energy leads to that of momentum and mass.

2.3 Modelling of inertia and advection

The material derivative $\boldsymbol{\gamma} = d\mathbf{V}/dt$ can make the derivative in time, and what will be called inertia, appear κ_i in the form $\boldsymbol{\gamma} = \partial \mathbf{V}/\partial t + \boldsymbol{\kappa}_i$. This last term represents the advection of the medium. In a Lagrangian description, by following the particle during its movement the material derivative is equal to the derivative in time. In discrete mechanics inertia has a specific form [12] which cannot be deduced from one of those from the continuous medium; material derivative reads:

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + \nabla \left(\frac{1}{2} \|\mathbf{V}\|^2 \right) - \nabla \times \left(\frac{1}{2} \|\mathbf{V}\|^2 \, \mathbf{n} \right) \tag{6}$$

The inertia $\boldsymbol{\kappa}_i$ is presented as the mean curvature of the inertial potential $\phi_i = \|\mathbf{V}\|^2/2$ defined both on the vertices and on the barycentres of the facets of the geometric topology of the figure (1). Taking into account the form of the inertial terms and those of the potentials of the acceleration it is possible to reveal the Bernoulli potential$\phi_B^o = \phi^o + \phi_i$. The superposition of several velocity fields poses no problem for the linear terms of the second member of the equation of motion, but leads to additional non-linear terms for inertia. Let us consider the components $\mathbf{V}$ and $\mathbf{W}$ of vectors carried by the edge Γ of unit vector $\mathbf{t}$. The norm of this sum is equal to:

$$\|\mathbf{V} \pm \mathbf{W}\|^2 = \|\mathbf{V}\|^2 \pm 2 \, \|\mathbf{V} \cdot \mathbf{W}\| + \|\mathbf{W}\|^2 \tag{7}$$

Since the vectors $\mathbf{V}$ and $\mathbf{W}$ are collinear, we have $\|\mathbf{V} \cdot \mathbf{W}\| = \|\mathbf{V}\| \, \|\mathbf{W}\|$ and the vector $\mathbf{P} = (2 \, \mathbf{V} \cdot \mathbf{W}) \, \mathbf{t}$ is also collinear with the other two. These non-linear terms appear during the coupling of several phenomena, in magnetohydrodynamics for example.

3 Extension to electromagnetism

3.1 Unification of variables

The equation of discrete mechanics (4) can be considered as generic, the field $\mathbf{V}$ can be the velocity in fluid and solid mechanics, the electric current in electromagnetism, etc. The celerities c_l and c_t, legitimate and different in solid mechanics, are replaced by the celerity of light in the medium $c_l = c_t = c$ in the field of electromagnetism or physical optics. The objective is to show that the application of the equation to cases of mechanics can be transposed to those of electromagnetism. The number of variables and units to describe them is high, and some of them sometimes overlap. In order to unify the terminology in these different fields it is necessary to define new variables by restricting the number of units with which they are expressed. This study shows that it is possible to limit the independent variables to two, one of space and one of time.

Table (1) presents the correspondence between the unified variables described by equation (4) and those conventionally used in electromagnetism. The units of variables of the discrete equation are $[\mathbf{V}] = m\ s^{-1}$ for velocity, $[\phi^o] = [\boldsymbol{\psi}^o] = m^2\ s^{-2}$ for potentials and $[c] = m\ s^{-1}$ for celerity. With the usual units of electromagnetism, $\boldsymbol{j}$ is the density of electrical current, e is the electric potential and $\mathbf{B}$ the induction magnetic field. The other quantities are respectively charge electrical density ρ_m, permittivity ε_m, magnetic permeability μ_m and electrical conductivity σ_m.

unified	$\mathbf{V}$	ϕ	$\boldsymbol{\psi}$	$dt\ c_l^2$	$dt\ c_t^2$
electromag.	$\boldsymbol{j}/\rho_m$	$(\rho_m/\rho)\ e$	$(\rho_m/\rho)/(\mu_m\ \sigma_m)\ \mathbf{B}$	$dt/(\varepsilon_m\ \mu_m)$	$\nu_m = 1/(\mu_m\ \sigma_m)$

Table 1. *Correspondence of quantities, variables and properties, used in discrete mechanics and the usual quantities in electromagnetism whith* $\boldsymbol{j}$ *the current density,* e *the electric potential,* $\mathbf{B}$ *the induction magnetic field,* ρ_m *the electrical charge density,* ε_m *the permittivity,* μ_m *the magnetic permeability and* σ_m *the electrical conductivity.*

Maxwell's equations in their classical form give a system of four partial differential equations coupled via the electric fields $\mathbf{E}$ and magnetic fields $\mathbf{B}$; in all cases the latter is solenoidal. Consider these same properties in discrete mechanics:

- $\nabla \cdot \mathbf{B} = 0$; the rotation term $\nabla \times (\boldsymbol{\psi}^o - dt\ c_t^2\ \nabla \times \mathbf{V})$ shows the transverse celerity c_t which, by definition, is a quantity that is constant on each of the $\mathcal{S}$ facets of the primal topology, and therefore the corresponding term is divergence-free. As the discrete property $\nabla_h \cdot \boldsymbol{\psi} = 0$ is satisfied whatever the polygonal surface, the vector potential of equilibrium $\boldsymbol{\psi}^o$ (i.e. the magnetic field $\mathbf{B}$) is indeed solenoidal.

- $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ where the electric field is equal to $\mathbf{E} = -\nabla \phi - \partial \mathbf{A}/\partial t$ but the field $\mathbf{A}$ is not the vector potential of the acceleration but only of $\mathbf{B}$. In order to obtain an equivalent, the operator curl must be applied to the equation of motion to lead to $\nabla \times \boldsymbol{\gamma} = \nabla \times \nabla \times \boldsymbol{\psi}$. If the partial temporal derivative of $\mathbf{B}$ can be found, there remain other non-zero terms, notably a part of the inertia not included in Maxwell's linear equations.

The equations of electromagnetism in its different forms from the concept of continuous medium constitute a model recognized as representative of all physical phenomena, from electrostatics to relativistic electromagnetism. Two ways can therefore be envisaged to show the validity of the application of the equation (4) to electromagnetism: (i) a strict comparison to detect the differences and the equivalences between the two formalisms, (ii) direct application of the discrete equation to multiple emblematic cases of electromagnetism.

3.2 Attempt to compare with Maxwell's equations

Maxwell's equation can be written in different forms: (i) classical ways, as presented by JC Maxwell [1], (ii) in tensorial form or (iii) in the context of special relativity in the electromagnetic four-potential formulation. Whatever its form, the system of equations always derives from the concept of a continuum and the choice to satisfy Newton's second law.

The discrete topology in figure (1) cannot be reduced to a point, for the following reasons: (i) the reference frame $(\mathbf{m}, \mathbf{n}, \mathbf{t})$ local is not linked to a global reference frame, (ii) the dimension of the edge Γ can be reduced but the primal surface $\mathcal{S}$ will remain polygonal, (iii) the elapsed time dt is discrete, (iv) the derivation at one point, integration, analysis, etc. are all abandoned. So it is difficult to deduce a partial differential equation for electromagnetism comparable to Maxwell's equations. However, one can try to transpose the discrete equation (4) in a Lagrangian formulation where the material derivative is equal to the partial derivative. Furthermore, we suppose that the celerity c is constant and that the operator $(1/dt)\, \partial \mathbf{V} / \partial t$ is transposed into a second derivative; we also set $\widehat{\phi^o} = \phi^o / c^2 \, dt$ and $\widehat{\boldsymbol{\psi}^o} = \boldsymbol{\psi}^o / c^2 \, dt$. In order to reveal the Laplacian operator, we use:

$$\nabla^2 \mathbf{V} = \nabla (\nabla \cdot \mathbf{V}) - \nabla \times \nabla \times \mathbf{V} \tag{8}$$

The Laplacian operator $\nabla^2 \mathbf{V}$ has no legitimacy in discrete mechanics; only $\nabla(\nabla \cdot \,)$ and $\nabla \times (\nabla \times \,)$ represent admissible operations on the stencil in figure (1). Indeed, these operators physically represent different phenomena, the first related to the unidirectional compression of the longitudinal waves and the second the transverse propagation of the shear-rotation waves. Thus the relation $\nabla (\alpha \, \nabla \cdot \mathbf{V}) - \nabla \times (\beta \, \nabla \times \mathbf{V})$ cannot be reduced to a Laplacian if α and β are arbitrary functions. The equations of physics written in terms of Laplacians implicitly associate these phenomena with no further possible distinction. One of the only ones to keep the distinction is the Navier-Lamé equation in solid mechanics.

The discrete equation is then split by considering two cases, the first for a divergence-free and the second for a curl-free. The application of the relation (8) does however reveal the d'Alembertian:

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{9}$$

to obtain:

$$\begin{cases} \dfrac{1}{c^2} \dfrac{\partial^2 \mathbf{V}}{\partial t^2} - \nabla^2 \mathbf{V} = - \nabla \phi^o \quad & if \quad \nabla \times \mathbf{V} = 0 \\[2ex] \dfrac{1}{c^2} \dfrac{\partial^2 \mathbf{V}}{\partial t^2} - \nabla^2 \mathbf{V} = \nabla \times \boldsymbol{\psi}^o \quad & if \quad \nabla \cdot \mathbf{V} = 0 \end{cases} \tag{10}$$

where ϕ^o and $\boldsymbol{\psi}^o$ are the retarded potentials similar to those of Liénard-Wichert [13].

These equations are comparable to wave equations in vacuum $\Box \, \mathbf{E} = 0$ and $\Box \, \mathbf{B} = 0$, where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field or, in the covariant form, $\Box \, \mathcal{A}^\mu = 0$ where $\mathcal{A}^\mu$ is the electromagnetic four-potential. Significant differences remain; first of all the d'Alembertians of the system (10) relate only to the unified velocity variable $\mathbf{V}$ and not the electric and magnetic fields.

In electromagnetism, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ have the same status within wave equations. The electric field deduced from the potential ϕ is written $\mathbf{E} = -\nabla \phi$, giving rise to hope for a form close to the discrete formulation where this field corresponds to the curl-free component of the acceleration. On the other hand, the magnetic field is not its divergence-free component, it is itself what is described by a vector potential $\mathbf{A}$ in the form $\mathbf{B} = \nabla \times \mathbf{A}$. This vector potential is not that of acceleration; it is $\mathbf{B}$ which represents the solenoidal component of acceleration, $\boldsymbol{\gamma} = -\nabla \phi + \nabla \times \mathbf{B}$ to the nearest units ($\mathbf{B}$ is expressed in tesla while $\boldsymbol{\psi}^o$ is expressed in $m^2 \, s^{-2}$). Table (1) makes it possible to replace the variables of electromagnetism by the unified variables depending only on the units of length and time. The vector potential $\mathbf{A}$ of the magnetic field is defined only to ensure incompressibility in the field $\mathbf{B}$. The duality of classical electrodynamics between $\mathbf{E}$ and $\mathbf{B}$ does not ensure the consistency and symmetry of a true Hodge-Helmholtz decomposition. The induced magnetic field $\mathbf{B}$, like the potential ϕ^o for the compression effects, allows us to store the energy of rotation, in particular to represent the permanent magnetization with $\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right)$ where $\mathbf{H}$ is the magnetic field and $\mathbf{M}$ is the magnetization.

In discrete mechanics, the scalar potential ϕ^o is deduced from the divergence $\nabla \cdot \mathbf{V}$, and the vector potential $\boldsymbol{\psi}^o$ is upgraded by the primal curl $\nabla \times \mathbf{V}$ of the same variable velocity. In fluid mechanics or in mathematics, the construction of a field with zero divergence is carried out starting from a projection, as for the magnetic field, but that is not sufficient to consider that the corresponding potential is related to an absolute fundamental quantity like acceleration.

From the physical point of view, the components $\mathbf{V}$ of velocity represent the current directly produced by the potential difference at the terminals a and b of the segment Γ, but also that which is induced by the circulation of currents in each of the turns of the contour Γ^* in figure (1). The variable $\mathbf{V}$ is the quantity represented by the direct and induced currents in a unified description.

At the end of the confrontation of the two formalisms, it must be admitted that it is not possible to show that the discrete equation is reduced to Maxwell's equations. What inhibits transformation are the essential differences between the discrete and continuous approaches. This does not call into question the representativeness of the two formalisms, but there remains only the path of comparisons with experimental observations to decide on the merits of discrete mechanics to represent the phenomena of electromagnetism.

4 Numerical methodology

The numerical methodology presented for the application of the discrete model to fluid mechanics [14] is taken again for electromagnetism insofar as the equations are the same. This return to a discrete form of the equation of motion from its continuous expression (4) is necessary because the latter is not sufficiently explicit, as the primal and dual curls are represented by the same symbol $\nabla \times$; when applied to velocity vector $\mathbf{V}$ it is a primal curl and when applied to potential vector $\boldsymbol{\psi}$ it is a dual curl. To clarify this aspect and give a convincing geometric sense, the physical model is transformed using the concepts of mimetic methods [15].

This geometric view of the differential operators of physics was used by Hyman et al. [16, 17, 18] to rewrite Maxwell's equations. These discrete operators were derived to satisfy discrete analogs of the main vector calculus theorems. The authors described the mimetic discretizations of Maxwell's equations in terms of magnetic flux density $\mathbf{B}$ and electric field intensity $\mathbf{E}$, where the magnetic field satisfies $\nabla \times \mathbf{B} = 0$.

A very complete review of mimetic [19] methods encompasses applications from different fields of physics, fluid mechanics and the Navier-Stokes equation or Darcy's law, electromagnetism and Maxwell's equations, and Lagrangian mechanics for particle motions. This point of view aimed at unifying numerical methods for different branches of physics is the one adopted here for physical modeling.

It is possible to build a mimetic approximation of the Maxwell's equations from the classical continuous form; to do this we need to include the dielectric properties of the medium, magnetic permeability μ, and permittivity ε within the primal and dual operators to conserve certain properties, due to the specific form of the Helmholtz-Hodge decomposition.

Considering that $c_l = c_t = c$, the celerity of light, the model (4) is rewritten in terms of discrete operators in the form:

$$\begin{cases} \boldsymbol{\gamma} = -\mathcal{GRAD}\left(\phi^o - dt\, c_l^2\, \widetilde{\mathcal{DIV}}\, \widetilde{\mathbf{V}}\right) + \widetilde{\mathcal{CURL}}\left(\widetilde{\boldsymbol{\psi}^o} - dt\, c_t^2\, \mathcal{CURL}\, \mathbf{V}\right) + \boldsymbol{g}_s \\ (1-\alpha_l)\ \phi^o - c_l^2\, dt\, \widetilde{\mathcal{DIV}}\, \widetilde{\mathbf{V}} \longmapsto \phi^o \\ (1-\alpha_t)\ \widetilde{\boldsymbol{\psi}^o} - c_t^2\, dt\, \mathcal{CURL}\, \mathbf{V} \longmapsto \widetilde{\boldsymbol{\psi}^o} \end{cases} \tag{11}$$

where $\widetilde{\mathcal{GRAD}}$, $\widetilde{\mathcal{DIV}}$ and $\widetilde{\mathcal{CURL}}$ are the derived operators satisfying the properties $\mathcal{DIV}\mathcal{CURL} = 0$ and $\widetilde{\mathcal{DIV}}\widetilde{\mathcal{CURL}} = 0$ as well as $\mathcal{CURL}\mathcal{GRAD} = 0$ and $\widetilde{\mathcal{CURL}}\widetilde{\mathcal{GRAD}} = 0$, as demonstrated in [19]. Of the six operators, only four are retained in a vertex-based circulation approach. Mimetic discretizations can be applied in the same way to differential operators $\mathcal{GRAD}$, $\mathcal{DIV}$, $\mathcal{CURL}$ or their derived operators.

However, many aspects differentiate the proposed physical model from Maxwell's equations, (i) the absence of a global frame of reference in favor of local frames of reference interacting from cause to effect, (ii) the strong coupling in the form of two Lagrangians of direct and induced effects, (iii) the emergence of a single unknown, the current $\mathbf{V}$, instead of two fields, the electric potential $\mathbf{E}$ and induced magnetic field $\mathbf{B}$, (iv) the upgrade of the two acceleration potentials, ϕ and $\boldsymbol{\psi}$ respectively, by the operators $\widetilde{\mathcal{DIV}}\, \widetilde{\mathbf{V}}$ and $\mathcal{CURL}\, \mathbf{V}$.

The hypothesis $\nabla \cdot \mathbf{B} = 0$ expressing the absence of magnetic monopoles in classical electromagnetism [20, 21] is not taken up in discrete mechanics as it would be verified as a consequence only if the celerity of waves was infinite. The vector potential $\boldsymbol{\psi}^o$ would in this case be a curl and its divergence indeed equal to zero. In the general case, the upgrade of $c^2\, dt \mathcal{CURL}\, \mathbf{V}$ in time is not a curl. J.C. Maxwell's idea [1] of coupling the classical laws of electricity and magnetism through a time dependence translates here into an interlacing of direct and induced currents, which we can refer to as dynamic entanglement.

Despite these significant differences in the physical model, the numerical methodology used is very similar to the mimetic discretization and Discrete Exterior Calculus approaches. The system of equations (4) finds an ideal mathematical framework to build the discrete model.

5 Verifications

The discrete motion equation (4) is assumed to be an alternative model to Maxwell's equations. As it is impossible to reduce the differences between the two formalisms, it is proposed to submit the new model to emblematic test cases of electromagnetism and to classical validation cases in computational electromagnetics.

5.1 Magnetic field around a permanent magnet

The first magnetostatic example consists in making a permanent magnet and calculating the residual magnetic field. Equation (4) is a continuous memory model which includes the possibility of representing the permanent magnetization of a ferromagnetic metal. Consider the case of the magnetic field $\mathbf{B}$ created by a permanent magnet $\mathbf{M}$. The induced magnetic field $\mathbf{B} = \mu_0\, (\mathbf{H} + \mathbf{M})$, where $\mathbf{H}$ is the exitation magnetic field, $\mathbf{M}$, is the magnetization field and

μ_0 is the permeability of vacuum; in the case $\mathbf{H} = 0$, the magnetic field $\mathbf{B}$ is the residual flux density.

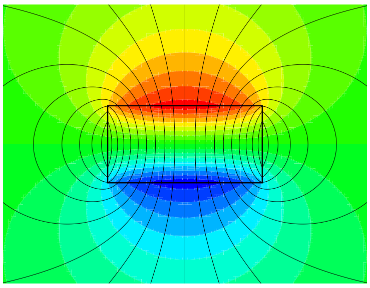


Figure 2. *A magnet made of a ferromagnetic material represented by the rectangle produces a magnetic field on the outside. The field imposed in the magnet is equal to $\psi^o = y\, \mathbf{e}_z$. The magnetic field throughout the field is represented by a gradient of colors and the equipotentials by lines.*

Magnetization is achieved by applying a magnetic field to a piece of ferromagnetic material. The system of equations (4) is solved by maintaining a constant field $\mathbf{B}$ and by ensuring the upgrade of the equilibrium vector potential $\boldsymbol{\psi}^o$ with the attenuation factor transverse $\alpha_t = 1$. At the end of this phase the magnet is permanent and $\boldsymbol{\psi}^o$ maintains a constant $\mathbf{M}$ field inside the magnet. The value of the field ψ^o can be obtained with the classical variables of electromagnetism, that is, $\boldsymbol{\psi}^o = (\rho_m/(\rho\, \sigma\, \mu_0))\, \mathbf{B} = (\rho_m/(\rho\, \sigma))\, \mathbf{M}$. The second phase of the experiment consists in looking for the induced magnetic field $\mathbf{B}$ outside the magnet in the vacuum or in the air, always with (4), by maintaining a field $\boldsymbol{\psi}^o = y\, \mathbf{e}_z$ generating a velocity equal to $\mathbf{V} = \nabla \times \psi^o = 1\, \mathbf{e}_x$ inside the solid domain. The scalar potential ϕ^o upgraded over time by the divergence of the velocity is not zero but it does not influence the magnetic field; this velocity field becomes divergence-free (Coulomb gauge) inside but also outside the magnet.

Figure (2) shows the magnetic field generated by a permanent magnet in a vacuum. This field $\boldsymbol{\psi}$ is indeed with zero divergence and corresponds to that which gave birth to it if the material is perfectly remanent.

We can see that the current lines of the magnetic flux $\boldsymbol{\psi}^o$ and the equipotentials of ϕ^o are indeed orthogonal.

5.2 Skin effect in a cylindrical conductor

When medium and high frequency alternating currents flow in a conductor, they present a skin effect due to opposing eddy currents induced by the changing magnetic field resulting from these alternating currents. The current density is found to be greatest at the conductor's surface. In good conductors such as metals, the skin depth is given by:

$$\delta = \sqrt{\frac{2}{\omega\, \mu_m\, \sigma_m}} \tag{12}$$

where μ_m is the magnetic permeability of material, σ_m is the conductivity of the conductor and $\omega = 2\, \pi\, f$ the angular frequency of current.

Let us consider the case of a cylindrical conductor of indefinite length and radius R subjected to an alternating current of frequency f. The solution to this problem is obtained from Maxwell's equations where time-dependent terms are used. The current density has three components in a cylindrical orthogonal coordinate system but only the component following z is non-zero, $\mathbf{I} = (0, 0, I_z)$. By setting $\nu = 1/\mu_m\,\sigma_m$, the equation reduced to a component of the vector Laplacian is written:

$$\frac{\partial I_z}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left(\nu\, r \frac{\partial I_z}{\partial r}\right) \tag{13}$$

For a temporal resolution in the general case, equation (13) can be solved by separating the variables of time and space to give a solution of the form:

$$I_z(z,t) = \sum_{n=1}^{\infty} \left[A_n J_0(k_n r) + B_n K_0(k_n r)\right]\; exp\{-k_n^2\, t\} \tag{14}$$

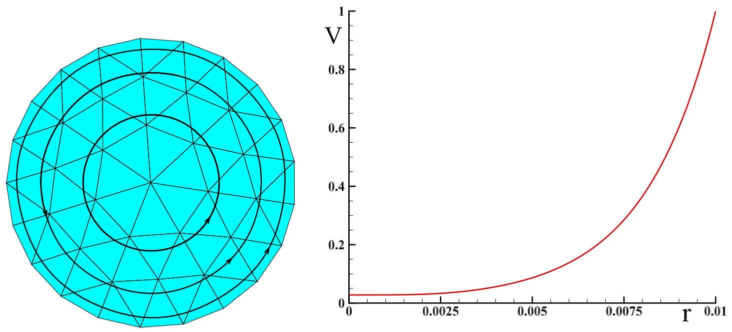
where the families of constants (A_n, B_n, k_n) are to be determined by the boundary conditions and the initial condition. In an established periodic regime it is possible to set $I_z(r,t) = \Re\{\underline{I_z}(r)\; exp(j\,\omega\, t)\}$ and the solution is then:

$$\underline{I_z}(r) = \underline{A} J_0(kr) + \underline{B} K_0(kr) \tag{15}$$

where $\underline{A}$ and $\underline{B}$ are complex constants to be fixed from the mean current to be imposed so as to obtain a current at the surface in $r = R$ equal to I_M, and of the boundary condition in $r = 0$. For the case of the cylindrical conductor, the solution is written:

$$I_z(r) = I_M\, \frac{J_0(kr)}{J_0(kR)} \tag{16}$$

The quantities of electromagnetism $\sigma_m\;\mu_m$ are replaced here by $\nu = dt\; c_t^2$. The nonlinear inertia terms are not retained in this presentation but they exist, even if they are not present in the Maxwell equations. This equation, deduced from the vector equation of the system (4), can be solved in time or in frequency space. The numerical solution of this problem is sought directly without the condition of a gauge from the equation of the discrete motion (4), because it serves to obtain the solution to the problem, the current density $\mathbf{V}$ and the magnetic field $\boldsymbol{\psi}$, while maintaining the field $\mathbf{V}$ with zero divergence $\nabla \cdot \mathbf{V} = 0$.



(a) (b)

Figure 3. *(a) Magnetic field current lines for an unstructured coarse mesh of* 63 *meshes based on regular triangles. (b) Reduced effective current density induced by an alternating electric current of frequency f in a cylindrical conductor.*

The simulation was carried out in time mode and the effective current density is calculated over a large number of periods. The parameters of the simulation are $\nu = 1/(\mu_m\,\sigma_m) = 10^{-2}$,

$f = 1000$, $R = 10^{-2}$ and $dt = 10^{-6}$. For these values, the skin depth is equal to $\delta = 1.78\,10^{-3}\,m$. The quantitative comparison was not carried out, but we expect a convergence in time and space of order two, like all the other simulations carried out with this formulation. The simulations can be performed in 1D, in (r, z), 2D in (r, θ), 3D in structured or unstructured mesh (polygonal, polyhedral) whose cells have any number of planar faces. The vector potential field $\boldsymbol{\psi}^o(r)$ in figure (3 a) represents the induced magnetic field $\mathbf{B}$ with divergence-free in the plane (r, θ) of a cylindrical conductor. The unstructured mesh based on very coarse regular triangles is used to show the appearance of magnetic flux lines. The effective current (velocity) in the conductor is associated with the mean magnetic field $\boldsymbol{\psi}^o$; the equation corresponding to the steady state is written:

$$\nabla \times (\boldsymbol{\psi}^o - \nu\, \nabla \times \mathbf{V}) = 0 \tag{17}$$

Figure (3 b) shows the skin effect, the non-uniform distribution of the current density for an alternating current of moderate frequency.

5.3 Trajectory of an electron in a magnetic field

The case treated here corresponds to the interaction between an electron beam emitted at constant velocity $\mathbf{V}$ and a constant magnetic field $\mathbf{B}$ produced by Maxwell coils. The device is that shown diagrammatically in figure (4), where a source emits electrons of charge e^- at an average velocity lower than the celerity of light in the direction orthogonal to the surface $\mathcal{S}$. The interaction between the electron beam and the magnetic field results in an acceleration $\boldsymbol{\gamma}_e$ orthogonal to the plane defined by the vector product $\mathbf{V} \times \mathbf{B}$. As the fields $\mathbf{V}$ and $\mathbf{B}$ are stationary, the acceleration $\boldsymbol{\gamma}_e$ is constant and the jet is deflected according to a circular trajectory whose direction depends on the direction of $\mathbf{B}$.

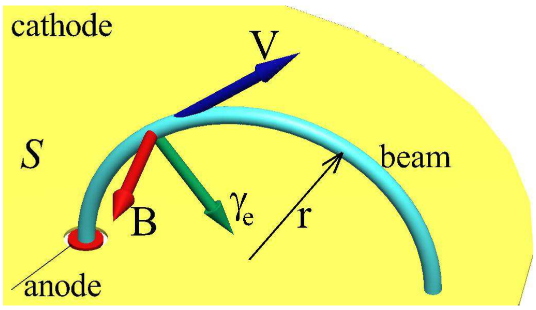


Figure 4. *Electron gun ejecting electrons at the velocity* $\mathbf{V}$*. A constant magnetic field* $\mathbf{B}$ *bends the beam along a circular trajectory of radius* r*.*

Consider the interaction between the magnetic field $\mathbf{B}$ and the velocity field $\mathbf{V}$. The magnetic field being with zero divergence, it is written according to the velocity $\mathbf{W}$ in the form:

$$\mathbf{B} = \nabla \times \left(\frac{\rho}{\rho_e}\, \mathbf{W} \right) \tag{18}$$

where ρ_e is the electric density and ρ mass density.

The equation of stationary motion (4) is first simplified by the elimination of viscous and compression effects, largely negligible within the electron beam; the non-linearities of the induced field $\mathbf{W}$ are also neglected. There remains the inertia specific to the electron beam and the interaction with the induced field:

$$\nabla \times \left(\frac{1}{2}\, \|\mathbf{V}\|^2\, \mathbf{n} \right) - \nabla \times (\mathbf{V} \cdot \mathbf{W}\, \mathbf{n}) = 0 \tag{19}$$

Let $\mathbf{V} = 2\,\mathbf{W} = \rho_e/\rho\,\mathbf{A}$ and $\mathbf{B} = \nabla \times (\rho/\rho_e\,\mathbf{V})$; the magnetic field in turn induces a current (a velocity) in the electron beam of equal intensity to keep the balance between the mechanical inertia of the jet and the action of the magnetic field. By setting $B = \mathbf{B} \cdot \mathbf{e}_z$ and integrating equation (18), we obtain the solution in a suitable frame:

$$v = \frac{\rho_e}{\rho}\, B\, r\, \mathbf{e}_\theta \tag{20}$$

Or, by setting $v = \|\mathbf{V}\|$, we obtain the radius r corresponding to the trajectory of the electron beam $r = \rho\, v/(\rho_e\, B)$ or $r = \rho\, v/([e^-/m]\, B)$ where e^- is the charge of the electron and m its mass; the accepted value of e^-/m is $1.758820 C/kg$. The experiment consists in measuring r for a constant field $\mathbf{B}$ and deducing from it e^-/m. This analytical example underlines the importance of the quantities defined per unit of mass and helps to justify the abandonment of this quantity for the equations of physics.

5.4 Electromagnet coil

The N turns of a cylindrical coil of axis z, of radius R and of length L are traversed by a current of intensity I creating a magnetic field $\mathbf{H}$ whose component on θ depends on the variables (r, z). The objective is to find the magnetic field and the electric field from the system of equations (4), in the context of unified variables transposing the magnetic field $\mathbf{H}$ into vector potential $\boldsymbol{\psi}$ and the electric field defined by the electric potential e transposed into scalar potential ϕ.

The current in the coil is represented by a source term S_b, constant on a rectangle of the plane (r, z) of length $L = 0.1\, m$ and of area $\mathcal{A} = 10^{-3}\, m^2$. The imposed source term is equal to $S_b = 10^3\, \mathbf{e}_z$, which serves to calculate a parameter named δ from the intensity of the current and the number of turns of the coil, that is, $\delta = N\, I/2\, L = 125$. This value is associated with the value of the linearly dependent magnetic field of r created in the coil of average radius $R = 0.025\, m$.

Figure (5) shows the axisymmetric solution of the component on θ of the magnetic field $\boldsymbol{\psi}(r, z)$ and of the electric potential $\phi(r, z)$ around the coil. The limits of the domain have been chosen sufficiently distant from the coil so as to obtain a solution independent of these.

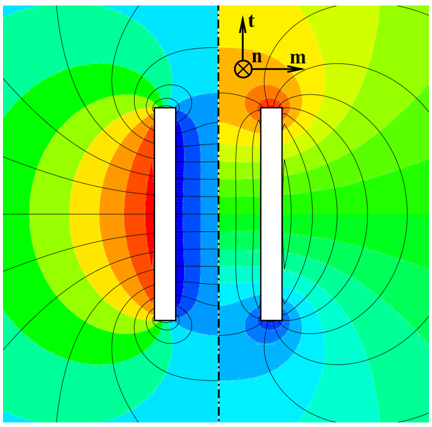


Figure 5. *Electromagnet coil where* $\mathbf{n} = \mathbf{t} \times \mathbf{m}$ *is the unit vector orthogonal to* (r, z) *plane;* $\mathbf{E}$ *following* $\mathbf{m}$, $\mathbf{B}$ *(*$\boldsymbol{\psi}$*) according to* $\mathbf{n}$ *and* $\mathbf{j}$ *(*$\mathbf{V}$*) following* $\mathbf{t}$

The solution converges in a few iterations to obtain a current field $\mathbf{V}$ with zero divergence at machine precision. At each point, the orthogonality of the electric current lines and the magnetic flux lines is verified. As expected, in the coil, the magnetic field is practically uniform according to z, and the electric potential varies linearly; as a first approximation their evolutions can be written in the form:

$$\begin{cases} \phi^o(z) = -\delta\, z \\ \\ \boldsymbol{\psi}(r) = \nu\, \nabla \times \mathbf{V} = \dfrac{\delta}{2}\, r\, \mathbf{e}_\theta \end{cases} \tag{21}$$

Throughout the domain the stationary solution $(\phi, \boldsymbol{\psi})$ depends on (r, z). It satisfies the following equation:

$$-\nabla \phi^o - \nabla \times (\nu\, \nabla \times \mathbf{V}) = 0 \tag{22}$$

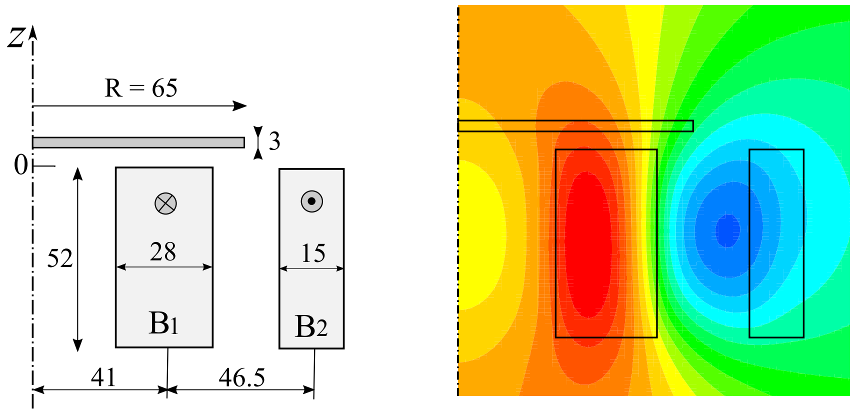
The solution obtained by the formalism presented is, once transposed, identical to that of the classical equations of electromagnetism. However, several fundamental conceptual differences appear from the derivation of the equations. The case presented or similar cases are generally associated with the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ where $\mathbf{A}$ is the vector potential of the magnetic field. Here the constraint $\nabla \cdot \mathbf{V} = 0$ is not strictly guaranteed; the order of magnitude of $\nabla \cdot \mathbf{V}$ depends on the celerity of the wave c; even if $c = c_0$ the celerity of light in a vacuum, the constraint is not imposed *a priori* but obtained during the resolution. Likewise, the generalized magnetic field $\boldsymbol{\psi}^o = \nu\, \nabla \times \mathbf{V}$ is indeed with zero divergence since ν is a constant on each facet of the primal geometric topology, but $\boldsymbol{\psi}^o$ is the vector potential of acceleration. During the resolution all the nonlinear terms of the equation of the motion were retained, even though they are negligible in the case test proposed. Finally, the solution to an electromagnetism problem in the discrete context is always represented by the gradient of a potential ϕ^o and the dual curl of a vector potential $\boldsymbol{\psi}^o$, two fields orthogonal on the whole of the selected geometric topology.

5.5 A levitation test case

The selected test problem was the subject of a code validation benchmark in 2008, based on the experiences of Kurz et al. dating from 1996 [22], [23]. An aluminum cylinder defined by the domain Ω_c of conductivity $\sigma = 3.4\, 10^7\, S\, m^{-1}$ and of mass $m = 0.107\, kg$ is set in levitation using two coaxial cylindrical coils; the electrodynamic levitation device is shown in figure (6 a). The internal coil produced from $N = 960$ turns and the external coil includes $N = 576$ turns; they are traversed by a sinusoidal current of opposite direction $I(t) = \pm I_0\, sin(2\pi f_c t)$ with $I_0 = 20\, A$ and $f_c = 50 Hz$. The source term in the coils is given by $S_b = N/\mathcal{A}\, I(t)$ where $\mathcal{A}$ is the area of the cross section of each coil in the plane (r, z). The plate is positioned at $z = 3.8\, 10^{-3} m$ at the initial time.

The simulation is performed on a domain Ω large enough to avoid disturbances caused by the boundary conditions; these are of Neumann type, homogeneous on the velocity. The mesh is composed of $n = 8\, 10^4$ quadrangles to represent the plane (r, z) in the axisymmetric cylindrical coordinate. The time step is that of the discrete model $dt = 10^{-4} s$ and the simulation represents a total time of $1.8\, s$.

Several strategies for modeling phenomena are possible, from direct monolithic simulation coupling all the equations of electromagnetism and mechanics to a more or less partial decoupling of these. It should first of all be noted that the frequency of the current $f_c = 50 Hz$ is much higher than that of the mechanical oscillations generated by the competition between the gravitational acceleration and the acceleration due to the electrodynamic effects. It is then possible to decouple electromagnetism from mechanical actions by making time averages over a sufficient number of periods on the magnitudes, the velocity and the scalar and vector potentials. The commonly



(a) (b)

Figure 6. *Levitation of a cylindrical plate; (a) electrodynamic levitation device with dimensions in mm; (b) average scalar potentiel* $\overline{\phi^o}(r,z)$.

accepted assumption is that the induced magnetic field is not impacted by mechanical oscillations. Consequently, the filtered potentials $\overline{\phi^o}$ and $\overline{\boldsymbol{\psi}^o}$ will serve to calculate the acceleration directed upwards, generated by the alternating current in the coils. The unsteady equations (4) including the nonlinear inertia terms are temporally integrated. Figure (6 b) shows the field of the average scalar potential $\overline{\phi^o}(r,z)$ for a coil-to-plate distance equal to $z = 1.1\,10^{-3}\,m$. The values of the potential which decreases with z in the zone of the plate clearly show that the force exerted goes down with increasing height z.

The mechanical force exerted by the magnetic field on the plate is conventionally expressed by the integral on Ω_c of the vector product $\mathbf{J} \times \mathbf{B}$. In discrete mechanics, this vector product corresponds to one of the nonlinear terms of the equation of motion $\nabla \times (\|\mathbf{V}\|^2/2)$, which is also an acceleration. These nonlinear effects are found in the vector potential $\overline{\boldsymbol{\psi}^o}$, but also in the scalar potential ϕ^o. The electromagnetic acceleration on the plate in coordinates (r,θ,z) can therefore be calculated from the acceleration $-\nabla\overline{\phi^o}$:

$$\boldsymbol{h}_m(z) = -\frac{1}{[\Omega_c]} \int_{\Omega_c} \nabla\overline{\phi^o}\, dv \tag{23}$$

Taking into account the mean field $\overline{\phi^o}(r,z)$ calculated previously, we find that $\boldsymbol{h}_m(z)$ decreases when the distance between the coils and the plate increases; at the first order we can estimate its variation by the law $\boldsymbol{h}(z) = -1.7 10^3\, z + 29.2$. The acceleration of gravity is equal to $\mathbf{g} = -9.81\,\mathbf{e}_z$. If $\boldsymbol{v}$ is the vertical velocity of the levitation cylindrical plate, we can calculate its evolutions over time with the fundamental law of mechanics in terms of accelerations $d\boldsymbol{v}/dt = \boldsymbol{h}_m(z) - \mathbf{g}$, with $d\boldsymbol{z}/dt = \boldsymbol{v}$. In this case the solution of the vertical position of the plate obtained with a second order scheme in time oscillates indefinitely around its equilibrium position, equal to $z = 11.4\,10^{-3}\,m$ with a frequency of $f = 6.5\,Hz$. Second-order induced electromagnetic effects or mechanical effects can cause the amplitude of the oscillations observed experimentally to decrease [22], [23], [24].

The geometric confuguration of the electrodynamic levitation device shows the proximity of the first coil and the plate; the average acceleration on the first cycle is of order of magnitude of $\boldsymbol{v} \approx 0.14\,m\,s^{-1}$, but an acceleration of the order of magnitude of $(2\pi f)^2$, or nearly 150 times that of gravity. It is likely that damping effects are due to positive and negative pressure variations between the plate and the coils.

In order to specify the influence of air flows on the motion of the levitating plate over time, it is necessary to solve the equation of movement by considering that the coils are solid obstacles and that the plate is animated by an upward motion of velocity $\boldsymbol{v}$. The solution in air is described by equation (4) where $\mathbf{V}$ is the velocity of the fluid, $\phi_B^o = p/\rho$ is the scalar potential of Bernoulli, p the pressure and ρ the density, $\nu = \mu/\rho$ is the kinematic viscosity; the celerity c_l is here that of sound in the air. The equation becomes:

$$\frac{\partial \mathbf{V}}{\partial t} - \nabla \times \left(\frac{\|\mathbf{V}\|^2}{2}\, \mathbf{n} \right) = -\nabla \left(\phi_B^o - dt\, c_l^2\, \nabla \cdot \mathbf{V} \right) - \nabla \times \left(\nu\, \nabla \times \mathbf{V} \right) \tag{24}$$

The potential field ϕ^o (pressure) in a limited area around the plate is given in figure (7) for a time $t_0 = 0.04\, s$, for which the plate is located at a height of $z = 11\, 10^{-3}\, m$. From the instant solution it is possible to evaluate the pressure difference generated by the rise of the plate at an imposed velocity of $\boldsymbol{v} = 0.05\, m\, s^{-1}$ for air considered as a compressible perfect gas of celerity $c_l \approx 340\, m\, s^{-1}$.

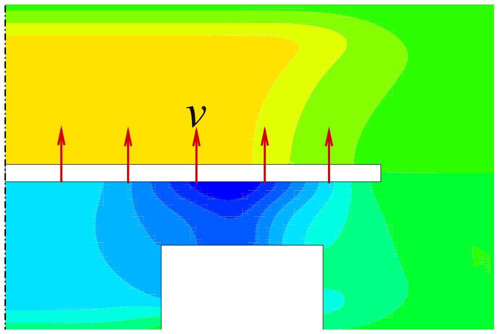


Figure 7. *Levitation of an aluminum plate; instantaneous pressure field at time $t = t_0 + 8\, 10^{-5}\, s$ in the vicinity of the plate during its upward motion. The average pressure difference on either side of the plate is equal to $\Delta\phi^o = [-35,\ 15]$ or $\Delta p = [-41,\ 18] \approx 60\, Pa$.*

The elapsed time $\tau = 8\, 10^{-5}\, s$ allows the wave to propagate beyond the distance separating the plate from the inner coil. We observe coherent phenomena where the pressure is higher on the upper part of the plate, but especially, a significant depression between the two elements. From the mass $m = 0.107\, kg$ of the plate and its surface it is possible to calculate the negative instantaneous acceleration due to the pressure difference Δp or $\gamma_p \approx 20\, m\, s^{-2}$. Compared to electromagnetic acceleration and that of gravity, that due to the pressure difference is not negligible. Thus the coupling between the plate and the internal coil acts as a damper: when the plate rises the coil exerts a negative acceleration and when it descends an overpressure induces a positive acceleration. Even if they only represent a small part of this contribution, viscous motions produce dissipation which is transformed into heat. As the velocities considered are low, these effects depend, as a first approximation, linearly on the velocity of the plate $\boldsymbol{v}$. In order to model these phenomena, the linearity hypothesis is adopted: the damping is described by a constant η. The oscillator is thus composed of the accelerations due to electromagnetic fields, gravity and that of damping in the form:

$$\left\{ \begin{aligned} & \frac{d\boldsymbol{v}}{dt} = \boldsymbol{h}_m - \mathbf{g} - \eta\, \boldsymbol{v} \\ & \frac{d\boldsymbol{z}}{dt} = \mathbf{v} \end{aligned} \right. \tag{25}$$

In the absence of a complete simulation of the mechanical and electromagnetic effects, the constant η has been chosen to best represent the decrease in oscillations; only the frequency

$f = 6.5\ Hz$ is calculated from the electromagnetic simulation. Figure (8) makes it possible to compare the result of the model with those obtained experimentally [22], [23].

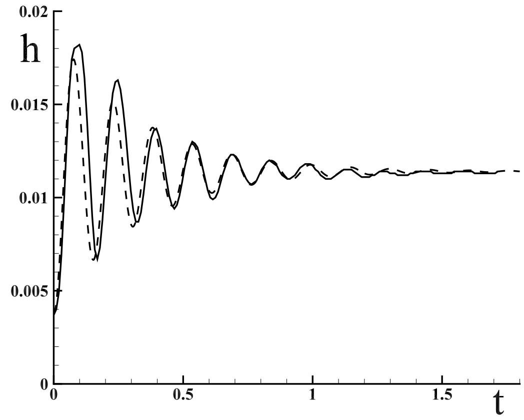


Figure 8. *Levitation of a cylindrical plate; height measured (solid line) [22], [23] versus computed (dash line)*

The results of the simulations carried out using the proposed physical model are consistent with those of the literature, for example [24]. This analysis makes it possible to understand the role of the unsteady compressible flows of air around the levitating plate, but it is only qualitative. Fluid structure interactions coupled with electromagnetism can be treated in a monolithic way by considering that the equations of electromagnetism and those of mechanics are the same. However, the differences in orders of magnitude between the different frequencies of the problem make this approach more difficult to implement and are no longer part of the framework of validation of the physical model presented.

6 Conclusions

The physical model of electromagnetism adapted from that of discrete mechanics [9, 8, 14] proves to be representative of the electrodynamic and magnetic effects, which are well formalized by Maxwell's equations. Unlike these, the discrete motion equation immediately integrates the nonlinear inertia terms represented separately by the Lorentz force in Maxwell's equations. Furthermore, the current $\mathbf{J}$ and the vector potential $\mathbf{A}$ of the magnetic field $\mathbf{B}$ are not independent; they are represented by a single quantity, the velocity $\mathbf{V}$, in the new formalism. The four equations of Maxwell's initial formalism are reduced to a single equation on acceleration and its potentials.

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