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▶ **To cite this version:**

Georgi Vodev. APPROXIMATION OF THE ELASTIC DIRICHLET-TO-NEUMANN MAP. Inverse Problems and Imaging , 2023, 17 (1), pp.297-318. hal-03523364

HAL Id: hal-03523364
https://hal.science/hal-03523364

Submitted on 12 Jan 2022

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APPROXIMATION OF THE ELASTIC DIRICHLET-TO-NEUMANN MAP

GEORGI VODEV

Abstract. We study the Dirichlet-to-Neumann map for the stationary linear equation of elasticity in a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary. We show that it can be approximated by a pseudodifferential operator on the boundary with a matrix-valued symbol and we compute the principal symbol modulo conjugation by unitary matrices.

Key words: linear equation of elasticity, Dirichlet-to-Neumann map.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded, connected domain with a C^∞ smooth boundary $\Gamma = \partial\Omega$, and consider the stationary isotropic linear equation of elasticity

$$\begin{cases} (\Delta_{\lambda,\mu} + \tau^2 n(x))u = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \Gamma, \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{C}$, $\operatorname{Re}\tau > 0$, $|\tau| \gg 1$, $u = (u_1, ..., u_d)$, $f = (f_1, ..., f_d)$, and $\Delta_{\lambda,\mu}$ denotes the elastic Laplacian defined by

$$(\Delta_{\lambda,\mu} u)_i = \sum_{j=1}^d \partial_{x_j} (\sigma_{ij}(u)), \quad i = 1, ..., d,$$

where

$$\sigma_{ij}(u) = \lambda \operatorname{div} u \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the stress tensor, $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. Here $\lambda, \mu \in C^\infty(\overline{\Omega})$ are scalar real-valued functions called Lamé parameters supposed to satisfy the condition

$$\mu(x) > 0, \quad \lambda(x) + \mu(x) > 0, \quad \forall x \in \overline{\Omega}. \tag{1.2}$$

The scalar function $n \in C^\infty(\overline{\Omega})$ in (1.1) is the density and is supposed to be strictly positive. It is easy to see that the elastic Laplacian can be written in the form

$$\Delta_{\lambda,\mu} u = \mu \Delta u + (\lambda + \mu) \nabla(\nabla \cdot u)$$

modulo a first-order matrix-valued differential operator, where Δ and ∇ denote the Euclidean Laplacian and gradient, respectively.

The natural Neumann boundary condition for the elastic equation is $B_{\lambda,\mu} u = 0$, where

$$(B_{\lambda,\mu} u)_i = \sum_{j=1}^d \sigma_{ij}(u) \nu_j, \quad i = 1, ..., d,$$

$\nu = (\nu_1, ..., \nu_d)$ being the Euclidean unit normal to Γ. We define the elastic Dirichlet-to-Neumann map

$$N(\tau) : H^1(\Gamma; \mathbb{C}^d) \to L^2(\Gamma; \mathbb{C}^d)$$

by

$$N(\tau)f = B_{\lambda,\mu}u|_\Gamma$$

where u and f satisfy the equation (1.1).

The equation (1.1) describes the propagation of elastic waves in Ω with a frequency τ. It is well-known that the elastic waves are superpositions of two waves, called S and P waves, mooving with speeds $\sqrt{\frac{\mu}{n}}$ and $\sqrt{\frac{2\mu+\lambda}{n}}$, respectively. From purely mathematical point of view, this is explained by the fact that the principal symbol, P, of the operator $-\Delta_{\lambda,\mu}$ can be decomposed as

$$P(x,\xi) = c_s(x)\Pi_s(\xi) + c_p(x)\Pi_p(\xi)$$

where $c_s = \mu$, $c_p = 2\mu+\lambda$, $\Pi_s(\xi)+\Pi_p(\xi) = \xi^2 I_d$, I_d being the identity $d\times d$ matrix, and $\Pi_p(\xi) = \xi\otimes\xi$. Throughout this paper, given two vectors $\xi = (\xi_1,...,\xi_d)\in\mathbb{C}^d$, $\eta = (\eta_1,...,\eta_d)\in\mathbb{C}^d$, we will denote by $\xi\otimes\eta$ the matrix defined by

$$(\xi\otimes\eta)g = \langle\xi, g\rangle\eta, \quad g\in\mathbb{C}^d.$$

Hereafter, $\langle\xi,g\rangle := \xi_1 g_1 + ... + \xi_d g_d$ and $\xi^2 := \langle\xi,\xi\rangle$.

Note that the existance of two different speeds implies that the boundary value problem (1.1) has two disjoint glancing regions Σ_s and Σ_p defined by

$$\Sigma_s = \left\{(x',\xi')\in T^*\Gamma : c_{s,0}(x')r_0(x',\xi') - n_0(x') = 0\right\},$$

$$\Sigma_p = \left\{(x',\xi')\in T^*\Gamma : c_{p,0}(x')r_0(x',\xi') - n_0(x') = 0\right\},$$

where $c_{s,0} = c_s|_\Gamma$, $c_{p,0} = c_p|_\Gamma$, $n_0 = n|_\Gamma$, and $r_0\geq 0$ is the principal symbol of the operator $-\Delta_\Gamma$. Here Δ_Γ denotes the negative Laplace-Beltrami operator on Γ with Riemannian metric induced by the Euclidean one. Set $h = (\operatorname{Re}\tau)^{-1}$ if $\operatorname{Re}\tau\geq|\operatorname{Im}\tau|$ and $h = |\operatorname{Im}\tau|^{-1}$ if $|\operatorname{Im}\tau|\geq\operatorname{Re}\tau$, $z = h\tau$ and $\theta = |\operatorname{Im} z|\leq 1$. Clearly, in the first case we have $z = 1+i\theta$, while in the second case we have $\theta = 1$. When $\theta > 0$ we introduce the functions

$$\rho_s(x',\xi',z) = \sqrt{-r_0(x',\xi') + z^2 n_0(x')/c_{s,0}(x')}, \quad \operatorname{Im}\rho_s > 0,$$

$$\rho_p(x',\xi',z) = \sqrt{-r_0(x',\xi') + z^2 n_0(x')/c_{p,0}(x')}, \quad \operatorname{Im}\rho_p > 0.$$

Our goal in the present paper is to approximate the operator

$$\mathcal{N}(z,h) := -ihN(z/h)$$

by a matrix-valued $h-\Psi$DO similarly to the Dirichlet-to-Neumann operator associated to the Helmholtz equation (see [6], [7]) or that one associated to the Maxwell equation (see [8]). We also compute the principal symbol in terms of the functions ρ_s, ρ_p (see Lemma 5.5). Denote by M_2 the 2×2 matrix with entries M_{ij} given by

$$M_{11} = \frac{z^2 n_0\rho_s}{r_0+\rho_s\rho_p}, \quad M_{22} = \frac{z^2 n_0\rho_p}{r_0+\rho_s\rho_p},$$

$$-M_{21} = M_{12} = -2\mu_0\sqrt{r_0} + \frac{z^2 n_0\sqrt{r_0}}{r_0+\rho_s\rho_p},$$

where $\mu_0 = \mu|_\Gamma = c_{s,0}$. When $d\geq 3$ we set

$$M_d = \widetilde{M}_2 + \mu_0\rho_s(I_d - \widetilde{I}_2).$$

Throughout this paper, given a 2×2 matrix M with entries M_{ij}, we denote by $\widetilde{M}$ the $d \times d$ matrix with entries $\widetilde{M}_{ij} = M_{ij}$ if $1 \leq i, j \leq 2$, $\widetilde{M}_{ij} = 0$ otherwise. Given a partition of the unity $\kappa_\ell \in C^\infty(T^*\Gamma \setminus 0)$, $0 \leq \kappa_\ell \leq 1$, $\ell = 1, ..., L$, $\sum_{\ell=1}^L \kappa_\ell = 1$, introduce the function

$$m_d = \sum_{\ell=1}^{L} \kappa_\ell J_\ell M_d J_\ell^{-1} \tag{1.3}$$

where $J_\ell(x', \xi') \in C^\infty(T^*\Gamma \setminus 0)$ are matrix-valued functions, homogeneous of order zero in ξ', and such that $J_\ell^{-1} = J_\ell^t$. Our main result is the following

Theorem 1.1. *Let $\theta \geq h^{2/5-\epsilon}$ and $0 < h \ll 1$, where $0 < \epsilon \ll 1$ is arbitrary. Then for every $f \in H^3(\Gamma; \mathbb{C}^d)$ we have the estimate*

$$\|\mathcal{N}(z,h)f - \mathrm{Op}_h(m_d)f\|_{L^2(\Gamma;\mathbb{C}^d)} \lesssim h\theta^{-2}\left(1 + (d-2)\theta^{-1/2}\right)\|f\|_{H_h^3(\Gamma;\mathbb{C}^d)} \tag{1.4}$$

where $m_d \in C^\infty(T^*\Gamma)$ is of the form (1.3) with a suitable partition of the unity κ_ℓ and matrix-valued functions J_ℓ independent of λ, μ and n. When $d = 2$ the functions J_ℓ do not depend on the variable ξ'.

Hereafter the Sobolev spaces are equipped with the h-semiclassical norm. Note that much better estimates for the Dirichlet-to-Neumann operator associated to the Helmholtz equation are proved in [6], [7]. This is due to the fact that one can construct a much better parametrix near the boundary for the Helmholtz equation than that one for the equation (1.1) we construct in the present paper. Indeed, such a parametrix is built in [6], [7] in the form of an oscilatory integral with a complex-valued phase function and an amplitude satisfying the eikonal and transport equations mod $\mathcal{O}(x_1^N)$, respectively, where $N \gg 1$ is arbitrary and $0 < x_1 \ll 1$ denotes the normal variable near the boundary, that is, the distance to Γ. Thus the parametrix satisfies the Helmholtz equation modulo an error term which is given by an oscilatory integral with amplitude of the form $\mathcal{O}(x_1^N) + \mathcal{O}(h^N)$. In the case of the equation (1.1), however, it is very hard to solve the transport equations, especially when the boundary data f is microlocally supported in a neighbourhood of the glancing regions. That is why we build in the present paper a less accurate parametrix for the equation (1.1) which does not require to solve transport equations. In this case the parametrix is a sum of two oscilatory integrals with two complex-valued phase functions corresponding to the two speeds of propagation of the elastic waves. Each of these phase functions satisfies the same eikonal equation as in the case of the Helmholtz equation solved in [6]. The parametrix satisfies the equation (1.1) modulo an error term which is given by a sum of two oscilatory integrals with amplitudes of the form $\mathcal{O}(x_1^N) + \mathcal{O}(h)$. To estimate the difference between the exact solution to equation (1.1) and its parametrix we use the a priori estimate (4.2). Most probably, the estimate (1.4) is not optimal and could be improved if one manages to build a better parametrix. In particular, for some applications it is better to have the L^2 norm in the right-hand side of (1.4) instead of the H_h^3 one. To do so, one needs to construct a better parametrix in the deep elliptic region, only, that is, in $\{r_0 \gg 1\}$. Recall that the approximation of the Dirichlet-to-Neumann map is usually used to get parabolic regions free of transmission eigenvalues (see [6], [7], [8]).

Note that microlocal parametrices have been recently constructed in [2], [3], [9] for the wave elastic equation in the case $d = 3$. All these parametrices, however, are very different from the parametrix we construct in the present paper. In particular, they are not valid near the glancing regions. In contrast, our parametrix remains valid even when the boundary data is microlocally supported near Σ_s and Σ_p, provided $\theta \geq h^{2/5-\epsilon}$. In [9] the principal symbol of the DN map has been computed explicitly still in the context of the wave elastic equation and $d = 3$. Note that

the formula in [9] agrees with that one we get in the present paper modulo a conjugation by a unitary matrix and after making a suitable change in the notations. In [4] a full parametrix was constructed for the stationary elastic equation in the exterior of a strictly convex body when $d = 3$ and the Lamé parameters being constants. In this case the parametrix for the elastic equation can be expressed in terms of the parametrix for the Helmholtz equation, which in turn is well-known. Similarly, one can construct a parametrix in the elliptic region for the stationary elastic equation in the exterior of an arbitrary compact body (see [5]).

2. Preliminaries

Throuhout this paper we will denote by $e_1, e_2, ..., e_d \in \mathbb{R}^d$ the vectors $(1, 0, ..., 0)$, $(0, 1, ..., 0)$, ..., $(0, 0, ..., 1)$, respectively. Given $\xi = (\xi_1, \xi_2, ..., \xi_d) \in \mathbb{C}^d$, introduce the $d \times d$ matrix

$$U_0(\xi) = \xi_1 I_d + \sum_{j=2}^{d} \xi_j (e_j \otimes e_1 - e_1 \otimes e_j).$$

Set $\xi^2 = \sum_{j=1}^{d} \xi_j^2$ and $|\xi|^2 = \sum_{j=1}^{d} |\xi_j|^2$. In this section we will prove the next two lemmas.

Lemma 2.1. *The matrix U_0 satisfies*

$$U_0(\xi)\xi = \xi^2 e_1. \tag{2.1}$$

Moreover, the matrices $e_1 \otimes e_1$ and $Z_0(\xi) := U_0(\xi)U_0^t(\xi)$ commute.

Proof. We have

$$U_0(\xi)\xi = \xi_1 \xi + \sum_{j=2}^{d} \xi_j (\langle e_j, \xi \rangle e_1 - \langle e_1, \xi \rangle e_j)$$

$$= \sum_{j=1}^{d} \xi_1 \xi_j e_j + \sum_{j=2}^{d} \xi_j (\xi_j e_1 - \xi_1 e_j) = \sum_{j=1}^{d} \xi_j^2 e_1$$

which proves (2.1). Set $\widetilde{\xi} = (0, \xi_2, ..., \xi_d)$. Then $U_0^t(\widetilde{\xi}) = -U_0(\widetilde{\xi})$ and hence

$$Z_0(\xi) = \xi_1^2 I_d - U_0(\widetilde{\xi})^2.$$

On the other hand, it is easy to see that

$$U_0(\widetilde{\xi})(e_1 \otimes e_1) = -\sum_{j=2}^{d} \xi_j e_1 \otimes e_j,$$

$$(e_1 \otimes e_1) U_0(\widetilde{\xi}) = \sum_{j=2}^{d} \xi_j e_j \otimes e_1.$$

Thus we get

$$(e_1 \otimes e_1) U_0(\widetilde{\xi})^2 = -\widetilde{\xi}^2 e_1 \otimes e_1 = U_0(\widetilde{\xi})^2 (e_1 \otimes e_1),$$

and hence $Z_0(\xi)$ and $e_1 \otimes e_1$ commute. □

Given a matrix A with entries A_{ij}, define its norm by $\|A\| = \max_{ij} |A_{ij}|$.

Lemma 2.2. *Given any $\eta_1 \in \mathbb{C}$ we have the formula*

$$\det(U_0(\xi) + \eta_1 e_1 \otimes e_1) = (\xi^2 + \xi_1\eta_1)\xi_1^{d-2}. \tag{2.2}$$

When $d \geq 3$ suppose that $\xi_1 \in \mathbb{C}$, $\xi_1 \neq 0$, and $\xi_k \in \mathbb{R}$, $k = 2, ..., d$. Then, if $\xi^2 + \xi_1\eta_1 \neq 0$, we have the estimate

$$\left\|(U_0(\xi) + \eta_1 e_1 \otimes e_1)^{-1}\right\| \lesssim (|\xi| + |\eta_1|)|\xi^2 + \xi_1\eta_1|^{-1} + (d-2)|\xi_1|^{-1}. \tag{2.3}$$

Proof. Denote the matrix $U_0(\xi) + \eta_1 e_1 \otimes e_1$ by $\mathcal{M}_d(\xi_1, ..., \xi_d, \eta_1)$. The lemma is very easy to prove when $d = 2$. Indeed, in this case we have

$$\mathcal{M}_2(\xi_1, \xi_2, \eta_1) = \begin{pmatrix} \xi_1 + \eta_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix}$$

and hence $\det\mathcal{M}_2 = \xi^2 + \xi_1\eta_1$. Moreover, if $\xi^2 + \xi_1\eta_1 \neq 0$, we have

$$\mathcal{M}_2^{-1}(\xi_1, \xi_2, \eta_1) = (\xi^2 + \xi_1\eta_1)^{-1} \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 + \eta_1 \end{pmatrix}$$

and (2.3) in this case is obvious. When $d \geq 3$ the formula (2.2) can be proved by induction. Indeed, we have

$$\det\mathcal{M}_d(\xi_1, ..., \xi_d, \eta_1) = \xi_1 \det\mathcal{M}_{d-1}(\xi_1, ..., \xi_{d-1}, \eta_1) + (-1)^d \xi_d \det\mathcal{P}_{d-1}(\xi_1, ..., \xi_d),$$

where $\mathcal{P}_{d-1}$ denotes the $(d-1)\times(d-1)$ matrix with lines $(\xi_2, ..., \xi_d)$, $(0, \xi_1, 0, ..., 0)$, ..., $(0, ..., \xi_1, 0)$. Hence

$$\begin{aligned}\det\mathcal{P}_{d-1}(\xi_1, \xi_2, ..., \xi_d) &= \xi_1^{d-2}\det\mathcal{P}_{d-1}(1, \xi_2, ..., \xi_d) \\ &= \xi_1^{d-2}\det\mathcal{P}_{d-1}(1, 0, ..., 0, \xi_d) = (-1)^d \xi_d \xi_1^{d-2}.\end{aligned}$$

Thus we get

$$\det\mathcal{M}_d(\xi_1, ..., \xi_d, \eta_1) = \xi_1 \det\mathcal{M}_{d-1}(\xi_1, ..., \xi_{d-1}, \eta_1) + \xi_d^2 \xi_1^{d-2}.$$

It is clear now that if (2.2) holds for $\det\mathcal{M}_{d-1}$, it holds for $\det\mathcal{M}_d$, as well. Thus we conclude that (2.2) holds for all d.

The proof of (2.3) when $d \geq 3$ is more delicate. Set $\widetilde{\xi} = (0, \xi') \in \mathbb{R}^d$, where $\xi' = (\xi_2, ..., \xi_d) \in \mathbb{R}^{d-1}$, and suppose that there exists a $d \times d$ matrix $\Theta(\xi')$, homogeneous of order zero, such that $\Theta(\xi')e_1 = e_1$, $\Theta(\xi')\widetilde{\xi} = |\xi'|e_2$, $\Theta(\xi')^{-1} = \Theta^t(\xi')$ and $\|\Theta(\xi')\| \leq C$ with a constant $C > 0$ independent of ξ'. Then, given any $g \in \mathbb{C}^d$, we have

$$U_0(\widetilde{\xi})\Theta^t(\xi')g = \langle\Theta(\xi')\widetilde{\xi}, g\rangle e_1 - \langle\Theta(\xi')e_1, g\rangle\widetilde{\xi} = |\xi'|\langle e_2, g\rangle e_1 - \langle e_1, g\rangle\widetilde{\xi}.$$

Hence

$$\begin{aligned}\Theta(\xi')U_0(\widetilde{\xi})\Theta^t(\xi')g &= |\xi'|\langle e_2, g\rangle\Theta(\xi')e_1 - \langle e_1, g\rangle\Theta(\xi')\widetilde{\xi} \\ &= |\xi'|\langle e_2, g\rangle e_1 - |\xi'|\langle e_1, g\rangle e_2 = U_0(|\xi'|e_2)g,\end{aligned}$$

which implies

$$U_0(\xi) = \Theta^t(\xi')U_0(\xi_1 e_1 + |\xi'|e_2)\Theta(\xi').$$

Since $\Theta^t(\xi')(e_1 \otimes e_1)\Theta(\xi') = (e_1 \otimes e_1)$, we obtain

$$(U_0(\xi) + \eta_1 e_1 \otimes e_1)^{-1} = \Theta(\xi')\left(U_0(\xi_1 e_1 + |\xi'|e_2) + \eta_1 e_1 \otimes e_1\right)^{-1}\Theta^t(\xi'). \tag{2.4}$$

This implies

$$\left\|(U_0(\xi) + \eta_1 e_1 \otimes e_1)^{-1}\right\| \lesssim \left\|\left(U_0(\xi_1 e_1 + |\xi'|e_2) + \eta_1 e_1 \otimes e_1\right)^{-1}\right\|. \tag{2.5}$$

On the other hand,

$$\big(U_0(\xi_1e_1+|\xi'|e_2)+\eta_1e_1\otimes e_1\big)^{-1}=\widetilde{\mathcal{M}_2^{-1}}(\xi_1,|\xi'|,\eta_1)+\xi_1^{-1}\sum_{j=3}^d e_j\otimes e_j. \tag{2.6}$$

It follows from (2.6) that

$$\Big\|\big(U_0(\xi_1e_1+|\xi'|e_2)+\eta_1e_1\otimes e_1\big)^{-1}\Big\|\le\big\|\mathcal{M}_2^{-1}(\xi_1,|\xi'|,\eta_1)\big\|+|\xi_1|^{-1}. \tag{2.7}$$

Clearly, in this case (2.3) follows from (2.5) and (2.7).

It remains to see that such a matrix $\Theta(\xi')$ exists. When $d=3$ it is easy to see that the matrix

$$\Theta(\xi')=\begin{pmatrix}1&0&0\\0&\xi_2/|\xi'|&\xi_3/|\xi'|\\0&-\xi_3/|\xi'|&\xi_2/|\xi'|\end{pmatrix}$$

has the desired properties. When $d\ge 4$, however, it is hard (and probably impossible) to find only one global matrix $\Theta(\xi')$ with these properties. Such a matrix, however, exists locally. Indeed, let $\mathcal{U}\subset\mathbb{S}^{d-2}$ be a small open domain in the unit sphere of dimension $d-2$. Then there exists a smooth $(d-1)\times(d-1)$ matrix-valued function $V(w)$, $w\in\mathcal{U}$, depending on $\mathcal{U}$, such that $V^{-1}(w)=V^t(w)$ and $V(w)w=\widetilde{e}_1=(1,0,...,0)\in\mathbb{R}^{d-1}$. Then we define the matrix $\Theta(\xi')$ for $\xi'/|\xi'|\in\mathcal{U}$ by

$$\Theta(\xi')=\begin{pmatrix}1&0\\0&V(\xi'/|\xi'|)\end{pmatrix}.$$

It is easy to see that $\Theta(\xi')$ has the desired properties as long as $\xi'/|\xi'|\in\mathcal{U}$. Thus we can cover $\mathbb{S}^{d-2}$ by a finite number of open sets $\mathcal{U}_k$, $k=1,...,K$, so that to each $\mathcal{U}_k$ we can associate a matrix-valued function $\Theta_k(\xi')$ having the desired properties for $\xi'/|\xi'|\in\mathcal{U}_k$. Then the identity (2.4) remains valid with $\Theta(\xi')$ replaced by $\Theta_k(\xi')$ as long as $\xi'/|\xi'|\in\mathcal{U}_k$. This implies the bounds (2.5) and (2.7) for $\xi'/|\xi'|\in\mathcal{U}_k$, $k=1,...,K$, and hence for all $\xi'/|\xi'|\in\mathbb{S}^{d-2}$. □

3. Some properties of the $h-\Psi$DOs

We will first introduce the spaces of symbols which will play an important role in our analysis and will recall some basic properties of the $h-\Psi$DOs. Given $k\in\mathbb{R}$, $\delta_1,\delta_2\ge 0$, we denote by $S^k_{\delta_1,\delta_2}$ the space of all functions $a\in C^\infty(T^*\Gamma)$, which may depend on the semiclassical parameter h, satisfying

$$\Big|\partial^\alpha_{x'}\partial^\beta_{\xi'}a(x',\xi',h)\Big|\le C_{\alpha,\beta}\langle\xi'\rangle^{k-\delta_1|\alpha|-\delta_2|\beta|}$$

for all multi-indices α and β, with constants $C_{\alpha,\beta}$ independent of h. More generally, given a function $\omega>0$ on $T^*\Gamma$, we denote by $S^k_{\delta_1,\delta_2}(\omega)$ the space of all functions $a\in C^\infty(T^*\Gamma)$, which may depend on the semiclassical parameter h, satisfying

$$\Big|\partial^\alpha_{x'}\partial^\beta_{\xi'}a(x',\xi',h)\Big|\le C_{\alpha,\beta}\omega^{k-\delta_1|\alpha|-\delta_2|\beta|}$$

for all multi-indices α and β, with constants $C_{\alpha,\beta}$ independent of h and ω. Thus $S^k_{\delta_1,\delta_2}=S^k_{\delta_1,\delta_2}(\langle\xi'\rangle)$. Given a matrix-valued symbol a, we will say that $a\in S^k_{\delta_1,\delta_2}$ if all entries of a belong to $S^k_{\delta_1,\delta_2}$. Also, given $k\in\mathbb{R}$, $0\le\delta<1/2$, we denote by $\mathcal{S}^k_\delta$ the space of all functions $a\in C^\infty(T^*\Gamma)$, which may depend on the semiclassical parameter h, satisfying

$$\Big|\partial^\alpha_{x'}\partial^\beta_{\xi'}a(x',\xi',h)\Big|\le C_{\alpha,\beta}h^{-\delta(|\alpha|+|\beta|)}\langle\xi'\rangle^{k-|\beta|}$$

for all multi-indices α and β, with constants $C_{\alpha,\beta}$ independent of h. Again, given a matrix-valued symbol a, we will say that $a \in \mathcal{S}^k_\delta$ if all entries of a belong to $\mathcal{S}^k_\delta$. The $h-\Psi$DO with a symbol a is defined by

$$\left(\mathrm{Op}_h(a)f\right)(x') = (2\pi h)^{-d+1}\int\!\!\int e^{-\frac{i}{h}\langle x'-y',\xi'\rangle}a(x',\xi',h)f(y')d\xi' dy'.$$

If $a \in S^k_{0,1}$, then the operator $\mathrm{Op}_h(a) : H^k_h(\Gamma) \to L^2(\Gamma)$ is bounded uniformly in h, where

$$\|u\|_{H^k_h(\Gamma)} := \left\|\mathrm{Op}_h(\langle\xi'\rangle^k)u\right\|_{L^2(\Gamma)}.$$

It is also well-known (e.g. see Section 7 of [1]) that, if $a \in \mathcal{S}^0_\delta$, $0 \le \delta < 1/2$, then $\mathrm{Op}_h(a) : H^s_h(\Gamma) \to H^s_h(\Gamma)$ is bounded uniformly in h. More generally, we have the following

Proposition 3.1. *Let $h^{1/2-\epsilon} \le \theta \le 1$, $\ell \ge 0$, and let*

$$a \in S^{-\ell}_{1,1}(\theta) + S^k_{0,1} \subset \theta^{-\ell}\mathcal{S}^k_{1/2-\epsilon}.$$

Then we have

$$\|\mathrm{Op}_h(a)\|_{H^k_h(\Gamma)\to L^2(\Gamma)} \lesssim \theta^{-\ell}. \tag{3.1}$$

Let $\eta \in C^\infty(T^*\Gamma)$ be such that $\eta = 1$ for $r_0 \le C_0$, $\eta = 0$ for $r_0 \ge 2C_0$, where $C_0 > 0$ does not depend on h. Let ρ denote either ρ_s or ρ_p. It is easy to see (e.g. see Lemma 3.1 of [6]) that taking C_0 big enough we can arrange

$$C_1\theta^{1/2} \le |\rho| \le C_2, \quad \mathrm{Im}\,\rho \ge C_3|\theta||\rho|^{-1} \ge C_4|\theta|$$

for $(x',\xi') \in \mathrm{supp}\,\eta$, and

$$|\rho| \ge \mathrm{Im}\,\rho \ge C_5|\xi'|$$

for $(x',\xi') \in \mathrm{supp}\,(1-\eta)$ with some constants $C_j > 0$. We will say that a function $a \in C^\infty(T^*\Gamma)$ belongs to $S^{k_1}_{\delta_1,\delta_2}(\omega_1) + S^{k_2}_{\delta_3,\delta_4}(\omega_2)$ if $\eta a \in S^{k_1}_{\delta_1,\delta_2}(\omega_1)$ and $(1-\eta)a \in S^{k_2}_{\delta_3,\delta_4}(\omega_2)$. It is shown in Section 3 of [6] (see Lemma 3.2 of [6]) that

$$\rho^k, |\rho|^k \in S^k_{2,2}(|\rho|) + S^k_{0,1}(|\rho|) \subset S^{-\widetilde{k}/2}_{1,1}(\theta) + S^k_{0,1} \subset \theta^{-\widetilde{k}/2}\mathcal{S}^{-N}_{1/2-\epsilon} + S^k_{0,1} \subset \theta^{-\widetilde{k}/2}\mathcal{S}^k_{1/2-\epsilon} \tag{3.2}$$

as long as $\theta \ge h^{1/2-\epsilon}$, uniformly in θ and h, where $\widetilde{k} = 0$ if $k \ge 0$, $\widetilde{k} = -k$ if $k \le 0$ and $N \gg 1$ is arbitrary.

4. A priori estimates

In this section we will prove a priori estimates for the solution to the equation

$$\begin{cases} (h^2\Delta_{\lambda,\mu} + z^2 n)u = hv & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma. \end{cases} \tag{4.1}$$

More precisely, we will prove the following

Theorem 4.1. *Let $\theta \ge h$ and $0 < h \ll 1$. Let $u \in H^2(\Omega;\mathbb{C}^d)$ satisfy equation (4.1). Then the function $g = hB_{\lambda,\mu}u|_\Gamma$ satisfies the estimate*

$$\|g\|_{L^2(\Gamma;\mathbb{C}^d)} \lesssim h^{1/2}\theta^{-1/2}\|v\|_{L^2(\Omega;\mathbb{C}^d)}. \tag{4.2}$$

Proof. We will first prove the following

Lemma 4.2. *We have the estimate*

$$\|u\|_{H^1_h(\Omega;\mathbb{C}^d)} \lesssim h\theta^{-1}\|v\|_{L^2(\Omega;\mathbb{C}^d)}. \tag{4.3}$$

Proof. The analog of the Green formula for the elastic Laplacian applied to the solution u of (4.1) takes the form

$$\langle -\Delta_{\lambda,\mu} u, u\rangle_{L^2(\Omega;\mathbb{C}^d)} = \int_\Omega E(u) \tag{4.4}$$

where

$$E(u) = \lambda \sum_{j=1}^d \left|\frac{\partial u_j}{\partial x_j}\right|^2 + \frac{\mu}{2} \sum_{1\leq i,j\leq d} \left|\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right|^2$$

$$= (\lambda+2\mu) \sum_{j=1}^d \left|\frac{\partial u_j}{\partial x_j}\right|^2 + \frac{\mu}{2} \sum_{i\neq j} \left|\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right|^2$$

$$\geq C_1 \sum_{1\leq i,j\leq d} \left|\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right|^2$$

with some constant $C_1 > 0$. On the other hand, since $u = 0$ on Γ, by Korn's inequality we have

$$\int_\Omega \sum_{1\leq i,j\leq d} \left|\frac{\partial u_i}{\partial x_j}\right|^2 \leq C_2 \int_\Omega \sum_{1\leq i,j\leq d} \left|\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right|^2$$

with some constant $C_2 > 0$. Combining the above inequalities with (4.4) we obtain the coercive estimate

$$\langle -\Delta_{\lambda,\mu} u, u\rangle_{L^2(\Omega;\mathbb{C}^d)} \geq C \int_\Omega \sum_{1\leq i,j\leq d} \left|\frac{\partial u_i}{\partial x_j}\right|^2 \tag{4.5}$$

with some constant $C > 0$. The Green formula (4.4) also gives the identity

$$\mathrm{Im}(z^2) \left\|n^{1/2}u\right\|^2_{L^2(\Omega;\mathbb{C}^d)} = \mathrm{Im}\ \langle hv, u\rangle_{L^2(\Omega;\mathbb{C}^d)},$$

which implies

$$\|u\|_{L^2(\Omega;\mathbb{C}^d)} \lesssim h\theta^{-1} \|v\|_{L^2(\Omega;\mathbb{C}^d)}. \tag{4.6}$$

On the other hand, we have

$$\left\langle -h^2\Delta_{\lambda,\mu} u, u\right\rangle_{L^2(\Omega;\mathbb{C}^d)} = \mathrm{Re}(z^2) \langle nu, u\rangle_{L^2(\Omega;\mathbb{C}^d)} - \mathrm{Re}\ \langle hv, u\rangle_{L^2(\Omega;\mathbb{C}^d)}$$

$$\lesssim \|u\|^2_{L^2(\Omega;\mathbb{C}^d)} + h^2 \|v\|^2_{L^2(\Omega;\mathbb{C}^d)}$$

which combinned with (4.5) leads to the estimate

$$\int_\Omega \sum_{1\leq i,j\leq d} h^2 \left|\frac{\partial u_i}{\partial x_j}\right|^2 \lesssim \|u\|^2_{L^2(\Omega;\mathbb{C}^d)} + h^2 \|v\|^2_{L^2(\Omega;\mathbb{C}^d)}. \tag{4.7}$$

Clearly, (4.3) follows from (4.6) and (4.7). □

Let $\mathcal{V} \subset \mathbb{R}^d$ be a small open domain such that $\mathcal{V}^0 := \mathcal{V} \cap \Gamma \neq \emptyset$. Let $(x_1, x') \in \mathcal{V}^+ := \mathcal{V} \cap \Omega$, $0 < x_1 \ll 1$, $x' = (x_2, ..., x_d) \in \mathcal{V}^0$, be the local normal geodesic coordinates near the boundary. Recall (e.g. see Section 2 of [8]) that the Euclidean gradient ∇ can be written in the coordinates $x = (x_1, x')$ as

$$\nabla = \gamma(x)\nabla_x = \nu(x') \frac{\partial}{\partial x_1} + \sum_{k=2}^d \gamma(x) e_k \frac{\partial}{\partial x_k},$$

where γ is a smooth matrix-valued function such that $\gamma(x)e_1 = \nu(x')$, $\gamma(x)e_k$ satisfy

$$\langle \nu(x'), \gamma(x)e_k\rangle = 0, \quad k = 2, ..., d. \tag{4.8}$$

Let $\xi = (\xi_1, \xi')$ be the dual variable of $x = (x_1, x')$. Then the symbol of the operator $-i\nabla$ in the coordinates (x, ξ) takes the form $\xi_1\nu(x') + \beta(x, \xi')$, where

$$\beta(x, \xi') = \sum_{k=2}^{d} \xi_k \gamma(x)e_k.$$

Note that (4.8) implies the identity

$$\langle \nu(x'), \beta(x, \xi')\rangle = 0 \quad \text{for all} \quad (x, \xi'). \tag{4.9}$$

Thus we get that the principal symbol of $-\Delta$ is equal to $\xi_1^2 + r(x, \xi')$, where $r = \langle \beta, \beta\rangle$. Therefore, the principal symbol of the positive Laplace-Beltrami operator on Γ is equal to $r_0(x', \xi') = r(0, x', \xi') = \langle \beta_0, \beta_0\rangle$, where $\beta_0 = \beta|_{x_1=0}$. Clearly, there exist constants $C_1, C_2 > 0$ such that $C_1|\xi'|^2 \le r_0 \le C_2|\xi'|^2$.

Let $\mathcal{V}_1 \subset \mathcal{V}$ be a small open domain such that $\mathcal{V}_1^0 := \mathcal{V}_1 \cap \Gamma \neq \emptyset$. Choose a function $\psi \in C_0^\infty(\mathcal{V})$, $0 \le \psi \le 1$, such that $\psi = 1$ on $\mathcal{V}_1$. Then the function $u^\flat := \psi u$ satisfies the equation

$$\begin{cases} (h^2\Delta_{\lambda,\mu} + z^2 n)u^\flat = hv^\flat \quad \text{in} \quad \Omega, \\ u^\flat = 0 \quad \text{on} \quad \Gamma, \end{cases} \tag{4.10}$$

where $v^\flat = \psi v + h[\Delta_{\lambda,\mu}, \psi]u$ satisfies

$$\|v^\flat\|_{L^2(\Omega;\mathbb{C}^d)} \lesssim \|v\|_{L^2(\Omega;\mathbb{C}^d)} + \|u\|_{H_h^1(\Omega;\mathbb{C}^d)}. \tag{4.11}$$

We will now write the elastic Laplacian in the coordinates $x = (x_1, x')$. To this end, we will write the principal symbol of $-\Delta_{\lambda,\mu}$ in the coordinates (x, ξ). We have

$$\begin{aligned} P(x, \xi) &= \mu(\gamma\xi)^2 I_d + (\lambda + \mu)(\gamma\xi) \otimes (\gamma\xi) \\ &= \mu(\xi_1^2 + r(x, \xi'))I_d + (\lambda + \mu)\gamma(\xi \otimes \xi)\gamma^t \\ &= \xi_1^2 Q_0(x) + \xi_1 Q_1(x, \xi') + Q_2(x, \xi'), \end{aligned}$$

where

$$\begin{aligned} Q_0 &= c_s\Pi_s(e_1) + c_p\Pi_p(e_1), \\ Q_1 &= (\lambda + \mu)\gamma\left(e_1 \otimes \xi' + \xi' \otimes e_1\right)\gamma^t, \\ Q_2 &= \mu r(x, \xi')I_d + (\lambda + \mu)\gamma(\xi' \otimes \xi')\gamma^t \end{aligned}$$

are symmetric matrices. Denote $\mathcal{D}_{x_j} = -ih\partial_{x_j}$. We can write

$$-h^2\Delta_{\lambda,\mu} = Q_0(x)\mathcal{D}_{x_1}^2 + \mathcal{Q}_1\mathcal{D}_{x_1} + \mathcal{Q}_2 + h\mathcal{R}(x, \mathcal{D}_x), \tag{4.12}$$

where $\mathcal{R}$ is a first-order matrix-valued differential operator, and

$$\mathcal{Q}_j = \frac{1}{2}\left(Q_j(x, \mathcal{D}_{x'}) + Q_j(x, \mathcal{D}_{x'})^*\right) = Q_j(x, \mathcal{D}_{x'}) + h\mathcal{R}_{j-1}(x, \mathcal{D}_{x'}), \quad j = 1, 2,$$

are self-adjoint operators on $L^2(\Gamma; \mathbb{C}^d)$. Here Q^* denotes the adjoint of Q with respect to the scalar product, $\langle \cdot, \cdot\rangle_0$, in $L^2(\Gamma; \mathbb{C}^d)$, and $\mathcal{R}_{j-1}$ is a $j-1$-order matrix-valued differential operator. Introduce the function

$$F(x_1) = \left\langle Q_0(x_1, \cdot)\mathcal{D}_{x_1}u^\flat, \mathcal{D}_{x_1}u^\flat \right\rangle_0 - \left\langle \mathcal{Q}_2(x_1, \cdot, \mathcal{D}_{x'})u^\flat, u^\flat \right\rangle_0 + \mathrm{Re}(z^2)\left\langle n(x_1, \cdot)u^\flat, u^\flat \right\rangle_0.$$

Clearly,

$$F(0) = \left\langle Q_0(0, \cdot)\mathcal{D}_{x_1}u^\flat|_{x_1=0}, \mathcal{D}_{x_1}u^\flat|_{x_1=0} \right\rangle_0 \ge C\left\|\mathcal{D}_{x_1}u^\flat|_{x_1=0}\right\|_0^2 \tag{4.13}$$

with some constant $C>0$, where $\|\cdot\|_0$ denotes the norm in $L^2(\Gamma;\mathbb{C}^d)$. On the other hand,

$$F(0)=-\int_0^\delta F'(x_1)dx_1 \tag{4.14}$$

for some constant $\delta>0$, where F' denotes the first derivative with respect to x_1. We will now use (4.14) to bound $F(0)$ from above. To this end we will compute $F'(x_1)$ using that $u^\flat$ satisfies (4.10) together with (4.12). We have

$$\begin{aligned}F'(x_1)&=-2\mathrm{Re}\left\langle(Q_0\mathcal{D}_{x_1}^2+\mathcal{Q}_2-\mathrm{Re}(z^2)n)u^\flat,\partial_{x_1}u^\flat\right\rangle_0\\&\quad+\left\langle Q_0'\mathcal{D}_{x_1}u^\flat,\mathcal{D}_{x_1}u^\flat\right\rangle_0-\left\langle(\mathcal{Q}_2{}'-\mathrm{Re}(z^2)n')u^\flat,u^\flat\right\rangle_0\\&=2h^{-1}\mathrm{Im}\left\langle(h^2\Delta_{\lambda,\mu}+\mathrm{Re}(z^2)n)u^\flat,\mathcal{D}_{x_1}u^\flat\right\rangle_0+2h^{-1}\mathrm{Im}\left\langle(Q_1\mathcal{D}_{x_1}+h\mathcal{R})u^\flat,\mathcal{D}_{x_1}u^\flat\right\rangle_0\\&\quad+\left\langle Q_0'\mathcal{D}_{x_1}u^\flat,\mathcal{D}_{x_1}u^\flat\right\rangle_0-\left\langle(\mathcal{Q}_2{}'-\mathrm{Re}(z^2)n')u^\flat,u^\flat\right\rangle_0\\&=2\mathrm{Im}\left\langle(v^\flat-ih^{-1}\mathrm{Im}(z^2)nu^\flat),\mathcal{D}_{x_1}u^\flat\right\rangle_0+2\mathrm{Im}\left\langle\mathcal{R}u^\flat,\mathcal{D}_{x_1}u^\flat\right\rangle_0\\&\quad+\left\langle Q_0'\mathcal{D}_{x_1}u^\flat,\mathcal{D}_{x_1}u^\flat\right\rangle_0-\left\langle(\mathcal{Q}_2{}'-\mathrm{Re}(z^2)n')u^\flat,u^\flat\right\rangle_0.\end{aligned}$$

Hence

$$|F'(x_1)|\lesssim h\theta^{-1}\|v^\flat\|_0^2+\theta h^{-1}\sum_{\ell=0}^1\|\mathcal{D}_{x_1}^\ell u^\flat\|_0^2+\sum_{|\alpha|\le1}\|\mathcal{D}_x^\alpha u^\flat\|_0^2.$$

Using this estimate together with (4.11), (4.14) and Lemma 4.2 we obtain

$$\begin{aligned}F(0)&\le\int_0^{2\delta}|F'(x_1)|dx_1\lesssim h\theta^{-1}\|v\|^2_{L^2(\Omega;\mathbb{C}^d)}+(1+\theta h^{-1})\|u\|^2_{H_h^1(\Omega;\mathbb{C}^d)}\\&\lesssim(h\theta^{-1}+h^2\theta^{-2})\|v\|^2_{L^2(\Omega;\mathbb{C}^d)}\lesssim h\theta^{-1}\|v\|^2_{L^2(\Omega;\mathbb{C}^d)}.\end{aligned}\tag{4.15}$$

Observe now that

$$\mathcal{D}_{x_1}u^\flat|_{x_1=0}=\psi_0\mathcal{D}_{x_1}u|_{x_1=0},\quad\mathcal{D}_{x'}u|_{x_1=0}=0,$$

where $\psi_0=\psi|_{x_1=0}$ is supported in $\mathcal{V}^0$ and such that $\psi_0=1$ on $\mathcal{V}_1^0$. Therefore, by (4.13) and (4.15),

$$\|\psi_0\mathcal{D}_{x_1}u|_{x_1=0}\|_0\lesssim h^{1/2}\theta^{-1/2}\|v\|_{L^2(\Omega;\mathbb{C}^d)},$$

which clearly implies

$$\|\psi_0 g\|_0\lesssim h^{1/2}\theta^{-1/2}\|v\|_{L^2(\Omega;\mathbb{C}^d)}.\tag{4.16}$$

Since Γ is compact, there exist a finite number of smooth functions ψ_i, $0\le\psi_i\le1$, $i=1,...,I$, such that $1=\sum_{i=1}^I\psi_i$ and (4.16) holds with ψ_0 replaced by each ψ_i. Therefore, the estimate (4.2) is obtained by summing up all such estimates (4.16). □

5. Parametrix construction

We keep the notations from the previous sections and will suppose that $\theta \geq h^{2/5-\epsilon}$, $0 < \epsilon \ll 1$. It suffices to build the parametrix locally since the global parametrix can be obtained by using a suitable partition of the unity and summing up the corresponding local parametrices. Let the function $\phi_0 \in C_0^\infty(\mathbb{R})$ be such that $\phi_0(\sigma) = 1$ for $|\sigma| \leq 1$, $\phi_0(\sigma) = 0$ for $|\sigma| \geq 2$. Let $(x_1, x') \in \mathcal{V}^+$ be the local normal geodesic coordinates near the boundary. Take a function $\chi \in C^\infty(T^*\Gamma)$, $0 \leq \chi \leq 1$, such that $\pi_{x'}(\operatorname{supp}\chi) \subset \mathcal{V}^0$, where $\pi_{x'} : T^*\Gamma \to \Gamma$ denotes the projection $(x', \xi') \to x'$. Moreover, we require that either χ is of compact support or $\chi \in S^0_{0,1}$ with $\operatorname{supp}\chi \subset \operatorname{supp}(1-\eta)$. When χ is of compact support we require that $\operatorname{supp}\chi$ has common points with at most one glancing region. Let $f \in H^3(\Gamma; \mathbb{C}^d)$. We will be looking for a parametrix of the solution to equation (1.1) in the form

$$\widetilde{u} = (2\pi h)^{-d+1} \int\int e^{\frac{i}{h}(\langle y', \xi'\rangle + \varphi_s(x,\xi',z))} \Psi(x,\xi') A_s(x,\xi',z)\chi(x',\xi') f(y') d\xi' dy'$$

$$+ (2\pi h)^{-d+1} \int\int e^{\frac{i}{h}(\langle y', \xi'\rangle + \varphi_p(x,\xi',z))} \Psi(x,\xi') A_p(x,\xi',z)\chi(x',\xi') f(y') d\xi' dy',$$

where

$$\Psi = \phi_0(x_1\langle\xi'\rangle^\varepsilon/\delta)\phi_0\left(x_1/\left|\rho_s\right|^3\delta\right)\phi_0\left(x_1/\left|\rho_p\right|^3\delta\right), \quad 0 < \varepsilon \ll 1,$$

$0 < \delta \ll 1$ being a parameter independent of h and θ to be fixed in Lemma 5.1. We require that $\widetilde{u}$ satisfies the boundary condition $\widetilde{u} = \operatorname{Op}_h(\chi) f$ on $x_1 = 0$. The phase functions are of the form

$$\varphi_s = \sum_{k=0}^{N-1} x_1^k \varphi_{s,k}, \quad \varphi_{s,0} = -\langle x', \xi'\rangle,\ \varphi_{s,1} = \rho_s,$$

$$\varphi_p = \sum_{k=0}^{N-1} x_1^k \varphi_{p,k}, \quad \varphi_{p,0} = -\langle x', \xi'\rangle,\ \varphi_{p,1} = \rho_p,$$

$N \gg 1$ being an arbitrary integer, and satisfy the eikonal equations mod $\mathcal{O}(x_1^N)$:

$$\begin{cases} c_s(x)(\gamma\nabla_x\varphi_s)^2 - z^2 n(x) = x_1^N \Phi_s, \\ c_p(x)(\gamma\nabla_x\varphi_p)^2 - z^2 n(x) = x_1^N \Phi_p, \end{cases} \tag{5.1}$$

where Φ_s, Φ_p are smooth functions up to the boundary $x_1 = 0$. One can solve the eikonal equations above in the same way as in [6]. The functions $\varphi_{s,k}$, $\varphi_{p,k}$, $k \geq 2$, are determined uniquely, independent of x_1, and have the following properties (see Section 4 of [6]).

Lemma 5.1. *For $0 \leq x_1 \leq 2\delta \min\{1, |\rho_s|^3\}$ with $\delta > 0$ small enough, we have*

$$\varphi_{s,k} \in S^{4-3k}_{2,2}\left(|\rho_s|\right) + S^1_{0,1}, \quad k \geq 1, \tag{5.2}$$

$$\partial_{x_1}^k \Phi_s \in S^{2-3N-3k}_{2,2}\left(|\rho_s|\right) + S^2_{0,1}, \quad k \geq 0, \tag{5.3}$$

$$\operatorname{Im}\varphi_s \geq x_1 \operatorname{Im}\rho_s/2, \tag{5.4}$$

$$|\partial_{x_1}\varphi_s| \geq |\rho_s|/2, \tag{5.5}$$

and similarly for φ_p.

Set $\widetilde{\varphi}_s = \varphi_s - \varphi_{s,0}$, $\widetilde{\varphi}_p = \varphi_p - \varphi_{p,0}$. The next lemma is proved in Section 4 of [8].

Lemma 5.2. *There exists a constant $C>0$ such that we have the estimates*

$$\left|\partial_{x'}^\alpha\partial_{\xi'}^\beta\left(e^{i\widetilde{\varphi}_s/h}\right)\right|\le\begin{cases}C_{\alpha,\beta}\theta^{-|\alpha|-|\beta|}e^{-Cx_1\theta/h} & on\quad \operatorname{supp}\eta,\\ C_{\alpha,\beta}|\xi'|^{-|\beta|}e^{-Cx_1|\xi'|/h} & on\quad \operatorname{supp}(1-\eta),\end{cases}\tag{5.6}$$

for $0\le x_1\le 2\delta\min\{1,|\rho_s|^3\}$ and all multi-indices α and β with constants $C_{\alpha,\beta}>0$ independent of x_1, θ, z and h. Similar bounds hold for $\widetilde{\varphi}_p$ as well.

The amplitudes A_s and A_p are matrix-valued functions which will be chosen so that on $\operatorname{supp}\chi$ we have

$$A_s+A_p=I_d\quad\text{on}\quad x_1=0,\tag{5.7}$$

and

$$\Pi_p(\gamma\nabla_x\varphi_s)A_s=0,\quad \Pi_s(\gamma\nabla_x\varphi_p)A_p=0.\tag{5.8}$$

If $\mathcal{V}^0$ is small enough, there exists a matrix-valued function $\Lambda(x')\in C^\infty(\mathcal{V}^0)$ such that $\Lambda(x')\nu(x')=e_1$ and $\Lambda^t=\Lambda^{-1}$ in $\mathcal{V}^0$. Set

$$U(\xi)=\Lambda^{-1}U_0(\Lambda\xi)\Lambda,$$

where U_0 is the matrix introduced in Section 2. It follows from (2.2) that $U(\xi)$ is invertible if $\langle\xi,\nu\rangle\ne0$. Clearly, $U(\nu)=I_d$. Moreover, by Lemma 2.1 we have $U(\xi)\xi=\xi^2\nu$. Therefore,

$$U(\xi)\Pi_p(\xi)U^t(\xi)=U(\xi)(\xi\otimes\xi)U^t(\xi)=(U(\xi)\xi)\otimes(U(\xi)\xi)=\xi^4\nu\otimes\nu=\xi^4\Pi_p(\nu).\tag{5.9}$$

Moreover, we have

$$Z(\xi):=U(\xi)U^t(\xi)=\Lambda^{-1}Z_0(\Lambda\xi)\Lambda,\quad \Pi_p(\nu)=\Lambda^{-1}\Pi_p(e_1)\Lambda.$$

Hence the matrices $Z(\xi)$ and $\Pi_p(\nu)$ commute. Set

$$A_s=U^t(\gamma\nabla_x\varphi_s)\Pi_s(\nu)T,\quad A_p=U^t(\gamma\nabla_x\varphi_p)\Pi_p(\nu)T,$$

where T is a matrix-valued function independent of x_1 to be defined below in such a way that (5.7) holds. Let us see that A_s and A_p satisfy (5.8). In view of (5.5) we have

$$\langle\gamma\nabla_x\varphi_s,\nu\rangle=\partial_{x_1}\varphi_s\ne0,\quad \langle\gamma\nabla_x\varphi_p,\nu\rangle=\partial_{x_1}\varphi_p\ne0.$$

Hence the matrices $U^t(\gamma\nabla_x\varphi_s)$ and $U^t(\gamma\nabla_x\varphi_p)$ are invertible, and by (5.9), we have

$$\begin{aligned}\Pi_p(\gamma\nabla_x\varphi_s)A_s&=\Pi_p(\gamma\nabla_x\varphi_s)U^t(\gamma\nabla_x\varphi_s)\Pi_s(\nu)T\\&=(\gamma\nabla_x\varphi_s)^4U^{-1}(\gamma\nabla_x\varphi_s)\Pi_p(\nu)\Pi_s(\nu)T=0,\\A_p=U^{-1}(\gamma\nabla_x\varphi_p)Z(\gamma\nabla_x\varphi_p)\Pi_p(\nu)T&=U^{-1}(\gamma\nabla_x\varphi_p)\Pi_p(\nu)Z(\gamma\nabla_x\varphi_p)T\\&=(\gamma\nabla_x\varphi_p)^{-4}\Pi_p(\gamma\nabla_x\varphi_p)Z(\gamma\nabla_x\varphi_p)T,\end{aligned}$$

which imply (5.8). We will now find the matrix T so that $WT=I_d$ with

$$W=U^t(\gamma\nabla_x\varphi_s|_{x_1=0})\Pi_s(\nu)+U^t(\gamma\nabla_x\varphi_p|_{x_1=0})\Pi_p(\nu).$$

Observe that

$$\gamma\nabla_x\varphi_s|_{x_1=0}=\rho_s\nu-\beta_0,\quad \gamma\nabla_x\varphi_p|_{x_1=0}=\rho_p\nu-\beta_0.$$

Therefore,

$$W=\rho_s\Pi_s(\nu)+\rho_p\Pi_p(\nu)-U^t(\beta_0).$$

We will derive from Lemma 2.2 the following

Lemma 5.3. *The matrix W is invertible with an inverse $T = W^{-1}$ satisfying the bounds*

$$\|T\| \lesssim \begin{cases} \theta^{-\ell} & on \quad \operatorname{supp}\eta, \\ \sqrt{r_0+1} & on \quad \operatorname{supp}(1-\eta), \end{cases} \tag{5.10}$$

where $\ell = 0$ if $d = 2$, $\ell = 1/2$ if $d \geq 3$. More generally, we have

$$T \in S^{-\ell}_{1,1}(\theta) + S^1_{0,1}. \tag{5.11}$$

Proof. Set $\zeta(x',\xi') = -\Lambda(x')\beta_0(x',\xi')$. Clearly, $\zeta^2 = \beta_0^2 = r_0$. In view of (4.9) we have $\langle \zeta, e_1\rangle = 0$ and hence $\zeta = (0,\zeta_2,...,\zeta_d)$. Then the matrix $W_0 := \Lambda W \Lambda^{-1}$ can be written in the form

$$\begin{aligned} W_0 &= U_0^t(\zeta) + \rho_s \Pi_s(e_1) + \rho_p \Pi_p(e_1) \\ &= U_0^t(\rho_s e_1 + \zeta) + (\rho_p - \rho_s) e_1 \otimes e_1. \end{aligned}$$

By (2.2) we get

$$\det W_0 = (r_0 + \rho_s\rho_p)\rho_s^{d-2}. \tag{5.12}$$

We need now the following

Lemma 5.4. *There exists a constant $C > 0$ such that*

$$|r_0 + \rho_s\rho_p| \geq C. \tag{5.13}$$

More generally, we have

$$(r_0 + \rho_s\rho_p)^{-1} \in S^0_{1,1}(\theta) + S^0_{0,1}. \tag{5.14}$$

Proof. Recall that $\rho_s^2 = -r_0 + z^2 k_s$, $\rho_p^2 = -r_0 + z^2 k_p$ with some functions $k_s, k_p \in C^\infty(\Gamma)$, $k_s > k_p > 0$. Then we have the identity

$$r_0 + \rho_s\rho_p = r_0 + \rho_s^2 + \frac{\rho_s(\rho_p^2 - \rho_s^2)}{\rho_s + \rho_p} = z^2 \frac{k_p\rho_s + k_s\rho_p}{\rho_s + \rho_p}. \tag{5.15}$$

Hence

$$|r_0 + \rho_s\rho_p| \geq \frac{k_p \operatorname{Im}\rho_s + k_s \operatorname{Im}\rho_p}{|\rho_s| + |\rho_p|} \geq C_1 \frac{\operatorname{Im}\rho_s + \operatorname{Im}\rho_p}{\sqrt{r_0+1}} \tag{5.16}$$

with some constant $C_1 > 0$. On the other hand, there is a constant $C_2 > 0$ such that $\operatorname{Im}\rho_s \geq C_2\sqrt{r_0+1}$ on $\operatorname{supp}(1-\eta)$, $\operatorname{Im}\rho_s \geq C_2\theta$ on $\operatorname{supp}\eta$, and similarly for ρ_p (see Section 3). Therefore, (5.13) follows from (5.16) when either $(x',\xi') \in \operatorname{supp}(1-\eta)$ or $(x',\xi') \in \operatorname{supp}\eta$ and $\theta \geq \theta_0 > 0$. Thus, it remains to prove (5.13) when $(x',\xi') \in \operatorname{supp}\eta$ and $\theta \ll 1$. In this case we have that $|\rho_s|$ and $|\rho_p|$ are uniformly bounded from above by a constant and $z^2 = 1 + \mathcal{O}(\theta)$. We will make use of the identity

$$(k_p\rho_s - k_s\rho_p)(k_p\rho_s + k_s\rho_p) = (k_s - k_p)\left((k_s + k_p)r_0 - z^2 k_s k_p\right). \tag{5.17}$$

If

$$\left|(k_s + k_p)r_0 - z^2 k_s k_p\right| \geq \varepsilon > 0,$$

it follows from (5.17) that

$$|k_p\rho_s + k_s\rho_p| \geq C_3\varepsilon$$

with some constant $C_3 > 0$. Thus in this case (5.13) follows from (5.15). Let now

$$\left|(k_s + k_p)r_0 - z^2 k_s k_p\right| \leq \varepsilon$$

with $0<\varepsilon\ll 1$. Then

$$\rho_s=\sqrt{\frac{k_sr_0}{k_p}}+\mathcal{O}(\varepsilon),\quad \rho_p=\sqrt{\frac{k_pr_0}{k_s}}+\mathcal{O}(\varepsilon),\quad r_0=\frac{k_sk_p}{k_s+k_p}+\mathcal{O}(\varepsilon)+\mathcal{O}(\theta).$$

Clearly, there exists a constant $C_4>0$ such that $|k_p\rho_s+k_s\rho_p|\geq C_4$, provided ε and θ are taken small enough, which again implies (5.13).

To prove (5.14) note first that, in view of (3.2), we have

$$\rho_s+\rho_p,\, k_p\rho_s+k_s\rho_p\in S^0_{1,1}(\theta)+S^1_{0,1}. \tag{5.18}$$

Therefore, in view of (5.15), to prove (5.14) it suffices to show that

$$(k_p\rho_s+k_s\rho_p)^{-1}\in S^0_{1,1}(\theta)+S^{-1}_{0,1}. \tag{5.19}$$

In other words, we must show that given any multi-indices α and β we have the estimates

$$\left|\partial^\alpha_{x'}\partial^\beta_{\xi'}\left((k_p\rho_s+k_s\rho_p)^{-1}\right)\right|\leq\begin{cases}C_{\alpha,\beta}\theta^{-|\alpha|-|\beta|} & \text{on}\quad \operatorname{supp}\eta,\\ C_{\alpha,\beta}|\xi'|^{-1-|\beta|} & \text{on}\quad \operatorname{supp}(1-\eta).\end{cases} \tag{5.20}$$

Clearly, for $\alpha=\beta=0$ the bounds in (5.20) follow from the analysis above. To prove them for all α and β, we will proceed by induction in $|\alpha|+|\beta|$. Suppose that (5.20) holds for all α and β such that $|\alpha|+|\beta|\leq K-1$, $K\geq 1$. Let us see that (5.20) holds for all α and β such that $|\alpha|+|\beta|=K$. To this end, we will use the identity

$$\begin{aligned}0&=\partial^\alpha_{x'}\partial^\beta_{\xi'}\left((k_p\rho_s+k_s\rho_p)(k_p\rho_s+k_s\rho_p)^{-1}\right)\\&=(k_p\rho_s+k_s\rho_p)\partial^\alpha_{x'}\partial^\beta_{\xi'}\left((k_p\rho_s+k_s\rho_p)^{-1}\right)\\&+\sum_{|\alpha'|+|\beta'|\leq K-1}\partial^{\alpha'}_{x'}\partial^{\beta'}_{\xi'}\left((k_p\rho_s+k_s\rho_p)^{-1}\right)\partial^{\alpha-\alpha'}_{x'}\partial^{\beta-\beta'}_{\xi'}\left(k_p\rho_s+k_s\rho_p\right).\end{aligned}$$

Thus, in view of (5.18), we obtain

$$\begin{aligned}&|k_p\rho_s+k_s\rho_p|\left|\partial^\alpha_{x'}\partial^\beta_{\xi'}\left((k_p\rho_s+k_s\rho_p)^{-1}\right)\right|\\&\leq\sum_{|\alpha'|+|\beta'|\leq K-1}\left|\partial^{\alpha'}_{x'}\partial^{\beta'}_{\xi'}\left((k_p\rho_s+k_s\rho_p)^{-1}\right)\right|\left|\partial^{\alpha-\alpha'}_{x'}\partial^{\beta-\beta'}_{\xi'}\left(k_p\rho_s+k_s\rho_p\right)\right|\\&\lesssim\begin{cases}\theta^{-|\alpha|-|\beta|} & \text{on}\quad \operatorname{supp}\eta,\\ |\xi'|^{-|\beta|} & \text{on}\quad \operatorname{supp}(1-\eta).\end{cases}\end{aligned}$$

Since $|\xi'|+1\lesssim|k_p\rho_s+k_s\rho_p|$, we conclude from the above bounds that (5.20) holds for all α and β such that $|\alpha|+|\beta|=K$, as desired. □

It follows from (5.12) and (5.13) that the matrix W_0 is invertible, and hence so is W. Moreover, by (2.3) its inverse satisfies the bound

$$\begin{aligned}\|T\|\lesssim\left\|W_0^{-1}\right\|&\lesssim|\zeta|+|\rho_s|+|\rho_p|+(d-2)|\rho_s|^{-1}\\&\lesssim\begin{cases}1+(d-2)\theta^{-1/2} & \text{on}\quad \operatorname{supp}\eta,\\ \sqrt{r_0+1} & \text{on}\quad \operatorname{supp}(1-\eta),\end{cases}\end{aligned}$$

which implies (5.10). To prove (5.11) we need to show that the estimates

$$\left\|\partial^\alpha_{x'}\partial^\beta_{\xi'}T\right\|\leq\begin{cases}C_{\alpha,\beta}\theta^{-\ell-|\alpha|-|\beta|} & \text{on}\quad \operatorname{supp}\eta,\\ C_{\alpha,\beta}|\xi'|^{1-|\beta|} & \text{on}\quad \operatorname{supp}(1-\eta).\end{cases} \tag{5.21}$$

hold for all multi-indices α and β. Note that in view of (3.2) we have

$$W \in S^0_{1,1}(\theta) + S^1_{0,1}. \tag{5.22}$$

Now (5.21) can be derived from (5.10) and (5.22) by induction in $|\alpha| + |\beta|$ in the same way as above. □

To get a parametrix for the elastic DN map we need the following

Lemma 5.5. *There exist matrix-valued functions $m_d, q \in C^\infty(T^*\Gamma)$ such that*

$$-ihB_{\lambda,\mu}\widetilde{u}|_{x_1=0} = \mathrm{Op}_h(m_d\chi + hq\chi)f. \tag{5.23}$$

Proof. Given a scalar-valued function φ and a vector-valued function a, we have the identity

$$-ihe^{-i\varphi/h}B_{\lambda,\mu}\left(e^{i\varphi/h}a\right) = \lambda\langle\gamma\nabla_x\varphi, a\rangle\nu + \mu\langle\nu, a\rangle\gamma\nabla_x\varphi + \mu\langle\nu,\gamma\nabla_x\varphi\rangle a$$
$$-ih\lambda\langle\gamma\nabla_x, a\rangle\nu - ih\mu\langle\nu,\gamma\nabla_x a\rangle - ih\mu\langle\nu,\gamma\nabla_x\rangle a.$$

Set $a_s = A_s\chi f$, $a_p = A_p\chi f$, $a^0_s = A^0_s\chi f$, $a^0_p = A^0_p\chi f$, where $A^0_s = A_s|_{x_1=0}$, $A^0_p = A_p|_{x_1=0}$ satisfy $A^0_s + A^0_p = I$. Set also $a^1_s = A^1_s\chi f$, $a^1_p = A^1_p\chi f$, where $A^1_s = \partial_{x_1}A_s|_{x_1=0}$, $A^1_p = \partial_{x_1}A_p|_{x_1=0}$. Applying the above identity to φ_s, a_s and φ_p, a_p leads to

$$\begin{aligned}
&-ihe^{-i\varphi_s/h}B_{\lambda,\mu}\left(e^{i\varphi_s/h}a_s\right)|_{x_1=0} - ihe^{-i\varphi_p/h}B_{\lambda,\mu}\left(e^{i\varphi_p/h}a_p\right)|_{x_1=0}\\
&= \lambda\langle\rho_s\nu - \beta_0, a^0_s\rangle\nu + \mu\langle\nu, a^0_s\rangle(\rho_s\nu - \beta_0) + \mu\langle\nu,\rho_s\nu-\beta_0\rangle a^0_s\\
&+\lambda\langle\rho_p\nu - \beta_0, a^0_p\rangle\nu + \mu\langle\nu, a^0_p\rangle(\rho_p\nu - \beta_0) + \mu\langle\nu,\rho_p\nu-\beta_0\rangle a^0_p\\
&-ih(\lambda+\mu)\langle\nu, a^1_s + a^1_p\rangle\nu - ih\mu(a^1_s + a^1_p)\\
&= (\lambda+\mu)\langle\nu,\rho_s a^0_s + \rho_p a^0_p\rangle\nu - \lambda\langle\beta_0, f\rangle\nu - \mu\langle\nu, f\rangle\beta_0 + \mu(\rho_s a^0_s + \rho_p a^0_p)\\
&-ih(\lambda+\mu)\langle\nu, a^1_s + a^1_p\rangle\nu - ih\mu(a^1_s + a^1_p)\\
&= (m_d + hq)\chi f,
\end{aligned}$$

where

$$\begin{aligned}
q &= -i\left(c_s\Pi_s(\nu) + c_p\Pi_p(\nu)\right)\left(A^1_s + A^1_p\right)\\
&= -i\left(c_s\Pi_s(\nu) + c_p\Pi_p(\nu)\right)U^t(\gamma\widetilde{\nabla}_{x'}(\rho_p - \rho_s))\Pi_p(\nu)T,
\end{aligned}$$

where $\widetilde{\nabla}_{x'} = (0, \nabla_{x'})$, and

$$\begin{aligned}
m_d &= \left(c_s\Pi_s(\nu) + c_p\Pi_p(\nu)\right)\left(\rho_s A^0_s + \rho_p A^0_p\right) - \lambda\beta_0\otimes\nu - \mu\nu\otimes\beta_0\\
&= -\lambda\beta_0\otimes\nu - \mu\nu\otimes\beta_0\\
&+\left(c_s\Pi_s(\nu) + c_p\Pi_p(\nu)\right)\left(\rho_s U^t(\rho_s\nu - \beta_0)\Pi_s(\nu) + \rho_p U^t(\rho_p\nu - \beta_0)\Pi_p(\nu)\right)T\\
&= -\lambda\beta_0\otimes\nu - \mu\nu\otimes\beta_0 + \left(c_s\rho_s^2\Pi_s(\nu) + c_p\rho_p^2\Pi_p(\nu)\right)T\\
&-\left(c_s\Pi_s(\nu) + c_p\Pi_p(\nu)\right)U^t(\beta_0)\left(\rho_s\Pi_s(\nu) + \rho_p\Pi_p(\nu)\right)T.
\end{aligned}$$

Hence

$$\begin{aligned}
m^0_d(\zeta) := \Lambda m_d\Lambda^{-1} &= \lambda\zeta\otimes e_1 + \mu e_1\otimes\zeta + \left(c_s\rho_s^2\Pi_s(e_1) + c_p\rho_p^2\Pi_p(e_1)\right)T_d\\
&+\left(c_s\Pi_s(e_1) + c_p\Pi_p(e_1)\right)U^t_0(\zeta)\left(\rho_s\Pi_s(e_1) + \rho_p\Pi_p(e_1)\right)T_d,
\end{aligned} \tag{5.24}$$

where $T_d(\zeta) := \Lambda T\Lambda^{-1} = W_0^{-1}$. We will first compute m^0_d when $d = 2$. We have

$$U^t_0(\zeta) = \begin{pmatrix} 0 & -\zeta_2\\ \zeta_2 & 0\end{pmatrix},$$

$$W_0 = \begin{pmatrix} \rho_p & -\zeta_2 \\ \zeta_2 & \rho_s \end{pmatrix}$$

and hence

$$T_2(\zeta_2) = (r_0 + \rho_s\rho_p)^{-1} \begin{pmatrix} \rho_s & \zeta_2 \\ -\zeta_2 & \rho_p \end{pmatrix}.$$

Then we have

$$\lambda\zeta \otimes e_1 + \mu e_1 \otimes \zeta = \begin{pmatrix} 0 & \lambda\zeta_2 \\ \mu\zeta_2 & 0 \end{pmatrix},$$

$$\begin{aligned} &\left(c_s\rho_s^2\Pi_s(e_1) + c_p\rho_p^2\Pi_p(e_1)\right) T_2 \\ &= (r_0 + \rho_s\rho_p)^{-1} \begin{pmatrix} c_p\rho_p^2 & 0 \\ 0 & c_s\rho_s^2 \end{pmatrix} \begin{pmatrix} \rho_s & \zeta_2 \\ -\zeta_2 & \rho_p \end{pmatrix} \\ &= (r_0 + \rho_s\rho_p)^{-1} \begin{pmatrix} c_p\rho_p^2\rho_s & \zeta_2 c_p\rho_p^2 \\ -\zeta_2 c_s\rho_s^2 & c_s\rho_s^2\rho_p \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} &\left(c_s\Pi_s(e_1) + c_p\Pi_p(e_1)\right) U_0^t(\zeta) \left(\rho_s\Pi_s(e_1) + \rho_p\Pi_p(e_1)\right) T_2 \\ &= (r_0 + \rho_s\rho_p)^{-1} \begin{pmatrix} 0 & -\zeta_2 c_p\rho_s \\ \zeta_2 c_s\rho_p & 0 \end{pmatrix} \begin{pmatrix} \rho_s & \zeta_2 \\ -\zeta_2 & \rho_p \end{pmatrix} \\ &= (r_0 + \rho_s\rho_p)^{-1} \begin{pmatrix} \zeta_2^2 c_p\rho_s & -\zeta_2 c_p\rho_s\rho_p \\ \zeta_2 c_s\rho_s\rho_p & \zeta_2^2 c_s\rho_p \end{pmatrix}. \end{aligned}$$

Since in this case $\zeta_2 = \sqrt{r_0}$, an easy computation leads to the formula

$$m_2^0 := (r_0 + \rho_s\rho_p)^{-1} \begin{pmatrix} z^2 n\rho_s & -2\mu\sqrt{r_0}(r_0 + \rho_s\rho_p) + z^2 n\sqrt{r_0} \\ 2\mu\sqrt{r_0}(r_0 + \rho_s\rho_p) - z^2 n\sqrt{r_0} & z^2 n\rho_p \end{pmatrix}. \tag{5.25}$$

Let now $d \geq 3$. Then, in view of (2.4) and (2.6), we have

$$\Theta_k(\zeta)^{-1} T_d(\zeta) \Theta_k(\zeta) = \widetilde{T_2}(\sqrt{r_0}) + \rho_s^{-1} \sum_{j=3}^{d} e_j \otimes e_j$$

for $\zeta/|\zeta| \in \mathcal{U}_k$. Using this and (5.24) one can easily obtain the formula

$$\Theta_k(\zeta)^{-1} m_d^0(\zeta) \Theta_k(\zeta) = \widetilde{m_2^0} + c_s\rho_s \sum_{j=3}^{d} e_j \otimes e_j = M_d \tag{5.26}$$

for $\zeta/|\zeta| \in \mathcal{U}_k$. Let $\phi_k \in C^\infty(\mathbb{S}^{d-2})$, $0 \leq \phi_k \leq 1$, $k = 1, ..., K$, $\sum_{k=1}^K \phi_k = 1$, be a partition of the unity such that $\operatorname{supp} \phi_k \subset \mathcal{U}_k$. Then we conclude from (5.25) and (5.26) that

$$\chi m_d = \sum_{k=1}^{K} \phi_k(\zeta/|\zeta|) \chi \mathcal{J}_k M_d \mathcal{J}_k^{-1}$$

where

$$\mathcal{J}_k(x', \xi') = \Lambda(x')^{-1} \Theta_k(\zeta(x', \xi')).$$

□

In what follows we will bound the norm of the difference between the DN map and the operator $\mathrm{Op}_h(m_d)$. To this end, observe that $\widetilde{u}$ satisfies the equation

$$(h^2\Delta_{\lambda,\mu}+z^2n)\widetilde{u}=h\widetilde{v},$$

where the function $\widetilde{v}$ is of the form

$$\begin{aligned}\widetilde{v}&=(2\pi h)^{-d+1}\int\int e^{\frac{i}{h}\langle y'-x',\xi'\rangle}\left(e^{i\widetilde{\varphi}_s/h}B_s+e^{i\widetilde{\varphi}_p/h}B_p\right)f(y')d\xi'dy'\\&=\mathrm{Op}_h\left(e^{i\widetilde{\varphi}_s/h}B_s+e^{i\widetilde{\varphi}_p/h}B_p\right)f\end{aligned}$$

with some matrix-valued functions B_s and B_p. To find them we will use the identity

$$e^{-i\varphi/h}(h^2\Delta_{\lambda,\mu}+z^2n(x))\left(e^{i\varphi/h}a\right)=\left(-P(x,\gamma\nabla_x\varphi)+z^2n\right)a+h^2\Delta_{\lambda,\mu}a+hL(\varphi,A)f,$$

where a is a vector-valued function of the form $a=A(x,\xi')\chi(x',\xi')f(y')$ and L is a matrix-valued function of the form

$$L(\varphi,A)=\sum_{|\alpha|+|\beta|\leq 2,\,|\alpha|\geq 1}L_{\alpha,\beta}(x)\partial_x^\alpha\varphi\partial_x^\beta(\chi A),$$

$L_{\alpha,\beta}$ being smooth matrix-valued functions depending only on the variable x. Observe also that $\Delta_{\lambda,\mu}a=G(A)f$, where $G(A)$ is a matrix-valued function of the form

$$G(A)=\sum_{1\leq|\alpha|\leq 2}G_\alpha(x)\partial_x^\alpha(\chi A).$$

We would like to apply the above identity to φ_s, $a_s=A_s\chi f$ and φ_p, $a_p=A_p\chi f$. In view of (5.1) and (5.8), we have

$$\begin{aligned}\left(P(x,\gamma\nabla_x\varphi_s)-z^2n\right)a_s&=\left(c_s-z^2n(\gamma\nabla_x\varphi_s)^{-2}\right)\Pi_s(\gamma\nabla_x\varphi_s)a_s\\&=x_1^N\Phi_s(\gamma\nabla_x\varphi_s)^{-2}\Pi_s(\gamma\nabla_x\varphi_s)a_s.\end{aligned}$$

By the above identities we get

$$\begin{aligned}B_s=h\left[\Delta_{\lambda,\mu},\Psi\right]\chi A_s-h^{-1}x_1^N\Psi\Phi_s(\gamma\nabla_x\varphi_s)^{-2}\Pi_s(\gamma\nabla_x\varphi_s)\chi A_s\\+\Psi L(\varphi_s,A_s)+h\Psi G(A_s)\end{aligned}$$

and similarly for B_p. Let u satisfy equation (1.1) with $u|_\Gamma=\mathrm{Op}_h(\chi)f$. Then $u-\widetilde{u}$ satisfies equation (4.1) with v replaced by $\widetilde{v}$. Therefore, by (4.2) we get the estimate

$$\|\mathcal{N}(z,h)\mathrm{Op}_h(\chi)f+ihB_{\lambda,\mu}\widetilde{u}|_{x_1=0}\|_{L^2(\Gamma;\mathbb{C}^d)}\lesssim h^{1/2}\theta^{-1/2}\|\widetilde{v}\|_{L^2(\Omega;\mathbb{C}^d)}. \tag{5.27}$$

Theorem 1.1 follows from (5.27) together with Lemma 5.5 and the following

Lemma 5.6. *For N big enough depending on ϵ and ε we have the estimates*

$$\|\mathrm{Op}_h(\chi q)f\|_{L^2(\Gamma;\mathbb{C}^d)}\lesssim\theta^{-1/2-\ell}\|f\|_{H_h^2(\Gamma;\mathbb{C}^d)}, \tag{5.28}$$

$$\|\widetilde{v}\|_{L^2(\Omega;\mathbb{C}^d)}\lesssim h^{1/2}\theta^{-3/2-\ell}\|f\|_{H_h^3(\Gamma;\mathbb{C}^d)}. \tag{5.29}$$

Indeed, we have

$$\|\mathcal{N}(z,h)\mathrm{Op}_h(\chi)f-\mathrm{Op}_h(\chi m_d)f\|_{L^2(\Gamma;\mathbb{C}^d)}\lesssim h\theta^{-2-\ell}\|f\|_{H_h^3(\Gamma;\mathbb{C}^d)}. \tag{5.30}$$

We can now take a partition of the unity χ_j, $j=1,...,J$, $0\leq\chi_j\leq 1$, $\sum_{j=1}^J\chi_j=1$, such that (5.30) holds with χ replaced by each χ_j. Moreover, to each χ_j we can associate a smooth

matrix-valued function $\Lambda_j(x')$ such that $\Lambda_j(x')\nu(x')=e_1$ and $\Lambda_j^{-1}(x')=\Lambda_j^t(x')$ in $\pi_{x'}(\operatorname{supp}\chi_j)$. Thus, summing up all estimates (5.30) leads to (1.4) with

$$m_d=\sum_{j=1}^J\chi_j m_d=\sum_{j=1}^J\sum_{k=1}^K\chi_j\phi_{k,j}\mathcal{J}_{k,j}M_d\mathcal{J}_{k,j}^{-1}$$

where

$$\phi_{k,j}(x',\xi')=\phi_k\left(-\Lambda_j(x')\beta_0(x',\xi')/\sqrt{r_0(x',\xi')}\right),$$
$$\mathcal{J}_{k,j}(x',\xi')=\Lambda_j(x')^{-1}\Theta_k(-\Lambda_j(x')\beta_0(x',\xi')).$$

6. Proof of Lemma 5.6

In view of (3.2), we have

$$\nabla_{x'}\rho_s\in S^{-1}_{2,2}(|\rho_s|)+S^1_{0,1}\subset S^{-1/2}_{1,1}(\theta)+S^1_{0,1}$$

and similarly for ρ_p. Therefore, we have

$$U^t(\gamma\widetilde{\nabla}_{x'}(\rho_p-\rho_s))\in S^{-1/2}_{1,1}(\theta)+S^1_{0,1}$$

which together with (5.11) yield

$$\chi q\in S^{-1/2-\ell}_{1,1}(\theta)+S^2_{0,1}. \tag{6.1}$$

Now (5.28) follows from (6.1) and Proposition 3.1. Furthermore, it is easy to see that (5.29) is a consequence of the following

Lemma 6.1. *For N big enough depending on ϵ and ε we have the estimate*

$$\left\|\mathrm{Op}_h\left(e^{i\widetilde{\varphi}_s/h}B_s\right)\right\|_{H^3_h(\Gamma;\mathbb{C}^d)\to L^2(\Gamma;\mathbb{C}^d)}\lesssim h+\theta^{-1-\ell}e^{-Cx_1\theta/h} \tag{6.2}$$

and similarly for $e^{i\widetilde{\varphi}_p/h}B_p$, where $C>0$ is the same constant as in Lemma 5.2.

Indeed, we have

$$\|\widetilde{v}\|^2_{L^2(\Omega;\mathbb{C}^d)}\lesssim h^2\|f\|^2_{H^3_h(\Gamma;\mathbb{C}^d)}+\theta^{-2-2\ell}\|f\|^2_{H^3_h(\Gamma;\mathbb{C}^d)}\int_0^\infty e^{-2Cx_1\theta/h}dx_1$$
$$\lesssim h\theta^{-3-2\ell}\|f\|^2_{H^3_h(\Gamma;\mathbb{C}^d)}.$$

□

Proof of Lemma 6.1. Observe first that by (3.2) we have

$$\phi_0\left(x_1/|\rho_s|^3\delta\right)\phi_0(x_1\langle\xi'\rangle^\varepsilon/\delta)\in S^0_{2,2}(|\rho_s|)+S^0_{0,1}\subset S^0_{1,1}(\theta)+S^0_{0,1} \tag{6.3}$$

uniformly in x_1, and similarly with $|\rho_s|$ replaced by $|\rho_p|$. In view of the choice of the function χ, it is easy to see that (6.3) implies

$$\chi\Psi\in S^0_{1,1}(\theta)+S^0_{0,1} \tag{6.4}$$

uniformly in x_1. On the other hand, by Lemma 5.2,

$$e^{Cx_1\theta/h}e^{i\widetilde{\varphi}_s/h}\in S^0_{1,1}(\theta)+S^0_{0,1} \tag{6.5}$$

on $\operatorname{supp}\Psi$, uniformly in x_1, and similarly with $\widetilde{\varphi}_s$ replaced by $\widetilde{\varphi}_p$. By (6.4) and (6.5),

$$e^{Cx_1\theta/h}e^{i\widetilde{\varphi}_s/h}\chi\Psi\in S^0_{1,1}(\theta)+S^0_{0,1} \tag{6.6}$$

uniformly in x_1. Furthermore, it is easy to see that Lemma 5.1 yields

$$\partial_{x'}^{\alpha}\varphi_s \in S^0_{2,2}(|\rho_s|) + S^1_{0,1} \subset S^0_{1,1}(\theta) + S^1_{0,1}, \quad 1 \le |\alpha| \le 2, \tag{6.7}$$

$$\partial_{x_1}^{k}\varphi_s \in S^{4-3k}_{2,2}(|\rho_s|) + S^1_{0,1} \subset \begin{cases} S^0_{1,1}(\theta) + S^1_{0,1} & \text{if} \quad k = 1, \\ S^{2-3k/2}_{1,1}(\theta) + S^1_{0,1} & \text{if} \quad k \ge 2, \end{cases} \tag{6.8}$$

on $\operatorname{supp}\Psi$, uniformly in x_1, and similarly with φ_s replaced by φ_p. By (6.8),

$$\partial_{x_1}^{k}U^t(\gamma\nabla_x\varphi_s), \partial_{x_1}^{k}U^t(\gamma\nabla_x\varphi_p) \in \begin{cases} S^0_{1,1}(\theta) + S^1_{0,1} & \text{if} \quad k = 0, \\ S^{(1-3k)/2}_{1,1}(\theta) + S^1_{0,1} & \text{if} \quad k \ge 1, \end{cases} \tag{6.9}$$

on $\operatorname{supp}\Psi$, uniformly in x_1. By (6.9) and (5.11),

$$\partial_{x_1}^{k}A_s, \partial_{x_1}^{k}A_p \in \begin{cases} S^{-\ell}_{1,1}(\theta) + S^2_{0,1} & \text{if} \quad k = 0, \\ S^{-\ell+(1-3k)/2}_{1,1}(\theta) + S^2_{0,1} & \text{if} \quad k \ge 1, \end{cases} \tag{6.10}$$

on $\operatorname{supp}\Psi$, uniformly in x_1. It is easy to see that (6.4), (6.8) and (6.10) imply

$$\Psi L(\varphi_s, A_s), \Psi L(\varphi_p, A_p) \in S^{-1-\ell}_{1,1}(\theta) + S^3_{0,1}, \tag{6.11}$$

$$\Psi G(A_s), \Psi G(A_p) \in S^{-5/2-\ell}_{1,1}(\theta) + S^2_{0,1}. \tag{6.12}$$

By (6.6), (6.11) and (6.12) we conclude

$$e^{Cx_1\theta/h}e^{i\widetilde{\varphi}_s/h}\Psi\left(L(\varphi_s, A_s) + hG(A_s)\right) \in S^{-1-\ell}_{1,1}(\theta) + S^3_{0,1} \tag{6.13}$$

as long as $\theta \ge h^{2/5-\epsilon}$. Thus, by (6.13) and Proposition 3.1 we obtain

$$\left\|\operatorname{Op}_h\left(e^{i\widetilde{\varphi}_s/h}\Psi\left(L(\varphi_s, A_s) + hG(A_s)\right)\right)\right\|_{H^3_h(\Gamma;\mathbb{C}^d)\to L^2(\Gamma;\mathbb{C}^d)} \lesssim \theta^{-1-\ell}e^{-Cx_1\theta/h}. \tag{6.14}$$

Furthermore, since

$$x_1^N e^{-Cx_1\theta/h} \lesssim h^N\theta^{-N}, \quad x_1^N e^{-Cx_1|\xi'|/h} \lesssim h^N|\xi'|^{-N},$$

we deduce from Lemma 5.2 that

$$h^{-N}x_1^N e^{i\widetilde{\varphi}_s/h} \in S^{-N}_{1,1}(\theta) + S^{-N}_{0,1} \tag{6.15}$$

uniformly in x_1 and h. By (6.15) and (5.3),

$$h^{-N}x_1^N e^{i\widetilde{\varphi}_s/h}\Phi_s \in S^{1-5N/2}_{1,1}(\theta) + S^{2-N}_{0,1} \tag{6.16}$$

on $\operatorname{supp}\Psi$, uniformly in x_1 and h. On the other hand, it follows from (6.7) and (6.8) that

$$\Pi_s(\gamma\nabla_x\varphi_s) \in S^0_{1,1}(\theta) + S^2_{0,1} \tag{6.17}$$

on $\operatorname{supp}\Psi$. Taking N big enough, depending on ε, and δ small enough, we can arrange

$$\left|z^2 n + x_1^N\Phi_s\right| \ge C - x_1^N|\Phi_s| \ge C - \mathcal{O}(\delta) \ge C/2$$

on $\operatorname{supp}\Psi$, with some constant $C > 0$. Therefore, using the eikonal equation (5.1) we can write

$$(\gamma\nabla_x\varphi_s)^{-2} = c_s\left(z^2 n + x_1^N\Phi_s\right)^{-1}.$$

In view of (5.3) we have

$$z^2 n + x_1^N\Phi_s \in S^0_{1,1}(\theta) + S^0_{0,1}$$

on $\operatorname{supp}\Psi$. Thus we obtain

$$(\gamma\nabla_x\varphi_s)^{-2} \in S^0_{1,1}(\theta) + S^0_{0,1} \tag{6.18}$$

on $\operatorname{supp}\Psi$. By (6.10), (6.16), (6.17) and (6.18) we conclude

$$h^{-N}x_1^N e^{i\widetilde{\varphi}_s/h}\Phi_s\Psi(\gamma\nabla_x\varphi_s)^{-2}\Pi_s(\gamma\nabla_x\varphi_s)\chi A_s\in S^{-5N/2}_{1,1}(\theta)+S^0_{0,1} \tag{6.19}$$

provided $N\geq 6$. It follows from (6.19) and Proposition 3.1 that

$$\Big\|\mathrm{Op}_h\Big(h^{-1}x_1^N e^{i\widetilde{\varphi}_s/h}\Phi_s\Psi(\gamma\nabla_x\varphi_s)^{-2}\Pi_s(\gamma\nabla_x\varphi_s)\chi A_s\Big)\Big\|_{L^2(\Gamma;\mathbb{C}^d)\to L^2(\Gamma;\mathbb{C}^d)}$$

$$\lesssim h^{N-1}\theta^{-5N/2}\lesssim h^{5\epsilon N/2-1}\lesssim h \tag{6.20}$$

as long as $\theta\geq h^{2/5-\epsilon}$ and $N\geq 4/5\epsilon$. Let χ be of compact support and suppose that $\operatorname{supp}\chi\cap\Sigma_s\neq\emptyset$, $\operatorname{supp}\chi\cap\Sigma_p=\emptyset$. Then $[\Delta_{\lambda,\mu},\Psi]\,\chi=0$ for $x_1\leq\delta_1|\rho_s|^3$ for some constant $\delta_1>0$. Therefore, on $\operatorname{supp}[\Delta_{\lambda,\mu},\Psi]\,\chi$ we have the bounds

$$e^{-Cx_1\theta/h}\leq e^{-C\delta_1|\rho_s|^3\theta/h}\leq e^{-\widetilde{C}\theta^{5/2}/h}\lesssim h^N\theta^{-5N/2}.$$

Clearly, we have similar bounds when $\operatorname{supp}\chi\cap\Sigma_p\neq\emptyset$, $\operatorname{supp}\chi\cap\Sigma_s=\emptyset$. When $\operatorname{supp}\chi\cap\Sigma_s=\emptyset$, $\operatorname{supp}\chi\cap\Sigma_p=\emptyset$, then $[\Delta_{\lambda,\mu},\Psi]\,\chi=0$ for $x_1\leq\delta_2$ for some constant $\delta_2>0$. So, in this case the above bounds still hold. Let now $\chi\in S^0_{0,1}$ be such that $\operatorname{supp}\chi\subset\operatorname{supp}(1-\eta)$. Then $[\Delta_{\lambda,\mu},\Psi]\,\chi=0$ for $x_1\leq\delta_3\langle\xi'\rangle^{-\varepsilon}$ for some constant $\delta_3>0$. Hence, on $\operatorname{supp}[\Delta_{\lambda,\mu},\Psi]\,\chi$ we have the bounds

$$e^{-Cx_1|\xi'|/h}\leq e^{-\widetilde{C}|\xi'|^{1-\varepsilon}/h}\lesssim h^N|\xi'|^{-N(1-\varepsilon)}.$$

Therefore, by Lemma 5.2 and (6.10) we get

$$h^{-N}e^{i\widetilde{\varphi}_s/h}\,[\Delta_{\lambda,\mu},\Psi]\,\chi A_s\in S^{-\ell-1-5N/2}_{1,1}(\theta)+S^0_{0,1} \tag{6.21}$$

for N big enough. It follows from (6.21) and Proposition 3.1 that

$$\Big\|\mathrm{Op}_h\Big(he^{i\widetilde{\varphi}_s/h}\,[\Delta_{\lambda,\mu},\Psi]\,\chi A_s\Big)\Big\|_{L^2(\Gamma;\mathbb{C}^d)\to L^2(\Gamma;\mathbb{C}^d)}$$

$$\lesssim h^{N+1}\theta^{-\ell-1-5N/2}\lesssim h^{5\epsilon N/2+1-(\ell+1)(2/5-\epsilon)}\lesssim h \tag{6.22}$$

as long as $\theta\geq h^{2/5-\epsilon}$ and N big enough. Now the estimate (6.2) follows from (6.14), (6.20) and (6.22). □

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Université de Nantes, Laboratoire de Mathématiques Jean Leray, 2 rue de la Houssinière, BP 92208, 44322 Nantes Cedex 03, France

Email address: Georgi.Vodev@univ-nantes.fr