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## Non-linear pre and post buckling behaviour of a Timoshenko beam

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*Summary* Plane bending of a Timoshenko beam is derived using Cosserat formulation by the mean of a material curve and a moving director frame. Equilibrium relations are derived in the case of linear stress-strain relations and for both large strains and large displacements. Beam bending is controlled by magnitude and direction of a force and a bending moment. By a series of transformations, equilibrium equations are rearranged where pre and post buckling solutions are given in terms of Jacobi elliptic functions. From our knowledge this is the first attempt where such solution are derived by taking account a control of the bending moment.

### KINEMATICS AND INTERNAL FORCES UNDER COSSERAT FORMULATION

We consider a Timoshenko model for plane motion of an homogeneous straight beam of length  $L$ . In a Cartesian frame, the beam axis is oriented along  $e_z$  in the stress-free configuration, and the motion lies in the  $(e_x, e_z)$ -plane. However, for such a Cosserat-like structure it is justified to use a moving director frame  $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$  for which  $\mathbf{d}_2 = e_y$  and  $\mathbf{d}_3$  is normal to the cross-section, because this basis is orthonormal  $\mathbf{d}_1 = \mathbf{d}_3 \wedge \mathbf{d}_2$ . In the stress-free configuration this basis coincide with the Cartesian frame, however during motion, the orientation of the section is not uniform,  $\{\mathbf{d}_i(S)\}$  depends on the curvilinear abscissa  $S$  of the beam.  $\mathbf{d}_3$  is not tangent to the center line as in the Euler-Bernoulli model. Kinematics of the beam are governed by the placement  $\varphi(S)$  of the center line and rotation  $\theta(S)$  of the section around  $\mathbf{d}_2$ . Following [6], deformation are described thanks to the curvature  $\kappa(S)$  and generalized strains  $\tilde{\varepsilon}(S)$  with

$$\kappa := \frac{\partial \theta}{\partial S} \mathbf{d}_2 = \kappa_2 \mathbf{d}_2, \quad \tilde{\varepsilon} := \frac{\partial \varphi}{\partial S} - \mathbf{d}_3 = \varepsilon_1 \mathbf{d}_1 + (\varepsilon_3 - 1) \mathbf{d}_3. \quad (1)$$

Hereafter the kinematical variables are the component  $\kappa_2(S)$  of the curvature and components  $\varepsilon_1(S)$  and  $\varepsilon_3(S)$  of the deformation  $\frac{\partial \varphi}{\partial S}$  of the center-line. In particular  $\varepsilon_1$  is the shear strain and  $\varepsilon_3 - 1$  is the longitudinal strain. Internal force acting on a section is  $\mathbf{N} = N_1 \mathbf{d}_1 + N_3 \mathbf{d}_3$  where  $N_1$  and  $N_3$  are the shear and normal components respectively. Internal moment is  $\mathbf{M} = M_2 \mathbf{d}_2$  where  $M_2$  is the bending moment.

Linear constitutive law is assumed :  $N_1 = GA\varepsilon_1$ ,  $N_3 = EA(\varepsilon_3 - 1)$  and  $M_2 = EI\kappa_2$  where  $A$  and  $I$  are the area and the quadratic moment of the section,  $E$  and  $G$  are the Young modulus and shear modulus (including eventually a shear correction factor) of the beam material.

### EQUILIBRIUM RELATIONS

Equilibrium relations for this static problem states [1, 2] :

$$\frac{\partial \mathbf{N}}{\partial S} = 0, \quad \frac{\partial \mathbf{M}}{\partial S} + \frac{\partial \varphi}{\partial S} \wedge \mathbf{N} = 0. \quad (2)$$

Non-dimensional kinematical variables are given thanks to the gyration radius  $\varrho = \sqrt{I/A}$ :

$$s = \frac{S}{\varrho}, \quad \ell = \frac{L}{\varrho}, \quad \varepsilon_i(s) = \varepsilon_i(S), \quad \kappa_2(s) = \varrho \kappa_2(S), \quad i = 1, 3. \quad (3)$$

Last, a material parameter  $g = E/G$  is introduced. Note that  $g \simeq 2(1 + \nu)$  where  $\nu$  is the Poisson's ratio, then  $2 \lesssim g \lesssim 3$ . Hence (2) become in a non-dimensional form (the prime stands for derivation over  $s$ ) :

$$\varepsilon_1' + g(\varepsilon_3 - 1)\kappa_2 = 0 \quad (4)$$

$$g \varepsilon_3' - \varepsilon_1 \kappa_2 = 0 \quad (5)$$

$$g \kappa_2' + \varepsilon_1 \varepsilon_3 - g \varepsilon_1 (\varepsilon_3 - 1) = 0 \quad (6)$$

In order to prevent rigid motion  $\theta(0)$  and  $\varphi(0)$  are set to zero. In this work boundary conditions are fixed by imposing force and moment at  $s = \ell$ . Let us denote  $\mathbf{N}_\ell := \mathbf{N}(\ell)$  and  $\mathbf{M}_\ell := \mathbf{M}(\ell)$ . In terms of force, either orientation  $\phi_\ell$  and magnitude  $N_\ell$  or equivalently shear and longitudinal components  $N_{1\ell}$  and  $N_{3\ell}$  may be prescribed. Here  $\phi_\ell$  is measured from the opposite normal of the section, then  $N_{1\ell} = -N_\ell \sin(\phi_\ell)$  and  $N_{3\ell} = -N_\ell \cos(\phi_\ell)$ .

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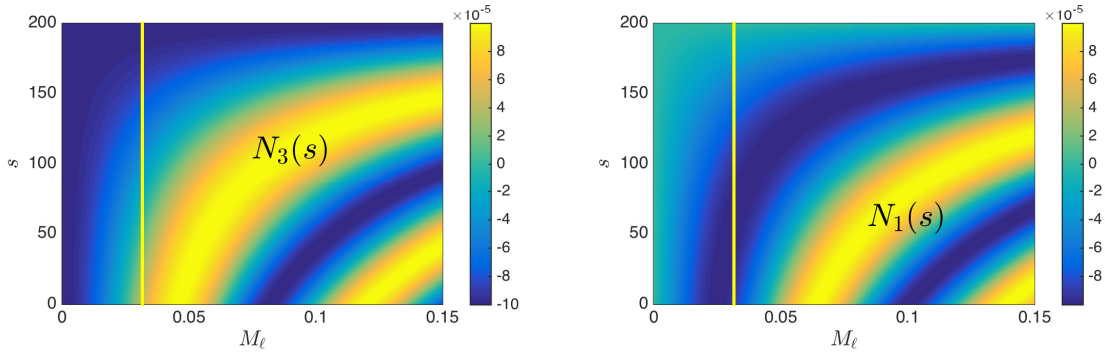


Figure 1: Internal longitudinal force  $N_3(s)$  and shear force  $N_1(s)$  for various bending moments at boundary  $M_2(\ell) = M_\ell$ . Here  $\ell = 200$ ,  $g = 2.5$  and the external load  $N_\ell = 10^{-4}$  is fixed and purely longitudinal ( $\phi_\ell = 0$  then  $N_{1\ell} = 0$ ). Note that for such material the Euler critical buckling is  $P_e = g\pi^2/\ell^2 \sim 6 \cdot 10^{-4}$  then a priori  $|N_{3\ell}| < P_e$ . The yellow line  $M_\ell = 2\sqrt{gN_\ell} \sim 0.03$  splits the  $M_\ell$  domain where  $a < 0$  (left) and  $a > 0$  (right).

### PRE AND POST BUCKLING SOLUTIONS

From the first relation of (2), the force is uniform  $N(s) = N_\ell$ . This motivate us to introduce  $\phi(s) \in ]-\pi; \pi[$  such that  $N_1(s) = -N_\ell \sin(\phi(s))$  and  $N_3(s) = -N_\ell \cos(\phi(s))$  with  $\phi(\ell) = \phi_\ell$ . According to (4) or (5) it is straight forward to obtain  $\phi' = -\kappa_2$ . Injecting this relation into (6) we obtain after integration

$$-(g\phi')^2 + 2gN_\ell \cos \phi + (g-1)N_\ell^2 \cos^2 \phi = \mu_\ell. \quad (7)$$

The constant of integration  $\mu_\ell$  is imposed by boundary conditions as  $\mu_\ell = -M_{2\ell}^2 - 2gN_{3\ell} + (g-1)N_{3\ell}^2$ . With the change of variable  $t(s) = \tan(\phi(s)/2)$ , (7) may be written as

$$(t')^2 = a(t^2 - \alpha_-)(t^2 - \alpha_+) \quad \text{with} \quad a = \frac{(g-1)N_\ell^2 - 2gN_\ell - \mu_\ell}{4g^2} \quad \alpha_\mp = \frac{(g-1)N_\ell^2 + \mu_\ell \mp 2N_\ell\sqrt{g^2 + (g-1)\mu_\ell}}{(g-1)N_\ell^2 - 2gN_\ell - \mu_\ell}$$

that is completely determined by  $\mu_\ell$  and magnitude  $N_\ell$ . Let us note  $\omega_\mp = |\alpha_\mp|$ . For practical application  $\alpha_+ < 0$  but sign of  $\alpha_-$  and  $a$  may vary according to the set  $\{\mu_\ell, N_\ell\}$ . This would strongly affect the solution of the differential equation:

If  $a < 0$  and  $\alpha_- < 0$  no real-valued solution

If  $a < 0$  and  $\alpha_- > 0$   $t(s) = \pm\sqrt{\omega_-} \operatorname{sn}\left(\sqrt{a|\omega_+}(s-s_0) \mid -\frac{\omega_-}{\omega_+}\right)$  or  $t(s) = \pm\sqrt{\alpha_-}$

If  $a > 0$  and  $\alpha_- < 0$   $t(s) = \pm\sqrt{\omega_-} \operatorname{sc}\left(\sqrt{a|\omega_+}(s-s_0) \mid 1 - \frac{\omega_-}{\omega_+}\right)$

where  $s_0$  is a constant chosen in order to satisfy boundary conditions. Jacobian functions  $\operatorname{sn}(z|m)$  and  $\operatorname{sc}(z|m)$  are defined according to [3].

An illustration is given for a practical application in Fig.1. It must be emphasized that for moderates bending momentum at the boundary, some homogeneous solutions may appears in addition to the non-homogenous ones depicted in the figures. Hence three equilibrium positions are possible for moderate bending boundary condition. The non-homogeneous solution observed for moderate bending moment corresponds to solution observed after such iterative process (i) impose a large external load in order to reach buckling (ii) reduce progressively the external load.

It is then interesting to investigate the stability of all these solutions in order to complete the analysis.

### References

- [1] L. Le Marrec, J. Lerbet, L. R. Rakotomanana, Vibration of a Timoshenko beam supporting arbitrary large pre-deformation. Acta Mechanica, 229.1 (2018), 109-132.
- [2] Bažant, Z., Cedolin, L.: Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories. Oxford University Press, New York (1991)
- [3] Walker, P., 2003. The analyticity of Jacobian functions with respect to the parameter k. Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 459(2038), pp.2569-2574.
- [4] Batista, M., 2016. A closed-form solution for Reissner planar finite-strain beam using Jacobi elliptic functions. International Journal of Solids and Structures, 87, pp.153-166.
- [5] Humer, A., 2013. Exact solutions for the buckling and postbuckling of shear-deformable beams. Acta Mechanica, 224(7), pp.1493-1525.
- [6] Simo, J.C., 1985. A finite strain beam formulation. Comput Methods Appl Mech Eng, 49,70.