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Buckling of Timoshenko beam under two-parameters Winkler foundations

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Abstract

This paper exposes buckling solutions of a plane, quasi-static Timoshenko beam with small transformation subjected to a longitudinal force and surrounded by an elastic wall modeled by two-parameters Winkler foundation. A non-dimensional analysis of associated Haringx and Engesser model is performed and buckling stress and shape are exposed analytically. Relations for rigidity of the wall and buckling solutions were made for different regimes and for both models using asymptotic approach. Introducing the yield limit gives a simple criterion in terms of stiffness foundation and slenderness ratio for which buckling or irreversible transformation occur.

Keywords: Timoshenko beam, Buckling, Engesser model, Haringx model, Winkler foundation, Pasternak foundation.

1. Introduction

Beams are slender structures resistant to vertical loads, shear forces and bending moments. They are widely used in civil, mechanical, aerospace engineering. Even if the deformation of the beam is, in first approach, proportional to the load, it is known that a subsequent compressive load will suddenly deflect the beam to the side rather than crushing it. Such buckling depends on the material, geometry and boundary conditions of the beam.

Research on beam buckling has a long history initiated by Euler. In order to extend the validity domain of the Euler's prediction, different theories exist. Among others, Haringx's theory, initially stated for helical springs, explains the fact that springs of small slenderness do not buckle [1]. Another well-known theory proposed by F. Engesser considers the influence of shear on the buckling load of straight bars [2]. These two theories differ concerning how the normal and shear forces are defined with respect to the non-deformed or

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deformed cross-section. In Haringx's approach, the normal force is chosen normal to the deformed cross-section in the loaded state, whereas in Engesser's theory, the normal force is chosen parallel to the beam axis. During the last decades many researchers investigated beam behaviour, some of them supported the
15 Engesser approach while others adapted Haringx strategy. So far, no accord is not reached yet.

Nänni studied the effect of shear forces on the buckling load of simply supported beam with narrow rectangular or elliptic cross-section and concluded that Engesser approach was superior [3]. While Reissner computed a one-dimensional large-strain beam theory for plane beams and noticed that Haringx theory provides better agreement [4]. Ziegler used different formulation for bars and privileged the Engesser approach
20 to Haringx's, he also proposed an improvement by incorporating axial shortening [5]. However, Ziegler considers still that Haringx's model is more appropriate for helical springs. These conclusions were, short time later, refuted by Reissner who again supported Haringx's formulation, finding that both theories can be obtained by different forms of the one-dimensional stress-strain relations [6]. Bažant derived Engesser formula from the theoretical analysis of the problem [7]. He considers theoretically the buckling behaviour
25 of weak-in shear columns and, despite some correspondence between Engesser and Haringx formulations, concluded that Haringx formula gave better results [8]. However, for sandwich buckling and using finite element method he recovered results corresponding to Engesser-type theory [9].

This reversal shows how sensitive the models are to the problem under study. Accordingly, Blaauwendraad recommended Engesser's hypothesis in building and civil engineering [10]. But these results were rebutted
30 by Aristizabal who employed Haringx's for the static and dynamic studies of stability of uniform shear beam under generalized boundary conditions [11]. For sandwich structure the controversy exists too. On its study on the buckling behaviour of sandwich structures, Allen derived the Engesser formula [12] while Kardomateas *et al* made theoretical predictions and Finite Element Methods analysis with metallic laminated facings and foam or honeycomb cores under uniform axial compression, and noticed that Haringx
35 formula was closer to their results [13]. Attard computed an internal strain energy density for isotropic hyperelastic Hookean materials and showed that this formulation leads to a buckling load formula that agrees with Haringx's [14]. It was confirmed by Simo and Kelly who performed a two-dimensional buckling analysis of multilayer elastomeric beam. They concluded that, as long as the beam theory assumptions holds, their formulation was in closer agreement with the Haringx theory [15].

40 Banerjee and Williams analyzed shear-deformable uniform columns, derived the Engesser formula and buckling curves for the most common supporting conditions [16]. Xiang-Fang Li *et al.* studied the effects of Engesser's and Haringx's hypotheses on buckling of Timoshenko and higher-order shear deformable columns and they concluded that for shorter columns with weak shear rigidity Engesser beam theory gives the lowest estimate of the critical load, and it is more conservative [17]. Pedro Dias Simao [18] studied

45 the influence of shear deformations on the buckling of columns using the Generalized Beam Theory and deduced that Engesser load better agrees with their results.

Knowing that beam behaviour is also related to what surrounds it, many researchers examine beams supported on elastic foundation and their applications in modern engineering, such as soil foundation, strip foundation, foundation of the buildings. To describe the interactions of the beam and foundation, various kinds of foundation models have been proposed. The most common model for the elastic foundation is the Winkler model which regards the foundation as a series of separated springs without coupling effects between each other[19]. Pasternak generalized the Winkler model and assumed existence of shear interactions between the spring elements [20]. According to Kerr [21], all along the paper the terminology *two-parameter* Winkler foundation encompassed Pasternak proposal and Filonenko-Borodich model.

Many results have later been obtained using simple or generalized Winkler models. Zhang *et al.* investigated the bifurcations of a beam on nonlinear foundations [22], whereas G.Xia used a numerical method applied on beam on elastic foundation [23]. Challamel studied the buckling of elastic beam using Reissner model, and he also investigated the buckling of generic higher-order shear beam with elastic connections using local and nonlocal formulation [24]. In the dynamic regime, Rajesh examined the free vibration of uniform Timoshenko beams on Pasternak foundation using coupled displacement field method [25].

In this paper, we focus on an elastic, isotropic, homogenous straight Timoshenko beam with small, quasi-static plane transformation and linear constitutive load. A longitudinal load is imposed at the boundary but the beam is surrounded by *two-parameters* foundations. In many practical cases, this foundation has to increase the critical buckling stress. Furthermore, this foundation exerts an external force and moment per unit length (for example by means of springs that act on the boundary of the cross-section) in order to control displacement and rotation behaviour of the cross-section [26, 27].

Geometrical general problem under these hypotheses is presented in a dimensionless form that leads to Haringx approach in the first section. The Engesser model is presented in a dimensional form and for the same two-parameter foundation too. A comparison between the two problems is performed in the second section where buckling loads and buckling modes are first exhibited in a general way, then for pinned-pinned boundary conditions. The third section introduces a finest analysis of buckling load where both values of the foundation stiffness and slenderness ratio are involved. In the fourth section, the discussion is enriched by incorporating the yield limit of the material constituent of the beam. It leads to a simple criterion that may be exploited by engineering analyses. The conclusion underlines the main points of the work and proposes some further considerations.

2. Problem statement

2.1. General ingredients

The kinematics of the Timoshenko beam are described in a Cartesian frame $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ where \mathbf{e}_z is oriented along the beam axis in the stress-free configuration, and the motion of the beam lies in the $(\mathbf{e}_x, \mathbf{e}_z)$ -plane. However, for such a Cosserat-like structure it is justified to use a moving director frame basis $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ for which $\mathbf{d}_2 = \mathbf{e}_y$ and \mathbf{d}_3 is normal to the cross-section, as this basis is orthonormal $\mathbf{d}_1 = \mathbf{d}_3 \wedge \mathbf{d}_2$ and $(\mathbf{d}_1, \mathbf{d}_2)$ defines the plane of a rigid cross-section. As the orientation of the beam is not uniform, this basis $\{\mathbf{d}_i(S)\}$ depends on the curvilinear abscissa S of the beam (it is mainly implicit hereafter). In contrary to Euler-Bernoulli model \mathbf{d}_3 is not necessarily tangent to the center line. Of course for linearized theory $\mathbf{d}_1 \simeq \mathbf{e}_x$ and $\mathbf{d}_3 \simeq \mathbf{e}_z$.

The kinematics of the beam is governed by the displacement $\mathbf{u}(S) = \underline{u}_1(S)\mathbf{d}_1(S) + \underline{u}_3(S)\mathbf{d}_3(S)$ of any point of the center-line and rotation $\phi(S)$ of the section around \mathbf{d}_2 . With the same formalism, the internal force acting on the beam is of the form $\mathbf{N}(S) = N_1\mathbf{d}_1 + N_3\mathbf{d}_3$ where $N_1(S)$ is the shear force and $N_3(S)$ is the normal force and the moment $\mathbf{M}(S) = M_2\mathbf{d}_2$ where $M_2(S)$ is the bending moment. Linear constitutive laws are assumed:

$$N_1 = GA\left(\frac{\partial \underline{u}_1}{\partial S} - \phi\right), \quad N_3 = EA\frac{\partial \underline{u}_3}{\partial S}, \quad M_2 = EI\frac{\partial \phi}{\partial S},$$

where A and I are the area and the quadratic moment of the cross-section, E and G are the Young modulus and shear modulus (including eventually a shear correction factor) of the beam material. The external compression load is:

$$\mathbf{P} = -P\mathbf{d}_3, \quad P \geq 0.$$

This load causes the buckling of the beam. This is countered by the foundations. For elastic foundation, the external applied force per unit length and external applied moment per unit length are respectively:

$$\mathbf{q}(S) = -K_1\underline{u}_1\mathbf{d}_1, \quad \mathbf{m}(S) = -K_2\phi\mathbf{d}_2,$$

80 where K_1 and K_2 are translational and rotational stiffness [28, 29]. The motivation of the applied moment $\mathbf{m}(S)$ may be surprising. In Fig.1 a discrete construction of such external load is justified. If the distance between each rigid transversal bar is small enough, a continuous modeling leads to incorporate $\mathbf{m}(S) = -K_2\phi\mathbf{d}_2$ in the model. This stiffness can be caused by the tension in an elastic membrane connecting the ends of the Winkler springs (Filonenko-Borodich model), or as the shear stiffness of a shear layer (Pasternak
85 model) [26]. Such load may be observed in real life as for auger blade embedded into the soil.

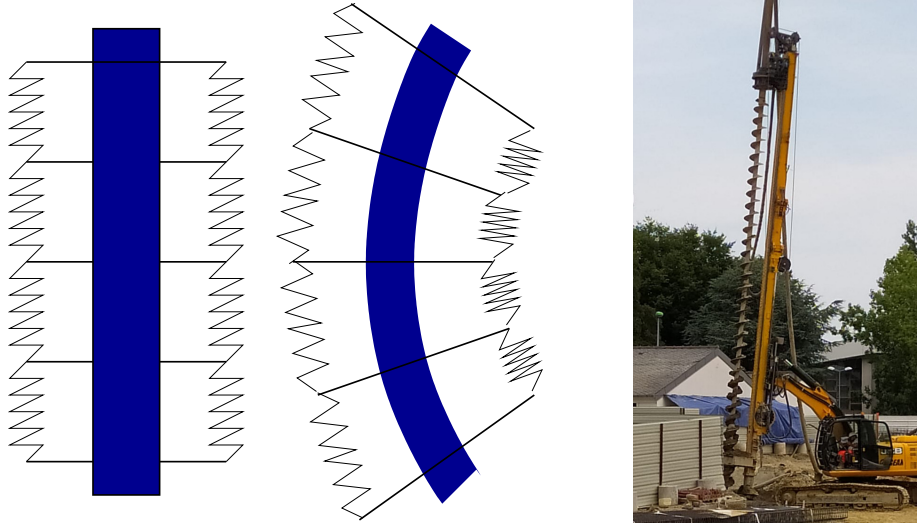


Figure 1: Motivation for the rotational reaction of the soil. Left: a beam structure in blue with glued rigid transversal blades embedded in an elastic media is modeled by a discrete system of springs. Right: an auger blade (Photo courtesy of N.Belhabchi.)

2.2. Equilibrium relations

Even for large transformation, equilibrium relations for this static problem states [30] :

$$\frac{\partial \mathbf{N}}{\partial S} + \mathbf{q} = 0, \quad \frac{\partial \mathbf{M}}{\partial S} + \frac{\partial \mathbf{u}}{\partial S} \wedge \mathbf{N} + \mathbf{m} = 0,$$

where, according to [31]:

$$\frac{\partial \mathbf{u}}{\partial S} = \boldsymbol{\varepsilon} + \mathbf{d}_3 ,$$

and for which $\boldsymbol{\varepsilon} = \varepsilon_1 \mathbf{d}_1 + \varepsilon_3 \mathbf{d}_3$, where $\varepsilon_1(S) = \frac{\partial u_1}{\partial S} - \phi$ is the shear strain and $\varepsilon_3(S) = \frac{\partial u_3}{\partial S}$ is the longitudinal strain. As $\frac{\partial \mathbf{d}_i}{\partial S} = \frac{\partial \phi}{\partial S} \mathbf{d}_2 \wedge \mathbf{d}_i$ and projecting along directors, one obtains the following system:

$$\begin{aligned} \frac{\partial N_3}{\partial S} - \frac{\partial \phi}{\partial S} N_1 &= 0 , \\ \frac{\partial N_1}{\partial S} + \frac{\partial \phi}{\partial S} N_3 - K_1 u_1 &= 0 , \\ \frac{\partial M_2}{\partial S} + (1 + \varepsilon_3) N_1 - \varepsilon_1 N_3 - K_2 \phi &= 0 . \end{aligned} \tag{1}$$

The present analysis concerns an infinitesimal perturbation superimposed on a finite longitudinal compression induced by a finite force \mathbf{P} . In such case \underline{u}_1 , ε_1 , ϕ and their derivatives are infinitesimal quantities. Neglecting $\varepsilon_3 N_1$ in (1) is usually interpreted as an *inextensible* approximation. Accordingly a linearized

version of (1) is obtained:

$$\begin{aligned} GA\left(\frac{\partial^2 u_1}{\partial S^2} - \frac{\partial \phi}{\partial S}\right) - \frac{\partial \phi}{\partial S}P - K_1 u_1 &= 0, \\ EI\frac{\partial^2 \phi}{\partial S^2} + \left(\frac{\partial u_1}{\partial S} - \phi\right)(GA + P) - K_2 \phi &= 0, \end{aligned} \quad (2)$$

what corresponds to the application of Haringx model for Winkler foundation [7, 32]. More precisely, if P is set to zero we recover the model from [26] in its static version.

2.3. Non-dimensionalization procedure

90 Non-dimensional formulation of the problem (2) is introduced thanks to the following variables:

$$\varrho = \sqrt{\frac{I}{A}}, \quad g = \frac{E}{G}, \quad \kappa_1 = \frac{K_1}{E} \frac{I}{A^2}, \quad \kappa_2 = \frac{K_2}{EA}, \quad \epsilon = \frac{P}{EA}. \quad (3)$$

For any material $g \simeq 2(1 + \nu)$ where ν is the Poisson ratio then $2 \lesssim g \lesssim 3$, For compression in the elastic regime $0 < \epsilon < \epsilon_{yield}$ where ϵ_{yield} is nothing else than the strain limit for which irreversible transformation occurs. For translational and rotational stiffness $\kappa_i, \kappa_i = 0$ in absence of foundation and $\kappa_i = \infty$ for a rigid foundation. This paper focuses on the cases for which $\kappa_i < 1$ that corresponds to foundations softer than the structures. The kinematical variables become in a non-dimensional form:

$$s = \frac{S}{\varrho}, \quad \ell = \frac{L}{\varrho}, \quad u(s) = \frac{u_1(S)}{\varrho}, \quad \theta(s) = \phi(S).$$

Note that $\varrho = R/2$ for circular cross-section of radius R therefore ℓ is twice the standard slenderness ratio. More generally, beam model is justified for $\ell \gtrsim 20$. The terminology *slender beam* is used if $\ell \gtrsim 40$ whereas $20 \lesssim \ell \lesssim 40$ characterizes a *thick beam*.

Therefore (2) takes the form:

$$\begin{aligned} u'' - g\kappa_1 u - (1 + g\epsilon)\theta' &= 0, \\ g\theta'' + (1 + g\epsilon)u' - (1 + g\epsilon + g\kappa_2)\theta &= 0, \end{aligned} \quad (4)$$

95 where the convention $f' := \frac{\partial f}{\partial s}$ for any function $f(s)$ is used.

Another model widely used for buckling is proposed by Engesser [2] for which the non-dimensional equilib-

rium relations are in case of two-parameter foundations:

$$\boxed{\begin{aligned} (1 - g\epsilon)u'' - g\kappa_1 u - \theta' &= 0, \\ g\theta'' + u' - (1 + g\kappa_2)\theta &= 0. \end{aligned}} \quad (5)$$

3. Buckling modes

3.1. Secular relations and eigenfunctions

For harmonic solution $u(s) = Ue^{iks}$ and $\theta(s) = \Theta e^{iks}$, the linear differential system becomes $\mathbb{K}\mathbf{V} = 0$ where $\mathbf{V} = (U, \Theta)^T$ and the (hermitian) rigidity matrix is, for (4) and (5) respectively:

$$\begin{aligned} \mathbb{K}^H &= \begin{pmatrix} k^2 + g\kappa_1 & ik(1 + g\epsilon) \\ -ik(1 + g\epsilon) & 1 + g(k^2 + \epsilon + \kappa_2) \end{pmatrix}, \\ \mathbb{K}^E &= \begin{pmatrix} k^2(1 - g\epsilon) + g\kappa_1 & ik \\ -ik & 1 + g(k^2 + \kappa_2) \end{pmatrix}. \end{aligned} \quad (6)$$

Non-trivial solutions arise if $\det(\mathbb{K}) = 0$ what may be written as a secular equation:

$$\begin{aligned} \mathcal{P}^H(\epsilon, k) &= g(k^2(\kappa_2 + k^2 - \epsilon(g\epsilon + 1)) + \kappa_1(g\kappa_2 + g(\epsilon + k^2) + 1)), \\ \mathcal{P}^E(\epsilon, k) &= g(k^2(\kappa_2(1 - g\epsilon) - \epsilon(gk^2 + 1) + k^2) + \kappa_1(g\kappa_2 + gk^2 + 1)). \end{aligned} \quad (7)$$

By solving $\mathcal{P}(\epsilon) = 0$ (for a fixed k), one finds a polynomial with respect to ϵ whose real positive roots are

$$\begin{aligned} \epsilon^H &= \frac{g\kappa_1 + \sqrt{(g\kappa_1 + k^2)(g\kappa_1 + (1 + 4g\kappa_2)k^2 + 4gk^4) - k^2}}{2gk^2}, \\ \epsilon^E &= \frac{\kappa_2 + k^2}{1 + g(\kappa_2 + k^2)} + \frac{\kappa_1}{k^2}. \end{aligned} \quad (8)$$

100 Hence explicit expressions of the critical strain are obtained for a given structure (the material parameter g and the stiffnesses κ_i of the foundations) and a given wavenumber k . This latter will be fixed by boundary conditions. Without foundation the Euler critical load is $\underline{k}^2 EI/L^2$, where $\underline{k} = k/\rho$ is the dimensional wavenumber. In a non-dimensional form the critical strain associated to Euler critical load is $\epsilon^{eu} = k^2$. By taking $\kappa_1 = \kappa_2 = 0$ and applying Taylor expansion to (8) for small k and taking first approximation one

105 gets:

$$\epsilon = k^2 + \mathcal{O}(k^4), \quad (9)$$

for both models. Hence the two models are in first approximation equal to Euler's critical load if no foundation is present and if $k \ll 1$.

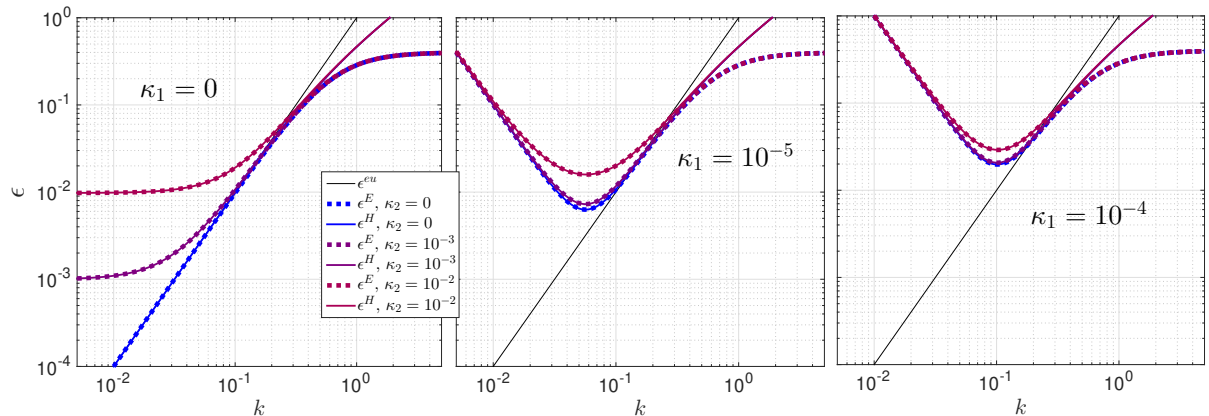


Figure 2: Secular equations (8) for various κ_1 and κ_2 where $g = 5/2$.

Figure 2 presents the variation of ϵ^H and ϵ^E for various stiffnesses ; the non-dimensional Euler's critical strain $\epsilon^{eu} = k^2$ is presented too. For $k \ll 1$ Haringx and Engesser's models give the same estimation of critical strain ϵ . Of course, this estimation is sensible to the stiffnesses κ_i . On the other hand, for $1 \lesssim k$, the Haringx and Engesser's models gives distinct estimations of ϵ but curiously these estimations are in first approximation independent of the stiffness of the foundation. Note that $\epsilon^E < \epsilon^H$ in this regime as mentioned in [17].

Conversely, for fixed ϵ , \mathcal{P} is a second degree polynomial with respect to k^2 . Hence $u(s)$ and $\theta(s)$ have the general form:

$$\begin{aligned} u(s) &= ae^{ik_1s} + be^{-ik_1s} + ce^{ik_2s} + de^{-ik_2s}, \\ \theta(s) &= \Xi(k_1)(ae^{ik_1s} - be^{-ik_1s}) + \Xi(k_2)(ce^{ik_2s} - de^{-ik_2s}), \end{aligned} \quad (10)$$

where a, b, c and d will be defined by boundary conditions. Here, $(k_1)^2$ and $(k_2)^2$ are the two roots of $\mathcal{P}(k^2) = 0$. The expression of $\theta(s)$ is obtained by solving $\mathbb{K}_{11}U + \mathbb{K}_{12}\Theta = 0$ therefore $\Theta = i\Xi U$ where $i\Xi = -\mathbb{K}_{11}/\mathbb{K}_{12}$. In details:

$$\Xi^H = k \frac{1}{1 + g\epsilon} + \frac{g\kappa_1}{k} \frac{1}{1 + g\epsilon}, \quad \Xi^E = k(1 - g\epsilon) + \frac{g\kappa_1}{k}. \quad (11)$$

Taylor expansion in terms of ϵ gives:

$$\Xi^H = \Xi^E + \mathcal{O}\left(\frac{\kappa_1 \epsilon}{k}, k\epsilon^2\right). \quad (12)$$

By construction κ_2 is not explicit in Ξ but intervenes implicitly through ϵ .

110 3.2. Pinned-pinned beam

Without loss of generality, we consider a pinned-pinned beam for which (non-dimensional) boundary conditions are in terms of kinematical variables $u(s) = 0$, $\theta'(s) = 0$ at $s = 0$ and $s = \ell$. This forms a set of four linear equations in terms of $X = (a, b, c, d)^T$ that may be written algebraically as $\mathbb{M}X = 0$ with

$$\mathbb{M} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ k_1 \Xi(k_1) & k_1 \Xi(k_1) & k_2 \Xi(k_2) & k_2 \Xi(k_2) \\ e^{ik_1 \ell} & e^{-ik_1 \ell} & e^{ik_2 \ell} & e^{-ik_2 \ell} \\ k_1 \Xi(k_1) e^{ik_1 \ell} & k_1 \Xi(k_1) e^{-ik_1 \ell} & k_2 \Xi(k_2) e^{ik_2 \ell} & k_2 \Xi(k_2) e^{-ik_2 \ell} \end{pmatrix}. \quad (13)$$

Again, non trivial solutions exist if $\det(\mathbb{M}) = 0$ that gives the following relation:

$$(k_1 \Xi(k_1) - k_2 \Xi(k_2))^2 \sin(k_1 \ell) \sin(k_2 \ell) = 0. \quad (14)$$

115 Direct computation shows that $k_1 \Xi(k_1) \neq k_2 \Xi(k_2)$. First it is observed that k_1 and k_2 play a similar role therefore one focuses on $k \equiv k_1$ in the following. According to (14) non-trivial solutions exist only if $\sin(k\ell) = 0$, this implies that k must be chosen among an infinite countable set of values, more precisely:

$$k_n = \frac{n\pi}{\ell}, \quad n \in \mathbb{N}^*. \quad (15)$$

One should note that in order to respect beam hypotheses the dimensional wavelength $\underline{\lambda}$ has to satisfy $\underline{\lambda} > 2R = 4\rho$ for circular cross-sections, or in a non-dimensional way $\lambda = \underline{\lambda}/\rho > 4$ and since $\lambda_n = 2\pi/k_n > 4$

120 so the following conditions are obtained:

$$\boxed{n < \frac{\ell}{2}, \quad k_n \leq 1.} \quad (16)$$

From this information, it means that even for a thick beam, k must be explored in a variation domain $[0, 1]$. According to equation (13), the modal amplitude may be obtained up to an arbitrary constant, by

solving:

$$\mathbb{A}Y = -aZ, \quad \text{where} \quad Y = (b, c, d)^T,$$

with $\mathbb{A}_{ij} = \mathbb{M}_{ij}$ and $Z_i = \mathbb{M}_{i1}$ for $1 \leq i \leq 3$ and $2 \leq j \leq 4$. Fixing $a = 1$ and $k_n = n\pi/\ell$ one obtains:

$$\begin{aligned} u(s) &= \sin\left(\frac{n\pi}{\ell}s\right), \\ \theta(s) &= \Xi\left(\frac{n\pi}{\ell}\right) \cos\left(\frac{n\pi}{\ell}s\right). \end{aligned} \tag{17}$$

Hence ϵ and κ_i intervenes in the modes shapes only through the parameter $\Xi(k, \epsilon, \kappa)$. In other words, the general shape of the eigenmode is not qualitatively modified by the loads ϵ and the stiffnesses κ_i of the foundations. As Ξ is distinct for Haringx and Engesser models, the mode shapes may differ for the two models even if the value of critical load obtained for each model is similar.

125 3.3. General case

For a general boundary conditions k could still be organized as an increasing countable sequence $\{k_n\}_n$ where $k_n = \mathcal{O}\left(\frac{n}{\ell}\right)$. In the following, it is considered that the variation domain $]0, 1]$ of k , obtained for pinned-pinned boundary conditions, is valid for other boundary conditions. From Fig.2, it is observed that in this domain, the estimation $\epsilon(k)$ mainly coincides for Engesser and Haringx models.

130 The expression of the eigenmodes still refers to (10) but the detailed expression may be less clear than (17). However the general methodology explained above (17) holds.

4. Buckling limit

Till now the buckling mode is not fixed. As this latter must respect the boundary conditions, the buckling mode is associated to a wavenumber k_b among the list $\{k_n\}$. Even if $\{k_n\}$ is an increasingly sorted list, k_b is not a priori k_1 . Indeed, as k_b prescribes the *first* buckling mode, it must be chosen in such a way that it corresponds to the minimal critical strain:

$$\epsilon_b := \epsilon(k_b) = \min_n(\epsilon(k_n)) . \tag{18}$$

4.1. Continuous approximation of the critical strain

Solving (18) consists on a discrete optimisation. In order to have a first overview of the solution, the problem is here explored in its continuous version. Indeed, from Fig.2, $\epsilon(k)$ is a strictly convex function for

any model (and for fixed κ_i). This ensures the existence and uniqueness of a global minimum of $\epsilon(k)$.

Finding in \mathbb{R}^{*+} such position k_{min} of the minimum leads to calculate $\frac{\partial \epsilon}{\partial k}|_{k_{min}} = 0$. According to (8) one obtains for Haringx and Engesser respectively:

$$\begin{aligned} \frac{\partial \epsilon^H}{\partial k} &= \frac{2k^6 - \kappa_1 \left(g\kappa_1 + 2g\kappa_2 k^2 + \sqrt{(g\kappa_1 + k^2)(g\kappa_1 + 4gk^4 + 4g\kappa_2 k^2 + k^2)} + k^2 \right)}{k^3 \sqrt{(g\kappa_1 + k^2)(g\kappa_1 + k^2(4g\kappa_2 + 4gk^2 + 1))}} = 0, \\ \frac{\partial \epsilon^E}{\partial k} &= \frac{2k}{(g\kappa_2 + gk^2 + 1)^2} - \frac{2\kappa_1}{k^3} = 0. \end{aligned} \quad (19)$$

Knowing that $g > 0$, $\kappa_1 > 0$ and $\kappa_2 > 0$, we deduce the solution of (19)

$$\begin{aligned} k_{min}^H &= \sqrt{\frac{1}{2} \sqrt{\kappa_1} \left(1 + \sqrt{1 + 4g(\kappa_2 + \sqrt{\kappa_1})} \right)}, \\ k_{min}^E &= \sqrt{\frac{\sqrt{\kappa_1}(1 + g\kappa_2)}{1 - g\sqrt{\kappa_1}}}. \end{aligned} \quad (20)$$

As $\kappa_i < 1$, leading terms approximation with respect to κ_1 and κ_2 gives, for both models:

$$\boxed{k_{min} \sim \kappa_1^{1/4}} \quad (21)$$

This behaviour of k_{min} is illustrated in Figure (3).

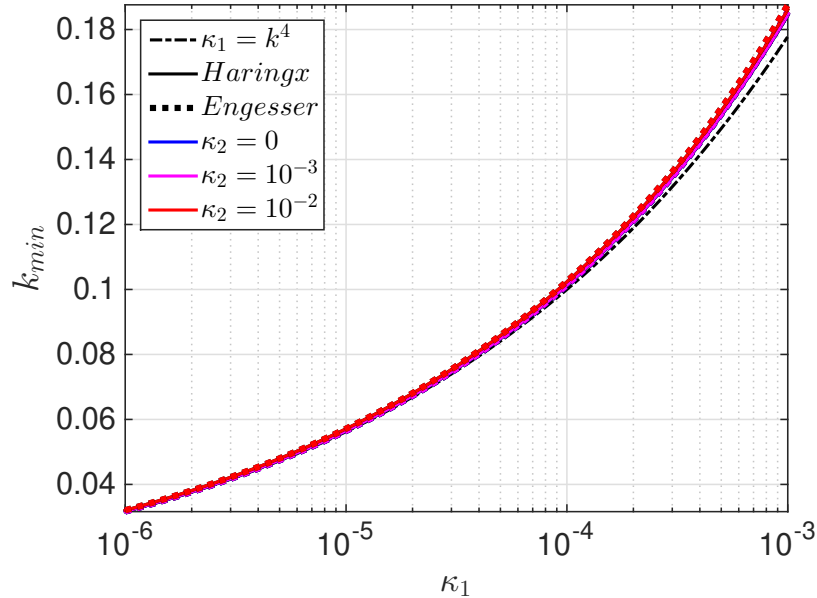


Figure 3: Solutions k_{min} of (20) versus κ_1 for various κ_2 . Comparison between Haringx and Engesser models with $g = 5/2$.

4.2. Discrete analysis of the critical strain

The preceding analysis may be used to define the buckling wavenumber $k_b \in \{k_n\}$ and then ϵ_b . Practically, two cases have to be studied.

4.2.1. Thick beam on soft foundations

Considering first the case where $k_{min} \leq k_1$ (and $k_1 \leq 1$). Since $\epsilon(k)$ is an increasing function for $k \geq k_{min}$ then the criterion (18) is satisfied for $k_b = k_1$. Note that k_b could be far larger than k_{min} and the only attainable information is $k_b = \mathcal{O}(\pi/\ell) < 1$.

According to (21) and the discussion in the section 3.3, $k_{min} \leq k_1$ is satisfied only if $\kappa_1^{1/4} < \mathcal{O}(\pi/\ell)$, or in a more proper (non-dimensional or dimensional) version, if:

$$\kappa_1 \ell^4 \ll 1, \quad \frac{K_1 L^4}{EI} \ll 1 .$$

Such situation appears if the foundation is soft or if the beam has a moderate slenderness ratio. This regime is called *small-regime* hereafter, as both the length and the stiffness are small numbers.

From (8) and since $\kappa_1 \sim k_{min}^4 < k_b^4 \leq 1$ and $\kappa_2 < 1$, the buckling strains estimated by the two model are similar. It may be approached by:

$$\epsilon_b(k_b) = \frac{k_b^2}{1 + gk_b^2} + \frac{\kappa_1}{k_b^2} + \frac{\kappa_2}{(1 + gk_b^2)^2} , \quad (22)$$

for thick or slender beam. For thin structure, $k_b \ll 1$, allows some simplification:

$$\epsilon_b(k_b) = k_b^2 + \frac{\kappa_1}{k_b^2} + \kappa_2 . \quad (23)$$

140 4.2.2. Thin beam on stiff foundations

For $k_{min} \gg k_1$, the situation is different. This regime appears for slender structure or for stiff foundations. It is called *large-regime*, and is controlled by $\kappa_1 \ell^4 \gg 1$. In that case $k_b = \mathcal{O}(k_{min})$ in order to minimize the buckling strain $\epsilon(k_b)$. Note that one observes the following hierarchy $k_b = \mathcal{O}(k_{min}) = \mathcal{O}(\kappa_1^{1/4}) \gg \mathcal{O}(1/\ell)$, however as $\kappa_1 < 1$ then $1/\ell \ll k_b \leq 1$.

145 The wavelength $\lambda_b = 2\pi/k_b$ of the buckling mode satisfies

$$\lambda_b \sim \frac{2\pi}{\kappa_1^{1/4}} . \quad (24)$$

It is controlled by $\kappa_1 = K_1 I / (EA^2)$ only, in particular it is independent of the length of the structure. The buckling appears with a pattern having several arches. The number of arches observed on a buckled structure is $N := 2\ell/\lambda_b$ (a wavelength is considered as composed of two arches). The order of magnitude of this parameter is:

$$N \sim \left\lfloor \frac{\ell}{\pi} \kappa_1^{1/4} \right\rfloor, \quad (25)$$

150 where $\lfloor \cdot \rfloor$ denotes the floor function.

In terms of strain criterion, since $\kappa_i \ll gk$, one obtains:

$$\boxed{\epsilon_b = \mathcal{O}(2\sqrt{\kappa_1} + \kappa_2)}, \quad (26)$$

for both models.

For pinned-pinned beam the modal amplitudes are approached by (12) that gives:

$$\Xi \sim \kappa_1^{1/4} (1 - g\kappa_2). \quad (27)$$

This shows the high influence of foundation stiffness in this *large-regime*: it uniquely defines the buckling load (26), the modal amplitude (27) and the buckling's wavelength (24). An illustration of the buckled shape in the *large-regime* is given in Fig.4 for pinned-pinned boundary conditions.

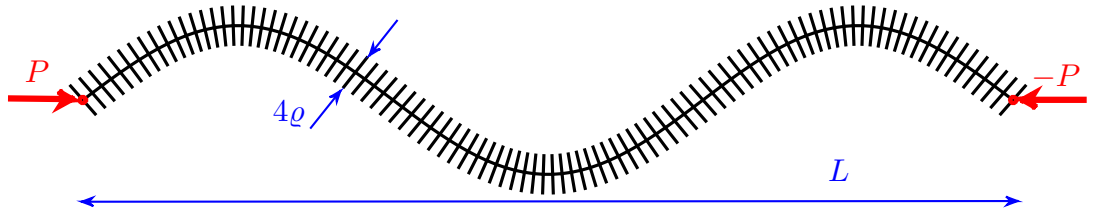


Figure 4: Shape of a pinned-pinned beam under a Winkler foundation. $g = 5/2$, $\ell = 100$, $\kappa_1 = 10^{-4}$, $\kappa_2 = 10^{-2}$. The buckling mode associated to the smallest strain that is the third one: $k_b = 3\pi/\ell$ with a critical strain $\epsilon_b = 3 \cdot 10^{-2}$.

4.2.3. Discussion

The estimations of ϵ_b for *large* and *small* regimes (in (26) and (23), respectively) are given in Fig.5. The given approximations are clearly justified for both models. It is the occasion to highlight that the *large-regime* approximation (26) of ϵ_b is included in (23) as:

$$\lim_{k_b \rightarrow \kappa_1^{1/4}} \left(k_b^2 + \frac{\kappa_1}{k_b^2} + \kappa_2 \right) = 2\sqrt{\kappa_1} + \kappa_2.$$

This fundamental equation (23) corresponds to the differential equation:

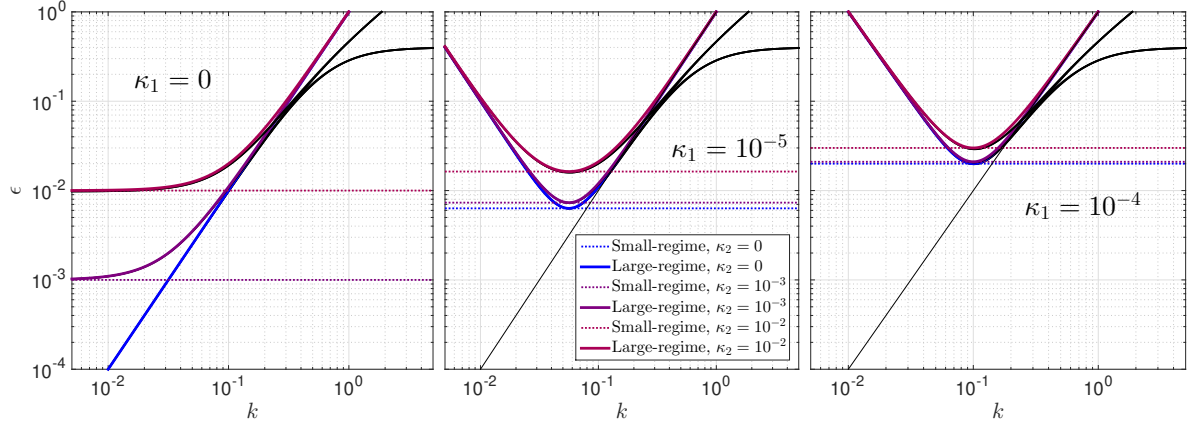


Figure 5: Report of Fig.2 where all previous curves are in black. The new superimposed curves are the *small-regime* and *large-regime* estimations of ϵ_b given in (26) and (23), respectively.

$$u^{(4)} + (\epsilon_b - \kappa_2) u'' + \kappa_1 u = 0 .$$

This corresponds to the superposition of the Euler-buckling equation with the Pasternak foundation model.

In the non-dimensional form used in this paper, these latter are respectively:

$$u^{(4)} + \epsilon_b u'' = 0, \quad u^{(4)} - \kappa_2 u'' + \kappa_1 u = 0 .$$

5. Yield limit

It must be recalled that the proposed models are accompanied by some physical hypotheses such as $\epsilon <$
160 ϵ_{yield} . For steel-like material $\epsilon_{yield} \simeq 2 \cdot 10^{-3}$ and for fiber reinforced composite $\simeq 5 \cdot 10^{-2}$. If this hypothesis is respected the Engesser and Haringx predictions of ϵ for a given wavenumber k coincide (see Fig.5).

Let us consider first the *small-regime* where $\kappa_1 \ell^4 \ll 1$ (*sec-4.2.1*). As (23) is the sum of three positive terms, the criterion $\epsilon < \epsilon_{yield}$ implies that three constrains have to be satisfied *a minima*: $k_b^2 < \epsilon_{yield}$, $\kappa_1/k_b^2 < \epsilon_{yield}$ and $\kappa_2 < \epsilon_{yield}$. The first involves $1 < \epsilon_{yield} \ell^2$ that is already present in the Euler model
165 without foundation: this criterion is satisfied for most of the slender structure but could be violated for thick beam. The second may be rewritten as $\kappa_1 \ell^2 < \epsilon_{yield}$, however if the preceding criterion is satisfied the following hierarchy is observed $\kappa_1 \ell^4 < 1 < \epsilon_{yield} \ell^2$ then $\kappa_1 \ell^2 < \epsilon_{yield}$ is verified. Lastly, the third criterion $\kappa_2 < \epsilon_{yield}$ gives a constraint on κ_2 for a given material but independently of the length of the beam.

For the situation where $1 \ll \kappa_1 \ell^4$ (26) gives two constraints on the foundation stiffness that have to be
170 satisfied a minima : $\sqrt{\kappa_1} < \epsilon_{yield}$ and $\kappa_2 < \epsilon_{yield}$.

If the material of the structure (and then ϵ_{yield}) is known, these analyses exhibit non-trivial necessary condition on κ_i and ℓ for-which buckling may appear before irreversible transformation:

$$\kappa_2 < \epsilon_{yield} \quad \text{in all the cases and} \quad \begin{array}{ll} \ell > \frac{1}{\sqrt{\epsilon_{yield}}} & \text{if } \kappa_1 \ell^4 \ll 1 \\ \kappa_1 < \epsilon_{yield}^2 & \text{if } 1 \ll \kappa_1 \ell^4 \end{array} \quad (28)$$

In the *small-regime*, combining $\ell > 1/\sqrt{\epsilon_{yield}}$ and $\kappa_1 \ell^4 \ll 1$ leads to $\kappa_1 < \epsilon_{yield}^2$. Hence the preceding constraints may be easily synthesized, regardless of the regime, by:

$$\boxed{\begin{array}{l} \ell > \frac{1}{\sqrt{\epsilon_{yield}}}, \quad \kappa_1 < \epsilon_{yield}^2, \quad \kappa_2 < \epsilon_{yield} \\ \text{equivalently} \quad \max\left(\frac{1}{\ell^2}, \sqrt{\kappa_1}, \kappa_2\right) < \epsilon_{yield} \end{array}} \quad (29)$$

175 From the point of view of the authors, such bounds may be of particular interest for engineering design of slender structures subjected to elastic foundation.

The constraints (29) have motivated the choice of the values of $\kappa_1 \leq 10^{-4}$ and $\kappa_2 \leq 10^{-2}$ in Fig.2, Fig.3 and Fig.5 as $\epsilon_{yield} \leq 5 \cdot 10^{-2}$ for most of the standard material.

Of course, because (29) are *necessary* conditions, for a given problem the more restrictive *sufficient* condition
 180 $\epsilon < \epsilon_{yield}$ has to be checked even if (29) is respected. Hence, note that even if (29) are satisfied in Fig.5, the criterion $\epsilon < \epsilon_{yield}$ doesn't look systematically satisfied. This is particularly true if $\kappa_1 = 10^{-4}$ where ϵ_b is systematically larger than 10^{-2} while $\epsilon_{yield} > 10^{-2}$ is not necessarily obtained for large range of material. This illustrates the *non-sufficient* character of conditions (29). Note also that Engesser and Haringx models differ in a regime for which $\epsilon > \epsilon_{yield}$ for most of the standard material.

185 At last, for $\kappa_1 < 10^{-4}$, (24) induces $\lambda_b > 20\pi$ in the *small-regime*. In a dimensional point of view the wavelength is $\lambda_b > 20\pi\rho$ (for a cylindrical beam of radius R , $\lambda_b > 10\pi R$).

6. Conclusion

Analytical study of buckling of a beam supported by foundation has been performed. The problem was stated for plane, quasi-static and infinitesimal motion of a straight and uniform Timoshenko beam superim-
 190 posed to a finite longitudinal force such as Haringx and Engesser models were used. Foundations operate external densities of moment and transversal force. These densities are linearly related to rotation and transverse displacement of the beam in accordance with the two-parameters Winkler model of foundations. The problem was formulated in a non-dimensional way in order to reduce the number of independent parameters such as allowing a general statement.

195 For both models, buckling analysis was performed by investigating, in an analytical way, the relation between the critical buckling load and the wavenumber of buckling modes according to the foundation parameters. The explicit expression of the buckling mode and critical stress were given for pinned-pinned beam and general formulation was exhibited for general boundary conditions.

Introducing the yield stress of the material of the beam completes the discussion by introducing a criterion for which buckling prevails over irreversible transformation. The non-dimensional form of this simple criterion makes it particularly suitable for engineering design. In the buckling-regime the Engesser and Haringx model converge to the same estimation of the critical buckling stress and buckling modes have the same behaviour.

In the regime for which the effect of foundation is relevant, the equation used to determine buckling stress and eigenmodes coincides with the superposition of the Euler-buckling equation with the Pasternak foundation model. These two models were build on Euler-Bernoulli beam theory and then the Timoshenko theory proposed through the Engesser or Haringx models looked too sophisticated for the description of the problem of buckling of beam supported by two-parameter foundations. As a corollary consequence, the discussion about the meaning of the second parameter of the generalized Winkler foundations (whether it is associated to rotation θ of the section or slope u' of the centerline) is not relevant as the kinematical constraint $u' - \theta = 0$ holds for Euler-Bernoulli beam.

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