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# Constructive theory of ordinals

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## Abstract

**Martin-Löf [1970]** describes recursively constructed ordinals. He gives a constructively acceptable version of Kleene’s computable ordinals. In fact, the Turing definition of computable functions is not needed from a constructive point of view. We give in this paper a constructive theory of ordinals that is similar to Martin-Löf’s theory, but based only on the two relations “ $x \leq y$ ” and “ $x < y$ ”, i.e. without considering sequents whose intuitive meaning is a classical disjunction. In our setting, the operation “supremum of ordinals” plays an important rôle through its interactions with the relations “ $x \leq y$ ” and “ $x < y$ ”. This allows us to approach as much as we may the notion of linear order when the property “ $\alpha \leq \beta$  or  $\beta \leq \alpha$ ” is provable only within classical logic. Our aim is to give a formal definition corresponding to intuition, and to prove that our constructive ordinals satisfy constructively all desirable properties.

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# 1 Introduction

In classical mathematics, a natural definition for an ordinal is to be an order-type of a well-ordered set (see e.g. [Bourbaki, 1968, III.2.Ex.14]). Nevertheless it is more convenient to use von Neumann ordinals, for which many results can be proved without using choice (see e.g. [Krivine, 1998, Chapitre 2] and [Dehornoy, 2017, Chapitre II]).

Let us now propose a constructive approach. A binary relation  $<$  on a set  $X$  is said to be *well-founded* if for any family of sets  $(E_x)_{x \in X}$  indexed by  $X$ , it is possible to construct elements of  $\prod_{x \in X} E_x$  by  $<$ -induction. Precisely, each time a construction  $\gamma$  is given which from an element  $a \in X$  and an element  $\varphi \in \prod_{x \in X, x < a} E_x$  constructs an element  $\gamma(a, \varphi) \in E_a$ , there exists a unique  $\Phi \in \prod_{x \in X} E_x$  such that for all  $a \in X$  we have  $\Phi(a) = \gamma(a, \Phi|_{x \in X, x < a})$ .

This notion has a clear constructive meaning.

In particular, let us consider a property for elements in  $X$ . If the property is  $<$ -hereditary, i.e. if it is true for  $a \in X$  as soon as it is true for all  $x \in X$  with  $x < a$ , then this property is true for all elements in  $X$ .

In constructive mathematics, Mines, Richman, and Ruitenburg [1988, Section I.6] define an ordinal as a linearly ordered set for which the order relation

is well-founded. So all subsets of  $\mathbb{N}$  are ordinals, even if we don't know whether they have a smallest element.

The **Univalent Foundations Program** [2013, Section 10.3] proceeds similarly in the framework of univalent homotopy type theory; the ordinals of a given universe turn out to form a set (and not a groupoid).

Among other constructive points of view there are descriptions of countable ordinals constructed by induction in the works **Brouwer** [1926], **Gentzen** [1936], **Church** [1938], **Kleene** [1938], **Heyting** [1961], and **Martin-Löf** [1970, Chapter 3].

Brouwer proposes an inductive construction based on the idea that when ordinals  $\alpha_n$  are defined for all  $n \in \mathbb{N}$  and are linearly ordered well-founded sets, then we can describe the ordinal  $\alpha$  corresponding intuitively to  $\alpha_1$  followed by  $\alpha_2$  followed by  $\alpha_3$  followed by  $\dots$ . The ordered set  $\alpha$  defined by Brouwer will again be a linearly ordered well-founded set. And if the order relation on each  $\alpha_i$  is decidable, the same is true for  $\alpha$ .

Two Brouwer ordinals are in general not comparable (within intuitionistic logic): there is no general criterion allowing us to decide whether two ordinals have the same order-type, and, when this is not the case, which is isomorphic to an initial segment of the other.

Martin-Löf describes recursively constructed ordinals. He gives a constructively acceptable version of Kleene's computable ordinals. Intuitively, an ordinal à la Martin-Löf is inductively defined using two basic constructions:

- there is a minimum ordinal  $\underline{0}$ ;
- if  $(\alpha_n)$  is an explicit sequence of ordinals (indexed by  $\mathbb{N}$  or by an  $\mathbb{N}_k = \{n \in \mathbb{N} \mid n < k\}$ ), the supremum of the successors of the  $\alpha_n$ 's is an ordinal<sup>1</sup>.

To say that the definition is inductive is to say that every ordinal is constructed using the indicated rules.

In a constructive framework, we can drop Turing machines and replace Turing computability by intuitive (undefined) computability. In this case, the main difference between Brouwer and Martin-Löf ordinals is that Martin-Löf ordinals, being defined in a “parallel” way rather than in a “sequential” way, are more general: it is possible for any sequence of well-defined ordinals  $(\alpha_n)$  to construct the supremum of the successors of the ordinals  $\alpha_n$ . A drawback is that there is no way to associate to a Martin-Löf ordinal a linearly ordered well-founded set with the same order-type. E.g. if the  $\alpha_n$  are all equal to 0 or 1, it is a priori impossible to decide whether the supremum of the successors of the  $\alpha_n$ 's equals 1 or 2.

---

<sup>1</sup>Martin-Löf denotes this supremum by  $\text{sup}(\alpha_n)$ . In his setting,  $\underline{0}$  is in fact the supremum of the empty sequence. Except for this case, his  $\text{sup}(\alpha_n)$  is the supremum of the successors of the  $\alpha_n$ 's; we shall prefer the notation  $s(\alpha_n)$ .

# Ordinals as trees

Martin-Löf proposes to visualise an ordinal  $\alpha$  as a *well-founded tree with finite or countable branchings*. The ordinal  $\alpha$  is given with an index set denoted by  $\text{In}_\alpha$ ; in the sequel, it will be an element of the set  $\mathfrak{F}_2$  of index sets consisting of  $\mathbb{N}$  and its finite subsets  $\mathbb{N}_k$ .

- The tree with only its root represents  $\underline{0}$ .
- If  $(t_i)_{i \in \text{In}_\alpha}$  is a family of ordinal trees for a family of ordinals  $(\alpha_i)_{i \in \text{In}_\alpha}$ , the supremum  $\alpha = s(\alpha_i)_{i \in \text{In}_\alpha}$  of the successors of the  $\alpha_i$ 's is given by the ordinal tree for which there are  $\#\text{In}_\alpha$  branches above the root and a copy of  $t_i$  is attached to the branch indexed by  $i \in \text{In}_\alpha$ .

If  $n \in \mathbb{N}$ , the ordinal  $\underline{n}$  can be represented by the tree with  $n$  successive unary branchings at  $n$  nodes, so that it has  $n + 1$  nodes.

The first infinite ordinal  $\omega$  can be represented by the tree that has a countable branching above the root, the branches being the preceding trees (representing  $\underline{n}$ ,  $n \in \mathbb{N}$ ).

Its successor, denoted by  $\omega + \underline{1}$ , can be represented by the tree with unary branching above the root, the branch being the preceding tree.

The ordinal  $\omega + \underline{2}$  can be represented by the tree with unary branching above the root, the branch being the preceding tree.

The ordinal  $\omega + \omega$  can be represented by the tree that has a countable branching above the root, the branches being the trees representing  $\omega + \underline{n}$ ,  $n \in \mathbb{N}$ .

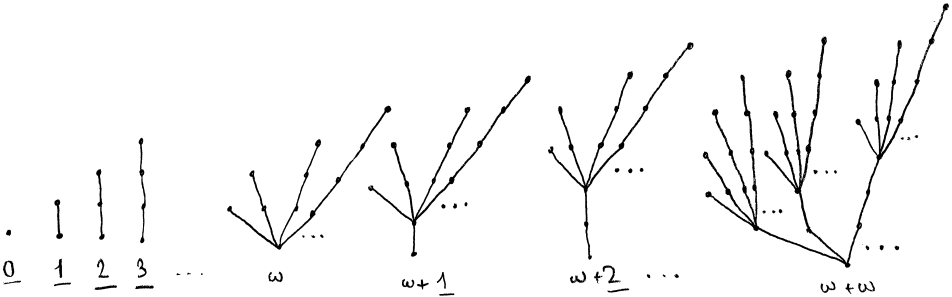


Figure 1: Ordinal trees.

More formally, such a tree can be defined as the set of its nodes, or branching points, suitably named. We may consider the set  $\text{Lst}(\mathbb{N})$  of finite lists of elements of  $\mathbb{N}$ . Let  $n \in \mathbb{N}$  and  $\ell, \ell' \in \text{Lst}(\mathbb{N})$ . We denote by  $n \frown \ell$  the list  $[n, \ell_1, \dots, \ell_k]$ , where  $\ell = [\ell_1, \dots, \ell_k]$ , by  $\ell \frown n$  the list  $[\ell_1, \dots, \ell_k, n]$ , and by  $\ell \frown \ell'$  the concatenation of the lists  $\ell$  and  $\ell'$ .

We remark that  $\text{Lst}(\mathbb{N})$  can be enumerated in a natural way<sup>2</sup> and that the notion of an  $\mathbb{N}$ -indexed family in  $\text{Lst}(\mathbb{N})$  corresponds, via such an enumeration, to the basic (undefined) notion of map from  $\mathbb{N}$  to  $\mathbb{N}$ .

A well-founded tree with finite or countable branchings can then be described as a detachable subset  $T$  of  $\text{Lst}(\mathbb{N})$  which is inductively constructed according to the previously indicated process.  $T$  is stable by initial segments: if  $\ell \in \text{Lst}(\mathbb{N})$ ,  $p \in \mathbb{N}$ , and  $\ell \hat{\ } p \in T$ , then  $\ell \in T$ . Thus, to each ordinal  $\alpha$ , we are associating a tree, defined as a suitable subset of  $\text{Lst}(\mathbb{N})$ , denoted by  $\text{Tree}(\alpha)$ .

If  $n \in \mathbb{N}$ , the ordinal  $\underline{n}$  can be described by the finite sequence of  $n + 1$  lists  $[], [0], [0, 0], \dots, [0, \dots, 0]$ .

The first infinite ordinal  $\omega$  can be described by the subset of  $\text{Lst}(\mathbb{N})$  enumerated by the infinite sequence  $[], [0], [1], [1, 0], [2], [2, 0], [2, 0, 0], [3], [3, 0], [3, 0, 0], [3, 0, 0, 0]$ , etc.

The ordinal  $\omega + \underline{1}$  can be described by the infinite sequence  $[], [0], [0, 0], [0, 1], [0, 1, 0], [0, 2], [0, 2, 0], [0, 2, 0, 0], [0, 3], [0, 3, 0], [0, 3, 0, 0], [0, 3, 0, 0, 0]$ , etc.

The ordinal  $\omega + \underline{2}$  can be described by the infinite sequence  $[], [0], [0, 0], [0, 0, 0], [0, 0, 1], [0, 0, 1, 0], [0, 0, 2], [0, 0, 2, 0], [0, 0, 2, 0, 0], [0, 0, 3], [0, 0, 3, 0], [0, 0, 3, 0, 0], [0, 0, 3, 0, 0, 0]$ , etc.

The ordinal  $\omega + \omega$  can be described by the doubly infinite sequence  $[], [0], [0, 0], [0, 1], [0, 1, 0], [0, 2], [0, 2, 0], [0, 2, 0, 0], [0, 3], [0, 3, 0], [0, 3, 0, 0], [0, 3, 0, 0, 0]$ , etc.,  $[1], [1, 0], [1, 0, 0], [1, 0, 1], [1, 0, 1, 0], [1, 0, 2], [1, 0, 2, 0], [1, 0, 2, 0, 0], [1, 0, 3], [1, 0, 3, 0], [1, 0, 3, 0, 0], [1, 0, 3, 0, 0, 0]$ , etc.,  $[2], [2, 0], [2, 0, 0], [2, 0, 0, 0], [2, 0, 0, 1], [2, 0, 0, 1, 0], [2, 0, 0, 2], [2, 0, 0, 2, 0], [2, 0, 0, 2, 0, 0], [2, 0, 0, 3], [2, 0, 0, 3, 0], [2, 0, 0, 3, 0, 0]$ , etc., etc.

\* \* \*

We give in this paper a constructive theory of ordinals that is similar to Martin-Löf's theory, but based only on the two relations " $x \leq y$ " and " $x < y$ ", i.e. without considering sequents whose intuitive meaning is a classical disjunction.

In our setting, the operation "supremum of ordinals" plays an important rôle through its interactions with the relations " $x \leq y$ " and " $x < y$ ". This allows us to approach as much as we may the notion of linear order when the property " $\alpha \leq \beta$  or  $\beta \leq \alpha$ " is provable only within classical logic.

Our problem is to give a formal definition corresponding to intuition, and to prove that our constructive ordinals satisfy constructively all desirable properties.

\* \* \*

The first step in Section 2 is to describe which these desirable properties are.

---

<sup>2</sup>For example, for  $\ell = [\ell_1, \dots, \ell_k] \in \text{Lst}(\mathbb{N})$ , we let  $\mu(\ell) = \sum_{i=1}^k (\ell_i + 1)$  and we enumerate the lists by increasing  $\mu(\ell)$ .

## 2 Linear orders associated to a set of index sets

We define in this section the structure of linear orders associated to a set  $\mathfrak{F}$  of index sets,  $\mathfrak{F}$ -orders for short.

### 2.1 Index sets

First we need a set  $\mathfrak{F}$  of index sets. An index set will be denoted by  $I, J, K, I', I'', J', I_a, I_b$ , etc.

An *index set* is simply a set that will be used as a set of indices for the families we shall consider. In the sequel a finitely enumerated subset of  $A$  is always a subset of  $A$  defined à la Bishop by a map  $\mathbb{N}_k \rightarrow A$ . If  $A$  is discrete, a finitely enumerated subset of  $A$  is a detachable subset.

**Properties of the set  $\mathfrak{F}$  of index sets.** *We will assume that*

- $\mathbb{N}$  and the finite sets  $\mathbb{N}_k = \{n \in \mathbb{N} \mid n < k\}$  ( $k \geq 0$ ) are elements of  $\mathfrak{F}$ ;
- any finitely enumerated subset<sup>3</sup> of an element of  $\mathfrak{F}$  is isomorphic<sup>4</sup> to an element of  $\mathfrak{F}$ ;
- if  $J \in \mathfrak{F}$ , the set of finitely enumerated subsets of  $J$  is isomorphic to an element of  $\mathfrak{F}$ ;
- $\mathfrak{F}$  is stable by disjoint unions indexed by  $\mathfrak{F}$ : we will denote by  $I+J$  a disjoint union of  $I$  and  $J$ , and by  $\sum_{i \in I} J_i$  a disjoint union of the family  $(J_i)_{i \in I}$ .

Disjoint unions are to be understood as direct sums in the category of sets. The disjoint union  $J = \sum_{i \in I} J_i$  comes with a family  $\iota_\ell : J_\ell \rightarrow J$  of injective maps realising  $J$  as the direct sum of the  $J_i$ 's in the category of sets.

If we restrict ourselves to countable ordinals, we can take for  $\mathfrak{F}$  the set

$$\boxed{\mathfrak{F}_2 = \{\mathbb{N}_k \mid k \in \mathbb{N}, k \geq 0\} \cup \{\mathbb{N}\}}$$

with convenient operations for the set of finite subsets of an  $I \in \mathfrak{F}$  and for disjoint unions of elements of  $\mathfrak{F}$  indexed by an element of  $\mathfrak{F}$ . Any other set  $\mathfrak{F}$  of index sets will contain  $\mathfrak{F}_2$ .

An  $\mathfrak{F}$ -indexed family of elements of  $E$  is a family  $(x_i)_{i \in I}$  where  $I \in \mathfrak{F}$  and the  $x_i$ 's  $\in E$ . The set of  $\mathfrak{F}$ -indexed families of elements of  $E$  is denoted by  $\text{Fam}(\mathfrak{F}, E)$ .

We shall restrict the use of subscripts for ordinal variables to this meaning, and use superscripts for all other uses.

<sup>3</sup>By definition this is a subobject given by a function  $\mathbb{N}_k \rightarrow \mathfrak{F}$ .

<sup>4</sup>In the category of sets.

## 2.2 Axioms

A structure of  $\mathfrak{F}$ -order on a set  $(E, =)$  is given as  $(E, <, \leq, 0_E, \text{sup}, \text{s})$ , where

- $<$  and  $\leq$  are binary relations defined on  $(E, =)$ ;
- $0_E$  is an element of  $E$  and we let  $E^* = \{\alpha \in E \mid 0_E < \alpha\}$ ;
- $\text{sup}$  is a map from  $\text{Fam}(\mathfrak{F}, E^*)$  to  $E^*$ : taking as input an element  $(\alpha_i)_{i \in I}$  of  $\text{Fam}(\mathfrak{F}, E^*)$ , it constructs an element of  $E^*$  denoted by  $\alpha = \text{sup}(\alpha_i)_{i \in I}$ ;
- $\text{s}$  is a unary map from  $E$  to  $E^*$ : taking as input an element  $\beta \in E$ , it constructs an element of  $E^*$  denoted by  $\text{s}(\beta)$ .

**Definition 2.1.** In order to write axioms with finite sup's we define  $\text{sup}(\alpha, \beta)$  for  $\alpha, \beta \in E$  in the following way (using implicitly Axiom 15):  $\text{sup}(0_E, \alpha) = \alpha = \text{sup}(\alpha, 0_E)$ ; if  $\alpha, \beta \in E^*$ ,  $\text{sup}(\alpha, \beta)$  is already defined.

These data are to satisfy the following axioms.

### Axioms for $\mathfrak{F}$ -orders.

1.  $\alpha = \beta$  if and only if  $\alpha \leq \beta$  and  $\beta \leq \alpha$  (reflexivity and antisymmetry);
2.  $0_E \leq \alpha$ ;
3. if  $\alpha < \alpha$  then  $0_E = \beta$  (irreflexivity);
4. if  $\alpha < \beta$  then  $\alpha \leq \beta$ ;
5. if  $\alpha \leq \beta$  and  $\beta \leq \gamma$ , then  $\alpha \leq \gamma$  (transitivity 1);
6. if  $\alpha < \beta$  and  $\beta \leq \gamma$ , then  $\alpha < \gamma$  (transitivity 2);
7. if  $\alpha \leq \beta$  and  $\beta < \gamma$ , then  $\alpha < \gamma$  (transitivity 3);
8.  $\alpha < \text{s}(\beta)$  if and only if  $\alpha \leq \beta$  (using Axiom 1 this gives  $\alpha < \text{s}(\alpha)$ );
9.  $\text{s}(\beta) \leq \alpha$  if and only if  $\beta < \alpha$ ;
10. if  $\alpha < \gamma$  and  $\beta < \gamma$ , then  $\text{sup}(\alpha, \beta) < \gamma$ ;
11. if  $\alpha < \text{sup}(\alpha, \beta)$  then  $\alpha < \beta$ ;
12. if  $\gamma < \alpha$  and  $\alpha \leq \text{sup}(\beta, \gamma)$ , then  $\alpha \leq \beta$ ;
13. for  $(\alpha_i)_{i \in I} \in \text{Fam}(\mathfrak{F}, E^*)$  and  $\beta \in E$  we have (characteristic property of sup)

$$\alpha_i \leq \beta \text{ for all } i \in I \text{ if and only if } \text{sup}(\alpha_i)_{i \in I} \leq \beta;$$



14. if  $\gamma < \beta$  for all  $\gamma < \alpha$ , then  $\alpha \leq \beta$ ;

15. either  $\alpha \leq 0_E$  or  $0_E < \alpha$ .

The category of  $\mathfrak{F}$ -orders is defined by its morphisms

$$(E, <_E, \leq_E, 0_E, \text{sup}_E, s_E) \longrightarrow (F, <_F, \leq_F, 0_F, \text{sup}_F, s_F),$$

which are maps from  $E$  to  $F$  preserving the structure (in the usual meaning).

*Comments.* 1) Let  $\gamma \in E^*$  and  $(\alpha_n)_{n \in \mathbb{N}}$  such that  $\alpha_n = \gamma$  or  $\alpha_n = s(\gamma)$  for each  $n$ . The element  $\text{sup}(\alpha_n)_{n \in \mathbb{N}}$  hesitates between  $\gamma$  and  $s(\gamma)$ . Thus there is no hope that the disjunction “ $\alpha \leq \beta$  or  $\beta < \alpha$ ” be constructive for arbitrary elements  $\alpha, \beta \neq 0_E$ . Consequently, we have introduced the sup map together with its axioms in order to best describe in what sense the order can be thought of as linear. Perhaps this is not optimal (reasonable axioms, satisfied for the set **Ord**<sub>2</sub> of ordinals of the second class constructed in Section 3, might be missing).

2) The irreflexivity is given a form that, instead of stating a negation, allows  $E$  to reduce to a singleton. This happens if and only if  $0_E = s(0_E)$ , which implies  $0_E < 0_E$  using Axiom 8.

3) Axiom 15 expresses that  $\{0_E\}$  is detachable. This contrasts with the fact that elements other than  $0_E$  do not define detachable singletons. We have defined sup on  $E^*$  rather than on  $E$  in order to satisfy constructively the disjunction of Axiom 15.

4) The characteristic property of sup shows that this law satisfies idempotence as well as generalised associativity and commutativity.  $\diamond$

## 2.3 Some properties

**Proposition and definition 2.2** (generalising Definition 2.1). For  $\alpha^1, \dots, \alpha^r \in E$  we let

$$\text{sup}(\alpha^1, \dots, \alpha^r) \stackrel{\text{def}}{=} \begin{cases} 0_E & \text{if } \alpha^1 = \dots = \alpha^r = 0_E \\ \text{the sup of the } \alpha^k \neq 0_E & \text{otherwise.} \end{cases}$$

The characteristic property of sup is satisfied:

$$\alpha^1 \leq \beta \text{ and } \dots \text{ and } \alpha^r \leq \beta \text{ if and only if } \text{sup}(\alpha^1, \dots, \alpha^r) \leq \beta.$$

**Fact 2.3.** Let  $\alpha, \beta$  be elements of  $E$ .

- $s(\alpha) < s(\beta)$  if and only if  $\alpha < \beta$ .

- $s(\alpha) \leq s(\beta)$  if and only if  $\alpha \leq \beta$ .

*Proof.* Use Axioms 8 and 9. □

**Fact 2.4.** Axioms 10 to 12 and 14 are in fact equivalences:

- 10.  $\alpha < \gamma$  and  $\beta < \gamma$  hold simultaneously if and only if  $\sup(\alpha, \beta) < \gamma$ ;
- 11.  $\alpha < \sup(\alpha, \beta)$  if and only if  $\alpha < \beta$ ;
- 12. if  $\gamma < \alpha$ , then  $\alpha \leq \sup(\beta, \gamma)$  holds if and only if  $\alpha \leq \beta$ ;
- 14.  $\alpha \leq \beta$  if and only if  $\gamma < \beta$  for all  $\gamma < \alpha$ .

*Proof.* Use the transitivities and the characteristic property of sup. □

**Fact 2.5** ( $s$  commutes with finite sup's, notation as in Proposition and definition 2.2).

We have  $\sup(s(\alpha), s(\beta)) = s(\sup(\alpha, \beta))$ . More generally,  $\sup(s(\alpha^1), \dots, s(\alpha^r)) = s(\sup(\alpha^1, \dots, \alpha^r))$ .

In particular, if  $\alpha^1 < \gamma, \dots, \alpha^r < \gamma$ , then  $\sup(\alpha^1, \dots, \alpha^r) < \gamma$ .

*Proof.* It suffices to prove  $s(\sup(\alpha, \beta)) = \sup(s(\alpha), s(\beta))$ . We have the following chain of equivalences:  $s(\sup(\alpha, \beta)) \leq \gamma \iff \sup(\alpha, \beta) < \gamma \iff (\alpha < \gamma \text{ and } \beta < \gamma) \iff (s(\alpha) \leq \gamma \text{ and } s(\beta) \leq \gamma) \iff \sup(s(\alpha), s(\beta)) \leq \gamma$ . □

**Proposition and definition 2.6** (definition of infinitary  $s$  and its characteristic property). For any  $(\alpha_i)_{i \in J} \in \text{Fam}(\mathfrak{F}, E)$ , we define  $s(\alpha_i)_{i \in J} = \sup(s(\alpha_i))_{i \in J}$ . Then we get the following equivalence:

$$\alpha_i < \beta \text{ for all } i \in J \text{ if and only if } s(\alpha_i)_{i \in J} \leq \beta.$$

*Proof.* Use Axioms 9 and 13. □

We write  $\boxed{F \subseteq_f I}$  in order to express that  $F$  is a finitely enumerated subset of  $I$ .

**Fact 2.7.** Let  $\alpha, \beta^1, \dots, \beta^m \in E$ .

1. Assume that  $\alpha = s(\alpha_i)_{i \in J}$  with  $(\alpha_i)_{i \in J} \in \text{Fam}(\mathfrak{F}, E)$  and that  $\alpha_i < \sup(\beta^1, \dots, \beta^m)$  for all  $i \in J$ . Then  $\alpha \leq \sup(\beta^1, \dots, \beta^m)$ .
2. Assume that  $\beta^k = s((\beta^k)_i)_{i \in J_k}$  with  $((\beta^k)_i)_{i \in J_k} \in \text{Fam}(\mathfrak{F}, E)$  for  $k \in \llbracket 1..m \rrbracket$ . Let  $F_1 \subseteq_f J_1, \dots, F_m \subseteq_f J_m$  not all be empty. If

$$\alpha \leq \sup((\beta^k)_j)_{k \in \llbracket 1..m \rrbracket, j \in F_k},$$

then  $\alpha < \sup(\beta^1, \dots, \beta^m)$ .

*Proof.* 1. This is Proposition and definition 2.6.

2. Suppose e.g. that  $F_1$  is nonempty. Then  $\alpha \leq \sup((\beta^1)_j)_{j \in F_1} < \beta^1 \leq \sup(\beta^1, \dots, \beta^m)$ . The strict inequality comes from Fact 2.5 because all  $(\beta^1)_j$ 's are  $< \beta^1$  by Proposition and definition 2.6.  $\square$

### 3 Inductive construction of ordinals

In Sections 3 and 4, the set  $\mathfrak{F}$  of index sets is fixed but often implied.

We shall define a set of ordinals **Ord** (more precisely **Ord** $_{\mathfrak{F}}$ ) and we shall prove that it is an initial object in the category of  $\mathfrak{F}$ -orders.

First we define a set **ord** of names for  $\mathfrak{F}$ -indexed ordinals by an inductive definition. The simplest inductive definition of an infinite set is that of  $\mathbb{N}$ : it admits an element 0 and a successor map  $x \mapsto s(x) : \mathbb{N} \rightarrow \mathbb{N}$ . The inductive definition of **ord** is very similar to that of  $\mathbb{N}$ . In  $\mathbb{N}$  each element is either 0 or an  $s(x)$  for an  $x \in \mathbb{N}$ . Similarly, in **ord**, each element is either  $\underline{0}$  or the  $s$  of an  $\mathfrak{F}$ -indexed family in **ord**; we denote by **ord** $^*$  the set of elements of this second type.

**Definition 3.1.** The set **ord** (more precisely **ord** $_{\mathfrak{F}}$ ) is defined in an inductive way: it is to admit a distinguished element  $\underline{0}$  and a map

$$s : \text{Fam}(\mathfrak{F}, \mathbf{ord}) \rightarrow \mathbf{ord}.$$

N.b.: the only constraint in this inductive definition is that  $s$  be indeed a map from  $\text{Fam}(\mathfrak{F}, \mathbf{ord})$  to **ord**.

An element of **ord** will be called [*name of an*] ordinal in the sequel.

When  $\mathfrak{F} = \mathfrak{F}_2$  we get the set of names of countable ordinals, denoted by **ord** $_2$ .

*Remark 3.2.* Each element  $\alpha \in \mathbf{ord}^*$  is given with two data:

- the index set used in the definition of  $\alpha$ : it will be denoted by  $\text{In}_{\alpha}$ ;
- the family  $\chi_{\mathbf{ord}}(\alpha, i)_{i \in \text{In}_{\alpha}}$  of its *definitional subordinals*, i.e. the element of  $\text{Fam}(\mathfrak{F}, \mathbf{ord})$  such that  $\alpha = s(\chi_{\mathbf{ord}}(\alpha, i))_{i \in \text{In}_{\alpha}}$ .

Thus the inductive definition of **ord** implies the existence of a map  $\alpha \mapsto \text{In}_{\alpha} : \mathbf{ord}^* \rightarrow \mathfrak{F}$  and the existence of a dependent family  $(\alpha, i) \mapsto \chi_{\mathbf{ord}}(\alpha, i)$  which is defined for  $\alpha \in \mathbf{ord}^*$  and  $i \in \text{In}_{\alpha}$ . In order to make the text more readable we will perform a slight abuse of notation: we shall not mention the construction of the dependent family  $\chi_{\mathbf{ord}}$ , and the notation  $\alpha_i$  will be an abbreviation for  $\chi_{\mathbf{ord}}(\alpha, i)$ .

With these conventions we may write

$$\alpha = s(\alpha_i)_{i \in \text{In}_{\alpha}}.$$

$\diamond$

For  $\alpha^1, \dots, \alpha^r \in \mathbf{ord}$  we define  $\boxed{s(\alpha^1, \dots, \alpha^r) = s(\alpha^i)_{i \in [1..r]}}$ .

In particular, if  $\alpha \in \mathbf{ord}$ , its *immediate successor*  $s(\alpha)$  is the element  $\beta = s(\beta_i)_{i \in \text{In}_\beta}$  where  $\text{In}_\beta = \mathbb{N}_1 = \{0\}$  and  $\beta_0 = \alpha$ . The sequence  $(\underline{m})_{m \in \mathbb{N}}$  in  $\mathbf{ord}$  is defined inductively by  $\underline{m+1} = s(\underline{m})$ . Then we can define  $\boxed{\omega = s(\underline{n})_{n \in \mathbb{N}}}$ .

In order to prove a property of  $\alpha = s(\alpha_i)_{i \in \text{In}_\alpha}$ , it is sufficient to prove the property for each  $\alpha_i$ . In a similar way we can construct inductively a map whose domain is  $\mathbf{ord}$ , or define inductively a predicate on  $\mathbf{ord}$ . This is stated precisely in Fact 3.4.

### 3.1 Subordinals

Here is a correct inductive definition.

**Definition 3.3.** Let  $\alpha = s(\alpha_i)_{i \in \text{In}_\alpha} \in \mathbf{ord}^*$ . An element  $\beta$  of  $\mathbf{ord}$  is a *definitional subordinal* of  $\alpha$  if  $\beta = \alpha_i$  for an  $i \in \text{In}_\alpha$ : we write this  $\beta \triangleleft_1 \alpha$ . An element  $\gamma$  is a *subordinal* of  $\alpha$  if it is a definitional subordinal of  $\alpha$  or a subordinal of a definitional subordinal of  $\alpha$ . We write this  $\gamma \triangleleft \alpha$ .

Thus  $\underline{0}$  is the only element of  $\mathbf{ord}$  which has no subordinal.

**Fact 3.4.** *The relations  $\triangleleft_1$  and  $\triangleleft$  on  $\mathbf{ord}$  are well-founded.*

Consequently there is no infinite branch in the tree of subordinals of an element of  $\mathbf{ord}$ , in the following sense.

**Fact 3.5.** *A sequence  $(\alpha^j)_{j=1,2,\dots}$  in  $\mathbf{ord}$ , where each  $\alpha^{j+1}$  is a subordinal of  $\alpha^j$ , reaches in a finite number of steps  $\alpha^r = \underline{0}$ .*

Remark that in order to perform a construction (or a proof) by  $\triangleleft_1$ -induction or by  $\triangleleft$ -induction, the case  $\underline{0}$  has to be dealt with separately since it has no subordinal. Nevertheless, we shall be able to avoid this case distinction until considering ordinal arithmetic on page 23.

### 3.2 Definition of the sup law

**Definition 3.6.**

1. The law  $\text{sup} : \text{Fam}(\mathfrak{F}, \mathbf{ord}^*) \rightarrow \mathbf{ord}^*$  is defined in the following way.

Let  $(\alpha^j)_{j \in J}$  be a family in  $\mathbf{ord}^*$  with  $J \in \mathfrak{F}$ .

If  $\alpha^j = s((\alpha^j)_i)_{i \in I_j}$ , then  $\boxed{\text{sup}(\alpha^j)_{j \in J}}$  is the element  $\varepsilon = s(\varepsilon_k)_{k \in K}$  where

- $K$  is the disjoint union of the  $I_j$ 's;
- $(\varepsilon_k)_{k \in K}$  is the family defined by  $\varepsilon_k = (\alpha^j)_i$  if  $\iota_j(i) = k$

(here  $\iota_j : I_j \rightarrow K$  is the injective map from  $I_j$  to the disjoint union of the  $I_j$ 's).

We shall write  $\sup(\alpha^j)_{j \in \llbracket 1..r \rrbracket} = \sup(\alpha^1, \dots, \alpha^r)$ .

2. The sup of a finite family in **ord** is defined in the following way.

$$\sup(\alpha^1, \dots, \alpha^r) \stackrel{\text{def}}{=} \begin{cases} \underline{0} & \text{if } \alpha^1 = \dots = \alpha^r = \underline{0} \\ \text{the sup of the } \alpha^k \in \mathbf{ord}^* & \text{otherwise.} \end{cases}$$

We note that Item 2 is formally included in Item 1 if we adopt the convention  $\text{In}_{\underline{0}} = \mathbb{N}_0$ . However, this convention would not allow us to define an arbitrary  $\mathfrak{F}$ -indexed sup in **ord**.

### 3.3 Definition of $\leq$ and $<$

The main job remains to be done, i.e. to define two binary relations  $\leq$  and  $<$  on **ord** with the required properties, viz.

- the relation “ $\alpha \leq \beta$  and  $\beta \leq \alpha$ ” has to be an equivalence relation (we shall denote by **Ord** the quotient set);
- the relations  $\leq$  and  $<$  and the maps sup and s have to descend to the quotient (we shall not change their names);
- with these maps and relations, **Ord** has to be an  $\mathfrak{F}$ -order.

Moreover, since the map  $s: \text{Fam}(\mathfrak{F}, \mathbf{ord}) \rightarrow \mathbf{ord}^*$  is defined before the map  $\text{sup}: \text{Fam}(\mathfrak{F}, \mathbf{ord}^*) \rightarrow \mathbf{ord}^*$ , we have to verify in our construction that Proposition and definition 2.6 is satisfied in **Ord**. This will be a consequence of Fact 3.11 below.

For our job, we define inductively two asymmetric relations between, on the left side, an element of **ord** and, on the right side, a *nonempty finitely enumerated set* of elements of **ord**, written as a list:

$$\boxed{\alpha \leq \beta^1, \dots, \beta^m \quad \text{and} \quad \alpha < \beta^1, \dots, \beta^m \quad (m \geq 1).}$$

#### Conventions.

- The letters  $\alpha, \beta, \gamma, \varepsilon$ , possibly with exponents, indices or primes, are used for elements of **ord**.
- If  $\alpha$  is an element of **ord** and if  $F$  is a finite list, possibly empty, in  $\text{In}_\alpha$ , we denote by  $\alpha_F$  the list of the  $\alpha_i$ 's with  $i$  in  $F$ .

The two relations are defined by simultaneous induction in the following way.

Particular cases involving  $\underline{0}$  are avoided by using the convention  $\text{In}_{\underline{0}} = \mathbb{N}_0$ .

$\alpha \leq \beta^1, \dots, \beta^m$  ( $m \geq 1$ ) is defined as  $\alpha_i < \beta^1, \dots, \beta^m$  for all  $i \in \text{In}_{\alpha}$ .

$\alpha < \beta^1, \dots, \beta^m$  ( $m \geq 1$ ) is defined as there are  $F_1 \subseteq_f \text{In}_{\beta^1}, \dots, F_m \subseteq_f \text{In}_{\beta^m}$  not all empty with  $\alpha \leq \beta_{F_1}^1, \dots, \beta_{F_m}^m$ .

This definition is correct since elements of **ord** are inductively defined and the pair of clauses is inductive.

Without the convention that  $\text{In}_{\underline{0}} = \mathbb{N}_0$ , we would have had to include Fact 3.8 below in the definition. This convention is a little miracle allowing us to avoid a case-by-case reasoning with respect to the disjunction “ $\alpha = \underline{0}$  or  $\alpha \in \mathbf{ord}^*$ ” in the proofs.

The meaning of the two relations is  $\alpha \leq \sup(\beta^1, \dots, \beta^m)$  and  $\alpha < \sup(\beta^1, \dots, \beta^m)$ .

**Lemma 3.7.** *We have  $\alpha < \beta^1, \dots, \beta^m$  if and only if  $\alpha < \sup(\beta^1, \dots, \beta^m)$ . Similarly, we have  $\alpha \leq \beta^1, \dots, \beta^m$  if and only if  $\alpha \leq \sup(\beta^1, \dots, \beta^m)$ .*

*Proof.* Let us write

$$\alpha \prec \beta^1, \dots, \beta^m \text{ for } \alpha < \sup(\beta^1, \dots, \beta^m),$$

$$\alpha \preceq \beta^1, \dots, \beta^m \text{ for } \alpha \leq \sup(\beta^1, \dots, \beta^m).$$

Let  $\varepsilon = \sup(\beta^1, \dots, \beta^m)$ . Then  $\alpha \prec \beta^1, \dots, \beta^m$  if and only if  $\alpha \leq \varepsilon_F$  with  $F$  a nonempty finitely enumerated subset of the disjoint union  $K$  of the  $\text{In}_{\beta^j}$ 's and  $\varepsilon_k = (\beta^j)_i$  if  $k$  is the image of  $i$  in  $K$ ; letting  $F_j = F \cap \text{In}_{\beta^j}$ , not all  $F_j$  are empty and this may be rewritten as  $\alpha \leq \beta_{F_1}^1, \dots, \beta_{F_m}^m$ . This holds if and only if  $\alpha < \beta^1, \dots, \beta^m$ .

We have  $\alpha \preceq \beta^1, \dots, \beta^m$  if and only if, for all  $i \in \text{In}_{\alpha}$ ,  $\alpha_i < \varepsilon$ , i.e.  $\alpha_i \prec \beta^1, \dots, \beta^m$ , i.e.  $\alpha_i < \beta^1, \dots, \beta^m$ ; this holds if and only if  $\alpha \leq \beta^1, \dots, \beta^m$ .  $\square$

The relation  $\alpha =_{\mathbf{Ord}} \beta$  is defined as meaning “ $\alpha \leq \beta$  and  $\beta \leq \alpha$ ”.

We shall show in Section 4 that the relation  $\cdot =_{\mathbf{Ord}} \cdot$  is an equivalence relation and we shall define the set **Ord** as the quotient of **ord** by this relation.

Let us note that until Theorem 4.8, the symbol  $=$  between two elements of **ord** is the equality in **ord** and has not the meaning of  $=_{\mathbf{Ord}}$ . Nevertheless, after having shown that the relations and the laws of **ord** descend to the quotient **Ord**, the statements with the symbol  $=$  will also work for the symbol  $=_{\mathbf{Ord}}$ .

### 3.4 Finite ordinals, bounded ordinals

We start with a few properties of  $\underline{0}$ .

**Fact 3.8.** Let  $m$  be an integer  $\geq 1$ ,  $\alpha, \beta^1, \dots, \beta^m \in \mathbf{ord}$ , and  $\gamma \in \mathbf{ord}^*$ . We have

1.  $\underline{0} \leq \beta^1, \dots, \beta^m$ ;
2.  $\underline{0} < \gamma, \beta^2, \dots, \beta^m$ ;
3.  $\alpha < \underbrace{\underline{0}, \dots, \underline{0}}_{m \text{ times}}$  is impossible.

*Proof.* This is straightforward from the definitions.  $\square$

*Remark 3.9.* Axiom 15 will be valid in  $\mathbf{Ord}$  because every element of  $\mathbf{ord}$  is given either as  $\underline{0}$  or as an element  $\gamma \in \mathbf{ord}^*$ , so that always  $\underline{0} < \gamma$  by Item 2 of Fact 3.8.  $\diamond$

**Fact 3.10.** Let  $m, n \in \mathbb{N}$ . Then

1.  $m \leq n$  if and only if  $\underline{m} \leq \underline{n}$ ;
2.  $m < n$  if and only if  $\underline{m} < \underline{n}$ ;
3.  $\underline{m} \leq \underline{n}$  and  $\underline{n} < \underline{m}$  are incompatible.

*Proof.* Concerning the direct implications in 1 and 2, we write  $n = m + r$  and we do an induction on  $r$ . For the reverse implications, cases  $m = 0$  and  $n = 0$  are already known. Next we see that  $\underline{m+1} \leq \underline{n+1}$  implies  $\underline{m} \leq \underline{n}$ , and that  $\underline{m+1} < \underline{n+1}$  implies  $\underline{m} < \underline{n}$ . This allows us to conclude by induction on  $m$ . Item 3 follows from items 1 and 2.  $\square$

An element  $\alpha \in \mathbf{ord}$  is said to be *finite* if  $\alpha =_{\mathbf{Ord}} \underline{m}$  for an  $m \in \mathbb{N}$ , *bounded* if  $\alpha \leq \underline{m}$  for an  $m \in \mathbb{N}$ . Bounded ordinals are much more complicated than finite ordinals (see Examples 3.17 and 3.18).

In Section 3.7, we shall discuss what the relations  $\leq$  and  $<$  on the set  $\mathbf{ord}_{\mathfrak{F}}$  become in classical mathematics.

### 3.5 First consequences

The following fact shows that the s law will satisfy the characteristic property given in Proposition and definition 2.6 when we shall know that it descends to the quotient  $\mathbf{Ord}$ .

**Fact 3.11 (sdef).** We have  $\alpha \leq \beta$  if and only if  $\alpha_i < \beta$  for all  $i \in \text{In}_\alpha$ .

*Proof.* This property is tautological: this is the definition of  $\alpha \leq \beta$ .  $\square$

Similarly, the following fact shows that the sup law will satisfy the characteristic property given in Axiom 13 when we shall know that it descends to the quotient  $\mathbf{Ord}$ .

**Fact 3.12 (supdef).** *Let  $(\alpha^j)_{j \in J}$  be a family in  $\mathbf{ord}^*$  with  $J \in \mathfrak{F}$ ,  $\gamma = \sup(\alpha^j)_{j \in J}$ , and  $\beta \in \mathbf{ord}$ .*

*We have  $\gamma \leq \beta$  if and only if  $\alpha^j \leq \beta$  for all  $j \in J$ . In particular,  $\sup(\alpha, \beta) \leq \beta$  if and only if  $\alpha \leq \beta$ .*

N.b.: the result is equally true for the sup of a finite family in  $\mathbf{ord}$ .

*Proof.* This is another linguistic tautology. We have  $\alpha^j = s((\alpha^j)_i)_{i \in I_j}$  for an  $I_j \in \mathfrak{F}$ . By the definition of  $\gamma$  and of  $\leq$ , the inequality  $\gamma \leq \beta$  means that for each  $j \in J$  and each  $i \in I_j$  we have  $(\alpha^j)_i < \beta$ , i.e. that for each  $j \in J$  we have  $\alpha^j \leq \beta$ .  $\square$

The following fact shows that Axioms 8 and 9 will be valid when we shall descend to the quotient  $\mathbf{Ord}$ .

**Fact 3.13.** 1. **ax8.** *We have  $\alpha < s(\beta)$  if and only if  $\alpha \leq \beta$ .*

2. **ax9.** *We have  $\beta < \alpha$  if and only if  $s(\beta) \leq \alpha$*

*Proof.* Recall that the element  $\gamma = s(\beta)$  is defined by  $\text{In}_\gamma = \{0\}$  and  $\gamma_0 = \beta$ .

1. By definition,  $\alpha < \gamma$  means that  $\alpha \leq \gamma_F$  for a nonempty list  $F \subseteq_f \{0\}$ . This forces  $F = [0]$  and  $\gamma_F = \beta$ .

2. By definition,  $\gamma \leq \alpha$  means that  $\gamma_0 < \alpha$ , i.e.  $\beta < \alpha$ .

Thus, better than equivalences, these are tautologies.  $\square$

The following fact will allow us to shorten certain proofs by induction.

**Fact 3.14.**

a. *We have an inequality  $\alpha \leq \beta$  if and only if for each  $i \in \text{In}_\alpha$ , there exists a nonempty  $F_i \subseteq_f \text{In}_\beta$  such that  $\alpha_i \leq \beta_{F_i}$ .*

b. *We have an inequality  $\alpha < \beta$  if and only if there exists a nonempty  $F \subseteq_f \text{In}_\beta$  such that for each  $i \in \text{In}_\alpha$  we have  $\alpha_i < \beta_F$ .*

*Proof.* Straightforward from the definitions.  $\square$

Now we leave behind tautological proofs and turn to inductive proofs.



**Fact 3.15.**

- **weakening.** If  $\alpha \leq \beta^1, \dots, \beta^m$ , then for each  $\beta$  we have  $\alpha \leq \beta, \beta^1, \dots, \beta^m$ .
- **contraction.** If  $\alpha \leq \beta^1, \beta^1, \beta^2, \dots, \beta^m$  then  $\alpha \leq \beta^1, \beta^2, \dots, \beta^m$ .
- *The same properties hold with  $<$  instead of  $\leq$ .*

*Proof.* Use induction applying the definitions. □

The following lemma is a corollary of Fact 3.14. Item 1 (resp. 2) will imply that the  $s$  (resp.  $\sup$ ) map descends to the quotient in **Ord** (resp. **Ord\***). Item 3 will imply that the relations  $\leq$  and  $=$  are reflexive in **Ord**; Items 5 and 7 will imply Axioms 3 and 14 for **Ord**.

**Lemma 3.16.**

1. **s0.** Let  $\alpha, \beta \in \mathbf{ord}$  with  $\text{In}_\alpha = \text{In}_\beta$  and  $\alpha_i \leq \beta_i$  for all  $i \in \text{In}_\alpha$ . Then  $\alpha \leq \beta$ .
2. **sup0.** Let  $\alpha, \beta \in \mathbf{ord}^*$  with  $\text{In}_\alpha = \text{In}_\beta$  and  $\alpha_i \leq \beta_i$  for all  $i \in \text{In}_\alpha$ . Then

$$\sup(\alpha_i)_{i \in \text{In}_\alpha} \leq \sup(\beta_i)_{i \in \text{In}_\beta}.$$

*The result works also for the sup of a finite family in **ord**.*

3. **rf1.** For all  $\alpha \in \mathbf{ord}$ , we have  $\alpha \leq \alpha$ . A fortiori,  $\alpha \leq \alpha, \beta^1, \dots, \beta^m$ .
4. **s1.** For all  $\alpha \in \mathbf{ord}^*$  and all  $i \in \text{In}_\alpha$ , we have  $\alpha_i < \alpha$ . A fortiori,  $\alpha_i < \alpha, \beta^1, \dots, \beta^m$ .
5. **irf1.** For all  $\alpha \in \mathbf{ord}$ ,  $\alpha < \alpha$  is impossible.
6.  $\alpha < s(\alpha)$ .
7. **ax14.** If  $\gamma < \beta$  for all  $\gamma < \alpha$ , then  $\alpha \leq \beta$ .

*Proof.* 1. Straightforward from Fact 3.14a. We take  $F = \{i\}$ .

2. Let  $\gamma = \sup(\alpha_i)_{i \in \text{In}_\alpha}$  and  $\epsilon = \sup(\beta_i)_{i \in \text{In}_\beta}$ . By Fact 3.14a, for each  $j \in \text{In}_{\alpha_i}$  there exists a nonempty  $F_{i,j} \subseteq_f \text{In}_{\beta_i}$  with  $(\alpha_i)_j \leq (\beta_i)_{F_{i,j}}$ ;  $F_{i,j}$  is a fortiori in the disjoint union of the  $\text{In}_{\beta_i}$ 's, so that  $(\alpha_i)_j < \epsilon$  by definition of  $\epsilon$ . By definition,  $\alpha_i \leq \epsilon$ , so that by Fact 3.12  $\gamma \leq \epsilon$ .

3. By induction: we use Fact 3.14a, we take  $F = \{i\}$  and  $\alpha \leq \alpha$  reduces to  $\alpha_i \leq \alpha_i$ .

4. By induction: we use Fact 3.14b, we take  $F = \{i\}$  and  $\alpha_i < \alpha$  reduces to  $(\alpha_i)_j < \alpha_i$ .

5. By induction: we use Fact 3.14b, we take  $F = \{i\}$  and “ $\alpha < \alpha$  is impossible” reduces to: “ $\alpha_i < \alpha_i$  is impossible”.

6. Apply  $\mathbf{s1}$  to  $\beta = s(\alpha)$ .

7. If  $\alpha = \underline{0}$ , the conclusion is clear. If  $\alpha = s(\alpha_i)_{i \in I}$ , as  $\alpha_i < \alpha$  for each  $i \in \text{In}_\alpha$  (Item 4), the hypothesis that  $\gamma < \beta$  for all  $\gamma < \alpha$  shows that  $\alpha_i < \beta$  for all  $i \in \text{In}_\alpha$ . We conclude by Fact 3.11 that  $\alpha \leq \beta$ .  $\square$

### 3.6 Ordinals and limited principles of omniscience

*Example 3.17.* Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence in  $\{0, 1\}$  which takes at most once the value 1. The lesser limited principle of omniscience **LLPO** says that we have

$$\exists k \in \{0, 1\} \quad \forall n \quad (v_n = 1 \Rightarrow n \equiv k \pmod{2}). \quad (*)$$

From such a sequence  $(v_n)_{n \in \mathbb{N}}$  let us define  $\varepsilon$ ,  $\varepsilon^1$  and  $\varepsilon^2 \in \mathbf{ord}$  in the following way:

$$\varepsilon = s(\underline{v_n})_{n \in \mathbb{N}}, \quad \varepsilon^1 = s(\underline{v_{2m}})_{m \in \mathbb{N}}, \quad \varepsilon^2 = s(\underline{v_{2m+1}})_{m \in \mathbb{N}}.$$

Then we have  $\varepsilon \leq \sup(\varepsilon^1, \varepsilon^2)$ . But  $\varepsilon \leq \varepsilon^1$  gives  $k = 0$  in  $(*)$  and  $\varepsilon \leq \varepsilon^2$  gives  $k = 1$  in  $(*)$ . Thus, the disjunction  $\varepsilon \leq \varepsilon^1$  or  $\varepsilon \leq \varepsilon^2$  has no constructive proof: assuming the disjunction for an arbitrary  $(v_n)$  would imply **LLPO**.

*Example 3.18.* Let  $(u_n)_{n \in \mathbb{N}}$  be a nondecreasing sequence in  $\{0, 1\}$ . The limited principle of omniscience **LPO** says that such a sequence is eventually constant:

$$\exists n \in \mathbb{N} \quad \forall m \in \mathbb{N}, \quad u_m \leq u_n. \quad (*)$$

From such a sequence  $(u_n)_{n \in \mathbb{N}}$  let us define  $\varepsilon$  and  $\varepsilon' \in \mathbf{ord}$  in the following way:

$$\varepsilon = s(\underline{u_n})_{n \in \mathbb{N}}, \quad \varepsilon' = s(\underline{u_n + 1})_{n \in \mathbb{N}}.$$

We notice that the strict inequality  $\varepsilon < \varepsilon'$  is equivalent (using Facts 3.10 and 3.14 and Lemma 3.7) to

$$\exists n \in \mathbb{N} \quad \forall m \in \mathbb{N}, \quad u_m < u_n + 1,$$

which is the same thing as  $(*)$ . In fact,  $\varepsilon$  hesitates between 1 and 2,  $\varepsilon'$  hesitates between 2 and 3, and the inequality  $\varepsilon < \varepsilon'$  is valid if we assume **LPO**. But asserting  $\varepsilon < \varepsilon'$  for all sequences  $(u_n)$  implies **LPO** in constructive mathematics. Here we see that hesitating between 1 and 2 for an infinite sequence has the same flavour as hesitating (in a classical setting) between bounded and unbounded for an infinite sequence of natural numbers: adding 1 to each term of the sequence increases strictly the sup only if the sequence is bounded.

### 3.7 In classical mathematics

Proposition 3.19 shows that the law of excluded middle (**LEM**) simplifies and/or obscures dramatically the structure of the set  $\mathbf{ord}_{\mathfrak{F}}$  with respect to the relations  $<$  and  $\leq$ .

**Proposition 3.19.** *Assume **LEM**. Then for  $\alpha, \beta \in \mathbf{ord}$ , we have  $\alpha \leq \beta$  or  $\beta < \alpha$ . Moreover, if  $\beta < \alpha$ , there exists an  $i \in \text{In}_\alpha$  such that  $\beta \leq \alpha_i$ .*

*Proof.* We prove by simultaneous induction the two following properties.

$$“\alpha \leq \beta \text{ or } \beta < \alpha” \quad \text{and} \quad “\beta \leq \alpha \text{ or } \alpha < \beta”.$$

By induction hypothesis, we have for all  $i \in \text{In}_\alpha$  and all  $j \in \text{In}_\beta$ , “ $\alpha \leq \beta_j$  or  $\beta_j < \alpha$ ”, and also “ $\beta \leq \alpha_i$  or  $\alpha_i < \beta$ ”.

The first disjunction implies by **LEM** that either  $\beta_j < \alpha$  for all  $j \in \text{In}_\beta$  or there is  $j \in \text{In}_\beta$  such that  $\alpha \leq \beta_j$ . In the first case, we have  $\beta \leq \alpha$  by definition of  $\cdot \leq \dots$ . In the second case, we have  $\alpha < \beta$  by definition of  $\cdot < \dots$ , with for  $F \subseteq_f \text{In}_\beta$  the list  $[j]$ .

The symmetric reasoning yields the second disjunction. □

N.b.: for countable ordinals, the limited principle of omniscience (**LPO**) suffices to prove the proposition.

**Corollary 3.20.** *Assume **LEM**. Any ordinal  $\alpha \neq \underline{0}$  is either an immediate successor or the sup of the ordinals  $\gamma < \alpha$ .*

*Proof.* Consider  $\alpha = \text{s}(\alpha_i)_{i \in \text{In}_\alpha}$  and compare  $\alpha$  with  $\text{sup}(\alpha_i)_{i \in \text{In}_\alpha}$ . The details are left to the reader. □

**Corollary 3.21.** *Assume **LEM**. Any bounded ordinal is finite.*

*Proof.* Left to the reader: use Fact 3.13. □

## 4 Fundamental results

### 4.1 $\text{Ord}_{\mathfrak{F}}$ is an initial object in the category of $\mathfrak{F}$ -orders

**Lemma 4.1.** *For  $\alpha^1, \dots, \alpha^r$  in  $\mathbf{ord}$  ( $r \geq 1$ ), we have*

$$\boxed{\text{sup}(\alpha^j)_{j \in \llbracket 1..r \rrbracket} < \text{s}(\alpha^j)_{j \in \llbracket 1..r \rrbracket}.}$$

*Proof.* Let us show e.g. that  $\epsilon = \text{sup}(\alpha, \beta) < \gamma = \text{s}(\alpha, \beta)$ . We have  $\text{In}_\epsilon = \text{In}_\alpha + \text{In}_\beta$ , with  $\epsilon_k = \alpha_i$  if  $\iota_1(i) = k$ , and  $\epsilon_k = \beta_j$  if  $\iota_2(j) = k$ . We have  $\text{In}_\gamma = \{1, 2\}$  with  $\gamma_1 = \alpha$  and  $\gamma_2 = \beta$ . We apply Fact 3.14b with  $F = \{1, 2\}$ . For an arbitrary  $k$  in  $\text{In}_\epsilon$ , we have  $\epsilon_k < \alpha, \beta$  since  $\epsilon_k$  is  $\alpha_i$  or  $\beta_j$  and, by **s1**, we have  $\alpha_i < \alpha$  (a fortiori  $\alpha_i < \alpha, \beta$ ) and  $\beta_j < \beta$  (a fortiori  $\beta_j < \alpha, \beta$ ). □

Let us note that the preceding proof relies on the fact that the definitions of  $\leq$  and  $<$  have been given with lists on the right hand side.

**Lemma 4.2** (transitivities). *1. trans1. If  $\alpha \leq \beta^1, \dots, \beta^m$  and, for each  $j \in \llbracket 1..m \rrbracket$ ,  $\beta^j \leq \gamma^1, \dots, \gamma^r$ , then  $\alpha \leq \gamma^1, \dots, \gamma^r$ .*

2. **trans2.** If  $\alpha < \beta^1, \dots, \beta^m$  and, for each  $j \in \llbracket 1..m \rrbracket$ ,  $\beta^j \leq \gamma^1, \dots, \gamma^r$ , then  $\alpha < \gamma^1, \dots, \gamma^r$ .
3. **trans3.** If  $\alpha \leq \beta^1, \dots, \beta^m$  and, for each  $j \in \llbracket 1..m \rrbracket$ ,  $\beta^j < \gamma^1, \dots, \gamma^r$ , then  $\alpha < \gamma^1, \dots, \gamma^r$ .

As particular cases, Axioms 5 to 7 will be valid when we shall descend to the quotient **Ord**:

- if  $\alpha \leq \beta$  and  $\beta \leq \gamma$  then  $\alpha \leq \gamma$ ;
- if  $\alpha < \beta$  and  $\beta \leq \gamma$  then  $\alpha < \gamma$ ;
- if  $\alpha \leq \beta$  and  $\beta < \gamma$  then  $\alpha < \gamma$ .

*Proof.* The three transitivities are being proved by simultaneous induction.

In order to prove **trans1**, we note that the hypothesis means that we have  $\alpha_i < \beta^1, \dots, \beta^m$  for all  $i \in \text{In}_\alpha$ . Let us fix such an  $i$ . We use **trans2** with this  $\alpha_i$  instead of  $\alpha$  and we get  $\alpha_i < \gamma^1, \dots, \gamma^r$ . Since this works for all  $i \in \text{In}_\alpha$ , this gives the desired conclusion  $\alpha \leq \gamma^1, \dots, \gamma^r$ .

In order to prove **trans2**, we note that the hypothesis implies that there are  $G_j \subseteq_f \text{In}_{\beta^j}$  not all empty such that  $\alpha \leq \beta_{G_1}^1, \dots, \beta_{G_m}^m$ . We have also for  $j \in \llbracket 1..m \rrbracket$  and for all  $h \in \text{In}_{\beta^j}$ ,  $\beta_h^j < \gamma^1, \dots, \gamma^r$ . A fortiori, this is true for the  $h$ 's  $\in G_j$ . We use **trans3** with these  $\beta_h^j$ 's instead of the  $\beta^j$ 's. This gives the desired conclusion  $\alpha < \gamma^1, \dots, \gamma^r$ .

In order to prove **trans3**, we note that the hypothesis implies (by weakening) that there are  $F_k \subseteq_f \text{In}_{\gamma^k}$  not all empty such that  $\beta^j \leq \gamma_{F_1}^1, \dots, \gamma_{F_r}^r$  for  $j \in \llbracket 1..m \rrbracket$ . This time we use **trans1** with the  $\gamma_{F_k}^k$ 's instead of the  $\gamma^k$ 's and we deduce that  $\alpha \leq \gamma_{F_1}^1, \dots, \gamma_{F_r}^r$ , which implies  $\alpha < \gamma^1, \dots, \gamma^r$ .  $\square$

The following lemma shows that when descending to the quotient, Axiom 4 will be valid in **Ord**.

**Lemma 4.3 (ax4).** Let  $\alpha, \beta^1, \dots, \beta^m \in \text{ord}$ . If  $\alpha < \beta^1, \dots, \beta^m$ , then  $\alpha \leq \beta^1, \dots, \beta^m$ .

*Proof.* Proof by induction on  $\alpha$ . We have  $\alpha < \beta^1, \dots, \beta^m$  if and only if we can find  $F_k \subseteq_f \text{In}_{\beta^k}$  not all empty such that, for each  $i \in \text{In}_\alpha$ , we have  $\alpha_i \leq \beta_{F_1}^1, \dots, \beta_{F_m}^m$ . Let us fix an  $i \in \text{In}_\alpha$ . For  $j \in F_k$ , we have  $\beta_j^k < \beta^k$ , and by weakening  $\beta_j^k < \beta^1, \dots, \beta^m$ . By **trans3**, we get  $\alpha_i < \beta^1, \dots, \beta^m$ . Finally, since this is true for all  $i \in \text{In}_\alpha$ , we have  $\alpha \leq \beta^1, \dots, \beta^m$ .  $\square$

The following fact shows that Axiom 10 will be valid when we shall descend to the quotient **Ord**.

**Lemma 4.4** (ax10). *If  $\alpha < \gamma$  and  $\beta < \gamma$ , then  $\sup(\alpha, \beta) < \gamma$ .*

*Proof.* By definition, we have  $s(\alpha, \beta) \leq \gamma$ . Lemma 4.1 gives  $\sup(\alpha, \beta) < s(\alpha, \beta)$ . By transitivity, we get  $\sup(\alpha, \beta) < \gamma$ .  $\square$

**Lemma 4.5.** *Let  $n$  be a positive integer and  $\alpha^1, \dots, \alpha^n \in \mathbf{ord}$ . It is impossible that, for each  $i \in \llbracket 1..n \rrbracket$ , we have  $\alpha^i < \alpha^1, \dots, \alpha^n$ .*

*Proof.* By induction. Using weakening, the hypothesis to be proven impossible gives finite lists

$$F_1 \subseteq_f \text{In}_{\alpha_1}, \dots, F_n \subseteq_f \text{In}_{\alpha_n},$$

not all empty, such that

$$\alpha^i \leq \alpha_{F_1}^1, \dots, \alpha_{F_n}^n \text{ for } i \in \llbracket 1..m \rrbracket.$$

In particular for  $j \in F_i$  (if  $F_i$  is nonempty) we have

$$\alpha_j^i < \alpha_{F_1}^1, \dots, \alpha_{F_n}^n.$$

This reduces to the hypothesis with the nonempty list  $\alpha_{F_1}^1, \dots, \alpha_{F_n}^n$  instead of the list  $\alpha^1, \dots, \alpha^n$ .  $\square$

**Lemma 4.6.** *Let  $\alpha^1, \dots, \alpha^n, \beta^1, \dots, \beta^m \in \mathbf{ord}$  ( $n, m \geq 1$ ).*

1. *If  $\alpha^i < \alpha^1, \dots, \alpha^n, \beta^1, \dots, \beta^m$  for  $i \in \llbracket 1..n \rrbracket$ , then  $\alpha^i < \beta^1, \dots, \beta^m$  for each  $i$ .*
2. *Let  $F_1 \subseteq_f \text{In}_{\alpha_1}, \dots, F_n \subseteq_f \text{In}_{\alpha_n}$ . If  $\alpha^i \leq \alpha_{F_1}^1, \dots, \alpha_{F_n}^n, \beta^1, \dots, \beta^m$  for  $i \in \llbracket 1..n \rrbracket$ , then  $\alpha^i \leq \beta^1, \dots, \beta^m$  for each  $i$ .*

*Proof.* 1. The hypothesis yields finite lists

$$F_1 \subseteq_f \text{In}_{\alpha^1}, \dots, F_n \subseteq_f \text{In}_{\alpha^n}, G_1 \subseteq_f \text{In}_{\beta^1}, \dots, G_m \subseteq_f \text{In}_{\beta^m},$$

not all empty, such that

$$\alpha^i \leq \alpha_{F_1}^1, \dots, \alpha_{F_n}^n, \beta_{G_1}^1, \dots, \beta_{G_m}^m \text{ for } i \in \llbracket 1..n \rrbracket. \quad (*)$$

Thus we have for  $i \in \llbracket 1..n \rrbracket$  and  $j \in \text{In}_{\alpha^i}$

$$\alpha_j^i < \alpha_{F_1}^1, \dots, \alpha_{F_i}^i, \dots, \alpha_{F_n}^n, \beta_{G_1}^1, \dots, \beta_{G_m}^m.$$

Let us fix  $i$  and  $j$ : a fortiori, with  $F'_i = F_i \cup \{j\}$

$$\alpha_j^i < \alpha_{F_1}^1, \dots, \alpha_{F'_i}^i, \dots, \alpha_{F_n}^n, \beta_{G_1}^1, \dots, \beta_{G_m}^m.$$

We have also by weakening, for  $k \in \llbracket 1..n \rrbracket$  and  $\ell \in F_k$

$$\alpha_\ell^k < \alpha_{F_1}^1, \dots, \alpha_{F'_i}^i, \dots, \alpha_{F_n}^n, \beta_{G_1}^1, \dots, \beta_{G_m}^m.$$

Thus by induction  $\alpha_j^i < \beta_{G_1}^1, \dots, \beta_{G_m}^m$ . Since  $j$  is arbitrary, we get  $\alpha^i \leq \beta_{G_1}^1, \dots, \beta_{G_m}^m$ . This gives the desired conclusion,  $\alpha^i < \beta^1, \dots, \beta^m$ , if at least one list  $G_k$  is nonempty, for an arbitrary  $i \in \llbracket 1..n \rrbracket$ . If this is not the case, (\*) yields  $\alpha^i \leq \alpha_{F_1}^1, \dots, \alpha_{F_n}^n$  for  $i \in \llbracket 1..n \rrbracket$ , with lists  $F_i$  not all empty. By definition, this implies  $\alpha^i < \alpha^1, \dots, \alpha^n$  for  $i \in \llbracket 1..n \rrbracket$ , which is impossible by Lemma 4.5.

2. We have for  $i \in \llbracket 1..n \rrbracket$  and  $j \in \text{In}_\alpha^i$

$$\alpha_j^i < \alpha_{F_1}^1, \dots, \alpha_{F'_i}^i, \dots, \alpha_{F_n}^n, \beta^1, \dots, \beta^m.$$

Let us fix  $i$  and  $j$ : a fortiori, with  $F'_i = F_i \cup \{j\}$ ,

$$\alpha_j^i < \alpha_{F_1}^1, \dots, \alpha_{F'_i}^i, \dots, \alpha_{F_n}^n, \beta^1, \dots, \beta^m.$$

We have also by weakening, for  $k \in \llbracket 1..n \rrbracket$  and  $\ell \in F_k$

$$\alpha_\ell^k < \alpha_{F_1}^1, \dots, \alpha_{F'_i}^i, \dots, \alpha_{F_n}^n, \beta^1, \dots, \beta^m.$$

Item 1 then yields  $\alpha_j^i < \beta^1, \dots, \beta^m$ . As  $j$  is arbitrary, we get what we wanted:  $\alpha^i \leq \beta^1, \dots, \beta^m$  for an arbitrary  $i \in \llbracket 1..n \rrbracket$ .  $\square$

The following fact shows that Axioms 11 and 12 will be valid when we shall descend to the quotient **Ord**.

**Lemma 4.7.**

1. **ax11.** If  $\alpha < \sup(\alpha, \beta)$ , then  $\alpha < \beta$ ;
2. **ax12.** If  $\gamma < \alpha$  and  $\alpha \leq \sup(\beta, \gamma)$ , then  $\alpha \leq \beta$ .

*Proof.* 1. Assume  $\alpha < \sup(\alpha, \beta)$ . Lemma 3.7 gives  $\alpha < \alpha, \beta$ . Item 1 of Lemma 4.6 gives  $\alpha < \beta$ .

2. Assume  $\gamma < \alpha$  and  $\alpha \leq \sup(\beta, \gamma)$ . The first hypothesis gives  $\gamma \leq \alpha_F$  for a nonempty  $F \subseteq_f \text{In}_\alpha$ . The second hypothesis gives  $\alpha \leq \gamma, \beta$  (by Lemma 3.7). By transitivity we have  $\alpha \leq \alpha_F, \beta$ . Item 2 of Lemma 4.6 gives  $\alpha \leq \beta$ .  $\square$

**Theorem 4.8.** We have constructed **Ord** as an  $\mathfrak{F}$ -order.

*Proof.* Using **rf1** and **trans1**, we first show that the equality is indeed an equivalence relation, and then that the relation  $\leq$  descends to the quotient in **Ord**.

Similarly, **trans2** and **trans3** imply that the relation  $<$  descends to the quotient in **Ord**.

The sup map descends to the quotient by Lemma 3.16, Item 2.

The unary s map descends to the quotient by Fact 3.13.

It remains to note that Axioms 1 to 15 of  $\mathfrak{F}$ -orders have been proved above. See, respectively: Lemma 3.16 (Item 3); Fact 3.8 (Item 1); Lemma 3.16 (Item 5); Lemma 4.3; Lemma 4.2; Fact 3.13; Lemma 4.4; Lemma 4.7; Fact 3.12; Lemma 3.16 (Item 7); Remark 3.9.  $\square$

The following theorem generalises Fact 3.10.

**Theorem 4.9.** *The set  $\mathbf{Ord}$  is not reduced to a point. More precisely:*

- for all  $\alpha, \beta \in \mathbf{Ord}$ ,  $\beta \leq \alpha$  and  $\alpha < \beta$  are incompatible;
- the map  $n \mapsto \underline{n} : \mathbb{N} \rightarrow \mathbf{Ord}$  is injective ( $m < n$  if and only if  $\underline{m} < \underline{n}$ ).
- for all  $\alpha \in \mathbf{Ord}$  and  $n > m$  in  $\mathbb{N}$ , it is impossible that  $s^{(n)}(\alpha) =_{\mathbf{Ord}} s^{(m)}(\alpha)$ ;

*Proof.* The first item is a consequence of `irfl` and of `trans2`. The rest follows.  $\square$

**Theorem 4.10.**  *$\mathbf{Ord}$  is an initial object in the category of  $\mathfrak{F}$ -orders.*

*Sketch of proof.* The structure is purely algebraic and in order to construct  $\mathbf{Ord}$ , we have only used the axioms of the structure.

In fact, let us consider an object  $(E, <_E, \leq_E, 0_E, \text{sup}_E, s_E)$  in the category. Elements of  $\mathbf{ord}$  do have their copies in  $E$ . Furthermore, the relations  $\cdot < \cdots$  and  $\cdot \leq \cdots$  defined in  $\mathbf{ord}$  are valid in  $E$  by Fact 2.7 if interpreted in  $E$  with finite sup's on the right hand side (as we may by Lemma 3.7). This implies that there is a unique morphism from  $\mathbf{Ord}$  to  $E$  in the category.  $\square$

## 4.2 More properties

**Proposition 4.11.** *The binary relation  $<$  on  $\mathbf{Ord}$  is well-founded.*

*Proof.* This is a direct consequence of Fact 3.4.  $\square$

**Lemma 4.12** (weak forms of the disjunction “ $\alpha \leq \beta$  or  $\beta < \alpha$ ”). *Let  $r \geq 1$  and  $\alpha, \beta^1, \dots, \beta^r, \gamma \in \mathbf{ord}$ .*

1. *If  $\alpha \leq \beta$  and  $\beta < \alpha, \gamma^1, \dots, \gamma^r$ , then  $\beta < \gamma^1, \dots, \gamma^r$ .*
2. *If  $\beta < \alpha$  and  $\alpha \leq \beta, \gamma^1, \dots, \gamma^r$ , then  $\alpha \leq \gamma^1, \dots, \gamma^r$ .*

*Proof.* Introduce  $\gamma = \text{sup}(\gamma^1, \dots, \gamma^r)$ . Using Lemma 3.7, both items reduce to already established properties.  $\square$

**Definition 4.13.** An element  $\beta = s(\beta_i)_{i \in \text{In}_\beta} \in \mathbf{ord}$  is said to be *filtering* if for each  $F \subseteq_f \text{In}_\beta$  there exists  $j \in \text{In}_\beta$  such that  $\text{sup}(\beta_i)_{i \in F} \leq \beta_j$ .

**Lemma 4.14.** *For each  $\alpha \in \mathbf{ord}$ , there exists  $\beta \in \mathbf{ord}$  such that  $\alpha =_{\mathbf{Ord}} \beta$  and  $\beta$  is filtering.*

*Proof.* If  $\alpha = s(\alpha_i)_{i \in J}$ , we let  $K$  be the set of finitely enumerated subsets of  $J$ , and for  $F \subseteq_f J$  we let  $\beta_F = \sup(\alpha_j)_{j \in F}$ . Then  $\beta =_{\mathbf{Ord}} s(\beta_F)_{F \in K}$ .  $\square$

### 4.3 Elementary ordinal arithmetic

#### (Sequential) addition

The sequential addition  $\alpha + \beta$  ( $\alpha$  followed by  $\beta$ : addition is not commutative) is defined by induction on  $\beta$ :

$$\alpha + \underline{0} = \alpha \quad \text{and} \quad \alpha + \beta = s(\alpha + \beta_j)_{j \in \text{In}_\beta} \quad \text{if} \quad \beta = s(\beta_j)_{j \in \text{In}_\beta} \in \mathbf{ord}^*.$$

This formula works only for the case  $\text{In}_\beta \neq \mathbb{N}_0$ : it would yield  $\alpha + \underline{0} = \underline{0}$  for  $\text{In}_\beta = \mathbb{N}_0$ .

We also have  $\alpha + \beta = \sup((\alpha + \beta_j) + \underline{1})_{j \in \text{In}_\beta}$  if  $\text{In}_\beta \neq \mathbb{N}_0$ .

The following properties can be proved by induction:

- if  $\alpha \leq \alpha'$  and  $\beta \leq \beta'$ , then  $\alpha + \beta \leq \alpha' + \beta'$ ;
- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ ;
- $\alpha + \underline{0} = \underline{0} + \alpha = \alpha$ ;
- $\alpha + \beta \leq \alpha + \gamma$  if and only if  $\beta \leq \gamma$ ;
- $\alpha + \beta < \alpha + \gamma$  if and only if  $\beta < \gamma$ ;
- $\alpha = \underline{1} + \alpha$  if and only if  $\omega \leq \alpha$ ;
- if  $\alpha \leq \gamma$ , then there is  $\beta$  such that  $\gamma = \alpha + \beta$ ;
- if  $\alpha < \gamma$ , then there is  $\beta \neq \underline{0}$  such that  $\gamma = \alpha + \beta$ .

#### Sequential sum

Let  $J \in \mathfrak{F}$  and consider a well-founded linear order relation  $\prec$  on  $J$  with a detachable minimal element  $0_J$ . Let  $(\beta^j)_{j \in J}$  be an element of  $\text{Fam}(J, \mathbf{ord})$ . The  $\prec$ -indexed sequential sum  $\sum_{j \prec \ell} \beta^j$  is defined by induction on  $\ell \in (J, \prec)$ :

$$\sum_{j \prec 0_J} \beta^j = 0_J \quad \text{and} \quad \sum_{j \prec \ell} \beta^j = \sup \left( \left( \sum_{j \prec k} \beta^j \right) + \beta^k \right)_{k \prec \ell} \quad \text{if} \quad 0_J \prec \ell.$$

We show by induction on  $\prec$  that, given two families  $(\beta^j)_{j \in J}$  and  $(\gamma^j)_{j \in J}$  such that  $\beta^j \leq \gamma^j$  for all  $j \in J$ , we have  $\sum_{j \prec \ell} \beta^j \leq \sum_{j \prec \ell} \gamma^j$  for all  $\ell \in J$ . This construction descends therefore to the quotient  $\mathbf{Ord}$ .



*Remark 4.15.* This construction allows us to define a map  $\mathbf{ord}_2^{\text{Br}} \rightarrow \mathbf{ord}_2$ , where  $\mathbf{ord}_2^{\text{Br}}$  is the set of names of Brouwer ordinals. See [Troelstra \[1969\]](#) and [Brouwer \[1918, 1926\]](#). Troelstra only treats countable Brouwer ordinals.

## Multiplication

We define  $\alpha \cdot \beta$  by induction on  $\beta \in \mathbf{ord}$ :

$$\alpha \cdot \underline{0} = \underline{0} \quad \text{and} \quad \alpha \cdot \beta = \sup(\alpha \cdot \beta_j + \alpha)_{j \in \text{In}_\beta} \quad \text{if } \beta = s(\beta_j)_{j \in \text{In}_\beta} \in \mathbf{ord}^*.$$

The following properties can be proved by induction:

- if  $\alpha \leq \alpha'$  and  $\beta \leq \beta'$ , then  $\alpha \cdot \beta \leq \alpha' \cdot \beta'$ ;
- $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ ;
- $\alpha \cdot \underline{1} = \underline{1} \cdot \alpha = \alpha$ ;
- $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$ ;
- if  $\underline{1} \leq \alpha$ , then  $\alpha \cdot \beta \leq \alpha \cdot \gamma$  if and only if  $\beta \leq \gamma$ ;
- if  $\underline{1} \leq \alpha$ , then  $\alpha \cdot \beta < \alpha \cdot \gamma$  if and only if  $\beta < \gamma$ .

## Exponentiation

We define  $\alpha^\beta$  by induction on  $\beta \in \mathbf{ord}$ :

$$\alpha^{\underline{0}} = \underline{1} \quad \text{and} \quad \alpha^\beta = \sup(\alpha^{\beta_j} \cdot \alpha)_{j \in \text{In}_\beta} \quad \text{if } \beta = s(\beta_j)_{j \in \text{In}_\beta} \in \mathbf{ord}^*.$$

## Ackermann

It is possible to continue this elementary arithmetic à la Ackermann. We define by induction an ordinal  $\underline{\text{Ack}}(\alpha, \beta, \gamma)$  that we get by iterating  $\gamma$  times the preceding map, initialised at  $\alpha$ , i.e. more precisely,

$$\begin{aligned} \underline{\text{Ack}}(\alpha, \beta, \underline{0}) &= \alpha + \beta \\ \underline{\text{Ack}}(\alpha, \underline{0}, \gamma) &= \alpha \quad \text{if } \gamma \in \mathbf{ord}^* \\ \underline{\text{Ack}}(\alpha, \beta, \gamma) &= \sup(\sup(\underline{\text{Ack}}(\alpha, \underline{\text{Ack}}(\alpha, \beta_j, \gamma), \gamma_k))_{j \in \text{In}_\beta})_{k \in \text{In}_\gamma} \\ &\quad \text{if } \beta = s(\beta_j)_{j \in \text{In}_\beta} \quad \text{and} \quad \gamma = s(\gamma_k)_{k \in \text{In}_\gamma}. \end{aligned}$$

In particular,  $\varepsilon_0 = \underline{\text{Ack}}(\omega, \omega, \underline{1})$ .

# 5 Countable ordinals

## 5.1 First steps

As previously indicated, we get countable ordinals when we choose as set of index sets

$$\mathfrak{F}_2 = \{\mathbb{N}_k \mid k \in \mathbb{N}, k \geq 0\} \cup \{\mathbb{N}\}$$

with convenient operations for the set of finite subsets of an  $I \in \mathfrak{F}_2$  and for disjoint unions. We write  $\mathbf{ord}_2$  and  $\mathbf{Ord}_2$  for  $\mathbf{ord}_{\mathfrak{F}_2}$  and  $\mathbf{Ord}_{\mathfrak{F}_2}$ . Thus  $\mathbf{Ord}_2$  is the set of ordinals of the second class and  $\mathbf{ord}_2$  is a set of names for elements of  $\mathbf{Ord}_2$ .

**Lemma 5.1.** *Any countable ordinal is the s of a nondecreasing sequence of countable ordinals.*

*Proof.* This is Lemma 4.14. □

**Proposition 5.2.** *Assume LPO. Then, for  $\alpha, \beta \in \mathbf{Ord}$ , we have  $\alpha \leq \beta$  or  $\beta < \alpha$ .*

*Proof.* Proceed as for Proposition 3.19, in the countable case. □

## 5.2 Comparison with Martin-Löf ordinals

We present a variation of the theory of ordinals in the book *Notes on Constructive Mathematics* [Martin-Löf, 1970, Chapter 3]. We write “variation” since Martin-Löf’s theory is formulated in the framework of Markov’s recursive mathematics, while we take as primitive intuitionistic logic with generalised inductive definitions, as does the work Heyting [1961] (the fact that this setting can provide a more elegant treatment than the one in recursive mathematics is stressed in Kreisel’s review of this work).

### 5.2.1 Martin-Löf’s formal system

In this system, ordinals are described inductively: if we have a finite or infinite sequence of ordinals  $u = u_0, \dots, u_n, \dots$  (maybe empty), then  $s(u)$  is an ordinal.

The (classical) semantics of this operation is the following: to  $(\alpha_n)$  sequence of ordinals we associate the supremum of the sequence of the successors of the  $\alpha_n$ ’s.

In particular  $\underline{0}$  is defined as  $s(u)$ , where  $u$  is the empty sequence.

We write simply  $s(a)$  for  $s(u)$ , where  $u$  is the sequence with one element  $u_0 = a$ .

In constructive mathematics, the set of all such ordinals is an example of a nondiscrete set.

As stated in the introduction, to any ordinal  $a$  we associate, by induction on  $a$ , a tree  $\text{Tree}(a)$ :  $\text{Tree}(a)$  always contains the empty sequence, and  $\text{Tree}(s(u))$  contains  $n \hat{\ } \sigma$  if  $\sigma$  is in  $\text{Tree}(u_n)$ .

This set  $\text{Tree}(a)$  does not contain any infinite branch: if  $f$  is a numerical function we can always find  $n$  such that  $[f(0), \dots, f(n-1)]$  is not in  $\text{Tree}(a)$ . This is proved directly by induction on  $a$ . In other words, the tree  $\text{Tree}(a)$  is well-founded. The fact that we get in this way all well-founded trees is the content of Brouwer's bar theorem, which holds neither in Bishop's set theory nor in dependent type theory. This follows from the fact that both systems have an interpretation in recursive mathematics, while the bar theorem does not hold in recursive mathematics, as shown by an example due to Kleene [Kleene and Vesley, 1965].

We define next what an atomic formula is: a formula of the form  $a < b$  or  $a \leq b$ .

Finally, we can define when a sequent  $\Gamma$  is provable, where  $\Gamma$  is a finite set of atomic formulae. The formulation is quite elegant!

$$\frac{\Gamma, a \leq u_n}{\Gamma, a < s(u)} \quad \frac{\dots \Gamma, u_n < b \dots}{\Gamma, s(u) \leq b}$$

Note that there is a direct proof of  $0 \leq b$  by the second rule.

The intuitive meaning of a sequent is the classical disjunction of the atomic formulae it contains.

Martin-Löf shows that we can prove for instance  $a < b, b \leq a$  by induction on  $b$  and  $a$ . He also shows that the following rule is admissible by induction on  $a$ :

$$\frac{\Gamma, a < a}{\Gamma}$$

which implies in particular that  $a < a$  is not provable.

Let us give an example of such proofs by induction.

**Lemma 5.3.** *The sequents  $a \leq a$  and  $a < s(a)$  are provable for all  $a$ .*

*Proof.* We prove  $a \leq a$  by induction on  $a$ . If  $a = s(u)$  we have to show  $u_n < s(u)$  for all  $n$ , which follows from  $u_n \leq u_n$ , which holds by induction.

It follows that we have  $a < s(a)$  by the first derivation rule.  $\square$

## 5.2.2 Comparison with our system

Let us explain now why this definition does not coincide with ours by giving an example of the form  $a < b$  which is provable in this sequent calculus but implies **LPO** in our system.

Let us return to Example 3.18: define  $a = s(\underline{u})$ , where  $u$  is a nondecreasing sequence  $(u_n)$  of 0's and 1's, and  $b = s(\underline{v})$ , where  $v$  is the sequence  $(v_n)$ ,  $v_n = u_n + 1$ .

**Lemma 5.4.**  *$a < b$  is provable.*

*Proof.* We have to prove  $a < s(\underline{v})$ .

By the second rule, it is enough to show  $a < s(\underline{v})$ ,  $a \leq \underline{v}_0$ . And for this we have to show  $\underline{u}_n < \underline{v}_0$ ,  $a < s(\underline{v})$  for all  $n$ . We fix  $n$  and we show  $\underline{u}_n < \underline{v}_0$ ,  $a < s(\underline{v})$ .

If we do have  $\underline{u}_n < \underline{v}_0 = s(\underline{u}_0)$  this is fine. Note that we can *test* whether or not  $\underline{u}_n < \underline{v}_0$  holds or not since both  $u_n$  and  $v_0$  are of the form 0 or 1 or 2.

Otherwise we get explicitly  $n$  such that  $\underline{u}_n \geq s(\underline{u}_0)$  and we have then  $\underline{u}_m \leq \underline{u}_n$  and so  $\underline{u}_m < \underline{v}_n$  for all  $m$ . We can prove  $\underline{u}_n < \underline{v}_0$ ,  $a < s(\underline{v})$  by  $\underline{u}_n < \underline{v}_0$ ,  $a \leq \underline{v}_n$  which holds since  $\underline{u}_n < \underline{v}_0$ ,  $\underline{u}_m < \underline{v}_n$  holds for all  $m$ .  $\square$

Note that we prove  $a < s(\underline{v})$  by proving  $a < s(\underline{v})$ ,  $a \leq \underline{v}_0$ , and we have to “keep”  $a < s(\underline{v})$ : maybe  $a \leq \underline{v}_0$  does not hold (it may happen that the sequence  $(u_n)$  takes the value 1 and  $v_0 = 1$ ).

In Example 3.18, we note that  $a < b$  implies **LPO** in our system. Therefore, in the set  $\mathbf{Ord}_2^{\text{ML}}$  of Martin-Löf ordinals, the equality is coarser than in the set  $\mathbf{Ord}_2$ , though both are based on the set  $\mathbf{ord}_2$ .

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