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# An ordered family of consistency measures of belief functions

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**Abstract.** We propose a family of measures of consistency (and induced conflict) derived from a definition of consistent belief functions introduced previously. Besides satisfying the desired properties of monotonicity, boundedness, and extreme values, the novel family encompasses the existing probabilistic and logical consistency measures which are shown to correspond to two extremes of the family (lower sharp bound and upper asymptotic limit respectively). We illustrate the definitions and measures of consistency within an example of vessel destination estimation with inconsistent sources.

## 1 Introduction

In maritime security, the measurement of inconsistency may reveal maritime anomalies such as vessels deviating from normalcy (*e.g.*, “off-route vessels”, “too fast vessels”) and those possibly spoofing the Automatic Identification System (AIS) signal (by, *e.g.*, changing their actual type, concealing their current position, hiding their actual destination) to hide suspect behaviour [1, 2]. Having a sound and proper measurement of inconsistency or, equivalently, consistency is thus of paramount importance for such intelligent systems.

Theoretical research on (in)consistency was pioneered by the artificial intelligence community working on knowledge bases over logical languages. Classical logic is explosive, *i.e.*, everything is a consequence of an inconsistency, so solving inconsistent knowledge bases is a major challenge. A variety of approaches have been proposed in the literature. Hunter and Konieczny [3] introduced the minimal inconsistent sets, while some other authors [4, 5] proposed to attach probabilities or degrees of beliefs to propositions rather than truth values.

The consistency notion plays also a central role in the belief function setting [6, 7] as it is directly related to the way conflict between pieces of evidence may be defined: as the inconsistency yielded by their conjunctive combination [8]. The two notions of inconsistency and conflict have been subject to studies whose starting point was often the logical interpretation of belief functions. Cuzzolin [9] provided a definition of consistent belief functions as a counterpart of consistent knowledge bases [10]. Destercke and Burger [8] proposed an axiomatic approach

to conflict which extends the properties of conflict between sets. Recently, Pichon and al. [11] revisited and extended some of Destercke and Burger’s results. In particular, they proposed a novel family of consistency definitions that encompasses the so-called probabilistic and logical definitions proposed in [8].

In this paper, we pursue this work and propose a new parametrised family of consistency (and their associated conflict) measures, study their properties and illustrate their use and interest on a vessel destination estimation problem. It is organized as follows. Necessary concepts of belief function theory as well as classical consistency and conflict notions and measures are first recalled. In Section 3, a parameterised family of consistency (and their associated conflict) measures is unveiled and its special cases and properties are discussed. We conclude and sketch the steps for future work in Section 4.

## 2 Background

In this section, we provide a brief reminder of necessary concepts on belief functions and recall the existing consistency definitions and measures in this setting.

### 2.1 Uncertainty representation with belief functions

**Preliminaries** Let the belief about the actual value of an uncertain variable  $\boldsymbol{x}$  defined over a frame  $\mathcal{X}$  be represented by a mass function which is a mapping  $m : 2^{\mathcal{X}} \rightarrow [0, 1]$  such that  $\sum_{A \subseteq \mathcal{X}} m(A) = 1$ .  $\mathcal{M}$  denotes the set of mass functions defined over  $\mathcal{X}$ . The set of focal sets of  $m$  is denoted  $\mathcal{F}(m)$  and its cardinality is denoted  $\mathfrak{F} := |\mathcal{F}(m)|$ . We allow  $m(\emptyset)$ , the mass associated to the empty set, to be strictly positive, which captures the fact that the true value of  $\boldsymbol{x}$  may be outside the frame of discernment.

*Example 1.* We denote by  $\mathcal{X} = \{\text{Imperia, Savona, Genova, La Spezia, Livorno}\} = \{d_1, d_2, d_3, d_4, d_5\}$  the set of possible destinations of a vessel. A cleaning-matching algorithm that “cleans” the AIS reported destination by formatting it in the standard format and matches it to a standard database (the World Port Index) of port names is applied. The algorithm identifies “SAVONA” as the closest name in the World Port Index with a confidence degree of 0.8, and identifies “SAVOONGA” (Alaska region) as a possible match, with a confidence of 0.2. This can be encoded by the following mass function:  $m_1(d_2) = 0.8; m_1(\emptyset) = 0.2$ .

Information encoded in a mass function  $m$  can be equivalently represented by different set-measures among which the plausibility  $Pl$  and the commonality  $q$  measures defined for every  $A \in 2^{\mathcal{X}}$  by:

$$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B); \quad q(A) = \sum_{B \supseteq A} m(B). \quad (1)$$

The contour function  $pl$  is defined over  $\mathcal{X}$  by:

$$pl(x) = Pl(\{x\}) = q(\{x\}). \quad (2)$$

When two mass functions provided by two sources inform on the same entity, their combination can be performed in several ways. In particular, if the sources are independent, the combination is performed using the conjunctive operator:

$$m_1 \odot_2(A) = \sum_{B \cap C = A} m_1(B)m_2(C) \quad \forall A \subseteq \mathcal{X},$$

The commonality function satisfies:

$$q_1 \odot_2(A) = q_1(A)q_2(A) \quad \forall A \subseteq \mathcal{X}. \quad (3)$$

**Conflict and Consistency** Different definitions of (total) consistency have been proposed in the belief function literature, among which the so-called probabilistic and logical definitions [8]:

**Definition 1 (Logical consistency [8]).** A mass function  $m$  is logically consistent iff  $\bigcap_{A \in \mathcal{F}} A \neq \emptyset$ .

**Definition 2 (Probabilistic consistency [8]).** A mass function  $m$  is probabilistically consistent iff  $m(\emptyset) = 0$ .

Two desirable properties of consistency measures for a mass function  $m$  are also provided in [8]:

1. Property 1 (Bounded): A measure of consistency should be bounded.
2. Property 2 (Extreme consistent values): A measure of consistency should reach its maximal value if and only if  $m$  is totally consistent (according to the considered definition of total consistency), and its minimal value if and only if  $m$  is totally inconsistent, *i.e.*,  $m(\emptyset) = 1$ .

Two consistency measures satisfying these properties for, respectively, Definitions 1 and 2 of total consistency, are [8]:

$$\phi_\pi(m) = \max_{x \in \mathcal{X}} pl(x); \quad \phi_m(m) = 1 - m(\emptyset). \quad (4)$$

Other measures have been proposed in the literature, such as Yager's [12]:

$$\phi_Y(m) = \sum_{A \cap B \neq \emptyset} m(A)m(B). \quad (5)$$

A conflict measure  $\kappa_x : \mathcal{M} \times \mathcal{M} \rightarrow [0, 1]$  between two mass functions can be defined from a consistency measure by:

$$\kappa_x(m_1, m_2) := 1 - \phi_x(m_1 \odot_2), \quad (6)$$

where  $x \in \{m, Y, \pi\}$ . These three conflict measures have been proved in [11, 8] to satisfy a set of desirable axioms for a conflict measure proposed in [8].

## 2.2 $N$ -consistency definition and measures

Probabilistic and logical consistency definitions require, respectively, each of the focal sets, and the intersection of all focal sets, to be non-empty. In-between properties of the focal sets may also be useful to capture as illustrated by the following example.

*Example 2.* We consider two other sources that inform about the destination of the vessel: a Track-to-Route algorithm ( $S_2$ ) which associates a vessel to pre-computed maritime routes based on its kinematics features, and a Vessel Traffic Service (VTS) operator ( $S_3$ ) who is monitoring the marine traffic for some port authority, and who based on experience provides a subjective assessment of the vessels destination.  $S_2$  and  $S_3$  provide the following assessments:  $m_2(\{d_3\}) = 0.2$ ;  $m_2(\{d_1, d_2, d_3\}) = 0.6$ ;  $m_2(\{d_1, d_2\}) = 0.2$ , and  $m_3(\{d_4\}) = 0.1$ ;  $m_3(\{d_3\}) = 0.1$ ;  $m_3(\{d_1, d_2\}) = 0.8$ .

Both mass functions are equally consistent according to the probabilistic and logical measures as we can see that they satisfy:  $\phi_m(m_2) = \phi_m(m_3) = 1$  and  $\phi_\pi(m_2) = \phi_\pi(m_3) = 0.8$ . We can therefore not compare the two assessments in terms of internal consistency using these measures. However, if we refine the analysis and consider for instance the pairwise intersection of the focal sets,  $m_2$  appears “less inconsistent” than  $m_3$  since its focal sets are “more” (pairwise) intersecting. This suggests the definition of other measures of internal consistency that can capture refined notions of consistency based on the degree of intersection of the focal sets.

To this aim, we proposed recently in [11] a family of definitions of consistency:

**Definition 3 ( $N$ -consistency [11]).** *A mass function  $m$  is said to be consistent of order  $N$  ( $N$ -consistent for short), with  $1 \leq N \leq \mathfrak{F}$ , iff its focal sets are  $N$ -wise consistent, i.e., if  $\forall \mathcal{F}_N \subseteq \mathcal{F}$  s.t.  $|\mathcal{F}_N| = N$ , we have:*

$$\bigcap_{A \in \mathcal{F}_N} A \neq \emptyset.$$

In addition, we proposed in [11] an associated family of consistency measures  $\phi_N$  from  $\mathcal{M}$  to  $[0, 1]$  defined for any  $m \in \mathcal{M}$  and  $1 \leq N \leq \mathfrak{F}$  by:

$$\phi_N(m) = 1 - m^{(N)}(\emptyset), \quad (7)$$

where  $m^{(N)}$  denotes the result of the conjunctive combination of  $m$  with itself  $N$  times ( $m^{(1)} = m$ ).

The family satisfies the following properties [11]:

- For every  $N \in [1, \mathfrak{F}]$ ,  $\phi_N$  satisfies Properties 1 and 2 in the case where total consistency is understood according to the  $N$ -consistency definition.
- Probabilistic and Yager consistency (definition and measure) coincide, respectively, with 1-consistency and 2-consistency.
- $m$  is logically consistent iff it is  $\mathfrak{F}$ -consistent.

- The family is monotonic in  $N$  for any given mass function  $m$ :

$$\phi_1(m) = \phi_m(m) \geq \phi_2(m) \geq \dots \geq \phi_{\mathfrak{F}}(m).$$

*Example 3.* Going back to the previous example, we have:

$\phi_1(m_2) = 1 > \phi_2(m_2) = 0.92 > \phi_3(m_2) = \phi_{\mathfrak{F}}(m_2) = 0.88 > \phi_{\pi}(m_2) = 0.8$ ; and  $\phi_1(m_3) = 1 > \phi_{\pi}(m_3) = 0.8 > \phi_2(m_3) = 0.66 > \phi_3(m_3) = \phi_{\mathfrak{F}}(m_3) = 0.514$ . Using these measures, it becomes possible to compare  $m_2$  and  $m_3$  in terms of internal consistency: in particular, the intuition that  $m_2$  appears less inconsistent than  $m_3$  when considering pairwise consistency is captured by measure  $\phi_2$ .

It appears that the measure  $\phi_{\pi}$  does not belong to and can not be ordered within the  $\phi_N$  family. In particular, it is not possible to compare the two measures that capture the same notion of logical consistency:  $\phi_{\pi}$  and  $\phi_{\mathfrak{F}}$ . In the following, we show that such a comparison becomes possible through a simple transformation of the  $\phi_N$  family.

### 3 Monotonically ordered consistency measures

In the following, we first propose a new family of consistency measures derived from the  $\phi_N$  family (Section 3.1), and then show that the probabilistic  $\phi_m$  and logical  $\phi_{\pi}$  consistency measures belong to the family and can be ordered within it (Section 3.2).

#### 3.1 A new family of consistency measures

**Definition 4.** Let  $m$  be a mass function of  $\mathcal{M}$ . The  $\psi_N$  measure of  $N$ -consistency of  $m$ , for  $N \in \mathbb{N}_{>0}$ , is the measure  $\psi_N : \mathcal{M} \rightarrow [0, 1]$  defined for any  $m \in \mathcal{M}$  by:

$$\psi_N(m) := (1 - m^{(N)}(\emptyset))^{\frac{1}{N}}. \quad (8)$$

Note that the new proposed family is simply the  $N$ -th root of the probabilistic consistency of the family of mass functions  $m^{(N)}$ ,  $N \in \mathbb{N}_{>0}$ , since:

$$\psi_N(m) = (\psi_1(m^{(N)}))^{\frac{1}{N}}. \quad (9)$$

It encompasses the probabilistic consistency measure which is retrieved when  $N = 1$  and  $\psi_1(m) = \phi_m(m)$ . However, contrary to the  $\phi_N$  family, the 2-consistency measure  $\psi_2$  does not coincide anymore with Yager's  $\phi_Y$ . We however have that  $\psi_2 = \sqrt{\phi_Y}$ .

**Proposition 1.**  $\psi_N$  measures satisfy Properties 1 and 2 for the  $N$ -consistency definition.

*Proof Sketch.* This stems from the result that the measure  $\phi_m$ , or equivalently  $\psi_1$ , satisfies both properties and the relation (9) between the measures  $\psi_N$  and  $\psi_1$ .

Although it is obvious that the family  $\phi_N$  is monotonic in  $N$  for all  $m \in \mathcal{M}$ , the equivalent result for the family  $\psi_N$  still holds but is less trivial, as stated by the proposition below and following proof.

**Proposition 2.** *For all  $m \in \mathcal{M}$  and  $N \geq 1$ :*

$$\psi_N(m) \geq \psi_{N+1}(m).$$

*Proof Sketch.* For any two mass functions  $m_1$  and  $m_2$  in  $\mathcal{M}$  we have:

$$m_{1 \odot 2}(\emptyset) \geq 1 - (1 - m_1(\emptyset))(1 - m_2(\emptyset)).$$

The right-hand value is reached when the non-empty focal sets of both mass functions intersect, in which case there is no creation of empty focal sets in the combination, and  $m_{1 \odot 2}(\emptyset)$  is solely due to the propagation of the masses of the empty sets of both mass functions. When  $m_1 = m_2$ , it is easy to prove by recursion and using the previous inequality that the following relation holds between  $m(\emptyset)$  and  $m^{(N)}(\emptyset)$ :

$$m^{(N)}(\emptyset) \geq 1 - (1 - m(\emptyset))^N, \text{ which is equivalent to } \phi_1(m) \geq (\phi_N(m))^{\frac{1}{N}}.$$

When  $m_1 = m$  and  $m_2 = m^{(N)}$ , the first inequality yields:  $m_{1 \odot 2}(\emptyset) = m^{(N+1)}(\emptyset) \geq 1 - (1 - m^{(N)}(\emptyset))(1 - m(\emptyset))$ . Since  $m(\emptyset)$  and  $m^{(N)}(\emptyset)$  are related by the recursive relation, we can deduce that:  
 $m^{(N+1)}(\emptyset) \geq 1 - (1 - m^{(N)}(\emptyset))(1 - m^{(N)}(\emptyset))^{\frac{1}{N}}$ , i.e.,  $\psi_{N+1}(m) \leq \psi_N(m)$ .

In the following, we study the relation between the existing and the new measures.

### 3.2 Relation with the existing measures

We are interested in studying the relation between the logical consistency measure  $\phi_\pi$  and the proposed family, in particular  $\psi_{\mathfrak{F}}$  since, we recall,  $\psi_{\mathfrak{F}}$  captures the same definition of total consistency as  $\phi_\pi$ .

We start by reporting a result on the relation between the first term of the family, i.e., the probabilistic consistency measure, and the logical one  $\phi_\pi$ .

**Lemma 1.** *Every mass function  $m \in \mathcal{M}$  with  $\mathfrak{F}$  focal sets satisfies:*

$$\psi_1(m) \geq \phi_\pi(m) \geq \frac{\psi_1(m)}{\mathfrak{F}^*},$$

where  $\mathfrak{F}^*$  denotes the number of non-empty focal sets of  $m$ .

*Proof Sketch.* The left-hand side of the inequality is a known result [8]. The right-hand part stems from observing that:

$$\phi_\pi(m) \geq \max_{A \in \mathcal{F}} (m(A)) \geq \frac{1 - m(\emptyset)}{\mathfrak{F}^*}.$$

Actually, the result in Lemma 1 holds between measures  $\phi_\pi$  and  $\psi_N$  for all  $N \geq 1$ :

**Proposition 3.** For every  $m \in \mathcal{M}$  with  $\mathfrak{F}$  focal sets, and for every  $N \geq 1$ :

$$\psi_N(m) \geq \phi_\pi(m) \geq \frac{\psi_N(m)}{(\mathfrak{F}_N^*)^{\frac{1}{N}}},$$

where  $\mathfrak{F}_N^*$  denotes the number of non-empty focal sets of  $m^{(N)}$ . Also, the series  $\psi_N(m)$  converges asymptotically to  $\phi_\pi(m)$ :

$$\lim_{N \rightarrow \infty} \psi_N(m) = \phi_\pi(m).$$

*Proof Sketch.* The inequalities stem from Lemma 1 applied to the mass function  $m^{(N)}$ , together with equation (9) and the relation:  $\phi_\pi(m^{(N)}) = (\phi_\pi(m))^N$  which stems from equations (3) and (2). For the second part of the proposition,  $\psi_N(m)$  is a decreasing bounded series, so it converges. Since the number of focal sets of  $m^{(N)}$  stops increasing after  $\mathfrak{F}$  auto-combinations of  $m$ , then  $\lim_{N \rightarrow \infty} (\mathfrak{F}_N^*)$  is a constant and  $\lim_{N \rightarrow \infty} (\mathfrak{F}_N^*)^{\frac{1}{N}} = 1$ .

By combining Propositions 2 and 3, it appears that the logical consistency measure corresponds to the upper asymptotic limit of the  $\psi_N$  family:

**Proposition 4.** For every mass function  $m$  defined over  $\mathcal{X}$  with  $\mathfrak{F}$  focal sets:

$$\phi_m(m) = \psi_1(m) \geq \psi_2(m) \geq \dots \geq \psi_{\mathfrak{F}}(m) \geq \phi_\pi(m) = \lim_{N \rightarrow \infty} \psi_N(m).$$

and for every pair  $m_1$  and  $m_2$ :

$$\kappa_m(m_1, m_2) \leq \kappa_2(m_1, m_2) \leq \dots \leq \kappa_{\mathfrak{F}_{12}}(m_1, m_2) \leq \kappa_\pi(m_1, m_2) = \lim_{N \rightarrow \infty} \kappa_N(m_1, m_2).$$

where  $\kappa_N(m_1, m_2) = 1 - \psi_N(m_1 \odot_2 m_2)$  and  $\mathfrak{F}_{12}$  the number of focal sets of  $m_1 \odot_2 m_2$ .

The analysis of the internal consistency of the belief function resulting from the conjunctive combination of the belief functions issued by some sources, *i.e.*, of their conflict, can be used in several ways to improve the estimation confidence on the fusion output. This can be done by discounting or discarding the most conflicting sources [13], or re-questioning those that are inconsistent with a certain reference source. A deep analysis of the conflict is therefore necessary as illustrated hereafter.

*Example 4.* To estimate the destination of the vessel, both the Track-to-Route and VTS operator rely on some extra contextual information ( $S_4$ ) encoded by a mass function  $m_4$ . The mass functions  $m_2$  and  $m_3$  are actually the results of the conjunctive combination of some mass functions  $m_{2b}$  and  $m_{3b}$  encoding the specific sources knowledge and  $m_4$ :  $m_2 = m_{2b} \odot_4 m_4$ ;  $m_3 = m_{3b} \odot_4 m_4$ . We are interested in determining which of the sources is more in conflict with the contextual knowledge which is highly reliable and trusted. The conflict values are, using  $\kappa_N(m_{2b}, m_4) = 1 - \psi_N(m_2)$  and  $\kappa_N(m_{3b}, m_4) = 1 - \psi_N(m_3)$ :

$$\kappa_m(m_{2b}, m_4) = 0; \kappa_2(m_{2b}, m_4) = 0.041; \kappa_3(m_{2b}, m_4) = 0.042; \kappa_\pi(m_{2b}, m_4) = 0.2;$$

$\kappa_m(m_{3b}, m_4) = 0; \kappa_2(m_{3b}, m_4) = 0.187; \kappa_3(m_{3b}, m_4) = 0.198; \kappa_\pi(m_{3b}, m_4) = 0.2$ . The probabilistic and logical conflict measures do not allow one to identify which of  $S_2$  and  $S_3$  is more in conflict with  $S_4$ , while the in-between conflict measures do, and suggest that  $S_3$  is more conflicting with  $S_4$  than  $S_2$ .

## 4 Conclusions and future work

In this paper, we proposed a parametrised family of consistency measures and illustrated its properties and interest with an example of multi-source vessel destination estimation problem. The family satisfies desired consistency measures properties such as boundedness and extreme values, and is monotonic. In addition, it subsumes the probabilistic and logical measures as, respectively, the lower sharp bound and the upper asymptotic limit.

In a future work, we will investigate the geometric interpretation of the proposed family of measures as well as the partial order induced by the vector  $(\psi_1, \dots, \psi_{\mathfrak{F}})$  on the mass functions space. It will also be interesting to know whether the new conflict measures introduced in this paper satisfy the conflict axioms of [8]. Other open questions such as the choice of the “best” measure (or level of consistency) will also be addressed considering theoretical and practical aspects such as the computational cost or some user’s expectations about the measures semantics.

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