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# The natural extension of the random beta-transformation

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## Abstract

We construct a geometrico-symbolic version of the natural extension of the random  $\beta$ -transformation introduced by Dajani and Kraaikamp in [5]. This construction provides a new proof of the existence of a unique absolutely continuous invariant probability measure for the random  $\beta$ -transformation, and an expression for its density. We then prove that this natural extension is a Bernoulli automorphism, generalizing to the random case the result of Smorodinsky [15] about the greedy transformation.

## Introduction

### Expansions in base $\beta$

In all the paper, we fix a real number  $\beta$ ,  $1 < \beta < 2$ . For  $x \in [0, 1)$ , an expansion of  $x$  in base  $\beta$  is a sequence  $(x_n)_{n \geq 1}$  of  $\{0, 1\}$  such that

$$x = \sum_{n=1}^{+\infty} \frac{x_n}{\beta^n}.$$

Rényi [14] introduced the *greedy map*  $T_\beta$  defined on  $[0, 1)$  by  $T_\beta(x) = \beta x \bmod 1$ , which provides a particular expansion of any real number in  $[0, 1)$ , given by the sequence  $x_n := \mathbb{1}_{[1/\beta, 1)}(T_\beta^{n-1}(x))$ , for all  $n \geq 1$ . This expansion is called the *greedy expansion* of  $x$ . In fact, almost every  $x \in [0, 1)$  has an infinite number of expansions in base  $\beta$ , and the greedy expansion is the greatest in the lexicographic order [7].

Rényi proved that  $T_\beta$  has a unique absolutely continuous invariant probability measure  $\nu_\beta$ , and Parry [12] proved that its density is proportional to the function

$$\sum_{n \geq 0} \frac{1}{\beta^n} \mathbb{1}_{[0, T_\beta^n(1)]},$$

and that the associated measure-preserving system is weakly mixing.

Smorodinsky then showed in [15] that the natural extension of this system is a Bernoulli automorphism. This result was also obtained by Dajani, Kraaikamp and Solomyak in [6] through a geometric construction of this natural extension, in the form of a tower.

The greedy map can be extended to a map  $T_g$  defined on the interval  $I_\beta := [0, \frac{1}{\beta-1}]$  by setting  $T_g(x) := \begin{cases} T_\beta(x) & \text{for } x \in [0, 1[ \\ \beta x - 1 & \text{for } x \in [1, \frac{1}{\beta-1}] \end{cases}$ . We will still refer to this extended map as the greedy map. The measure  $\nu_\beta$  extended to  $I_\beta$  by setting  $\nu_\beta([1, \frac{1}{\beta-1}]) = 0$  is still the unique absolutely continuous invariant probability measure of  $T_g$  on  $I_\beta$ .

We can obtain the smallest expansion of any real number of the interval  $I_\beta$  with the *lazy map*  $T_\ell$  [7], defined by

$$T_\ell : I_\beta \rightarrow I_\beta \\ x \mapsto \begin{cases} \beta x & \text{if } x \leq \frac{1}{\beta(\beta-1)} \\ \beta x - 1 & \text{otherwise} \end{cases} .$$

For  $x \in I_\beta$ , the *lazy expansion* of  $x$  is given by the sequence  $(x'_n)$  defined for all  $n \geq 1$  by  $x'_n = \mathbf{1}_{(\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}]}(T_\ell^{n-1}(x))$ .

Let  $s$  be the symmetry on  $I_\beta$ , defined by  $s(x) = \frac{1}{\beta-1} - x$ . The pushforward measure  $\tilde{\nu}_\beta := \nu_\beta \circ s^{-1}$  is the unique absolutely continuous invariant measure for  $T_\ell$ , and  $s$  conjugates  $T_g$  and  $T_\ell$ , hence the systems  $(I_\beta, \mathcal{B}, \nu_\beta, T_g)$  and  $(I_\beta, \mathcal{B}, \tilde{\nu}_\beta, T_\ell)$  are isomorphic [4].

In 2003, Dajani and Kraaikamp [5] introduced the random  $\beta$  transformation. We set  $\Omega := \{g, \ell\}^{\mathbb{N} \cup \{0\}}$ , the set of sequences of  $g$  and  $\ell$ . We then define the transformation  $K_\beta$  on  $\Omega \times I_\beta$  by

$$K_\beta : \Omega \times I_\beta \rightarrow \Omega \times I_\beta \\ (\omega, x) \mapsto (\sigma(\omega), T_{\omega_0}(x)) ,$$

where  $\sigma$  is the left shift on  $\Omega$ . The sequence  $\omega \in \Omega$  describes the successive transformations that will be applied to the real  $x \in I_\beta$ : once  $T_{\omega_0}$  is applied to  $x$ , we shift the sequence  $\omega$  to the left and then apply  $T_{\omega_1}$ , and so on. If we fix  $\omega \in \Omega$ , we obtain a particular expansion of  $x$  in base  $\beta$ , by setting, for any  $n \geq 1$ :

$$x_n^\omega := \begin{cases} \mathbf{1}_{\{\pi(K_\beta^{n-1}(\omega, x)) \geq \frac{1}{\beta}\}} & \text{if } \omega_{n-1} = g \\ \mathbf{1}_{\{\pi(K_\beta^{n-1}(\omega, x)) > \frac{1}{\beta(\beta-1)}\}} & \text{if } \omega_{n-1} = \ell \end{cases} ,$$

where  $\pi$  is the projection on  $I_\beta$ . Any expansion of  $x$  can be obtained with a sequence  $\omega \in \Omega$  [3].

In the model we consider, the sequence  $\omega$  will be drawn with the product Bernoulli measure  $m_p$ , where  $p \in (0, 1)$  is a fixed parameter. In other words, we draw each transformation independently with probability  $p$  for  $T_g$  and  $1-p$  for  $T_\ell$ . Dajani and De Vries proved in [3] that there is a unique absolutely continuous probability measure  $\mu_p$  on  $I_\beta$  such that  $m_p \otimes \mu_p$  is  $K_\beta$ -invariant. Kempton [11] obtained an expression of the density  $\rho_{1/2}$  of  $\mu_{1/2}$  with the construction of a natural extension of  $(\Omega \times I_\beta, m_{1/2} \otimes \mu_{1/2}, \mathcal{K}_\beta)$ , using two symmetric towers. Suzuki [16] then generalized this expression for any  $p$  with the use of the Perron-Frobenius operators.

**Roadmap of this paper** In this paper, we consider several extensions of the random  $\beta$ -transformation. In section 1, we construct a first geometrico-symbolic extension  $\mathcal{K}$  of the random transformation. This extension is defined on two towers (the greedy tower and the lazy tower), each tower having a ‘‘base’’.

By providing a simple invariant measure on this extension and projecting it on  $I_\beta$ , we obtain the expression of the density of  $\mu_p$  for any  $p \in (0, 1)$  (formula (8)). We then study the transformation  $\mathcal{K}_g$  induced by  $\mathcal{K}$  on the base of the greedy tower, and define a particular partition of this base. We prove that this partition is an independent generator of the induced system (Proposition 10), thus proving that this induced system is isomorphic to a unilateral Bernoulli shift. This provides a new proof of the ergodicity of the random system and of the uniqueness of  $\mu_p$ . This first extension is not yet invertible. In section 2, we construct a geometrico-symbolic version of the natural extension of this first extension. With the use of the relatively independent joining above a factor, we prove that this new extension is in fact the natural extension of the random  $\beta$ -transformation (Theorem 21). We then prove that this extension is isomorphic to a bilateral Bernoulli shift (Theorem 26), by “unfolding” the previous partition on the two towers.

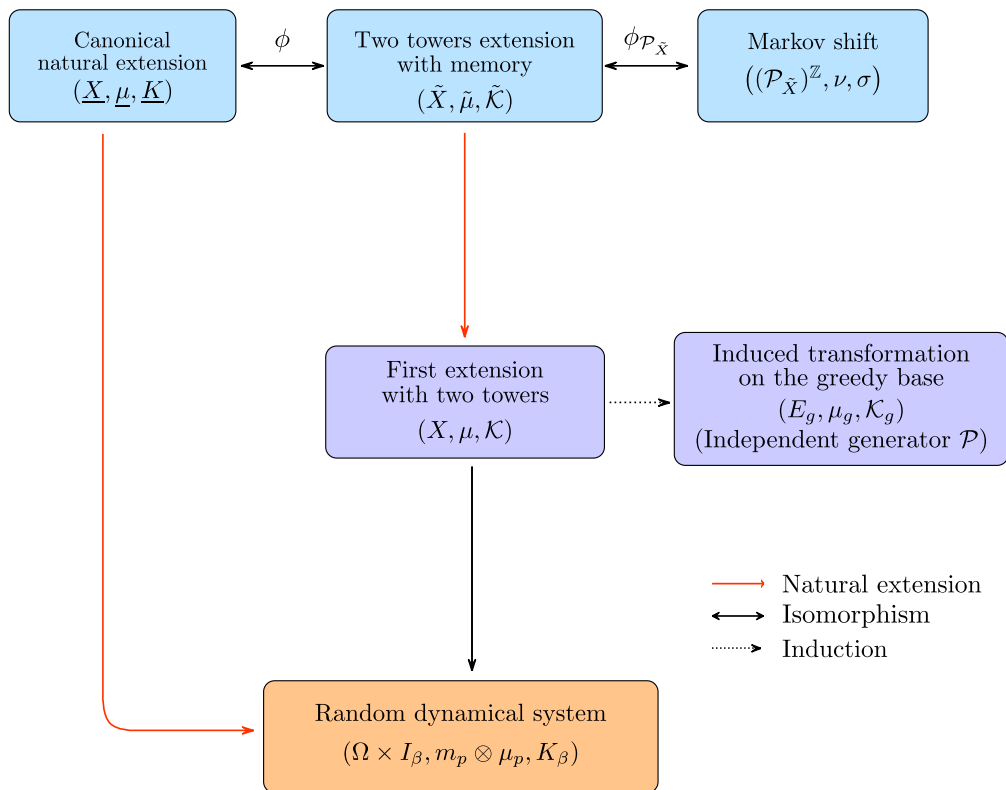


Figure 1: Extensions involved in this paper

## 1 The first extension

In this section, we extend the dynamics  $K_\beta$  on two towers. Kempton [11] implemented the same type of strategy in the case  $p = \frac{1}{2}$ , but his construction

does not generalize to any  $p$ . We construct a different extension, valid for any  $p \in (0, 1)$ .

### 1.1 The domain of the extension: the greedy and lazy towers

As in [11], the towers are made of floors. We first define the “base” of each tower. Let  $E_g$  be the base of the greedy tower  $\mathcal{G}$ , defined by:

$$E_g := \Omega \times \{g\} \times [0, 1].$$

The label of the greedy base is  $(g)$ , telling us that it is the first floor of the greedy tower  $\mathcal{G}$ . Likewise,  $E_\ell$  is the base of the lazy tower  $\mathcal{L}$ , defined by:

$$E_\ell := \Omega \times \{\ell\} \times \left[ s(1), \frac{1}{\beta-1} \right].$$

A floor in one of the towers is denoted by  $E_e$ , where  $e$  is the label of the floor. This floor is of the form:

$$E_e := \Omega \times \{e\} \times I_e,$$

where  $I_e$  is a sub-interval of  $I_\beta$  that will be specified later. Thus, a point in one of the two towers is of the form  $c = (\omega, e, x)$ , where:

- The sequence  $\omega \in \{g, \ell\}^{\mathbb{N} \cup \{0\}}$  describes the successive transformations  $T_g$  and  $T_\ell$  to be applied to  $x$ .
- The label  $e$  is of the form

$$e = (h, \omega_{-n}, \omega_{-n+1}, \dots, \omega_{-1})$$

and characterizes the floor containing the point  $c$ . The letter  $h$  indicates the tower in which the floor is: if  $h = g$ , it is a floor from the greedy tower, if  $h = \ell$ , from the lazy tower. The integer  $n$  is the *level* of the floor  $E_e$ . Therefore, there are  $2^n$  floors of level  $n$  in each tower, the bases being at level 0. Finally, the symbols  $\omega_{-n}, \dots, \omega_{-1}$  indicate the applied transformations since the last passage in one of the two bases: in other words, the maps  $T_{\omega_{-n}}$  to  $T_{\omega_{-1}}$  were applied to the real component of a point in  $E_e$  since the base  $E_h$ .

- $x$  represents the real component of the point  $c$ , on which will be applied the transformation  $T_{\omega_0}$ . We have  $x \in I_e$  where

$$\begin{cases} I_e := [0, T_{\omega_{-n}, \omega_{-n+1}, \dots, \omega_{-1}}(1^+)] & \text{if } h = g \\ I_e := \left[ T_{\omega_{-n}, \omega_{-n+1}, \dots, \omega_{-1}}(s(1)^-), \frac{1}{\beta-1} \right] & \text{if } h = \ell \end{cases} ,$$

where the  $+$  and  $-$  signs respectively refer to the right and left limit at the considered points, and  $T_{\omega_{-n}, \omega_{-n+1}, \dots, \omega_{-1}}$  is the composition  $T_{\omega_{-1}} \circ \dots \circ T_{\omega_{-n}}$ . For simplification, we will often note  $T_v$  with  $v \in \{g, \ell\}^n$  to refer to this type of composition.

We then define the greedy tower as the disjoint union of all floors whose label starts with a  $g$ , and the lazy tower as the disjoint union of all floors whose label starts with an  $\ell$ :

$$\mathcal{G} := \bigsqcup_{e' \in \{g, \ell\}^*} E_{g \cdot e'},$$

$$\mathcal{L} := \bigsqcup_{e' \in \{g, \ell\}^*} E_{\ell \cdot e'},$$

where the point  $\cdot$  represents the concatenation, and the set  $\{g, \ell\}^*$  is the set of finite sequences (including the empty sequence) of  $g$  and  $\ell$ . We then note  $X := \mathcal{G} \sqcup \mathcal{L}$  the disjoint union of the two towers, on which will be defined  $\mathcal{K}$ . Finally, given a floor of  $X$ , we call *length* of this floor the Lebesgue measure of the associated interval.

Applying the dynamics  $\mathcal{K}$  to a point in a tower consists in going up a level in the tower, or going back to the base of the tower, or going to the base of the other tower, depending on certain conditions that we will detail in the next section.

We represent the two towers in Figure 2, one above the other (the lazy tower is represented upside down). The  $\omega$ -component is represented vertically. Two points on a same vertical line have the same real coordinate  $x$ .

## 1.2 The dynamics on the two towers

We define the dynamics  $\mathcal{K}$  on the greedy tower first. We consider a floor of  $\mathcal{G}$ , with label  $e := (g, \omega_{-n}, \dots, \omega_{-1})$ . The floor  $E_e$  is split into two parts, depending on  $\omega_0$ :

$$E_e = (E_e \cap [g]_0) \sqcup (E_e \cap [\ell]_0),$$

where  $E_e \cap [g]_0$  abusively refers to the set of points  $(\omega, e, x)$  of  $E_e$  such that  $\omega_0 = g$  (and in the same way for  $\ell$ ). The dynamics  $\mathcal{K}$  is different on each of these parts. In the following, we note  $t$  the upper bound of  $I_e$ .

1. If  $\omega_0 = g$  and if  $t < \frac{1}{\beta}$  (see Figure 3):

$$\mathcal{K} : \begin{array}{ll} E_e \cap [g]_0 & \rightarrow E_{e \cdot g} \\ (\omega, e, x) & \mapsto (\sigma(\omega), e \cdot g, \beta x) \end{array} .$$

$\mathcal{K}$  sends  $E_e \cap [g]_0$  onto the floor  $E_{e \cdot g}$ .

2. if  $\omega_0 = g$  and if  $t \geq \frac{1}{\beta}$ , the left part of the floor is sent back onto the base  $E_g$  (see Figure 4):

$$\mathcal{K} : \begin{array}{ll} E_e \cap \{\omega_0 = g, x < \frac{1}{\beta}\} & \rightarrow E_g \\ (\omega, e, x) & \mapsto (\sigma(\omega), g, \beta x) \end{array} ;$$

and the right part goes up onto  $E_{e \cdot g}$ :

$$\mathcal{K} : \begin{array}{ll} E_e \cap \{\omega_0 = g, x \geq \frac{1}{\beta}\} & \rightarrow E_{e \cdot g} \\ (\omega, e, x) & \mapsto (\sigma(\omega), e \cdot g, \beta x - 1) \end{array} .$$

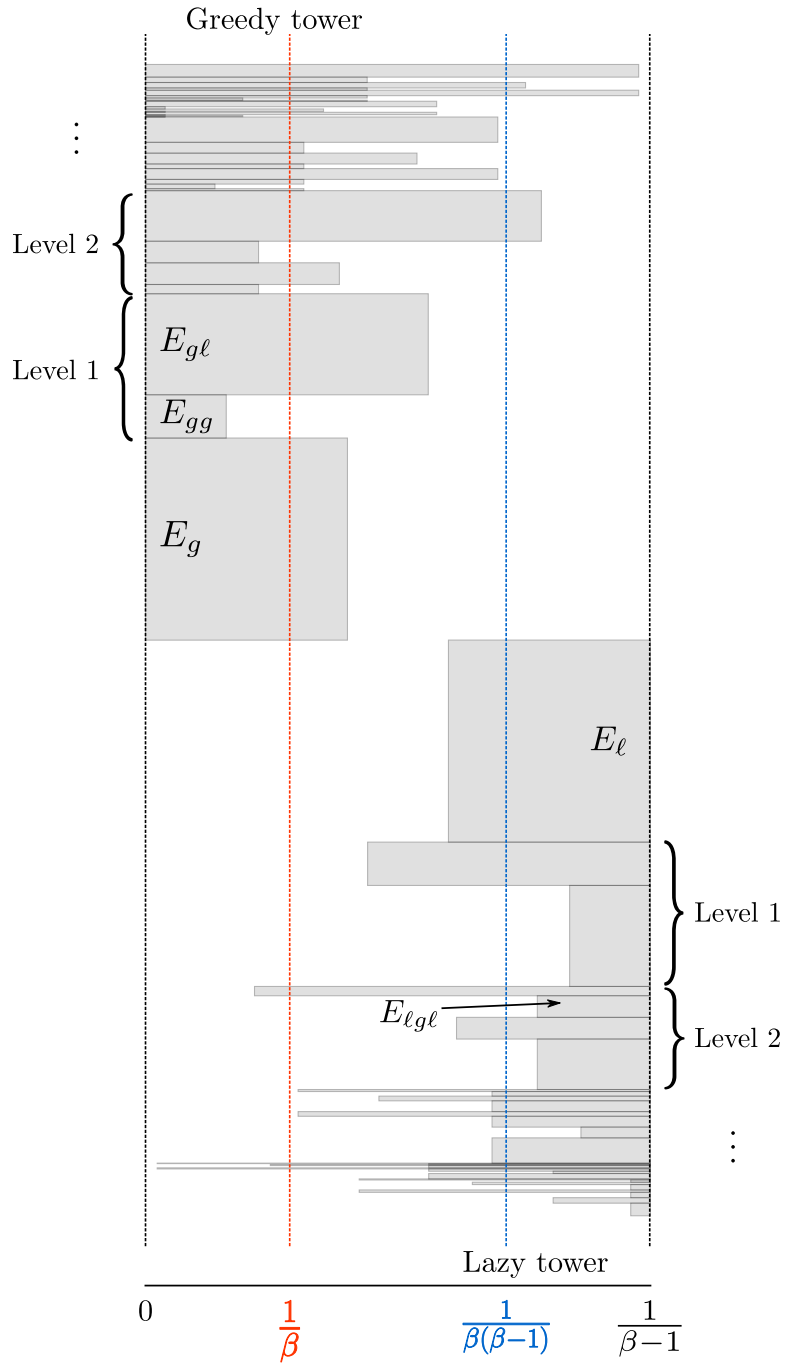


Figure 2: Greedy and lazy towers

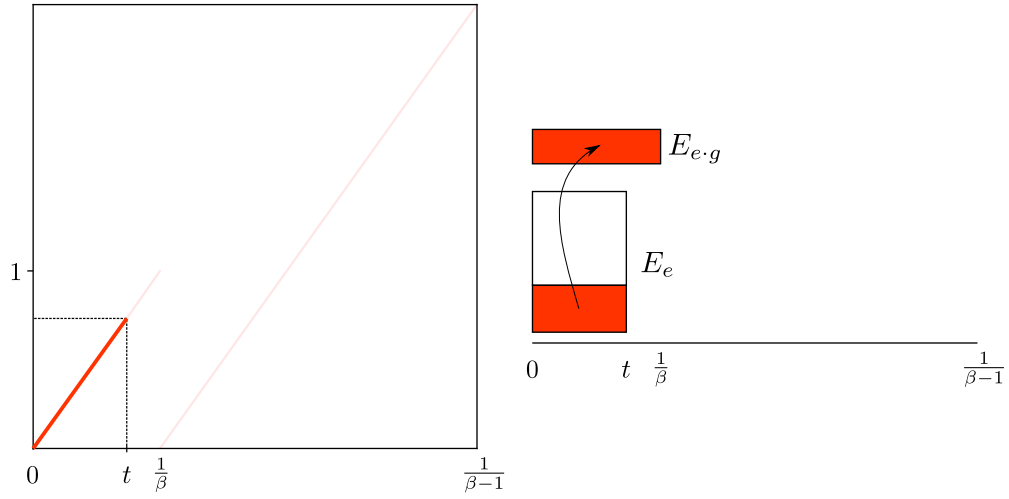


Figure 3:  $\omega_0 = g$  and  $t < \frac{1}{\beta}$

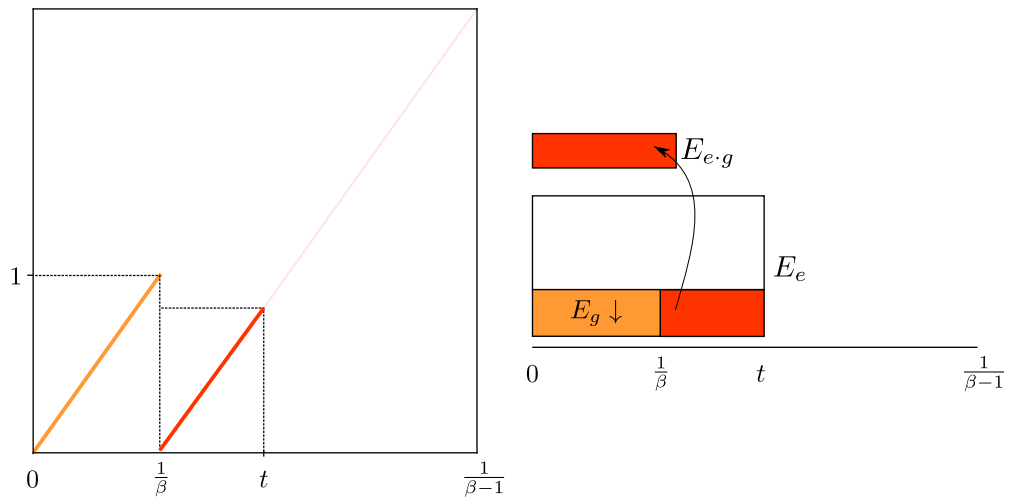


Figure 4:  $\omega_0 = g$  and  $t \geq \frac{1}{\beta}$



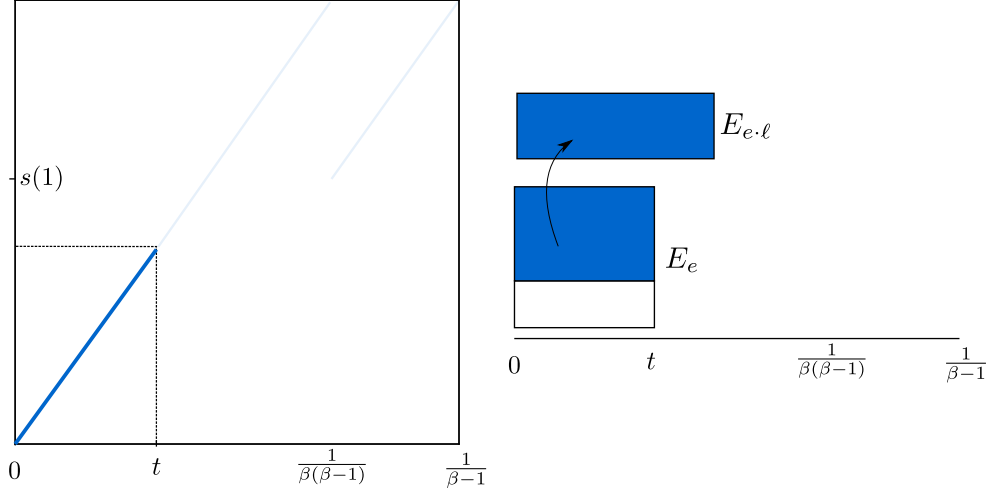


Figure 5:  $\omega_0 = \ell$  and  $t < \frac{1}{\beta(\beta-1)}$

3. if  $\omega_0 = \ell$  and if  $t < \frac{1}{\beta(\beta-1)}$  (see Figure 5):

$$\mathcal{K} : \begin{array}{l} E_e \cap [\ell]_0 \rightarrow E_{e.\ell} \\ (\omega, e, x) \mapsto (\sigma(\omega), e \cdot \ell, \beta x) \end{array} .$$

The part  $E_e \cap [\ell]_0$  is sent onto the floor  $E_{e.\ell}$ .

4. if  $\omega_0 = \ell$  and if  $t \geq \frac{1}{\beta(\beta-1)}$  (see Figure 6), the central part of the floor is sent onto the base  $E_\ell$ :

$$\mathcal{K} : \begin{array}{l} E_e \cap \{\omega_0 = \ell, \frac{s(1)}{\beta} < x \leq \frac{1}{\beta(\beta-1)}\} \rightarrow E_\ell \\ (\omega, e, x) \mapsto (\sigma(\omega), \ell, \beta x) \end{array} ;$$

and the left and right parts go up onto  $E_{e.\ell}$ :

$$\begin{array}{l} \mathcal{K} : \begin{array}{l} E_e \cap \{\omega_0 = \ell, x \leq \frac{s(1)}{\beta}\} \rightarrow E_{e.\ell} \\ (\omega, e, x) \mapsto (\sigma(\omega), e \cdot \ell, \beta x) \end{array} ; \\ \mathcal{K} : \begin{array}{l} E_e \cap \{\omega_0 = \ell, x > \frac{1}{\beta(\beta-1)}\} \rightarrow E_{e.\ell} \\ (\omega, e, x) \mapsto (\sigma(\omega), e \cdot \omega_0, \beta x - 1) \end{array} . \end{array}$$

On the lazy tower, the dynamics  $\mathcal{K}$  is defined in the same way by symmetry (see Figure 8).

We note  $\pi : X \rightarrow \Omega \times I_\beta$  the projection onto  $\Omega \times I_\beta$ . By construction, we have

$$\pi \circ \mathcal{K} = K_\beta \circ \pi.$$

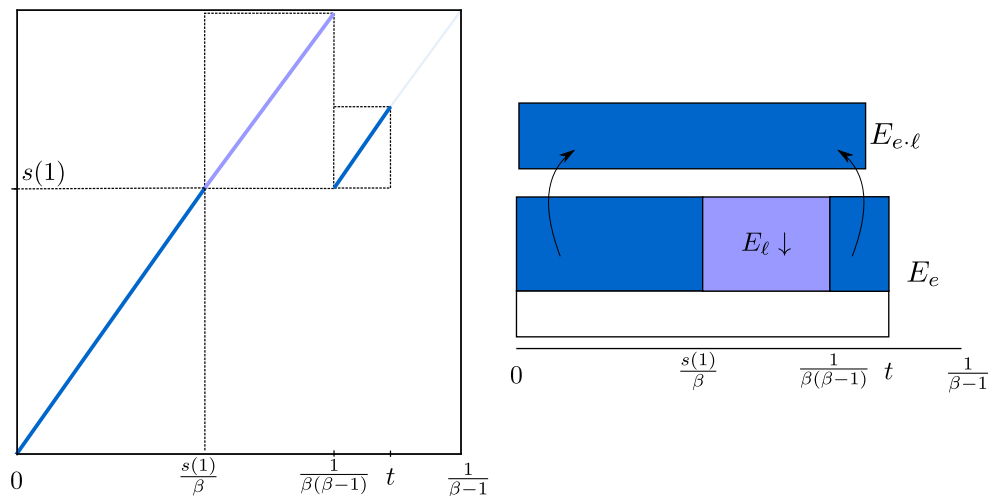


Figure 6:  $\omega_0 = \ell$  and  $t \geq \frac{1}{\beta(\beta-1)}$

### 1.3 Construction of an invariant measure

The goal of this section is to define a  $\mathcal{K}$ -invariant measure  $\mu$  on  $X$ , such that the projection of this measure on  $I_\beta$  is absolutely continuous. We denote the Lebesgue measure by  $\lambda$ , regardless of the interval on which we consider it. We recall that  $m_p$  is the product Bernoulli measure of parameter  $p$  on  $\Omega$  (we draw  $g$  with probability  $p$  and  $\ell$  with probability  $1 - p$ , independently at each step).

We first define the measures  $\mu_g$  and  $\mu_\ell$  with respective support  $\mathcal{G}$  and  $\mathcal{L}$  as follows.

On the bases, we set:

$$\mu_{g|E_g} := m_p \otimes \lambda,$$

$$\mu_{\ell|E_\ell} := m_p \otimes \lambda.$$

On the floor  $E_{g,\omega_{-n},\dots,\omega_{-1}}$ , the measure  $\mu_g$  is defined by:

$$\mu_{g|E_{g,\omega_{-n},\dots,\omega_{-1}}} := \frac{1}{\beta^n} m_p([\omega_{-n}, \dots, \omega_{-1}]_0^{n-1}) m_p \otimes \lambda,$$

where  $[\omega_{-n}, \dots, \omega_{-1}]_0^{n-1}$  is the cylinder of  $\Omega$  containing the sequences  $\omega$  whose  $n$  first terms are  $(\omega_{-n}, \dots, \omega_{-1})$ .

Likewise on the floor  $E_{\ell,\omega_{-n},\dots,\omega_{-1}}$ , the measure  $\mu_\ell$  is defined by:

$$\mu_{\ell|E_{\ell,\omega_{-n},\dots,\omega_{-1}}} := \frac{1}{\beta^n} m_p([\omega_{-n}, \dots, \omega_{-1}]_0^{n-1}) m_p \otimes \lambda.$$

On their respective tower, the measure  $\mu_g$  and  $\mu_\ell$  are preserved by  $\mathcal{K}$  when it goes up in the tower, but we look for a measure  $\mu$  on  $X$ , globally preserved by  $\mathcal{K}$ . The following theorem describes the situation in a more abstract framework, and shows the existence of the measure  $\mu$ .

**Theorem 1.** *Let  $(X, \mathcal{B})$  be a standard Borel space,  $\mathcal{K}$  a transformation on  $X$  and  $\mathcal{G}$  and  $\mathcal{L}$  two disjoint subsets of  $X$  such that:*

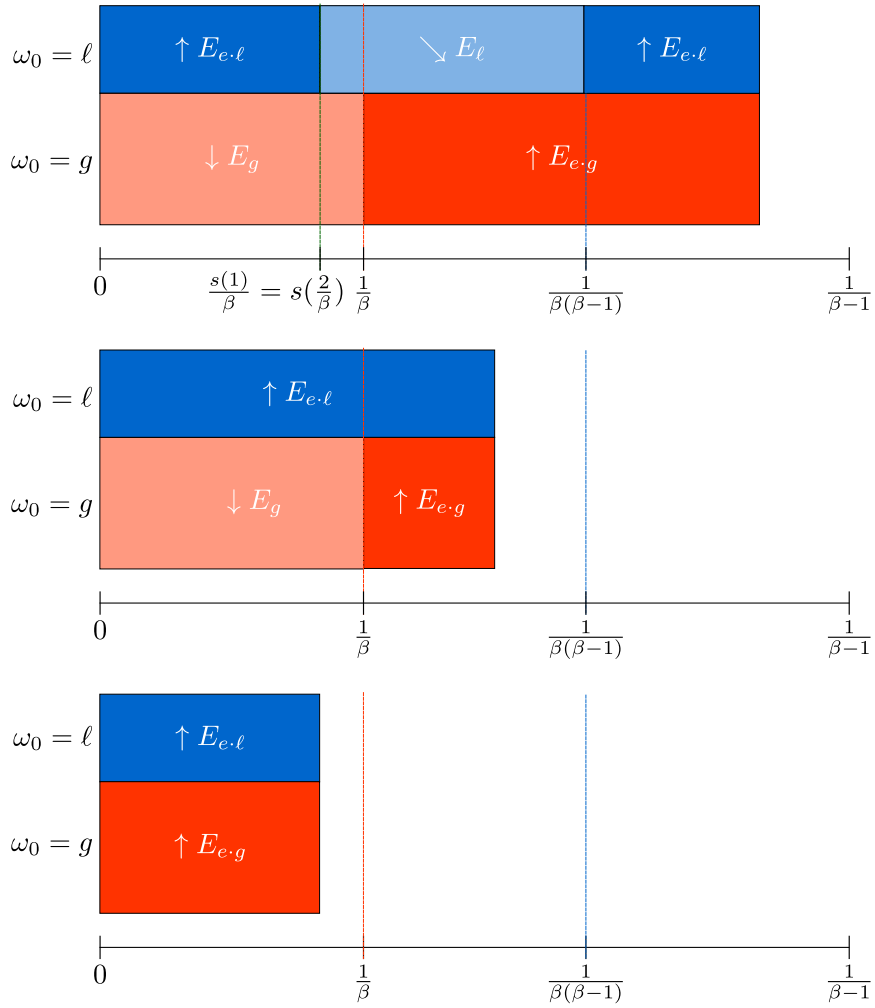


Figure 7: Overview of the dynamics on a floor  $E_e$  of the greedy tower, depending on the length of the floor

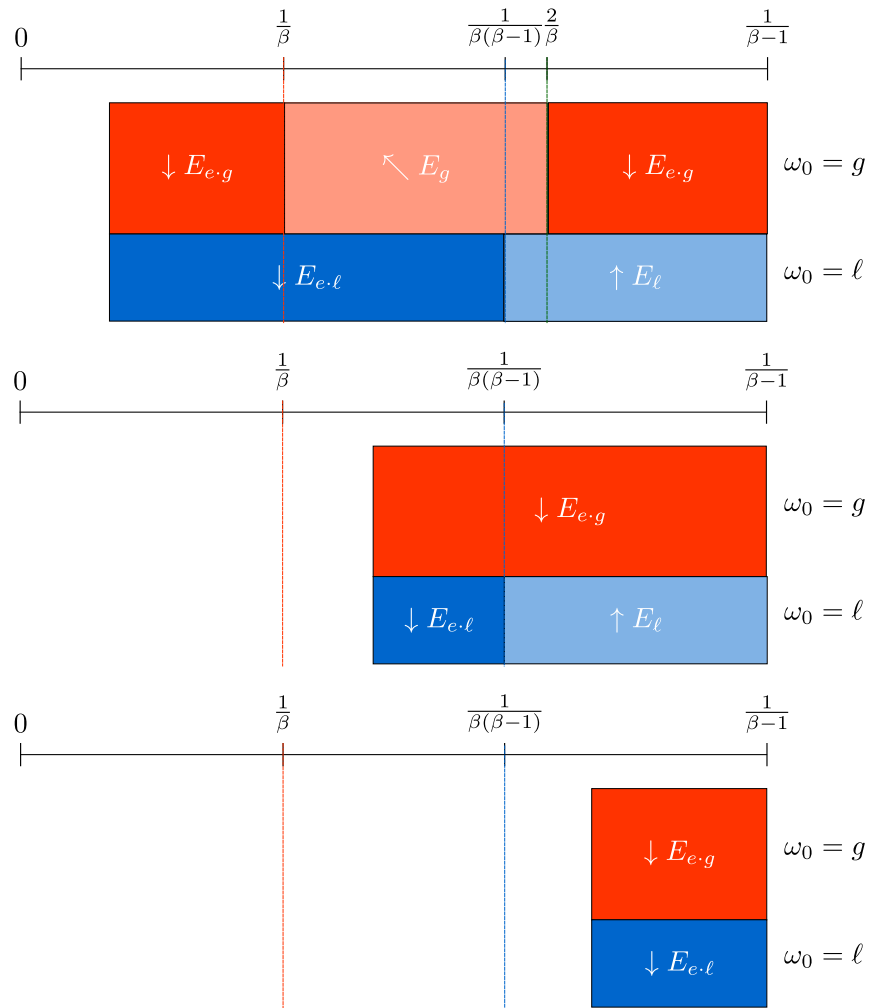


Figure 8: Overview of the dynamics on a floor  $E_e$  of the lazy tower (reversed), depending on the length of the floor

1.  $X = \mathcal{G} \sqcup \mathcal{L}$ .

2. There exist two sets  $E_g \subset \mathcal{G}$  and  $E_\ell \subset \mathcal{L}$  such that

$$\mathcal{K}(\mathcal{G}) \subset \mathcal{G} \cup E_\ell \tag{1}$$

and

$$\mathcal{K}(\mathcal{L}) \subset \mathcal{L} \cup E_g$$

3. There exist two finite measures  $\mu_g$  and  $\mu_\ell$  with respective support in  $\mathcal{G}$  and  $\mathcal{L}$  such that

$$\mu_g(E_g) = \mu_\ell(E_\ell) = 1$$

and for any measurable  $A$  of  $\mathcal{G} \setminus E_g$ , we have:

$$\mu_g(\mathcal{K}^{-1}(A)) = \mu_g(A), \tag{2}$$

and for any measurable  $A$  of  $\mathcal{L} \setminus E_\ell$ , we have:

$$\mu_\ell(\mathcal{K}^{-1}(A)) = \mu_\ell(A),$$

4. There exists positive constants  $g_0, \ell_0, g_1, \ell_1$  such that for any measurable  $A$  of  $X$  included in  $E_g$ , we have

$$\mu_g(\mathcal{K}^{-1}(A) \cap \mathcal{G}) = g_0 \mu_g(A) \tag{3}$$

and

$$\mu_\ell(\mathcal{K}^{-1}(A) \cap \mathcal{L}) = \ell_0 \mu_g(A), \tag{4}$$

and for any measurable  $A$  of  $X$  included in  $E_\ell$ , we have:

$$\mu_g(\mathcal{K}^{-1}(A) \cap \mathcal{G}) = g_1 \mu_\ell(A)$$

and

$$\mu_\ell(\mathcal{K}^{-1}(A) \cap \mathcal{L}) = \ell_1 \mu_\ell(A).$$

Then there exist a unique probability measure  $\mu$  on  $X$ , linear combination of  $\mu_g$  and  $\mu_\ell$ , such that  $\mu$  is  $\mathcal{K}$ -invariant.

*Proof.* Let  $\mu$  be a measure of the form

$$\mu = C_g \mu_g + C_\ell \mu_\ell,$$

with  $C_g$  and  $C_\ell$  two positive constants. We wish to determine the values of  $C_g$  and  $C_\ell$  such that  $\mu$  is a  $\mathcal{K}$ -invariant probability measure on  $X$ .

Let  $A$  be a measurable subset of  $X$  included in  $\mathcal{G} \setminus E_g$ . We have  $\mu(A) = C_g \mu_g(A)$ . Then, by assumptions (1) and (2), we have:

$$\mu(\mathcal{K}^{-1}(A)) = C_g \mu_g(\mathcal{K}^{-1}(A)) = C_g \mu_g(A) = \mu(A).$$

Similarly, for any measurable  $A$  of  $X$  included in  $\mathcal{L} \setminus E_\ell$ :

$$\mu(\mathcal{K}^{-1}(A)) = C_\ell \mu_\ell(\mathcal{K}^{-1}(A)) = C_\ell \mu_\ell(A) = \mu(A).$$

Let  $A$  be a measurable subset of  $X$  included in  $E_g$ . We have, by assumptions (3) and (4):

$$\begin{aligned}\mu(\mathcal{K}^{-1}(A)) &= \mu(\mathcal{K}^{-1}(A) \cap \mathcal{G}) + \mu(\mathcal{K}^{-1}(A) \cap \mathcal{L}) \\ &= C_g \mu_g(\mathcal{K}^{-1}(A) \cap \mathcal{G}) + C_\ell \mu_\ell(\mathcal{K}^{-1}(A) \cap \mathcal{L}) \\ &= g_0 C_g \mu_g(A) + \ell_0 C_\ell \mu_\ell(A) \\ &= (g_0 C_g + \ell_0 C_\ell) \mu_g(A).\end{aligned}$$

Therefore,  $\mu(\mathcal{K}^{-1}(A)) = \mu(A)$  if and only if

$$g_0 C_g + \ell_0 C_\ell = C_g. \quad (5)$$

Likewise, if  $A$  is a measurable subset of  $X$  included in  $E_\ell$ ,  $\mu(\mathcal{K}^{-1}(A)) = \mu(A)$  if and only if

$$g_1 C_g + \ell_1 C_\ell = C_\ell. \quad (6)$$

We need both the equations (5) and (6) to be satisfied. In other words, we solve the system:

$$\begin{cases} (g_0 - 1)C_g + \ell_0 C_\ell = 0 \\ g_1 C_g + (\ell_1 - 1)C_\ell = 0 \end{cases}. \quad (7)$$

From assumption (1), we have:

$$\mu_g(\mathcal{G}) = \mu_g(\mathcal{K}^{-1}(\mathcal{G} \setminus E_g)) + \mu_g(\mathcal{K}^{-1}(E_g) \cap \mathcal{G}) + \mu_g(\mathcal{K}^{-1}(E_\ell) \cap \mathcal{G}),$$

and since  $\mu_g(\mathcal{K}^{-1}(\mathcal{G} \setminus E_g)) = \mu_g(\mathcal{G} \setminus E_g)$ , we get

$$\mu_g(E_g) = \mu_g(\mathcal{K}^{-1}(E_g) \cap \mathcal{G}) + \mu_g(\mathcal{K}^{-1}(E_\ell) \cap \mathcal{G}).$$

Since  $\mu_g(E_g) = 1$ , we deduce that

$$g_0 + g_1 = 1.$$

Similarly,

$$\ell_0 + \ell_1 = 1.$$

Therefore, the system (7) can be reduced to the equation

$$C_g = \frac{\ell_0}{g_1} C_\ell.$$

In order for  $\mu$  to be a probability measure, we must also have

$$C_g \mu_g(\mathcal{G}) + C_\ell \mu_\ell(\mathcal{L}) = 1.$$

We finally obtain the (positive) values

$$C_g = \frac{\ell_0}{\ell_0 \mu_g(\mathcal{G}) + g_1 \mu_\ell(\mathcal{L})}$$

and

$$C_\ell = \frac{g_1}{\ell_0 \mu_g(\mathcal{G}) + g_1 \mu_\ell(\mathcal{L})}.$$

With this choice of constants, the measure  $\mu$  is a  $\mathcal{K}$ -invariant probability measure on  $X$ .  $\square$

Let us prove that the extension built in the previous section satisfies the assumptions of Theorem 1.

1. The greedy and lazy towers are disjoint.
2. Under the action of  $\mathcal{K}$ , any element of  $\mathcal{G}$  can go up in the tower  $\mathcal{G}$ , or go back to the base  $E_g$ , or be sent in the lazy base  $E_\ell$ , and the situation is symmetric for the elements of the lazy tower.
3. The measures  $\mu_g$  and  $\mu_\ell$  have their respective support in  $\mathcal{G}$  and  $\mathcal{L}$ , and  $\mu_g(E_g) = \mu_\ell(E_\ell) = 1$ . Let  $A$  be a measurable subset of  $E_{g, \omega_{-n}, \dots, \omega_{-1}}$  of the form  $A := [u]_0^{|u|-1} \times \{(g, \omega_{-n}, \dots, \omega_{-1})\} \times [a, b]$ , where  $u$  is a finite sequence of elements of  $\{g, \ell\}$ , and  $|u|$  denotes the length of this sequence. Let us prove that  $A$  and  $\mathcal{K}^{-1}(A)$  have the same measure. We have:

$$\begin{aligned} \mu_g(\mathcal{K}^{-1}(A)) &= \frac{1}{\beta^{n-1}} m_p([\omega_{-n}, \dots, \omega_{-2}]_0^{n-2}) m_p([\omega_{-1} \cdot u]_0^{|u|}) \frac{1}{\beta} (b-a) \\ &= \frac{1}{\beta^n} m_p([\omega_{-n}, \dots, \omega_{-1}]_0^{n-1}) m_p([u]_0^{|u|-1}) (b-a). \end{aligned}$$

We then deduce that for any measurable set  $A$  of  $\mathcal{G} \setminus E_g$ , we have  $\mu_g(\mathcal{K}^{-1}(A)) = \mu_g(A)$ . Similarly, for any measurable set  $A$  of  $\mathcal{L} \setminus E_\ell$ , we have  $\mu_\ell(\mathcal{K}^{-1}(A)) = \mu_\ell(A)$ .

4. We set:

$$\begin{aligned} g_0 &:= \mu_g(\mathcal{K}^{-1}(E_g) \cap \mathcal{G}) = \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{\substack{v \in \{g, \ell\}^n \\ T_v(1^+) \geq \frac{1}{\beta}}} m_p([v \cdot g]_0^n), \\ \ell_0 &:= \mu_\ell(\mathcal{K}^{-1}(E_g) \cap \mathcal{L}) = \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{\substack{w \in \{g, \ell\}^n \\ T_w(s(1)^-) \leq \frac{1}{\beta}}} m_p([w \cdot g]_0^n), \\ g_1 &:= \mu_g(\mathcal{K}^{-1}(E_\ell) \cap \mathcal{G}) = \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{\substack{v \in \{g, \ell\}^n \\ T_v(1^+) \geq \frac{1}{\beta(\beta-1)}}} m_p([v \cdot \ell]_0^n), \\ \ell_1 &:= \mu_\ell(\mathcal{K}^{-1}(E_\ell) \cap \mathcal{L}) = \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{\substack{w \in \{g, \ell\}^n \\ T_w(s(1)^-) \leq \frac{1}{\beta(\beta-1)}}} m_p([w \cdot \ell]_0^n). \end{aligned}$$

Let  $A \subset E_g$  of the form  $A = [u]_0^{|u|-1} \times \{g\} \times ]a; b[$ , with  $0 \leq a \leq b \leq 1$ . Then:

$$\mathcal{K}^{-1}(A) \cap \mathcal{G} = \bigsqcup_{n \geq 0} \bigsqcup_{\substack{v \in \{g, \ell\}^n \\ T_v(1^+) \geq \frac{1}{\beta}}} E_{g \cdot v} \cap \left\{ x \in \left] \frac{a}{\beta}; \frac{b}{\beta} \right[ ; \omega \in [g \cdot u]_0^{|u|} \right\}$$

Therefore

$$\begin{aligned}\mu_g(\mathcal{K}^{-1}(A) \cap \mathcal{G}) &= \sum_{n \geq 0} \sum_{\substack{v \in \{g, \ell\}^n \\ T_v(1^+) \geq \frac{1}{\beta}}} \frac{1}{\beta^n} \times m_p([v]_0^{n-1}) \times m_p([g \cdot u]_0^{|u|}) \times \frac{1}{\beta}(b-a) \\ &= g_0 \mu_g(A).\end{aligned}$$

We obtain the three other formulas with a similar computation.

Therefore, by setting  $C_g := \frac{\ell_0}{\ell_0 \mu_g(\mathcal{G}) + g_1 \mu_\ell(\mathcal{L})}$  and  $C_\ell := \frac{g_1}{\ell_0 \mu_g(\mathcal{G}) + g_1 \mu_\ell(\mathcal{L})}$ , the measure  $\mu := C_g \mu_g + C_\ell \mu_\ell$  is a  $\mathcal{K}$ -invariant probability measure on  $X$ .

On each floor of the two towers, the measure  $\mu$  can be represented as the product of  $m_p$  on  $\Omega$  and a multiple of  $\lambda$ . Thus, we get :

**Proposition 2.** *By projecting the measure  $\mu$  on  $\Omega \times I_\beta$ , we obtain the measure  $m_p \otimes \mu_p$ , where  $\mu_p$  has for density*

$$\begin{aligned}\rho_p(x) &:= \sum_{n=0}^{+\infty} \frac{1}{\beta^n} \left( C_g \sum_{\omega_1, \dots, \omega_n \in \{g, \ell\}^n} m_p([\omega_1, \dots, \omega_n]_0^{n-1}) \mathbb{1}_{[0, T_{\omega_1, \dots, \omega_n}(1^+)]}(x) \right. \\ &\quad \left. + C_\ell \sum_{\omega_1, \dots, \omega_n \in \{g, \ell\}^n} m_p([\omega_1, \dots, \omega_n]_0^{n-1}) \mathbb{1}_{[T_{\omega_1, \dots, \omega_n}(s(1^-)); \frac{1}{\beta-1}]}(x) \right). \quad (8)\end{aligned}$$

As a consequence, the natural projection  $\pi : X \rightarrow \Omega \times I_\beta$  is a factor map from the system  $(X, \mu, \mathcal{K})$  to  $(\Omega \times I_\beta, m_p \otimes \mu_p, K_\beta)$ .

## 1.4 Properties of the extension

The aim of this section is to prove that the system  $(X, \mu, \mathcal{K})$  is ergodic. For that, we firstly study the properties of  $\mathcal{K}$  when following paths on the towers. We then study the induced transformation of  $\mathcal{K}$  on the greedy base  $E_g$ , and prove that the induced system is Bernoulli. From this, we derive the ergodicity of the system  $(X, \mu, \mathcal{K})$ , which implies the ergodicity of the initial random system and the uniqueness of  $\mu_p$  (as an absolutely continuous probability measure on  $I_\beta$  such that  $m_p \otimes \mu_p$  is  $K_\beta$ -invariant).

To study the behaviour of the dynamics along paths in the towers, we first define a class of functions that will be useful in the following:

**Definition 3.** *Let  $n \in \mathbb{N}$ . We define the class of functions  $\mathfrak{F}_n$  as the set of functions  $f$  satisfying, up to a set of zero measure:*

- *There exists a finite number of disjoint intervals  $I_1, \dots, I_r$  and a sub-interval  $J$  of  $I_\beta$  such that  $f : I_1 \sqcup \dots \sqcup I_r \rightarrow J$ .*
- *The function  $f$  is nondecreasing on its definition domain.*
- *On each interval  $I_i$ ,  $f$  is a linear map with slope  $\beta^n$ .*
- *$f(\bigsqcup_{i=1}^r I_i) = \bigsqcup_{i=1}^r f(I_i) = J$ .*



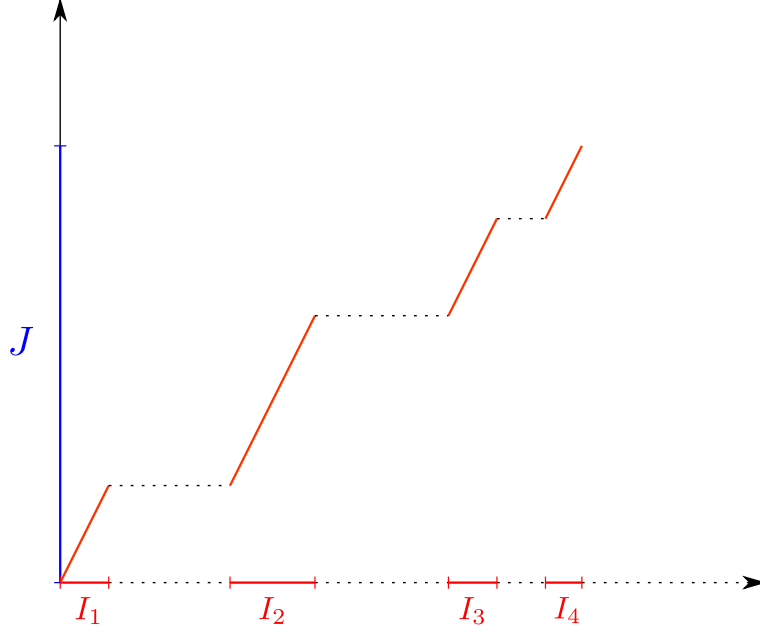


Figure 9: Function from the class  $\mathfrak{F}_n$

In particular, the intervals  $(I_i)$  are in the same order as the intervals  $(f(I_i))$ , and we have

$$\lambda(I_1 \sqcup \dots \sqcup I_r) = \frac{1}{\beta^n} \lambda(J). \quad (9)$$

**Lemma 4.** *Let  $n_1, n_2 \in \mathbb{N}$  and  $f \in \mathfrak{F}_{n_1}$ ,  $g \in \mathfrak{F}_{n_2}$  such that the definition domain  $D_f$  of  $f$  is included in the image of  $g$ . Then  $f \circ g|_{g^{-1}(D_f)} \in \mathfrak{F}_{n_1+n_2}$ .*

*Proof.* We note  $g : I_1 \sqcup \dots \sqcup I_{r_1} \rightarrow J$  and  $f : J_1 \sqcup \dots \sqcup J_{r_2} \rightarrow L$  and  $h = f \circ g|_{g^{-1}(D_f)}$ . Then  $g^{-1}(D_f)$  is the disjoint union of the intervals  $g^{-1}(J_i) \cap I_j$  (when this set is not empty). Therefore,  $h$  is defined on  $\bigsqcup_{i,j} g^{-1}(J_i) \cap I_j$  with values in  $L$ . As the composition of nondecreasing functions,  $h$  is nondecreasing on its definition domain. Moreover,  $h$  is the composition of two linear maps of slope  $\beta^{n_1}$  and  $\beta^{n_2}$  on each non-empty  $g^{-1}(J_i) \cap I_j$ , so it is a linear map of slope  $\beta^{n_1+n_2}$ . Finally, for all  $i \in \{1, \dots, r_2\}$ , we have

$$\bigsqcup_j g(g^{-1}(J_i) \cap I_j) = J_i$$

thus

$$\bigsqcup_{i,j} h(g^{-1}(J_i) \cap I_j) = L.$$

□

We note  $\mathcal{E} := \{g, \ell\}^* \setminus \{\emptyset\}$  the set of non-empty finite sequences of  $\{g, \ell\}$ , in other words the set of labels. For  $e \in \mathcal{E}$ , we note  $e(0)$  the first term of  $e$  and  $e(-1)$  its last term.

We define the oriented graph  $G$  on  $\mathcal{E}$  describing the set of admissible sequences of labels. A label  $e \in \mathcal{E}$  can be followed by the label  $e' \in \mathcal{E}$  (we then note  $e \rightarrow e'$ ) if  $\mu(\mathcal{K}^{-1}(E_{e'}) \cap E_e) > 0$ . In other words, we have  $e \rightarrow e'$  if the map  $\mathcal{K}$  sends a subset of positive measure of the floor  $E_e$  onto the floor  $E_{e'}$ .

We use the new class of functions  $\mathfrak{F}_n$  to describe the dynamics along paths in the two towers:

**Proposition 5.** *Let  $C = (e_0 = e, e_1, \dots, e_n = e')$  ( $n \geq 1$ ) be a finite sequence of labels of  $\mathcal{E}$  corresponding to a path in the graph  $G$ . We have*

$$E_C := \bigcap_{k=0}^n \mathcal{K}^{-k} E_k = [e_1(-1), \dots, e_n(-1)]_0^{n-1} \times \{e\} \times J_C$$

with:

1.  $J_C$  is a finite union of disjoint intervals  $I_1, \dots, I_r$ ,
2.  $T_{e_1(-1), \dots, e_n(-1)}|_{J_C} \in \mathfrak{F}_n$ .

In particular, the intervals  $(I_i)$  are in the same order as the intervals  $(T_{e_1(-1), \dots, e_n(-1)}(I_i))$ ,  $\lambda(J_C) = \frac{1}{\beta^n} \lambda(I_{e'})$  and the mapping  $\mathcal{K}^n : E_C \rightarrow E_{e'}$  is a bijection.

*Proof.* We prove each point by induction on the length of the path  $C$ . Suppose that  $C = (e, e')$  is an edge in the graph of transitions of  $\mathcal{E}$ . Set  $\omega_0 = e'(-1)$ . Without loss of generality, we can suppose that  $e(0) = g$ . Finally, set  $I_e = [0, t]$ , with  $t \in [0, \frac{1}{\beta-1}[$ .

- If  $\omega_0 = g$ , then  $E_C = [g]_0 \times \{e\} \times J_C$  with
  - if  $t < \frac{1}{\beta}$ , then  $J_C = I_e$ ,
  - if  $t \geq \frac{1}{\beta}$ , either  $e' = (g)$ , then  $J_C = [0, \frac{1}{\beta}[$ , or  $e' = e \cdot g$  then  $J_C = [\frac{1}{\beta}, t]$ .
- If  $\omega_0 = \ell$ , then  $E_C = [\ell]_0 \times \{e\} \times J_C$  with
  - if  $t < \frac{1}{\beta(\beta-1)}$ , then  $J_C = I_e$ ,
  - if  $t \geq \frac{1}{\beta(\beta-1)}$ , either  $e' = (\ell)$  then  $J_C = ]\frac{s(1)}{\beta}; \frac{1}{\beta(\beta-1}]$ , or  $e' = e \cdot \ell$  then  $J_C = [0, \frac{s(1)}{\beta}] \sqcup ]\frac{1}{\beta(\beta-1)}, t]$ .

In each case,  $J_C$  is a finite union of disjoint intervals. Moreover,  $T_{\omega_0} : J_C \rightarrow I_{e'}$  is a function of  $\mathfrak{F}_1$ .

Suppose now that  $C = (e = e_0, e_1, \dots, e_{n+1} = e')$ . Set  $C' = (e_1, \dots, e_{n+1})$ . The path  $C'$  is a path of length  $n$  in the graph of transitions of  $\mathcal{E}$ , such that  $E_{C'}$  satisfies the conclusions of the proposition, *i.e.* :

- $E_{C'} = [e_2(-1), \dots, e_{n+1}(-1)]_0^{n-1} \times \{e_1\} \times J_{C'}$  where  $J_{C'}$  is a finite union of disjoint intervals.
- $T_{e_2(-1), \dots, e_{n+1}(-1)}|_{J_{C'}} \in \mathfrak{F}_n$ .

We then have  $E_C = E_e \cap \mathcal{K}^{-1}(E_{C'})$ .

The component in  $\Omega$  is indeed  $[e_1(-1), e_2(-1), \dots, e_{n+1}(-1)]_0^n$ .

Set  $g = T_{e_1(-1)|_{J_{e_0, e_1}}}$  and  $f = T_{e_2(-1), \dots, e_{n+1}(-1)|_{J_{C'}}$ . We have  $g \in \mathfrak{F}_1$  and  $f \in \mathfrak{F}_n$ . By setting  $J_C := g^{-1}(J_{C'})$ , we do have that  $J_C$  is a finite union of disjoint intervals and

$$E_C = [e_1(-1), e_2(-1), \dots, e_{n+1}(-1)]_0^n \times \{e\} \times J_C.$$

Furthermore,  $T_{e_1(-1), e_2(-1), \dots, e_{n+1}(-1)|_{J_C}} = f \circ g|_{J_C}$ . Applying Lemma 4, the function  $T_{e_1(-1), e_2(-1), \dots, e_{n+1}(-1)|_{J_C}}$  is in  $\mathfrak{F}_{n+1}$ .  $\square$

We now study the induced transformation of  $\mathcal{K}$  on the base  $E_g$ . We note  $\mathcal{K}_g$  this induced transformation. The measure  $\mu(\cdot|_{E_g})$  is the measure  $\mu_g$ , and is preserved by the induced map  $\mathcal{K}_g$  [1, Prop. 3.6.1 p.58]. In the following, we identify the base  $E_g$  with  $\Omega \times [0, 1]$ .

We consider the set  $\mathcal{C}$  of possible paths for a first return to the base  $E_g$ . In other words,  $\mathcal{C}$  is the set of paths of the form  $C = (e_0 = (g), e_1, \dots, e_n = (g))$  (where  $|C| := n$  is the length of the path  $C$ ), with  $e_k \neq (g)$  for  $1 \leq k \leq n-1$  such that

$$\mu \left( \bigcap_{k=0}^n \mathcal{K}^{-k}(E_{e_k}) \right) > 0.$$

We then define the set  $\mathcal{P}$  of subsets of  $E_g$  by

$$\mathcal{P} := \{E_C, C \in \mathcal{C}\}.$$

**Lemma 6.** *The set  $\mathcal{P}$  is a countable partition of  $E_g$ , up to a set of zero measure.*

*Proof.* If we consider two distinct paths  $C, C' \in \mathcal{C}$ , then it is clear that  $E_C$  and  $E_{C'}$  are disjoint. And since  $\mu_g$ -almost every  $a$  in  $E_g$  has a finite return time, there exists  $C \in \mathcal{C}$  such that  $a \in E_C$ .  $\square$

For almost every  $a \in E_g$ , we call  $\mathcal{P}$ -name of  $a$  the unique sequence  $(P_n)_{n \geq 0}$  of atoms of  $\mathcal{P}$  such that for any  $n \geq 0$ ,  $K_g^n(a) \in P_n$ .

**Proposition 7.** *The sequence of partitions  $(\mathcal{K}_g^{-n}(\mathcal{P}))_{n \geq 0}$  is independent for the measure  $\mu_g$ , i.e. for any integer  $n \in \mathbb{N}$ , the partitions  $\mathcal{K}_g^{-n}\mathcal{P}$  and  $\bigvee_{0 \leq k \leq n-1} \mathcal{K}_g^{-k}\mathcal{P}$  are independent.*

*Proof.* Consider a path

$$C = (e_0 = (g), e_1, \dots, e_N = (g))$$

starting and ending with the label of the greedy base  $E_g$ . Then this path is a unique concatenation of paths  $C_0, \dots, C_k \in \mathcal{C}$ , and

$$E_C = \bigcap_{i=0}^k \mathcal{K}_g^{-i} E_{C_i} = \bigcap_{i=0}^N \mathcal{K}^{-i}(E_{e_i}) = [e_1(-1), \dots, e_N(-1)]_0^{N-1} \times J_C$$

with  $\lambda(J_C) = \frac{1}{\beta^N} \lambda(I_{(g)}) = \frac{1}{\beta^N}$ , from Proposition 5. Then, we have

$$\mu_g(E_C) = m_p([e_1(-1), \dots, e_N(-1)]_0^{N-1}) \times \frac{1}{\beta^N},$$

and we have a similar equality for all  $C_j$ ,  $0 \leq j \leq k$ . Therefore, it is clear that

$$\mu_g(E_C) = \prod_{j=0}^k \mu_g(E_{C_j}).$$

□

From Proposition 5, each atom of  $\mathcal{P}$  is of the form  $[\omega_0, \dots, \omega_{r-1}] \times J_C$ . Likewise, for each  $N \geq 0$ , each atom of  $\bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P}$  is again of this form.

**Definition 8.** Let  $N \in \mathbb{N} \cup \{0\}$  and  $(\omega, x) \in E_g$  such that there exists an atom of  $\bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P}$  containing  $(\omega, x)$ . We note  $C$  the associated path. Then, we define the set  $A_N(\omega, x)$  included in the interval  $[0, 1]$  by

$$A_N(\omega, x) := J_C.$$

**Lemma 9.** For almost every  $(\omega, x) \in E_g$ , for all  $n \in \mathbb{N} \cup \{0\}$  and  $m \in \mathbb{N}$ , there exists  $N \geq n$  and  $t \in [0, 1]$  such that

$$A_{N+m}(\omega, x) = A_N(\omega, x) \cap [0, t].$$

*Proof.* Let  $P_0 = [g]_0 \times [0, \frac{1}{\beta}[$ . The set  $P_0$  is an atom of the partition  $\mathcal{P}$ , of positive measure, corresponding to the set of points returning to the base  $E_g$  in one step. The process that maps a point in  $E_g$  to its  $\mathcal{P}$ -name (defined for almost every  $(\omega, x) \in E_g$ ) is a Bernoulli process. Therefore, for almost every  $(\omega, x) \in E_g$ , the orbit of  $(\omega, x)$  under  $\mathcal{K}_g$  meets infinitely often the atom  $P_0$ . Even better: for any  $m \in \mathbb{N}$ , the orbit of  $(\omega, x)$  meets the atom  $P_0$   $m$  times in a row, infinitely often.

Let  $n \in \mathbb{N} \cup \{0\}$ . For such a point  $(\omega, x)$ , let  $N \geq n$  be an integer such that

$$\mathcal{K}_g^N(\omega, x), \dots, \mathcal{K}_g^{N+m-1}(\omega, x) \in P_0.$$

Observe that the atom of  $\bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P}$  containing  $(\omega, x)$  is of the form

$$[\omega_0, \dots, \omega_{r-1}]_0^{r-1} \times A_N(\omega, x).$$

If we note  $A_N(\omega, x) = I_1 \sqcup \dots \sqcup I_m$ , Proposition 5 implies

$$[0, 1] = T_{\omega_0, \dots, \omega_{r-1}} I_1 \sqcup \dots \sqcup T_{\omega_0, \dots, \omega_{r-1}} I_m,$$

where  $T_{\omega_0, \dots, \omega_{r-1}} I_i$  is an interval for every  $1 \leq i \leq m$ , and the order of the intervals  $(T_{\omega_0, \dots, \omega_{r-1}} I_i)$  is the same as the order of the intervals  $(I_i)$ .

Since  $\mathcal{K}_g^N(\omega, x) \in P_0$ , we have

$$A_{N+1}(\omega, x) = \{y \in A_N(\omega, x), T_{\omega_0, \dots, \omega_{r-1}}(y) \in [0, \frac{1}{\beta}[\}$$

Then, there exists  $t_1 \in [0, 1]$  such that  $T_{\omega_0, \dots, \omega_{r-1}}(A_N(\omega, x) \cap [0, t_1]) = [0, \frac{1}{\beta}[$ . Hence

$$A_{N+1}(\omega, x) = A_N(\omega, x) \cap [0, t_1].$$

Finally, since  $\mathcal{K}_g^N(\omega, x), \dots, \mathcal{K}_g^{N+m-1}(\omega, x) \in P_0$ , we can apply the same reasoning  $m$  times, which implies the existence of  $t \in [0, 1]$  such that

$$A_{N+m}(\omega, x) = A_N(\omega, x) \cap [0, t].$$

This real number  $t$  is such that  $\lambda(A_N(\omega, x) \cap [0, t]) = \frac{1}{\beta^m} \lambda(A_N(\omega, x))$ .

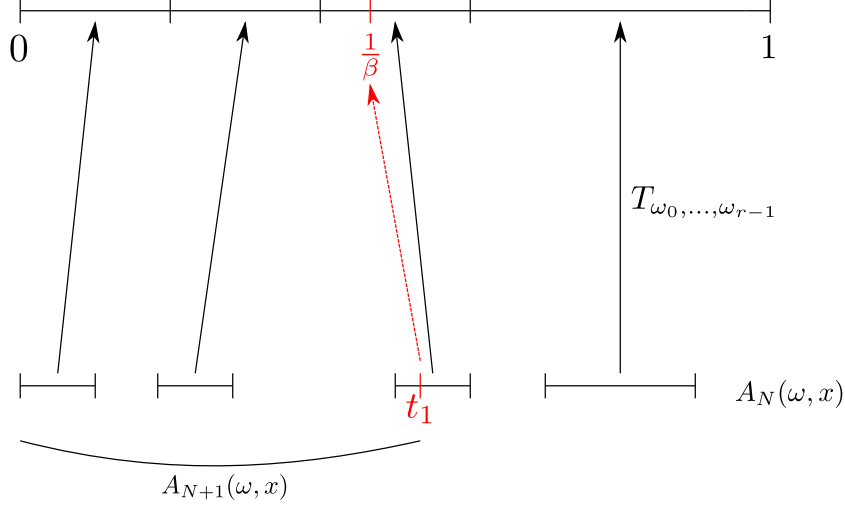


Figure 10: Passage from  $A_N(\omega, x)$  to  $A_{N+1}(\omega, x)$

□

Let  $\mathcal{F}$  be the factor  $\sigma$ -algebra  $\mathcal{F} := \bigvee_{j=0}^{+\infty} \mathcal{K}_g^{-j} \mathcal{P}$ . The partition  $\mathcal{P}$  provides a Bernoulli process, we want to show that this process generates the induced system by proving the following proposition:

**Proposition 10.** *The  $\sigma$ -algebra  $\mathcal{F}$  is (up to zero measure sets) the Borel  $\sigma$ -algebra on  $E_g$ . In other words, the partition  $\mathcal{P}$  is a generator of the induced system, which is isomorphic to a one-sided Bernoulli shift.*

**Lemma 11.**  *$\mathcal{F}$  contains the  $\sigma$ -algebra generated by the  $\omega$ -component.*

*Proof.* Let  $N \in \mathbb{N}$ . As said earlier, every atom of  $\bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P}$  is of the form  $[\omega_0, \dots, \omega_{r-1}]_0^{r-1} \times J_C$ , with  $r \geq N$ . Therefore, for any  $N \in \mathbb{N}$ ,  $(\omega_0, \dots, \omega_{N-1})$  is  $\mathcal{F}$ -measurable.

□

To prove Proposition 10, it suffices to show that for any continuous function  $\varphi : E_g \rightarrow \mathbb{R}_+$ , the conditional expectation  $E[\varphi|\mathcal{F}]$  satisfies

$$E[\varphi|\mathcal{F}] = \varphi, \quad \mu_g \text{ a.s.}$$

The existence of the regular conditional probability implies that for almost every  $(\omega, x) \in E_g$ , there exists a measure  $\mu_{(\omega, x)}$  on  $E_g$  such that for any positive continuous function  $\varphi : E_g \rightarrow \mathbb{R}_+$ ,

$$E[\varphi|\mathcal{F}](\omega, x) = \int \varphi d\mu_{(\omega, x)}.$$

We want to prove that for almost every  $(\omega, x) \in E_g$ , the measure  $\mu_{(\omega, x)}$  equals the Dirac measure on  $(\omega, x)$ , which leads to the announced result. Lemma 11 implies that, for almost every  $(\omega, x) \in E_g$ , the measure  $\mu_{(\omega, x)}$  is of the form  $\delta_\omega \otimes \tilde{\mu}_{(\omega, x)}$ , where  $\tilde{\mu}_{(\omega, x)}$  is a measure on  $[0, 1]$ . In the following, we identify the measures  $\mu_{(\omega, x)}$  and  $\tilde{\mu}_{(\omega, x)}$ . Therefore, it remains to prove that  $\mu_{(\omega, x)}$  is the Dirac measure on  $x$ .

Let  $\varphi : E_g \rightarrow \mathbb{R}_+$  be a continuous function depending only on the real variable  $x$ , i.e.  $\varphi(\omega, x) = f(x)$ , where  $f$  is a continuous function from  $[0, 1]$  into  $\mathbb{R}^+$ .

By the martingale convergence theorem, we have the almost sure convergence:

$$E \left[ \varphi \middle| \bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P} \right] \rightarrow E[\varphi | \mathcal{F}].$$

Thus, for almost every  $(\omega, x) \in E_g$ , we have the convergence

$$E \left[ \varphi \middle| \bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P} \right] (\omega, x) \rightarrow \int f(y) d\mu_{(\omega, x)}(y).$$

With  $N$  fixed, and for almost every  $(\omega, x) \in E_g$ , we have:

$$E \left[ \varphi \middle| \bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P} \right] (\omega, x) = \frac{1}{\lambda(A_N(\omega, x))} \int_{A_N(\omega, x)} f(y) dy.$$

Set  $\mu_{A_N(\omega, x)} := \frac{1}{\lambda(A_N(\omega, x))} \lambda|_{A_N(\omega, x)}$  the normalized Lebesgue measure on  $A_N(\omega, x)$ . For almost every  $(\omega, x) \in E_g$ , the sequence of measures  $(\mu_{A_N(\omega, x)})_N$  converges weakly to the measure  $\mu_{\omega, x}$ .

We define the set  $\bar{E}_g$  of full measure as the set of points  $(\omega, x) \in E_g$  satisfying:

- There exists a probability measure  $\mu_{(\omega, x)}$  on  $E_g$  such that for any continuous function  $\varphi : E_g \rightarrow \mathbb{R}_+$ , we have:

$$E[\varphi | \mathcal{F}](\omega, x) = \int \varphi d\mu_{(\omega, x)}.$$

- For any continuous function  $\varphi : E_g \rightarrow \mathbb{R}_+$ ,  $E \left[ \varphi \middle| \bigvee_{j=0}^n \mathcal{K}_g^{-j} \mathcal{P} \right] (\omega, x)$  tends to  $E[\varphi | \mathcal{F}](\omega, x)$ .
- For any integer  $N$ , the point  $(\omega, x)$  belongs to an atom of the partition  $\bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P}$  and satisfies Lemma 9.
- For any integer  $N$  and for any continuous function  $\varphi : E_g \rightarrow \mathbb{R}_+$  such that there exists a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  such that  $\varphi(\omega, x) = f(x)$  for any  $(\omega, x) \in E_g$ , we have:

$$E \left[ \varphi \middle| \bigvee_{j=0}^N \mathcal{K}_g^{-j} \mathcal{P} \right] (\omega, x) = \int f(y) d\mu_{A_N(\omega, x)}(y).$$

**Lemma 12.** Let  $(\omega, x) \in \bar{E}_g$ . Let  $[a, b] \subset [0, 1]$  such that  $\mu_{(\omega, x)}[a, b] = 1$ . Then one of these two propositions is true:

i)  $\mu_{(\omega, x)}[a, \frac{a+b}{2}] = 1$ ,

ii)  $\mu_{(\omega, x)}[\frac{a+b}{2}, b] = 1$ .

*Proof.* Suppose that ii) does not hold. Set  $\eta := \mu_{(\omega, x)}([a, \frac{a+b}{2}]) > 0$ . Let  $m$  be a large enough integer so that  $\frac{1}{\beta^m} < \frac{\eta}{2}$ . Let  $\varepsilon > 0$ . Let  $f_\varepsilon$  be the continuous function defined on  $[0, 1]$  by  $f_\varepsilon = 0$  on  $[a, \frac{a+b}{2}]$ ,  $f_\varepsilon = 1$  on  $[\frac{a+b}{2} + \varepsilon, b]$  and  $f_\varepsilon$  is a linear map on  $[\frac{a+b}{2}, \frac{a+b}{2} + \varepsilon]$ .

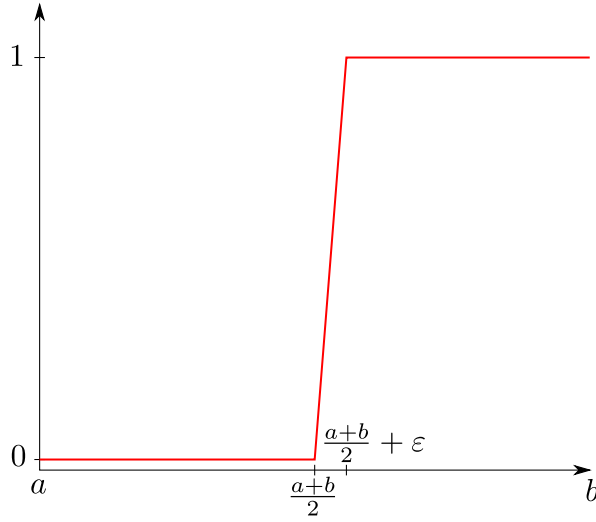


Figure 11: Graph of the function  $f_\varepsilon$

Then

$$\int f_\varepsilon d\mu_{(\omega, x)} \leq 1 - \eta.$$

Therefore, since the sequence  $(\mu_{A_n(\omega, x)})_n$  converges weakly to  $\mu_{(\omega, x)}$ , there exists  $N_0 \in \mathbb{N}$ , such that for any  $n \geq N_0$ ,

$$\int f_\varepsilon d\mu_{A_n(\omega, x)} \leq 1 - \frac{\eta}{2}.$$

Thus, for any  $n \geq N_0$ ,

$$\mu_{A_n(\omega, x)} \left( \left[ \frac{a+b}{2} + \varepsilon, b \right] \right) \leq 1 - \frac{\eta}{2} \leq 1 - \frac{1}{\beta^m}.$$

It then implies that

$$\mu_{A_n(\omega, x)} \left( \left[ a, \frac{a+b}{2} + \varepsilon \right] \right) > \frac{1}{\beta^m}.$$

Let  $n \geq N_0$  such that  $K_g^n(\omega, x), \dots, K_g^{n+m-1}(\omega, x) \in P_0$ . Then from Lemma 9, there exists a real number  $t \in [0, 1]$  such that  $A_{n+m}(\omega, x) = A_n(\omega, x) \cap [0, t]$ . Moreover,  $\lambda(A_{n+m}(\omega, x)) = \frac{1}{\beta^m} \lambda(A_n(\omega, x))$ . It implies that  $A_{n+m}(\omega, x) \subset [a, \frac{a+b}{2} + \varepsilon]$ , hence, for any  $n' \geq n + m$ ,

$$\mu_{A_{n'}(\omega, x)} \left( \left[ \frac{a+b}{2} + \varepsilon, b \right] \right) = 0.$$

Furthermore, we have

$$\int f_{2\varepsilon} d\mu_{A_n(\omega, x)} \rightarrow \int f_{2\varepsilon} d\mu_{(\omega, x)}.$$

We then get that for any  $\varepsilon > 0$ ,  $\mu_{(\omega, x)}([ \frac{a+b}{2} + 2\varepsilon, b]) = 0$ , hence  $\mu_{(\omega, x)}([ \frac{a+b}{2}, b]) = 0$ . □

By dichotomy, we deduce that for any  $(\omega, x) \in E_g$ , the measure  $\mu_{(\omega, x)}$  is the Dirac measure on  $x$ , which proves Proposition 10. In particular, the induced transformation  $\mathcal{K}_g$  is isomorphic to a one-sided Bernoulli shift.

**Lemma 13.** *There exists a floor of the greedy tower with length greater than or equal to  $\frac{1}{\beta(\beta-1)}$ . In particular, it is always possible to go from the tower  $\mathcal{G}$  to the tower  $\mathcal{L}$  under  $\mathcal{K}$ .*

*Proof.* We set:

$$n_0 = \min\{n \in \mathbb{N} \cup \{0\}, T_\ell^n(1) \geq \frac{1}{\beta(\beta-1)}.\}$$

Such integer  $n_0$  exists for any  $\beta > 1$ : if  $1 < \frac{1}{\beta(\beta-1)}$ , then  $T_\ell(1) = \beta \times 1$ . Iterating  $T_\ell$  from 1 is just a multiplication by  $\beta$ , as long as the images are less than  $\frac{1}{\beta(\beta-1)}$ . Thus, the length of the floor  $E_{g\ell^{n_0}}$  is greater than or equal to  $\frac{1}{\beta(\beta-1)}$ . Then we have

$$\mathcal{K}(E_{g\ell^{n_0}} \cap \{\omega_0 = \ell, x \in ]\frac{s(1)}{\beta}, \frac{1}{\beta(\beta-1)}]) = E_\ell.$$

□

**Theorem 14.** *The system  $(X, \mu, \mathcal{K})$  is ergodic.*

*Proof.* The induced transformation of  $\mathcal{K}$  on  $E_g$  is Bernoulli, hence ergodic. Therefore, it suffices to prove that, up to a zero measure set,

$$\bigcup_{n \geq 1} \mathcal{K}^{-n}(E_g) = X.$$

In other words, we want to prove that almost every point in  $X$  reaches the base  $E_g$  in a finite number of iterations of  $\mathcal{K}$ .

We first prove that almost every point in  $X$  can be reached from the base  $E_g$ . By construction of  $\mathcal{K}$ , every floor of the greedy tower is the image of a part



of  $E_g$  by a power of  $\mathcal{K}$ . More precisely, for any  $e \in \mathcal{E}$  such that  $e(0) = g$ , we have:

$$E_e = \mathcal{K}^{|e|-1} \left( \bigcap_{k=0}^{|e|-1} \mathcal{K}^{-k}(E_{e(0), \dots, e(k)}) \right).$$

Similarly, every floor of the lazy tower is the image by a power of  $\mathcal{K}$  of a part of the base  $E_\ell$ . Finally, from Lemma 13, we can pass from the greedy tower to the lazy tower.

Additionally, almost every point in the base  $E_g$  returns to  $E_g$  in a finite number of iterations of  $\mathcal{K}$  (from Poincaré recurrence theorem). Note  $\mathcal{N}_g$  the set of points of  $E_g$  which do not return to  $E_g$ . Then  $\mathcal{N}_g$  is a zero measure set. Therefore, the set  $\bigcup_{n \geq 0} \mathcal{K}^n(\mathcal{N}_g)$  is measurable, and of measure zero. Indeed, let  $n \in \mathbb{N} \cup \{0\}$ . Then

$$\mathcal{K}^n(\mathcal{N}_g) = \bigcup_{|C|=n} \mathcal{K}^n(\mathcal{N}_g \cap E_C).$$

For any path  $C$  of length  $n$ , the transformation  $\mathcal{K}^n$  is invertible on  $\mathcal{N}_g \cap E_C$  from Proposition 5. Therefore,  $\mu(\mathcal{K}^n(\mathcal{N}_g \cap E_C)) = 0$  and  $\mu(\mathcal{K}^n(\mathcal{N}_g)) = 0$ .

Let  $a \in X \setminus \bigcup_{n \geq 0} \mathcal{K}^n(\mathcal{N}_g)$ . There exist  $a_0 \in E_g$  and  $n \in \mathbb{N} \cup \{0\}$  such that  $\mathcal{K}^n(a_0) = a$ . Moreover,  $a_0 \notin \bigcup_{n \geq 0} \mathcal{K}^n(\mathcal{N}_g)$  so there exists a path  $C_0^a \in \mathcal{C}$  such that  $a_0 \in P_{C_0^a}$ . It implies that  $a$  will return to the greedy base in a finite number of iterations of  $\mathcal{K}$  (following the path  $C_0^a$ ).  $\square$

Since the system  $(\Omega \times I_\beta, m_p \otimes \mu_p, K_\beta)$  is a factor of the system  $(X, \mu, \mathcal{K})$ , we have:

**Corollary 15.** *The system  $(\Omega \times I_\beta, m_p \otimes \mu_p, K_\beta)$  is ergodic.*

**Corollary 16.** *The measure  $\mu_p$  is the unique absolutely continuous probability measure on  $I_\beta$  such that  $m_p \otimes \mu_p$  is  $K_\beta$ -invariant.*

*Proof.* From Lemma 13, for any  $1 < \beta < 2$ , there exists a floor of the greedy tower whose length is greater than or equal to  $\frac{1}{\beta(\beta-1)}$ . Symmetrically, there exists a floor of the lazy tower whose left limit is less than or equal to  $\frac{1}{\beta}$ . It implies that the support of  $\mu_p$  is the full interval  $I_\beta$ , and that  $m_p \otimes \mu_p$  is equivalent to  $m_p \otimes \lambda$ .

Let  $\nu$  be an absolutely continuous probability measure (with respect to  $\lambda$ , and so with respect to  $\mu_p$ ) such that  $m_p \otimes \nu$  is  $K_\beta$ -invariant. Then the measure  $m_p \otimes \nu$  is absolutely continuous with respect to the ergodic measure  $m_p \otimes \mu_p$ , hence  $m_p \otimes \nu = m_p \otimes \mu_p$ , and  $\nu = \mu_p$ .  $\square$

In [16], Suzuki proves that the density of  $\mu_p$  is proportional to the function

$$B^0(\beta, p) \left( \mathbf{1}_{[0,1]} + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{v \in \{g, \ell\}^n} m_p([v]_1^n) \mathbf{1}_{[0, T_v(1)]} \right) \\ + B^0(\beta, 1-p) \left( \mathbf{1}_{[s(1), \frac{1}{\beta-1}]} + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{w \in \{g, \ell\}^n} m_p([w]_1^n) \mathbf{1}_{[T_w(s(1)), \frac{1}{\beta-1}]} \right) \quad (10)$$

where

$$B^0(\beta, p) = p \left( 1 - \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{w \in \{g, \ell\}^{n-1}} \mathbb{1}_S(T_w(s(1))) m_p([w, \ell]_1^n) \right).$$

By uniqueness of  $\mu_p$ , we know that the density (8) and the expression obtained by Suzuki (10) are equal. This result can be easily proved in the case where  $\beta$  is not a root of a polynomial of the form  $X^{n_0} - X^{n_1} - \dots - X^{n_k} - 1$  with  $n_0 > n_1 > \dots > n_k$  (which is equivalent to say that 1 has no finite expansion in base  $\beta$ ).

Indeed, we prove that, in this case,  $\ell_0 = B^0(\beta, p)$ : on one hand, we recall that

$$\ell_0 = \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{w \in \{g, \ell\}^n, T_w(s(1)^-) \leq \frac{1}{\beta}} m_p([w \cdot g]_0^n).$$

Since  $\beta$  is not a root of a polynomial of the form  $X^{n_0} - X^{n_1} - \dots - X^{n_k} - 1$ , then  $T_w(s(1)^-) = T_w(s(1))$  for any  $w \in \{g, \ell\}^n$ . Moreover,  $T_w(s(1)^-) \leq \frac{1}{\beta}$  is equivalent to  $T_w(s(1)) < \frac{1}{\beta}$ . Therefore, after adjusting the index of summation, we have

$$\ell_0 = \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{w \in \{g, \ell\}^n} \mathbb{1}_R(T_w(1)) m_p([w \cdot g]_0^n).$$

On the other hand, after a new reindexing in  $\ell_1$ , we have:

$$\begin{aligned} \ell_0 = 1 - \ell_1 &= 1 - \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{w \in \{g, \ell\}^n} \mathbb{1}_{S \cup R}(T_w(1)) m_p([w \cdot \ell]_0^n) \\ &= 1 - \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{w \in \{g, \ell\}^n} \mathbb{1}_S(T_w(1)) m_p([w \cdot \ell]_0^n) - \frac{1-p}{p} \ell_0. \end{aligned}$$

Thus

$$\ell_0 = p \left( 1 - \sum_{n=0}^{+\infty} \frac{1}{\beta^{n+1}} \sum_{w \in \{g, \ell\}^n} \mathbb{1}_S(T_w(1)) m_p([w \cdot \ell]_0^n) \right) = B^0(\beta, p).$$

By symmetry, we also obtain that  $g_1 = B^0(\beta, 1-p)$ , and the equality of the two densities follows.

In the case where 1 has a finite expansion in base  $\beta$ , there exists  $n \in \mathbb{N}$  and a sequence  $w \in \{g, \ell\}^n$  such that  $T_w(1) = 0$ , and the right limit  $T_v(1^+)$  and the value  $T_v(1)$  can differ, thus complicating the identification of the two formulas. For the simple case where  $\beta = \frac{1+\sqrt{5}}{2}$ , we refer to the computations in the appendix of [17].

## 2 The natural extension

### 2.1 Construction of the new extension

Any point in  $X$  which is not in a base has a unique preimage by  $\mathcal{K}$ . However, a point in one of the two bases can come from several floors. Indeed, at each return

to a base, the map  $\mathcal{K}$  “forgets” where it comes from. This lack of information prevents  $\mathcal{K}$  from being invertible.

We wish to construct a natural extension of  $\mathcal{K}$ , denoted by  $\tilde{\mathcal{K}}$ . In some sense, it is the smallest invertible extension of  $\mathcal{K}$ . Therefore, to construct  $\tilde{\mathcal{K}}$ , we need to extend the set  $X$  such that the information of the past floors is available. We recall the definition of a natural extension of a system (see for example [2]).

**Definition 17.** *The system  $(Y, \mathcal{C}, \nu, F)$  is a natural extension of the system  $(X, \mathcal{B}, \mu, T)$  if there exist two sets  $X^* \in \mathcal{B}$  and  $Y^* \in \mathcal{C}$  such that  $\mu(X^*) = \nu(Y^*) = 1$  and a measurable mapping  $\pi : Y^* \rightarrow X^*$  such that:*

1.  $F$  is a bijection of  $Y^*$ ,
2.  $\mu = \nu \circ \pi^{-1}$ ,
3.  $\pi \circ F = T \circ \pi$  and
4.  $\mathcal{C} = \bigvee_{n \geq 0} F^n(\pi^{-1}(\mathcal{B}))$ .

Moreover, all natural extensions of a system are isomorphic [13]. We now construct the extension  $\tilde{\mathcal{K}}$  of  $\mathcal{K}$ , and then prove that it is one of its natural extensions. In the initial definition of the towers  $\mathcal{G}$  and  $\mathcal{L}$ , each floor has a label  $e$ . More precisely, a label  $e$  is a finite sequence of  $g$  and  $\ell$ , where the first term of  $e$  describes the present tower and the following terms of  $e$  (when there are) describe which transformations have been applied since the base: the label describes the recent past of a point in the floor. Therefore, instead of “erasing” the label at each return to a base, we will keep track of every past label, so that we can uniquely determine the past orbit of almost any point in the two towers.

We note  $Z$  the set of left-infinite sequences generated by the graph  $G$ , *i.e.*

$$Z := \{(e_j)_{j \in \mathbb{Z}_{\leq 0}} \in \mathcal{E}^{\mathbb{Z}_{\leq 0}} : \forall j \leq -1, e_j \rightarrow e_{j+1}\}.$$

A sequence of labels in  $Z$  contains in particular the information of the sequence of past transformations. Given a sequence  $\mathbf{e}$  in  $Z$ , we will note  $e_0$  its term of index 0.

For  $e \in \mathcal{E}$ , we set

$$Z_e := \{\mathbf{e} \in Z : e_0 = e\}.$$

We then define the base of the two towers by

$$\tilde{E}_g := \Omega \times Z_{(g)} \times [0, 1]$$

and

$$\tilde{E}_\ell := \Omega \times Z_{(\ell)} \times \left[ s(1), \frac{1}{\beta - 1} \right].$$

If  $e = (g, \omega_{-n}, \dots, \omega_{-1})$ ,  $n \in \mathbb{N}$ , we set

$$\tilde{E}_e := \Omega \times Z_e \times [0, T_{\omega_{-n}, \dots, \omega_{-1}}(1^+)].$$

If  $e = (\ell, \omega_{-n}, \dots, \omega_{-1})$ ,  $n \in \mathbb{N}$ , we set

$$\tilde{E}_e := \Omega \times Z_e \times \left[ T_{\omega_{-n}, \dots, \omega_{-1}}(s(1)^-), \frac{1}{\beta - 1} \right].$$

We note

$$\tilde{\mathcal{G}} := \bigsqcup_{e \in \mathcal{E}, e(0)=g} \tilde{E}_e$$

and

$$\tilde{\mathcal{L}} := \bigsqcup_{e \in \mathcal{E}, e(0)=\ell} \tilde{E}_e.$$

We finally note  $\tilde{X} := \tilde{\mathcal{G}} \sqcup \tilde{\mathcal{L}}$ . We set:

$$\pi_{\tilde{X}, X} : \begin{array}{ccc} \tilde{X} & \rightarrow & X \\ (\omega, \mathbf{e}, x) & \mapsto & (\omega, e_0, x) \end{array}$$

the projection from  $\tilde{X}$  onto  $X$ .

Let  $a = (\omega, \mathbf{e}, x) \in \tilde{X}$ . Then  $\pi_{\tilde{X}, X}(a) = (\omega, e_0, x)$ . We have  $\mathcal{K}(\omega, e_0, x) = (\sigma(\omega), e', T_{\omega_0}(x))$ , where the label  $e'$  depends on  $e_0, \omega_0$  and  $x$ , according to the previous construction. By construction of  $\mathcal{K}$ , we obviously have  $e_0 \rightarrow e'$  in the graph  $G$ .

Applying the new dynamics  $\tilde{\mathcal{K}}$  on  $a$  consists in concatenating the new label  $e'$  to the sequence  $\mathbf{e}$  (we note  $\mathbf{e} \cdot e'$  this new sequence), in addition to shifting the sequence  $\omega$  and applying  $T_{\omega_0}$  to  $x$ : we retain every past labels in memory.

In other words, the dynamics  $\tilde{\mathcal{K}}$  is defined on  $\tilde{X}$  by

$$\tilde{\mathcal{K}} : \begin{array}{ccc} \tilde{X} & \rightarrow & \tilde{X} \\ (\omega, \mathbf{e}, x) & \mapsto & (\sigma(\omega), \mathbf{e} \cdot e', T_{\omega_0}(x)) \end{array},$$

where  $e'$  is the label associated with the point  $\mathcal{K}(\omega, e_0, x) \in X$ .

The projection  $\pi_{\tilde{X}, X}$  is measurable and we have by construction

$$\mathcal{K} \circ \pi_{\tilde{X}, X} = \pi_{\tilde{X}, X} \circ \tilde{\mathcal{K}}.$$

Note  $\mathcal{B}$  the Borel  $\sigma$ -algebra on  $X$  and  $\tilde{\mathcal{B}}$  the Borel  $\sigma$ -algebra on  $\tilde{X}$ . Let us prove that

$$\tilde{\mathcal{B}} = \bigvee_{n \geq 0} \tilde{\mathcal{K}}^n(\pi_{\tilde{X}, X}^{-1}(\mathcal{B})). \quad (11)$$

The inclusion  $\bigvee_{n \geq 0} \tilde{\mathcal{K}}^n(\pi_{\tilde{X}, X}^{-1}(\mathcal{B})) \subset \tilde{\mathcal{B}}$  is clear.

Let  $k$  and  $n$  be two non-negative integers. We consider a set  $A$  of  $\tilde{\mathcal{B}}$  of the form  $A = [\omega_0, \omega_1, \dots, \omega_n]_0^n \times [e_{-k}, \dots, e_0]_{-k}^0 \times [a, b]$ . We then have:

$$A = (\Omega \times [e_{-k}, \dots, e_{-1}]_{-k}^{-1} \times [a, b]) \cap ([\omega_0, \omega_1, \dots, \omega_n]_0^n \times [e_0]_0 \times [a, b]).$$

Set  $A_- = \Omega \times [e_{-k}, \dots, e_{-1}]_{-k}^{-1} \times [a, b]$  and  $A_+ = [\omega_0, \omega_1, \dots, \omega_n]_0^n \times [e_0]_0 \times [a, b]$ .

Then  $A_- \in \tilde{\mathcal{K}}^k(\pi_{\tilde{X}, X}^{-1}(\mathcal{B}))$  and  $A_+ \in \pi_{\tilde{X}, X}^{-1}(\mathcal{B})$  which implies that  $A \in \bigvee_{n \geq 0} \tilde{\mathcal{K}}^n(\pi_{\tilde{X}, X}^{-1}(\mathcal{B}))$ , and proves (11).

We can now define the measure  $\tilde{\mu}$  on  $\tilde{X}$ . For  $A \in \bigvee_{n=0}^N \tilde{\mathcal{K}}^n(\pi_{\tilde{X}, X}^{-1}(\mathcal{B}))$ , we set

$$\tilde{\mu}(A) := \mu(\pi_{\tilde{X}, X}(\tilde{\mathcal{K}}^{-N}(A))).$$

In some sense, the measure  $\tilde{\mu}$  of this set  $A$  is obtained by “shifting” the set  $A$  to the future, so that it can be viewed as a set in the first extension  $X$ . No information is lost by projecting it on  $X$ , allowing us to use the measure  $\mu$ . The measure  $\tilde{\mu}$  is well defined: if  $A \in \bigvee_{n=0}^N \tilde{\mathcal{K}}^n(\pi_{\tilde{X},X}^{-1}(\mathcal{B}))$ , then for any integer  $k \in \mathbb{N}$ , we have

$$\mu(\pi_{\tilde{X},X}(\tilde{\mathcal{K}}^{-N}(A))) = \mu(\pi_{\tilde{X},X}(\tilde{\mathcal{K}}^{-N-k}(A)))$$

since  $\mu$  is  $\mathcal{K}$ -invariant. For any  $B \in \mathcal{B}$ , we have  $\tilde{\mu} \circ \pi_{\tilde{X},X}^{-1}(B) = \mu(B)$ , and the measure  $\tilde{\mu}$  is  $\tilde{\mathcal{K}}$ -invariant.

Finally, by construction,  $\tilde{\mathcal{K}}$  is one to one on  $\tilde{X}$ . It implies that  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  is a natural extension of  $(X, \mu, \mathcal{K})$ . Moreover, the natural extension of an ergodic system is ergodic, which provides the following result, as a direct consequence of Theorem 14:

**Corollary 18.** *The system  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  is ergodic.*

## 2.2 Natural extension of the initial system

The goal of this section is to prove that the previous extension is in fact a natural extension of the initial system  $(\Omega \times I_\beta, m_p \otimes \mu_p, K_\beta)$ . To do so, we first introduce a canonic way to construct a natural extension of the initial system, then prove that the two extensions are isomorphic.

We adapt the construction described in [1, p. 62], which provides a natural extension of  $K_\beta$ . We set

$$\underline{X} := \{(\underline{\omega}, \underline{x}) \in \{g, \ell\}^{\mathbb{Z}} \times I_\beta^{\mathbb{Z}} : \forall k \in \mathbb{Z}, T_{\omega_k} x_k = x_{k+1}\}.$$

Let  $\mathcal{B}$  be the  $\sigma$ -algebra generated by the cylinders of  $\underline{X}$ . We define the measure  $\underline{\mu}$  on every set of the form  $A = [\omega_k, \dots, \omega_{n+k}]_k^{n+k} \times [I_k, \dots, I_{n+k}]_k^{n+k}$ , with  $k \in \mathbb{Z}, n \in \mathbb{N}$ , and for any  $k \leq i \leq n+k$ ,  $\omega_i \in \{g, \ell\}$  and  $I_i$  sub-interval of  $I_\beta$ , by

$$\underline{\mu}(A) = m_p \otimes \mu_p \left( [\omega_k, \dots, \omega_{n+k}]_0^n \times \bigcap_{i=0}^n T_{\omega_{k+i}}^{-i}(I_{k+i}) \right).$$

We note  $\underline{K}$  the shift on  $\underline{X}$ .

**Proposition 19.** [1] *The dynamical system  $(\underline{X}, \underline{\mu}, \underline{K})$  is a natural extension of the system  $(\Omega \times I_\beta, m_p \otimes \mu_p, K_\beta)$ .*

Let  $(\omega, \mathbf{e}, x) \in \tilde{X}$ . We note  $\underline{\omega} = \underline{\omega}(\omega, \mathbf{e})$  the sequence of  $\{g, \ell\}^{\mathbb{Z}}$  defined by  $\underline{\omega}(k) = \omega_k$  if  $k \geq 0$  and  $\underline{\omega}(k) = \mathbf{e}_{k+1}(-1)$  if  $k \leq -1$ . The sequence  $\underline{\omega}$  is the bi-infinite sequence of applied transformations in the past, and transformations to be applied in the future. For any  $k \in \mathbb{Z}$ , we note  $x_k$  the real component of  $\tilde{\mathcal{K}}^k(\omega, \mathbf{e}, x)$ , and  $\underline{x} := (x_k)_{k \in \mathbb{Z}}$ . We define the application

$$\phi : \begin{array}{ccc} \tilde{X} & \rightarrow & \underline{X} \\ (\omega, \mathbf{e}, x) & \rightarrow & (\underline{\omega}, \underline{x}) \end{array}.$$

One easily checks that:

**Lemma 20.** *The application  $\phi$  is a factor map from  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  to  $(\underline{X}, \underline{\mu}, \underline{K})$ .*

**Theorem 21.** *The factor map  $\phi$  is an isomorphism. Therefore, the system  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  is a natural extension of  $(\Omega \times I_\beta, m_p \otimes \mu_p, K_\beta)$ .*

To prove this theorem, we use the notion of relatively independent joinings above a factor (see for example [8, pp. 126-127]).

Let  $\mathbb{P}$  be the relatively independent self-joining of the system  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  above its factor  $\underline{X}$  i.e. the measure on  $\tilde{X} \times \tilde{X}$  such that for two measurable sets  $A, B \in \tilde{\mathcal{K}}$ ,

$$\mathbb{P}(A \times B) = \int_{\tilde{X}} \tilde{\mu}(A|\mathcal{F})\tilde{\mu}(B|\mathcal{F})d\tilde{\mu},$$

where  $\mathcal{F} = \phi^{-1}(\mathcal{B}(\underline{X}))$  is the factor  $\sigma$ -algebra associated to the factor  $(\underline{X}, \underline{\mu}, \underline{K})$ . We have the following properties:

- $\mathbb{P}$  is a self-joining so its marginals on the first and second coordinate are  $\tilde{\mu}$ , and  $\mathbb{P}$  is  $\tilde{\mathcal{K}} \times \tilde{\mathcal{K}}$ -invariant;
- the measure  $\mathbb{P}$  is supported on the set  $\{(a, b) \in \tilde{X} \times \tilde{X} : \phi(a) = \phi(b)\}$ ;
- $\mathbb{P}$  is supported on the diagonal of  $\tilde{X} \times \tilde{X}$  if and only if the factor map  $\phi$  is an isomorphism.

Let  $a = (\omega, \mathbf{e}, x)$  and  $b = (\omega', \mathbf{e}', x')$  be two elements of  $\tilde{X}$  such that  $\phi(a) = \phi(b)$ . We then have  $\underline{\omega}(\omega, \mathbf{e}) = \underline{\omega}(\omega', \mathbf{e}')$  and  $\underline{x} = \underline{x}'$ . In other words,  $a$  and  $b$  describe two trajectories in the towers, starting from the same real number  $x$  and with the same transformations  $\underline{\omega}$ , but possibly following different floors  $\mathbf{e}$  and  $\mathbf{e}'$ . Proving that  $\mathbb{P}$  is supported by the diagonal of  $\tilde{X} \times \tilde{X}$  consists in proving that for  $\mathbb{P}$ -almost every  $(a, b) \in \tilde{X} \times \tilde{X}$ , the two sequences of floors  $\mathbf{e}$  and  $\mathbf{e}'$  are actually equal.

The idea of the proof consists in taking two trajectories in  $\tilde{X}$  starting from the same real number  $x$ , with the same transformations, and proving that they end up in the same floor. Once these trajectories come at the same time in the same floor, they coincide in the future, and in the past by injectivity.

We firstly work on the extension  $X$ . Let  $k \in \mathbb{N}$  and consider the set of positive measure

$$C_k := \left\{ (\omega, e, x) \in X : x < \frac{1}{\beta^k}, \omega_0 = \omega_1 = \dots = \omega_{k-1} = g \right\}.$$

The set  $C_k$  describes the points of  $X$  whose real component is close to 0, and on which will be applied  $k$  times in a row the greedy transformation. The label  $e$  does not take part in the definition of  $C_k$ , which means that the event  $C_k$  is measurable with respect to the factor  $\underline{X}$ : given two trajectories with the same real component and the same sequence of transformations, the event  $C_k$  happens almost surely (by ergodicity of  $\tilde{\mathcal{K}}$  and because  $C_k$  is of positive measure) and at the same time for both trajectories. Knowing that  $C_k$  is realized actually gives some information about the label. We will prove that for large  $k$  and conditionally to  $C_k$ , both trajectories are in a floor of the greedy tower whose interval is large, with high probability.

Let  $A_k$  be the union of the floors of  $\mathcal{G}$  whose length is less than  $\frac{1}{\beta^k}$ . Let  $B_k$  be the complementary set of  $A_k$  in  $\mathcal{G}$ . Then the set  $C_k$  can be divided into three parts:  $C_k \cap \mathcal{L}$ ,  $C_k \cap A_k$  and  $C_k \cap B_k$ . The set  $C_k \cap B_k$  is particularly interesting since a point in this set will end up in the greedy base  $E_g$  in at most  $k$  iterations:

such a point will remain in the left part of its floor under the action of  $\mathcal{K}$ , and the length of the floor will eventually be larger than  $\frac{1}{\beta}$ , allowing the point to go to  $E_g$  in less than  $k$  iterations (see Figures 3 and 4). In the following, we prove that for large  $k$  and knowing  $C_k$ , the event  $C_k \cap B_k$  happens with high probability.

We first introduce this technical lemma:

**Lemma 22.** *The intervals of the lazy tower  $\mathcal{L}$  never contain 0.*

*Proof.* This property holds by construction of the extension  $\mathcal{K}$ . Indeed, if an interval of a floor of  $\mathcal{L}$  contained 0, it would have to be the image by  $T_g$  of an interval whose lower bound is  $\frac{1}{\beta}$ . However, if the lower bound of an interval of  $\mathcal{L}$  equals  $\frac{1}{\beta}$ , we are in the case where the part of the floor satisfying  $x \in [\frac{1}{\beta}, \frac{2}{\beta}[$  is sent to the greedy base when applying  $T_g$ .  $\square$

For any label  $e \in \mathcal{E}$ , we set  $Y_e := \Omega \times \{e\}$ , and we have  $E_e = Y_e \times I_e$ . We then set  $Y := \bigsqcup_{e \in \mathcal{E}} Y_e$ , which is represented vertically in the figure 2. We note  $\nu$  the measure on  $Y$  such that, for any  $e \in \mathcal{E}$ ,

$$\mu|_{E_e} = \nu|_{Y_e} \otimes \lambda|_{I_e}.$$

Finally, we note  $\pi_{X,Y}$  the projection from  $X$  onto  $Y$ .

Let  $\varepsilon > 0$  be small enough, so that  $1 - \frac{1+2(\beta-1)}{(\beta-1)\mu(\mathcal{G})}\varepsilon > 0.5$ . Let  $N \in \mathbb{N}$  be such that the measure  $\mu$  of the floors beyond the level  $N$  in  $\mathcal{L}$  is less than  $\varepsilon$ . By the previous lemma, there exists  $k_0 \in \mathbb{N}$  such that no floor of  $\mathcal{L}$  up to the level  $N$  has an interval whose lower bound is less than  $\frac{1}{\beta^k}$ .

As previously said, we have:

$$\mu(C_k) = \mu(C_k \cap \mathcal{L}) + \mu(C_k \cap A_k) + \mu(C_k \cap B_k). \quad (12)$$

We want to estimate each term of the right hand side of equation (12). We will prove that for large  $k$ ,  $C_k \cap \mathcal{L}$  and  $C_k \cap A_k$  have a small measure, and that  $C_k \cap B_k$  represents more than half of the measure of  $C_k$ .

**Lemma 23.** *For  $k$  large enough, we have*

$$\mu(C_k \cap \mathcal{L}) \leq 2p^k \frac{\beta - 1}{\beta^k} \varepsilon.$$

*Proof.* Let  $k \geq k_0$ . A floor of  $\mathcal{L}$  can intersect  $C_k$  only if the lower bound of its interval is less than  $\frac{1}{\beta^k}$ . The integer  $N$  is fixed such that every floor below the level  $N$  does not contain  $\frac{1}{\beta^k}$ . Therefore, only the floors beyond the level  $N$  can intersect  $C_k$ . Additionally, for any  $e \in \mathcal{E}$ , we have, by definition of the measure  $\mu$  on each floor:

$$\mu \left( \left\{ (\omega, e, x) \in E_e : \omega_0 = \dots = \omega_{k-1} = g, x < \frac{1}{\beta^k} \right\} \right) = p^k \mu \left( E_e \cap \left\{ x < \frac{1}{\beta^k} \right\} \right).$$

We then have:

$$\mu(C_k \cap \mathcal{L}) = p^k \sum_{e \in \mathcal{E}, e(0)=\ell, |e| \geq N+1} \nu(Y_e) \lambda \left( I_e \cap \left[ 0; \frac{1}{\beta^k} \right] \right).$$

For a floor  $E_e$  of the lazy tower such that  $I_e \cap [0; \frac{1}{\beta^k}] \neq \emptyset$ , we have

$$\lambda(I_e) > \frac{1}{\beta-1} - \frac{1}{\beta^k}.$$

For  $k$  large enough, we then have

$$\lambda(I_e) > \frac{1}{2(\beta-1)}.$$

It implies that

$$\lambda\left(I_e \cap \left[0; \frac{1}{\beta^k}\right]\right) \leq \frac{1}{\beta^k} \leq 2\frac{\beta-1}{\beta^k}\lambda(I_e)$$

hence

$$\mu(C_k \cap \mathcal{L}) \leq 2p^k \frac{\beta-1}{\beta^k} \varepsilon.$$

□

**Lemma 24.** *For  $k$  large enough, we have*

$$\mu(C_k \cap A_k) \leq \frac{p^k}{\beta^k} \varepsilon.$$

*Proof.* Let  $k \in \mathbb{N}$ .

$$\mu(C_k \cap A_k) = p^k \sum_{e \in \mathcal{E}, E_e \subset A_k} \nu(Y_e) \lambda\left(I_e \cap \left[0; \frac{1}{\beta^k}\right]\right).$$

If  $E_e$  is a floor of  $A_k$ , then  $\lambda(I_e \cap [0; \frac{1}{\beta^k}]) \leq \frac{1}{\beta^k}$ . Therefore,

$$\mu(C_k \cap A_k) \leq \frac{p^k}{\beta^k} \nu(\pi_{X,Y}(A_k)).$$

We have

$$\bigcap_{k \geq 1} A_k = \emptyset,$$

hence

$$\pi_{X,Y}\left(\bigcap_{k \geq 1} A_k\right) = \emptyset.$$

Since the intersection of the  $A_k$  is decreasing, we have

$$\bigcap_{k \geq 1} \pi_{X,Y}(A_k) = \emptyset.$$

It implies that

$$\lim_{k \rightarrow \infty} \nu(\pi_{X,Y}(A_k)) = 0.$$

We can choose  $k$  large enough, so that  $\nu(\pi_{X,Y}(A_k)) < \varepsilon$ . Therefore,

$$\mu(C_k \cap A_k) \leq \frac{p^k}{\beta^k} \varepsilon.$$

□



**Proposition 25.** *For  $k$  large enough, we have*

$$\frac{\mu(C_k \cap B_k)}{\mu(C_k)} > 0.5.$$

*Proof.* Let  $k \in \mathbb{N}$  large enough such that Lemmas 23 and 24 are both satisfied. We have:

$$\frac{\mu(C_k \cap B_k)}{\mu(C_k)} = 1 - \frac{\mu(C_k \cap A_k) + \mu(C_k \cap \mathcal{L})}{\mu(C_k)}.$$

From Lemmas 23 and 24, we then have

$$\frac{\mu(C_k \cap B_k)}{\mu(C_k)} \geq 1 - \frac{\frac{p^k}{\beta^k} \varepsilon + 2p^k \frac{\beta-1}{\beta^k} \varepsilon}{\mu(C_k)} \geq 1 - \frac{\frac{p^k}{\beta^k} \varepsilon + 2p^k \frac{\beta-1}{\beta^k} \varepsilon}{\mu(C_k \cap \mathcal{G})}.$$

On the other hand,

$$\mu(C_k \cap \mathcal{G}) = p^k \sum_{e \in \mathcal{E}, e(0)=g} \nu(Y_e) \lambda \left( I_e \cap \left[ 0; \frac{1}{\beta^k} \right] \right).$$

For any floor  $E_e$  of  $\mathcal{G}$ , we have the minoration  $\lambda(I_e \cap [0, \frac{1}{\beta^k}]) \geq \lambda(I_e) \frac{\beta-1}{\beta^k}$  hence

$$\frac{\mu(C_k \cap B_k)}{\mu(C_k)} \geq 1 - \frac{\frac{p^k}{\beta^k} \varepsilon + 2p^k \frac{\beta-1}{\beta^k} \varepsilon}{p^k \frac{\beta-1}{\beta^k} \mu(\mathcal{G})} = 1 - \frac{1 + 2(\beta-1)}{(\beta-1)\mu(\mathcal{G})} \varepsilon.$$

The real  $\varepsilon$  is such that  $1 - \frac{1+2(\beta-1)}{(\beta-1)\mu(\mathcal{G})} \varepsilon > 0.5$ , which gives the result.  $\square$

All these computations are also valid on the extension  $\tilde{X}$ . Indeed,  $\pi_{\tilde{X}, X}^{-1}(C_k) \in \pi_{\tilde{X}, X}^{-1}(\mathcal{B})$ , which implies  $\tilde{\mu}(\pi_{\tilde{X}, X}^{-1}(C_k)) = \mu(C_k)$ . In the following we will still note  $C_k$  instead of  $\pi_{\tilde{X}, X}^{-1}(C_k)$ .

Since  $\tilde{\mathcal{K}}$  is ergodic and  $\mu(C_k) > 0$ , almost every trajectory in  $\tilde{X}$  encounters  $C_k$  with frequency  $\mu(C_k)$ . Therefore, for  $\mathbb{P}$ -almost every couple of trajectories in  $\tilde{X}$ , both trajectories simultaneously encounter  $C_k$  (since  $C_k$  is measurable with respect to the factor  $\underline{X}$ ) with frequency  $\mu(C_k)$  (since  $\mathbb{P}$  is a self-joining of  $\mu$ ). In other words, we have, for  $\mathbb{P}$ -almost every  $(a, b) \in \tilde{X} \times \tilde{X}$ , for any  $n \in \mathbb{N} \cup \{0\}$ ,

- $\tilde{\mathcal{K}}^n(a) \in C_k$  if and only if  $\tilde{\mathcal{K}}^n(b) \in C_k$ ,
- $\frac{1}{N} \sum_{i=1}^N \mathbb{1}_{C_k}(\tilde{\mathcal{K}}^i(a)) = \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{C_k}(\tilde{\mathcal{K}}^i(b)) \rightarrow \mu(C_k)$ .

The ergodicity of  $\tilde{\mathcal{K}}$  and Proposition 25 also imply that among the occurrences of  $C_k$ , the event  $C_k \cap B_k$  occurs with a frequency greater than 0.5 for the orbits of  $\mathbb{P}$ -almost every  $(a, b) \in \tilde{X} \times \tilde{X}$  under  $\tilde{\mathcal{K}}$ . Therefore, for  $\mathbb{P}$ -almost every  $(a, b) \in \tilde{X} \times \tilde{X}$ , there exists  $n_0 \in \mathbb{N} \cup \{0\}$  such that  $\tilde{\mathcal{K}}^{n_0}(a) \in C_k \cap B_k$  and  $\tilde{\mathcal{K}}^{n_0}(b) \in C_k \cap B_k$ . Once the two trajectories are in  $C_k \cap B_k$  simultaneously, they will end up in the greedy base in at most  $k$  iterations (note that even if one of the two points goes in the base before the other, it will “wait” for the second one to join it. Since its real component is small enough, the point is in the atom  $F_0$ ). In conclusion, for  $\mathbb{P}$ -almost every  $(a, b) \in \tilde{X} \times \tilde{X}$ , if we note

$a = (\omega, \mathbf{e}, x)$  and  $b = (\omega, \mathbf{e}', x)$ , there exists an integer  $M = M(\underline{\omega}, \underline{x}, \mathbf{e}, \mathbf{e}')$  such that  $e_M = e'_M = (g)$ .

Let  $\delta > 0$ . There exists an integer  $M_\delta$  such that

$$\mathbb{P}(M > M_\delta) < \delta.$$

Therefore, we have  $\mathbb{P}(e_{M_\delta} = e'_{M_\delta}) \geq 1 - \delta$ . Since  $\mathbb{P}$  is  $\tilde{\mathcal{K}} \times \tilde{\mathcal{K}}$ -invariant, we deduce that for any integer  $k \in \mathbb{Z}$  and any  $\delta > 0$ ,

$$\mathbb{P}(e_k = e'_k) \geq 1 - \delta$$

hence  $\mathbf{e} = \mathbf{e}'$ . We proved that  $\mathbb{P}$  is supported by the diagonal of  $\tilde{X} \times \tilde{X}$ , which implies that  $\phi$  is an isomorphism. Therefore,  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  is a natural extension of the initial system, which concludes the proof of Theorem 21.

### 2.3 Bernoullicity

We know that the natural extension of the initial random system is ergodic. In fact, we have the following stronger result:

**Theorem 26.** *The natural extension of the random  $\beta$ -transformation is isomorphic to a Bernoulli shift.*

To prove this theorem, we introduce a generating partition of the two towers. We then prove that this partition is weak-Bernoulli, by an argument on countable state space Markov chains of Ito, Totoki and Murata [9], which implies the theorem. In the following, we note  $\tilde{\mu}_g := \tilde{\mu}(\cdot | \tilde{E}_g)$ , and  $\tilde{\mathcal{K}}_g$  the induced transformation of  $\tilde{\mathcal{K}}$  on the base  $\tilde{E}_g$ .

**Lemma 27.** *Let  $(X, \mu, T)$  be an ergodic dynamical system, and  $A$  a measurable set of  $X$  with positive measure. Note  $(\tilde{X}, \tilde{\mu}, \tilde{T})$  a natural extension of the system, and  $\pi$  a factor map from  $\tilde{X}$  onto  $X$ . Then the system induced by  $\tilde{T}$  on  $\pi^{-1}(A)$  is a natural extension of the system induced by  $T$  on  $A$ .*

*Proof.* We start by introducing the necessary notations. The natural extension of the system  $(X, \mu, T)$  can be described by a two-sided shift  $\tilde{T}$  on the set

$$\tilde{X} := \{\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in X^{\mathbb{Z}}, \forall n \in \mathbb{Z}, T(x_n) = x_{n+1}\},$$

with the measure  $\tilde{\mu}$  defined by

$$\tilde{\mu}([A_k, A_{k+1}, \dots, A_{n+k}]_k^{n+k}) = \mu \left( \bigcap_{i=0}^n T^{-i}(A_{k+i}) \right),$$

where  $A_k, \dots, A_{n+k}$  are measurable sets of  $X$  and  $[A_k, A_{k+1}, \dots, A_{n+k}]_k^{n+k} := \{\mathbf{x} \in \tilde{X} : \text{for } k \leq i \leq n+k, x_i \in A_i\}$ .

Note  $\tilde{A} := \pi^{-1}(A) = \{(x_n) \in \tilde{X}, x_0 \in A\}$  and  $(\tilde{A}, \tilde{\mu}_{\tilde{A}}, \tilde{T}_{\tilde{A}})$  the system induced by  $\tilde{T}$  on  $\tilde{A}$ . The transformation  $\tilde{T}_{\tilde{A}}$  is defined by  $\tilde{T}_{\tilde{A}}(\mathbf{x}) = \tilde{T}^{r_A(x_0)}(\mathbf{x})$  for every  $\mathbf{x} \in \tilde{A}$ , where  $r_A(x_0)$  is the return time of  $x_0$  in  $A$ , which is the same as the return time of  $\mathbf{x}$  in  $\tilde{A}$ .

We denote by  $(A, \mu(\cdot|A), T_A)$  the system induced by  $T$  on  $A$ . The natural extension of this induced system can also be described by a two-sided shift  $\underline{T}$

on the set  $\underline{A} = \{\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in A^{\mathbb{Z}}, \forall n \in \mathbb{Z}, T_A(x_n) = x_{n+1}\}$ , with the measure  $\underline{\mu}$  defined in the same way as  $\tilde{\mu}$ .

We want to prove that the two systems  $(\tilde{A}, \tilde{\mu}_{\tilde{A}}, \tilde{T}_{\tilde{A}})$  and  $(\underline{A}, \underline{\mu}, \underline{T})$  are isomorphic. We define the map  $\varphi$  as:

$$\varphi : \begin{array}{l} \tilde{A} \rightarrow \underline{A} \\ \mathbf{x} \mapsto ((\tilde{T}_{\tilde{A}}^n(\mathbf{x}))_0)_{n \in \mathbb{Z}} \end{array} .$$

The map  $\varphi$  consists in keeping only the terms of the sequence  $x$  that belong to  $A$ .

We clearly have that  $\varphi \circ \tilde{T}_{\tilde{A}} = \underline{T} \circ \varphi$  and that  $\underline{\mu}$  is the pushforward measure  $\tilde{\mu}_{\tilde{A}} \circ \varphi^{-1}$ : the map  $\varphi$  is a factor map from  $(\tilde{A}, \tilde{\mu}_{\tilde{A}}, \tilde{T}_{\tilde{A}})$  to  $(\underline{A}, \underline{\mu}, \underline{T})$ .

We now prove that  $\varphi$  is an isomorphism. Let  $\mathbf{y} = (y_n)_{n \in \mathbb{Z}} \in \underline{A}$ . Let  $n \in \mathbb{Z}$ . From  $y_n$ , we can construct the finite sequence of the iterates of  $y_n$

$$s(y_n) := (y_n, T(y_n), \dots, T^{r_A(y_n)-1}(y_n)),$$

stopping just before  $y_{n+1} = T^{r_A(y_n)}(y_n)$ . Then the sequence  $\mathbf{x}$  obtained by concatenating the sequences  $s(y_n)$  for all  $n \in \mathbb{Z}$  is by construction a sequence of  $\tilde{A}$  such that  $\varphi(\mathbf{x}) = \mathbf{y}$ .

Let us prove that  $\varphi$  is one-to-one. Let  $\mathbf{w}, \mathbf{x} \in \tilde{A}$  such that  $\varphi(\mathbf{w}) = \varphi(\mathbf{x})$ . Then  $w_0 = x_0$ , which implies that for any  $n \in \mathbb{N}$ ,  $T^n(w_0) = T^n(x_0)$ , *i.e.*  $x_n = w_n$ . Let  $j_0 := \max\{j < 0, \tilde{T}^j(\mathbf{w}) \in \tilde{A}\}$ , and  $j_1 := \max\{j < 0, \tilde{T}^j(\mathbf{x}) \in \tilde{A}\}$ . The existence of the integers  $j_0$  and  $j_1$  is assured by Poincaré's recurrence theorem applied to  $\tilde{T}^{-1}$ . Note that  $\tilde{T}^{j_0}(\mathbf{w}) = \tilde{T}_{\tilde{A}}^{-1}(\mathbf{w})$ , and  $\tilde{T}^{j_1}(\mathbf{x}) = \tilde{T}_{\tilde{A}}^{-1}(\mathbf{x})$ . Since  $\varphi(\mathbf{w}) = \varphi(\mathbf{x})$ , we get that  $\tilde{T}^{j_0}(\mathbf{w}) = \tilde{T}^{j_1}(\mathbf{x})$ , *i.e.*  $w_{j_0} = x_{j_1}$ . The integer  $-j_0$  is the return time of  $w_{j_0}$  in  $A$ , which implies that  $j_0 = j_1$ , and  $w_n = x_n$  for  $n \geq j_0$ . By induction, we prove that  $\mathbf{w} = \mathbf{x}$ , and  $\varphi$  is an isomorphism, which concludes the proof.  $\square$

**Corollary 28.** *The system  $(\tilde{E}_g, \tilde{\mu}_g, \tilde{\mathcal{K}}_g)$  is a natural extension of the system  $(E_g, \mu_g, \mathcal{K}_g)$ .*

**Proposition 29.** *The partition  $\tilde{\mathcal{P}} := \pi^{-1}(\mathcal{P})$  is a generator of the system*

$$(\tilde{E}_g, \tilde{\mu}_g, \tilde{\mathcal{K}}_g).$$

*Proof.*  $\mu_g$ -almost every point  $a \in E_g$  is associated to a unique sequence  $(C_n^a)_{n \geq 0}$  of  $\mathcal{C}^{\mathbb{N} \cup \{0\}}$  such that for any  $n \in \mathbb{N} \cup \{0\}$ ,  $\mathcal{K}_g^n(a) \in E_{C_n^a}$ : the sequence  $(C_n^a)_{n \geq 0}$  can be viewed as the  $\mathcal{P}$ -name of the point  $a$ . Note  $\phi_{\mathcal{P}}$  the application defined almost everywhere on  $E_g$  by  $\phi_{\mathcal{P}}(a) = (C_n^a)_{n \geq 0}$ . Note  $\nu$  the image measure of  $\mu_g$  by  $\phi_{\mathcal{P}}$  and  $\sigma$  the left shift on  $\mathcal{C}^{\mathbb{N} \cup \{0\}}$ . The application  $\phi_{\mathcal{P}}$  is a factor map from the system  $(E_g, \mu_g, \mathcal{K}_g)$  to the one-sided Bernoulli shift  $(\mathcal{C}^{\mathbb{N} \cup \{0\}}, \nu, \sigma)$ . From Proposition 10, the partition  $\mathcal{P}$  is a generator of the system  $(E_g, \mu_g, \mathcal{K}_g)$ . Therefore, the application  $\phi_{\mathcal{P}}$  is an isomorphism.

We can construct a natural extension of the system  $(\mathcal{C}^{\mathbb{N} \cup \{0\}}, \nu, \sigma)$  the same way we did for  $\underline{K}$ . Doing so, we construct the system  $(\mathcal{C}^{\mathbb{Z}}, \tilde{\nu}, \sigma)$ , which is a two-sided Bernoulli shift on  $\mathcal{C}^{\mathbb{Z}}$ . Since the natural extensions of a system are isomorphic to one another, we deduce from the previous lemma that the systems  $(\mathcal{C}^{\mathbb{Z}}, \tilde{\nu}, \sigma)$  and  $(\tilde{E}_g, \tilde{\mu}_g, \tilde{\mathcal{K}}_g)$  are isomorphic, thus concluding the proof.  $\square$

From this partition  $\tilde{\mathcal{P}}$ , we construct the set

$$\mathcal{P}_{\tilde{X}} := \{\tilde{\mathcal{K}}^k(\tilde{E}_C), C \in \mathcal{C}, 0 \leq k \leq |C| - 1\}.$$

This family is obtained by “unfolding” the partition  $\tilde{\mathcal{P}}$  on the two towers: each path  $C$  of  $\mathcal{C}$  is not only described by the first atom  $\tilde{E}_C$ , but by the sequence of successive images of this atom by  $\tilde{\mathcal{K}}$ .

**Lemma 30.** *The set  $\mathcal{P}_{\tilde{X}}$  is a partition of  $\tilde{X}$ .*

*Proof.* Two distinct sets of  $\mathcal{P}_{\tilde{X}}$  are clearly disjoint.

On the other hand, let  $(\omega, \mathbf{e}, x) \in \tilde{X}$ . Almost surely, There exists  $k \in \mathbb{N} \cup \{0\}$  such that  $\mathbf{e}_{-k} = (g)$  (applying Poincaré recurrence theorem to  $\tilde{\mathcal{K}}^{-1}$ ). Let  $k_0 := \min\{k \in \mathbb{N} \cup \{0\}, \mathbf{e}_{-k} = (g)\}$ . The integer  $-k_0$  corresponds to the last visit of the greedy base in the past orbit of  $(\omega, \mathbf{e}, x)$ . Almost surely, there exists a unique  $C \in \mathcal{C}$  such that  $\tilde{\mathcal{K}}^{-k_0}(\omega, \mathbf{e}, x) \in \tilde{E}_C$ . Since  $k_0$  is minimal, we have  $0 \leq k_0 \leq |C| - 1$ , and  $(\omega, \mathbf{e}, x) \in \tilde{\mathcal{K}}^{k_0}(\tilde{E}_C)$ .  $\square$

**Proposition 31.** *The partition  $\mathcal{P}_{\tilde{X}}$  is a generator of the system  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$ .*

*Proof.* The goal is to prove that almost every point in  $\tilde{X}$  is uniquely determined by its  $\mathcal{P}_{\tilde{X}}$ -name.

Let  $(\omega, \mathbf{e}, x) \in \tilde{X}$ . Almost surely, we can define the integer  $k_0$  as in the previous proof:

$$k_0 := \min\{k \in \mathbb{N} \cup \{0\}, \mathbf{e}_{-k} = (g)\}.$$

We note  $a := \tilde{\mathcal{K}}^{-k_0}(e, x) \in \tilde{E}_g$ . Almost surely,  $a$  is characterized by its  $\tilde{\mathcal{P}}$ -name  $(C_n)_{n \in \mathbb{Z}}$ . To each path  $C_n$  corresponds a unique sequence of length  $|C_n|$  of atoms of  $\mathcal{P}_{\tilde{X}}$ : the sequence  $(\tilde{\mathcal{K}}^k(\tilde{E}_{C_n}), 0 \leq k \leq |C_n| - 1)$ . Therefore,  $a$  is characterized by the unique sequence  $(P_n)_{n \in \mathbb{Z}}$  of elements of  $\mathcal{P}_{\tilde{X}}$  such that for any  $n \in \mathbb{Z}$ ,

$$\tilde{\mathcal{K}}^n(a) \in P_n.$$

We deduce that the point  $(\omega, \mathbf{e}, x)$  is uniquely determined by the sequence  $(P_{n+k})_{n \in \mathbb{Z}}$ .  $\square$

We then define the isomorphism  $\varphi$ , which associates to almost every point in  $\tilde{X}$  its  $\mathcal{P}_{\tilde{X}}$ -name. We note  $\eta$  the image measure of  $\tilde{\mu}$  by  $\varphi$  and  $\sigma$  the left shift on  $\mathcal{P}_{\tilde{X}}^{\mathbb{Z}}$ . Then  $\varphi$  is a factor map from  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  to  $(\mathcal{P}_{\tilde{X}}^{\mathbb{Z}}, \eta, \sigma)$ .

The measure  $\eta$  is the Markov measure generated by the invariant distribution  $(\tilde{\mu}(P))_{P \in \mathcal{P}_{\tilde{X}}}$  and the following transition probabilities:

- From an atom  $\tilde{\mathcal{K}}^k(\tilde{E}_C)$  with  $C \in \mathcal{C}$  and  $0 \leq k \leq |C| - 2$ , the process goes to the atom  $\tilde{\mathcal{K}}^{k+1}(\tilde{E}_C)$  with probability 1.
- From an atom  $\tilde{\mathcal{K}}^{|C|-1}(\tilde{E}_C)$  with  $C \in \mathcal{C}$ , the process can go to  $\tilde{E}_{C'}$  for any  $C' \in \mathcal{C}$  with probability  $\mu_g(E_{C'})$ .

Therefore, we can identify the natural extension  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  with a Markov shift on  $\mathcal{P}_{\tilde{X}}^{\mathbb{Z}}$ . This Markov shift is irreducible and recurrent, since  $\tilde{\mathcal{K}}$  is ergodic. Moreover, it is aperiodic (the atom  $[g]_0 \times \{(g)\} \times [0, \frac{1}{\beta}[$  can appear twice in a row for example). Proposition 2 p. 579 of [9] implies that the partition  $\mathcal{P}_{\tilde{X}}$  is weak-Bernoulli, and the system  $(\tilde{X}, \tilde{\mu}, \tilde{\mathcal{K}})$  is isomorphic to a Bernoulli shift. This concludes the proof of Theorem 26.

## Open questions

1. Can we generalize the extensions of this paper to any  $\beta > 2$ ? We must take into account the different branches of the greedy and lazy transformations in the construction of the natural extension, especially the branches that induce a return to a base.
2. It seems natural to apply this kind of construction to other dynamical systems, either deterministic or random ones. It seems that it can be generalized to random systems with piecewise linear maps, even when the branches have different slopes. Indeed, with the construction of the towers, we can keep track of the exact branch that is being applied at each step.  
However, systems based on continued fraction expansions should pose way more difficulties. In [10], the authors highlight a bijection between the  $\beta$  parameter of the non-integer base expansions and the  $\lambda$  parameter of  $\lambda$ -continued fractions. The first question would be to identify the analog of the greedy and lazy transformations in this case, and then to closely study this bijection.
3. The description of the extension  $(X, \mu, \mathcal{K})$  provides a better understanding of the random  $\beta$ -transformation. The extension shows off some nice renewal times at each passage to the greedy base, which provided a proof of the Bernoullicity of the natural extension. We can hope to get limit theorems as well thanks to this property. Indeed, in [18], Young describes a certain class of dynamical systems with renewal properties on reference sets. Young shows that the tail distribution of the return time to this reference set is directly linked to convergence to equilibrium and limit theorems. In our setup, an estimation of the length of paths of  $\mathcal{C}$  should help in this direction. Could we then obtain a central limit theorem for the digit sequences in base  $\beta$  following a fixed sequence  $\omega$ ? Or on average on  $\omega$ ?
4. Our construction heavily relies on the independent choice of the transformations at each step. Could we adapt some of these constructions to more general stationary measures on  $\Omega$ ? For example Markov measures?

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