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An extension of specificity-based approximations to other belief function relations

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Abstract. Adopting a general framework to faithfully represent uncertainty, such as belief function theory, usually comes at a cost. In many real-life applications, we are constrained to handle mass functions that have too many focal elements. Fortunately, one can resort to approximation techniques to bypass this issue. In this paper, we extend the classical approximation techniques, which are mainly specificity-based, to other belief function relations such as lattice dominance. This allows to overcome the limits of classical techniques in some applications.

Keywords: Belief function · Approximation · Specificity · Relations.

1 Introduction

Belief function theory [15] is a rich and powerful uncertainty reasoning framework as it extends both the set and probability representations of uncertainty. Despite its successful application in many real-life problems, it has been criticized for its high computational complexity. Several techniques have been proposed to simplify the computations pertaining to this theory, either using exact [12] or approximate methods. We are particularly interested in the latter. Approximations can be computed by Monte-Carlo simulations [20], or by replacing the original mass function by a probability measure or a possibilistic one [19, 7]. Other approaches can be used where mass functions are combined on a coarsened frame of discernment [3] or where the number of focal sets is reduced [18, 1, 13, 8, 14, 2]. We draw a particular attention to this last family of methods. Besides simplicity, a good approximation has to be consistent and close enough to the original mass function [8]. Closeness is typically quantified by a distance measure, whereas consistency is unanimously based on comparing the specificity of the informative content of the original mass function and its approximation. Recently, Destercke and al. [4] introduced an approach that extends any set relation to belief functions. This approach generalizes the notion of comparison and allows, along with comparing the informative content of beliefs in terms of specificity, to establish other relations between them such as dominance. In this paper, we propose to extend this approach to approximation methods that reduce the number of focal sets of mass functions. We are motivated by the deficiency of classical approximation techniques in some applications. This deficiency arises from the use of approximate beliefs that are more or less specific than the original ones whilst the application requires rather to choose beliefs that are, for instance, dominant. We will develop this idea later in the paper.

The remainder of this paper is organized as follows. Section 2 gives a quick reminder on belief functions and set relations. Section 3 describes the notion of comparison in belief function theory. The proposed generalized approximation and a particular case study are presented in section 4. We conclude the paper in Section 5.

2 Basic definitions

In this section, we provide some basic definitions on belief functions and set relations that are required in our developments.

2.1 Theory of belief functions

Let x be an uncertain variable defined on finite set of values $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ called the frame of discernment. The available knowledge about x is represented by a mass function $m^{\mathcal{X}} : 2^{\mathcal{X}} \mapsto [0, 1]$ s.t. $\sum_{A \subseteq \mathcal{X}} m^{\mathcal{X}}(A) = 1$ and $m^{\mathcal{X}}(\emptyset) = 0$. $m^{\mathcal{X}}(A)$ quantifies the part of our belief that $x \in A$ without providing any further information about $x \in A' \subset A$. Each subset $A \subseteq \mathcal{X}$ such that $m^{\mathcal{X}}(A) > 0$ is called focal set or focal element of $m^{\mathcal{X}}$. Other knowledge representations can be obtained from $m^{\mathcal{X}}$, such as the *belief* $Bel^{\mathcal{X}}$ and the *plausibility* $Pl^{\mathcal{X}}$ function, defined for all $A \subseteq \mathcal{X}$:

$$Bel^{\mathcal{X}}(A) = \sum_{\emptyset \neq B \subseteq A} m^{\mathcal{X}}(B), \quad Pl^{\mathcal{X}}(A) = \sum_{B \cap A \neq \emptyset} m^{\mathcal{X}}(B). \quad (1)$$

$Bel^{\mathcal{X}}$ is the amount of evidence that supports $x \in A$ and $Pl^{\mathcal{X}}$ is interpreted as the amount of evidence that is consistent with $x \in A$.

2.2 Set relations

A relation \mathbf{R} between subsets of \mathcal{X} is a subset $\mathbf{R} \subseteq 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ that specifies which pair of subsets are related to each other [4]. Let A and B be two subsets of \mathcal{X} . We denote by $A\mathbf{R}B$ whenever $(A, B) \in \mathbf{R}$. A relation may have several properties such as: *reflexivity* ($A\mathbf{R}A, \forall A \subseteq \mathcal{X}$), *transitivity* ($A\mathbf{R}B$ and $B\mathbf{R}C \Rightarrow A\mathbf{R}C$, with $C \subseteq \mathcal{X}$), *antisymmetry* ($A\mathbf{R}B \wedge B\mathbf{R}A \Rightarrow A = B, \forall A, B \subseteq \mathcal{X}$), etc. Note that it is also possible to define more complex relations by combining those properties. For instance, the set-inclusion relation ($A\mathbf{R}B \Leftrightarrow A \subseteq B$) is reflexive, transitive and antisymmetric [4].

3 Comparing belief structures

According to the *Least Commitment Principle* [16], if we have to choose among multiple mass functions compatible with a set of constraints, the most appropriate one is the least informative. To use this principle, one has to define tools to compare the content of the available mass functions. This is commonly done via the notion of specialization [6]. Given two mass functions $m_1^{\mathcal{X}}$ and $m_2^{\mathcal{X}}$ defined on \mathcal{X} , $m_1^{\mathcal{X}}$ is said to be at least as *informative (specific)* as $m_2^{\mathcal{X}}$, which we denote by $m_1^{\mathcal{X}} \sqsubseteq m_2^{\mathcal{X}}$, if and only if $m_1^{\mathcal{X}}$ can be obtained from $m_2^{\mathcal{X}}$ by sharing each mass $m_2^{\mathcal{X}}(B)$ among subsets $A \subseteq B$.

Formally, there exists a non-negative square matrix, known as the specialization matrix $S = [S(A, B)]$, $A, B \in 2^{\mathcal{X}}$, verifying the conditions below:

$$\sum_{A \subseteq \mathcal{X}} S(A, B) = 1, \quad \forall B \subseteq \mathcal{X}, \quad (2)$$

$$S(A, B) > 0 \Rightarrow A \subseteq B, \quad \forall A, B \subseteq \mathcal{X}, \quad (3)$$

$$m_1^{\mathcal{X}}(A) = \sum_{B \subseteq \mathcal{X}} S(A, B)m_2^{\mathcal{X}}(B), \quad \forall A \subseteq \mathcal{X}. \quad (4)$$

$S(A, B) \in [0, 1]$ is the proportion of $m_2^{\mathcal{X}}(B)$ that flows into $A \subseteq B$. Note that if $m_1^{\mathcal{X}} \sqsubseteq m_2^{\mathcal{X}}$ then [6]:

$$[Bel_1^{\mathcal{X}}, Pl_1^{\mathcal{X}}] \subseteq [Bel_2^{\mathcal{X}}, Pl_2^{\mathcal{X}}]. \quad (5)$$

The recent work of Destercke and al. [4] highlighted the relevance of investigating other links, besides specificity, between mass functions, particularly those extending set relations such as equivalence or partial/total order. The authors introduced a more general definition of the comparison of belief function as follows:

Definition 1. Let $m_1^{\mathcal{X}}$ and $m_2^{\mathcal{X}}$ be two mass functions and let \mathbf{R} be a relation between subsets of \mathcal{X} . We say that $m_1^{\mathcal{X}} \tilde{\mathbf{R}} m_2^{\mathcal{X}}$ if there is a left stochastic matrix S , such that $\forall A, B \subseteq \mathcal{X}$.

$$m_1^{\mathcal{X}}(A) = \sum_{B \subseteq \mathcal{X}} S(A, B)m_2^{\mathcal{X}}(B), \quad (6)$$

$$\left(S(A, B) > 0 \right) \wedge \left(m_2^{\mathcal{X}}(B) > 0 \right) \Rightarrow \mathbf{R}A \mathbf{R}B. \quad (7)$$

$S(A, B)$ is the proportion of $m_2^{\mathcal{X}}(B)$ transferred to A , such that $\mathbf{R}A \mathbf{R}B$ [4].

Note that when $\tilde{\mathbf{R}}$ is replaced by \sqsubseteq and \mathbf{R} by \subseteq in (7), we obtain the specialization relation defined earlier. Furthermore, when \mathcal{X} is ordered, it is also possible to recover another relation that was studied in [9], by comparing two subsets $A, B \subseteq \mathcal{X}$ defined as $A = \{\underline{a}, \dots, \bar{a}\}$ ($\underline{a} \leq \bar{a}$) and $B = \{\underline{b}, \dots, \bar{b}\}$ ($\underline{b} \leq \bar{b}$) in terms of lattice dominance [4]. We say then that $m_1^{\mathcal{X}}$ is *at least as small as* $m_2^{\mathcal{X}}$, which we denote by $m_1^{\mathcal{X}} \preceq m_2^{\mathcal{X}}$, with $\tilde{\mathbf{R}}$ being replaced by \preceq and \mathbf{R} replaced by \leq_d where $A \leq_d B$ if $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$. The following property holds [9]:

$$m_1^{\mathcal{X}} \preceq m_2^{\mathcal{X}} \Rightarrow [Bel_2^{\mathcal{X}}, Pl_2^{\mathcal{X}}] \leq_d [Bel_1^{\mathcal{X}}, Pl_1^{\mathcal{X}}]. \quad (8)$$

4 Generalization of belief functions approximation

Usually, a mass function m is approximated by another mass function m' that is at most as specific as m , i.e., $m \sqsubseteq m'$. Assume that we want to approximate m by reducing the number of its focal sets. m' can be built from m by preserving the most significant focal sets, i.e., those with high mass values, and by aggregating or removing the redundant or the least significant ones as in [13]. It is also possible to reduce the number of focal sets iteratively as in [8, 14, 2]. These latter methods help to trade-off between the quality and the computational time required to determine m' .

In this section, we extend the previously stated techniques to other possible relations $\tilde{\mathbf{R}}$ between m and m' . Our motivation arises from the fact that specificity-based approximations may be inappropriate in some applications, such as in the combinatorial optimization problem that we studied in [17]. Specifically, we proposed in [17] a belief-constrained programming approach inspired from [9] to model the vehicle routing problem with time windows [11] and evidential service and travel times. In this kind of problems, each vehicle is compelled to start the service at any customer within his time availability interval (window). Arrivals after the closure of time windows are therefore forbidden. To fulfill such particular constraints, given the evidential time parameters, confidence levels are imposed on the belief and the plausibility functions of the arrival times which are combination of service and travel times. For instance, if x is the variable representing the arrival time at a given customer, C is the closure of his time window and $\alpha, \beta \in [0, 1]$ ($\alpha \leq \beta$) are two confidence levels, the time constraints for this customer can be expressed as:

$$Bel(x \leq C) \geq \alpha, \quad Pl(x \leq C) \geq \beta. \quad (9)$$

The use of belief functions adds more complexity to the problem that is already NP-hard. The problem involves indeed costly mass function combinations due to large numbers of focal sets. Consequently, we turned to classical approximation methods to overcome this issue. Nevertheless, we noticed that replacing the original service and travel time mass functions by less specific ones impacts inappropriately the set of feasible solutions, i.e., solutions that satisfy all the problem constraints. Indeed, a solution may be feasible when using approximations while it is rejected when using the original mass functions. Take for instance the variable x defined earlier, and suppose that uncertainty about the value of x is represented by the mass function: $m(\{15, 16\}) = 0.9$, $m(\{16, 17\}) = 0.05$, $m(\{16.30, 17.30\}) = 0.05$. Suppose that $C = 16$ and that $\alpha = 0.9, \beta = 1$. Using (1), we have $Bel(x \leq 16) = 0.9 = \alpha$ and $Pl(x \leq 16) = 0.95 < \beta$. The confidence level β is not met, thus the customer can not be served. Suppose now that uncertainty about x is represented using an approximation m' such that $m \sqsubseteq m'$. m' is given by $m'(\{15, 16\}) = 0.9$, $m'(\{16, 16.30, 17, 17.30\}) = 0.1$. We have $Bel'(x \leq 16) = 0.9 = \alpha$ and $Pl'(x \leq 16) = 1 = \beta$. Note that $Bel' = Bel$ and $Pl' > Pl$, this is due to the relation in (5). In this case, both of the confidence levels are verified and the customer in question can be served. Such a result is quite contradictory with the information we had originally. Hence, it is worthwhile to introduce a more general approach so that one can properly approximate a mass function by another one that is more/less specific or smaller/greater or equivalent, etc, to span a broad range of real-life applications.

4.1 Formalization

Building on the formal definition of approximations given in [2], we can introduce a generalized definition of an approximation as follows:

Definition 2. Let $\mathcal{P} = \{P_1, P_2, \dots, P_K\}$ be a partition of the set $\mathbb{N}_n = \{1, \dots, n\}$, i.e., $P_k \cap P_l = \emptyset$ and $\bigcup_{k=1}^K P_k = \mathbb{N}_n$ and let m be a mass function with focal elements

$\mathcal{F}(m) = \{A_1, A_2, \dots, A_n\}$ such that $m(A_i) \geq m(A_{i+1}), \forall i = 1, \dots, n - 1$. Let m' be another mass function with $\mathcal{F}(m') = \{B_1, \dots, B_K\}$ its focal sets verifying for each $k = 1, \dots, K$:

$$A_i \mathbf{R} B_k, \forall i \in P_k, \quad (10)$$

$$m'(B_k) = \sum_{i \in P_k} m(A_i). \quad (11)$$

m' is called $\tilde{\mathbf{R}}$ -approximation of m .

Definition 2 states that for a given relation $\tilde{\mathbf{R}}$, any mass function m' with fewer focal sets and that is related to m by $\tilde{\mathbf{R}}$, i.e., $m \tilde{\mathbf{R}} m'$, is an approximation of m . Note that m and m' verify the conditions of Definition 1 as it is possible, for any $P_k \in \mathcal{P}$ ($k = 1, \dots, K$), to retrieve $m(A_i)$ from $m'(B_k)$ by transferring a proportion $S(A_i, B_k) > 0$ of the mass $m'(B_k) > 0$ from the subset B_k to the subset A_i such that $A_i \mathbf{R} B_k$, with :

$$S(A_i, B_k) = \frac{m(A_i)}{m'(B_k)} = \frac{m(A_i)}{\sum_{j \in P_k} m(A_j)} \quad (12)$$

Particular cases: Definition 2 covers some well known cases that were already studied in the literature. For instance, if \mathbf{R} is an outer-inclusion relation, i.e., $A_i \subseteq B_k$, with $B_k = \bigcup_{i \in P_k} A_i$, then $\tilde{\mathbf{R}} = \sqsubseteq$, that is $m \sqsubseteq m'$, which corresponds to the outer approximations of the literature [2, 14, 8, 13].

We can also identify another sub-case when \mathbf{R} is an inner-inclusion relation, i.e., $A_i \supseteq B_k$, where $B_k = \bigcap_{i \in P_k} A_i$. In this case $\tilde{\mathbf{R}} = \supseteq$ that is $m \supseteq m'$, which is the inner approximation of Dencœux [2].

Furthermore, if an order is established on \mathcal{X} and $\tilde{\mathbf{R}} = \preceq$, it is also possible to approximate m by a mass function m' such that $m \preceq m'$, where \preceq is the generalized lattice dominance relation. This new approximation is detailed in Section 4.2.

To use the generalized approximation, one can for instance keep the first $K - 1$ most significant focal sets of m and replace the remaining focal sets by a set B such that $A_i \mathbf{R} B, \forall i = K, \dots, n$. This is the generalization of the summarization [13]. However, to provide a good quality approximation, we propose to combine the summarization with the hierarchical clustering procedure introduced in [2]. The main idea of our procedure is to preserve the first (most significant) p focal sets ($p < K < n$), then reduce iteratively, starting from $p + 1$, the number of the remaining focal sets, i.e., those with relatively small masses. At each iteration, a similarity measure or a distance is computed between each pair of focal sets A_i and A_j , then the most similar/nearest pair (A_{i^*}, A_{j^*}) is replaced by a set B_{iter} , such that $A_{i^*} \mathbf{R} B_{iter}$ and $A_{j^*} \mathbf{R} B_{iter}$ with B_{iter} being similar to A_{i^*} and A_{j^*} , and where $m(B_{iter}) = m(A_{i^*}) + m(A_{j^*})$. The process is repeated until we reach size K . The pseudo-code of the approach is explained in Algorithm 1 which runs in a time complexity of $\mathcal{O}(n^3)$. The worst case number of iterations in the repeat loop is $(n - 1)$ and the most expensive instruction inside this loop is the update of the similarity matrix $\mathcal{O}(n^2)$, this yields a total complexity of $\mathcal{O}(n^3)$.

Algorithm 1 The generalized approximation procedure $O(n^3)$

Require: a mass function $m = \{A_i, m(A_i), i = \overline{1, n}\}$, two integers $p, K : p < K < n$, a relation \mathbf{R} , and a similarity measure S .

Ensure: a mass function m' with K focal sets.

- 1: Initialization: $m' \leftarrow \emptyset$;
- 2: Add the most significant p focal sets to m' and remove them from m ;
- 3: Compute the similarity matrix M s.t $M(i, j) \leftarrow S(A_i, A_j) \forall i, j = \overline{1, n-p}$;
- 4: $iter \leftarrow n - p$;
- 5: **repeat**
- 6: Select the most similar pair (A_{i^*}, A_{j^*}) ;
- 7: Add B_{iter} to m s.t $A_{i^*} \mathbf{R} B_{iter}$ and $A_{j^*} \mathbf{R} B_{iter}$;
- 8: $m(B_{iter}) \leftarrow m(A_{i^*}) + m(A_{j^*})$;
- 9: Remove (A_{i^*}, A_{j^*}) from m ;
- 10: Update the similarity matrix M ;
- 11: $iter \leftarrow iter - 1$;
- 12: **until** ($iter = K - p$);
- 13: Add all the focal sets of m to m' ;

4.2 A lattice dominance-based approximation

This section studies a particular case of the general approximation where $\tilde{\mathbf{R}} = \preceq$. In other words, we want to approximate m with a mass function m' that is greater than m according to lattice dominance. Note that property (8) holds with this partial order relation.

Consider an ordered set $\mathcal{X} = \{x_1, \dots, x_n\}$ ($x_1 \leq \dots \leq x_n$) and a mass function m defined on \mathcal{X} having the following focal sets $\mathcal{F}(m) = \{A_1, \dots, A_n\}$ s.t $A_i = \{\underline{a}_i, \dots, \bar{a}_i\} (\underline{a}_i \leq \bar{a}_i)$, which we denote by $\llbracket \underline{a}_i, \bar{a}_i \rrbracket$, and where $m(A_i) \geq m(A_{i+1}), \forall i = 1, \dots, n-1$. Using Definition 2, we can build a lattice dominance-based approximation (\preceq -approximation) m' of m such that $m \preceq m'$ and where focal sets of m' are the subsets $B_k = \llbracket \underline{b}_k, \bar{b}_k \rrbracket = \{\underline{b}_k, \dots, \bar{b}_k\}$, with $\underline{b}_k \leq \bar{b}_k$, and verifying for each $i \in P_k$ and $k = 1, \dots, K$, $A_i \leq_d B_k$, i.e., $\underline{a}_i \leq \underline{b}_k$ and $\bar{a}_i \leq \bar{b}_k$.

To illustrate this approximation, we use Algorithm 1 with the lattice dominance relation \leq_d and Jaccard's similarity measure given by: $S_{\text{Jaccard}}(A_i, A_j) = \frac{|A_i \cap A_j|}{|A_i \cup A_j|}$ [10]. The pair of the most similar focal sets (A_{i^*}, A_{j^*}) is replaced by the subset B_{iter} that is the nearest to A_{i^*} and A_{j^*} and which is defined as follows:

$$B_{iter} = \llbracket \max(\underline{a}_{i^*}, \underline{a}_{j^*}), \max(\bar{a}_{i^*}, \bar{a}_{j^*}) \rrbracket \quad (13)$$

The process is repeated until m' reaches size K . Note that the choice of an adequate measure depends on the relation that is used as well as the application in hand. Jaccard's measure can be replaced by any other similarity measure such as Dice's measure [5] or others. Moreover, if m has disjoint focal sets, one can use a geometric distance [21] instead to capture the nearest focal sets.

Example 1. Let us use Algorithm 1 to build a \preceq -approximation for the mass function m defined such that: $\mathcal{F}(m) = \{A_1 = \llbracket 1, 3 \rrbracket, A_2 = \llbracket 2, 7 \rrbracket, A_3 = \llbracket 3, 9 \rrbracket, A_4 = \llbracket 1, 6 \rrbracket, A_5 = \llbracket 6, 8 \rrbracket, A_6 = \llbracket 2, 4 \rrbracket\}$ with $m(A_1) = 0.4, m(A_2) = 0.3, m(A_3) = 0.1, m(A_4) = 0.1, m(A_5) = 0.05, m(A_6) = 0.05$. Also let $K = 4$ and $p = 2$.

* *Step 1:* Add A_1 and A_2 to m' , then remove them from m . In this case, m becomes:

$\mathcal{F}(m) = \{A_3 = \llbracket 3, 9 \rrbracket, A_4 = \llbracket 1, 6 \rrbracket, A_5 = \llbracket 6, 8 \rrbracket, A_6 = \llbracket 2, 4 \rrbracket\}$ with $m(A_3) = 0.1, m(A_4) = 0.1, m(A_5) = 0.05, m(A_6) = 0.05$.

* *Step 2:* Compute the similarity matrix M for $m, iter = n - p = 4$.

F. sets	A_3	A_4	A_5	A_6^*
A_3	-	0.44	0.43	0.25
A_4^*	-	-	0.13	0.50
A_5	-	-	-	0
A_6	-	-	-	-

The most similar pair of focal sets (A_4, A_6) is replaced, in m , by the subset $B_{iter} = B_4 = \llbracket 2, 6 \rrbracket$ that is computed using equation (13), and $m(B_4) = m(A_4) + m(A_6) = 0.15$. Hence m becomes: $\mathcal{F}(m) = \{A_3 = \llbracket 3, 9 \rrbracket, B_4 = \llbracket 2, 6 \rrbracket, A_5 = \llbracket 6, 8 \rrbracket\}$ with $m(A_3) = 0.1, m(B_4) = 0.15$ and $m(A_5) = 0.05$.

* *Step 3:* Update the similarity matrix $M, iter = iter - 1 = 3$.

F. sets	A_3	B_4^*	A_5
A_3^*	-	0.50	0.43
B_4	-	-	0.14
A_5	-	-	-

The pair (A_3, B_4) is replaced, in m , by the subset $B_{iter} = B_3 = \llbracket 3, 9 \rrbracket$, given (13), and $m(B_3) = m(A_3) + m(B_4) = 0.25$. Hence m becomes: $\mathcal{F}(m) = \{B_3 = \llbracket 3, 9 \rrbracket, A_5 = \llbracket 6, 8 \rrbracket\}$, with $m(B_3) = 0.25$ and $m(A_5) = 0.05$.

* *Step 4:* $iter = iter - 1 = 2 = K - p$: Stop and add B_3 and A_5 to m' .

m' is the \preceq -approximation of m where $\mathcal{F}(m') = \{A_1 = \llbracket 1, 3 \rrbracket, A_2 = \llbracket 2, 7 \rrbracket, B_3 = \llbracket 3, 9 \rrbracket, A_5 = \llbracket 6, 8 \rrbracket\}$, with $m'(A_1) = 0.4, m'(A_2) = 0.3, m'(B_3) = 0.25$ and $m'(A_5) = 0.05$.

4.3 Preliminary tests:

The lattice dominance-based approximation method was incorporated within a meta-heuristic framework to accelerate the solution scheme of the combinatorial optimization problem studied in [17], while preventing the increase of the set of feasible solutions. Tests were conducted on an adaptation of medium to large-sized literature instances. The details about the solution scheme as well as the instances adaptation are explained in [17]. Table 1 presents average cost results for instances *Inst* of 50 customers, after performing 15 executions per instance. Note that the meta-heuristic algorithm stops after 50 iterations without improvement. Columns 2 (resp. 4) and 3 (resp. 5) show costs C (resp. C_{\preceq}) without (resp. with) \preceq -approximation and the corresponding execution time $CPU(s)$ (resp. $CPU_{\preceq}(s)$) recorded in seconds. The percentage of increase in solution cost induced by \preceq -approximation is displayed in column 6. Average costs C_{\preceq}^* of solutions using \preceq -approximation for the same amount of time as in column 3 are presented in column 7. The experiments show a significant decrease in $CPU_{\preceq}(s)$ when using the approximation, this is expected since the number of focal sets is reduced. Moreover, the increase in cost values when using approximation is around 7.18% which is quite acceptable given the gain in time. In addition, the highlighted costs in column 7, confirm that incorporating the \preceq -approximation in the meta-heuristic scheme helps

to enhance the solution quality. Specifically, providing fast solutions helps the meta-heuristic engine to explore, rapidly, further regions of the set of feasible solutions that might contain better quality solutions. Note that we chose to present results on medium-sized instances to highlight the advantage of using the proposed approximation as we were not even able to get results without approximation for large scale instances.

Table 1: Comparing results with and without \prec -approximation.

<i>Inst</i>	<i>C</i>	<i>CPU(s)</i>	C_{\preceq}	<i>CPU_⊆(s)</i>	<i>Inc(%)</i>	C_{\preceq}^*
C102	7549.88	260.70	7770.40	26.20	2.92	7238.41
C104	6052.45	668.50	6391.06	48.77	5.59	6421.58
C204	3580.75	356.71	3671.60	61.00	2.55	3474.26
R104	10479.60	970.83	11947.63	27.50	9.55	11573.78
R204	4026.19	742.13	4509.74	66.50	12.01	3842.19
R207	5246.60	299.40	5484.06	44.80	4.52	4294.79
R208	3399.43	1512.27	3734.58	70.50	9.85	3411.02
RC204	4368.41	494.40	4827.95	60.10	10.51	3874.12

5 Conclusions and perspectives

We proposed a general approach to approximate belief functions. This approach benefits from the generalization of set relations to belief functions and offers to simplify a mass function given any possible relation with its approximation. The presented approach includes some well known sub-cases, such that the inner and outer approximations of the literature. A lattice dominance-based case study was detailed and applied to a combinatorial optimization problem to accelerate the solution search. In future work, we will investigate other possible relations as well as the definition of other similarity measures that are problem-related to get more efficient results. An extension to approximations that are concerned with reducing the size of the frame of discernment is also an interesting perspective.

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