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# Reflected BSDE associated to jump Markov processes and application to PDE

Khaled Bahlali \*   Abdelkarim Oualaid †‡   Youssef Ouknine §¶

## Abstract

In this paper we study a class of reflected backward stochastic differential equations (RBSDE) driven by the compensated random measure associated to a given pure jump Markov process  $X$  on a general state space  $U$ . The "reflection" keeps the solution above a given càdlàg process. We prove the uniqueness and existence both by a combination of the Snell envelope theory and fixed point argument. We apply these results to represent probabilistically the value function of some quasi-variational inequalities associated to the Markov process  $X$ .

**Keywords** Backward stochastic differential equations, Jump Markov processes, Marked point processes, Quasi-variational inequalities.

## 1 Introduction

In this paper we introduce and solve a class of reflected BSDEs driven by a random measure associated to a given jump Markov process. Our results are applied to solve quasi-variational inequalities of no local-term.

Let us describe our framework. Our starting point is a pure jump Markov process  $X$  on a space  $(U, \mathcal{U})$ , such that  $\{x\} \in \mathcal{U}$  for every  $x \in U$ . As usual starting from a positive measure  $A \mapsto \nu(t, x, A)$ ,  $A \in \mathcal{U}$ , depending on  $t \geq 0$  and  $x \in U$  and called rate measure, that specifies  $\lambda(t, x) = \nu(t, x, K)$  the so called rate function and the jump measure  $\pi(t, x, A) =$

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$\nu(t, x, A)/\lambda(t, x)$ . If the process starts at time  $t$  from  $x \in U$ , it is well known that the distribution of its first jump time  $T_1$  is given by the formula

$$\mathbb{P}(T_1 > s) = \exp\left(-\int_t^s \lambda(r, x) dr\right),$$

and we have also

$$\mathbb{P}(X_{T_1} \in A | T_1 = s) = \pi(s, x, A).$$

Denoting by  $T_n$  the jump times of  $X$ , we consider the marked point process  $(T_n, X_{T_n})$  and the associated random measure  $p(dt dy) = \sum_n \delta_{(T_n, X_{T_n})}$  on  $(0, \infty) \times U$ , equivalently  $p(t, A) = p([0, t] \times A) = \sum_{n|T_n \leq t} \mathbb{1}_{\{X_{T_n} \in A\}}$ ; see, [6, 25, 28] as general references on the subject.

In this paper  $X$  is pure-jump Markov, Feller and càdlàg process, and we suppose that the measure  $\nu$  is uniformly bounded; see [7]. The boundedness of  $\nu$  implies that  $X$  is not explosive.

Since our process  $X$  is Markov, the predictable dual projection (or the compensator) of  $p$ , denoted by  $\tilde{p}$  is given by

$$\tilde{p}(dt dy) = \nu(t, X_{t-}, dy) dt.$$

Since  $\tilde{p}$  is absolutely continuous with respect to time and our filtration  $(\mathcal{F}_v)_{v \geq t}$  is complete and right continuous, then the stopping times  $(T_n)$  are inaccessible:  $\mathbb{P}[T_n = \tau < \infty] = 0$ , for all predictable stopping time  $\tau$ , and we have representation of  $\mathcal{F}$ -martingales by means of stochastic integral with respect to  $q(dt dy) := p(dt dy) - \tilde{p}(dt dy)$ .

We introduce a class of reflected BSDE with one obstacle:

$$\begin{cases} Y_s = g(X_T) + \int_s^T f(r, X_r, Y_r, Z_r(\cdot)) dr \\ \quad - \int_s^T \int_U Z_r(y) q(dr dy) + K_T - K_s, \quad \forall s \in [t, T] \\ Y_s \geq h(X_s), \quad \forall s \in [t, T] \\ \int_t^T (Y_{s-} - h(X_{s-})) dK_s = 0. \end{cases}$$

The data are  $f, g$  and  $h$ , and  $Y, Z$  and  $K$  are the unknown processes. The process  $K$  is increasing predictable keeps  $Y$  above the obstacle  $h$ . To our knowledge, the only paper dealing with reflected BSDEs with respect to point processes is [16] but not in a Markovian framework.

These equations are then used to solve some quasi-variational partial differential equations with no local term on the stat space  $U$  of the form

$$\begin{cases} \inf \{ \partial_t v(t, x) + \mathcal{L}_t v(t, x) + f(t, x, v(t, x), v(t, \cdot) - v(t, x)), v(t, x) - h(t, x) \} = 0, \\ \quad t \in [0, T], x \in U, \\ v(T, x) = g(x), \end{cases}$$

where  $\mathcal{L}_t$  denotes the generator of  $X$  and  $f, h$  and  $g$  are given functions. We construct the solution  $v$  by means of a family of reflected BSDE parametrized by  $(t, x) \in [0, T] \times U$ :

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}(\cdot)) dr \\ \quad - \int_s^T \int_U Z_r^{t,x}(y) q^t(dr dy) + K_T^{t,x} - K_s^{t,x}, \quad \forall s \in [t, T] \\ Y_s^{t,x} \geq h(X_s^{t,x}), \quad \forall s \in [t, T] \\ \int_t^T (Y_{s-}^{t,x} - h(X_{s-}^{t,x})) dK_s^{t,x} = 0. \end{cases}$$

We prove the existence and uniqueness of a triplet  $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{s \in [t, T]}$ . As a by-product we also obtain the representation formula

$$v(t, x) = Y_t^{t,x}, \quad Y_s^{t,x} = v(s, X_s^{t,x}), \quad Z_s^{t,x}(y) = v(s, y) - v(s, X_{s-}^{t,x}),$$

and

$$\begin{aligned} K_s^{t,x} = v(t, x) - v(s, X_s^{t,x}) - \int_t^s f(r, X_r^{t,x}, v(r, X_r^{t,x}), v(r, \cdot) - v(r, X_r^{t,x})) dr \\ + \int_t^s \int_U (v(r, y) - v(r, X_{r-}^{t,x})) q^t(dr, dy). \end{aligned}$$

Literaturely speaking, the subject of nonlinear BSDEs driven by a Brownian motion were first introduced in [31]. Later, many generalizations have been considered when the Brownian motion was replaced by more general processes. BSDEs with jumps and BSDEs driven by random measures have been studied in [23, 22, 33, 27, 35]. In particular, occurrence of marked point process in the equations has been considered in [34, 3]. Since, these equations have gradually become an important mathematical tool which is encountered in many fields such as financial mathematics, nonlinear PDEs and stochastic control; see, [14, 15, 4, 8, 30, 9, 19, 20, 21]. The subject of continuous reflected BSDEs has introduced in [13]. Actually, it is a backward equation, but the component  $Y$  of solution is forced to stay above a given barrier, which is an adapted continuous process. Since then there were many works on reflected BSDEs in [29, 18, 1, 2, 12, 21].

This paper is organized as follows, In section 2, we first recall some results on pure jump Markov process and its associated marked point process and describe the setting and the problem we want to solve. In section 3, we prove the existence and uniqueness of a reflected BSDE driven by a random measure associated to a given jump Markov process, when the generator is adapted only to the filtration generated by the Markov process, In the section 4, we solve the RBSDE in the general case with the help of a fixed point argument. finally, in section 5, we show that, provided the problem is formulated within a Markovian framework, the solution of the reflected BSDE provides a probabilistic representation of the unique viscosity solution of some quasi-variational inequalities.

## 2 Notations, preliminaries and basic assumptions

### 2.1 Jump Markov processes

Suppose we are giving a measurable space  $U$  endowed with its Borel  $\sigma$ -algebra  $\mathcal{U} = \mathcal{B}(U)$ , a set  $\Omega$  and a function  $X : \Omega \times [0, \infty[ \rightarrow U$ . For  $I \subset [0, \infty[$ , let  $\mathcal{F}_I = \sigma(X_t, t \in I)$ . We suppose that for every  $t \in [0, \infty[$  and  $x \in U$  a probability  $\mathbb{P}^{t,x}$  is given on  $(\Omega, \mathcal{F}_{[t, \infty[})$ . We suppose that  $X$  is a Feller process and càdlàg. Let

$$\mathcal{G}_v^{X,t,x} = \sigma(X_s, t \leq s \leq v) \vee \mathcal{N}; \quad \bar{\mathcal{G}}_v^{X,t} = \bigcap_{x \in U} \mathcal{G}_v^{X,t,x}.$$

$$\mathcal{F}_v = \bigcap_{u > v} \bar{\mathcal{G}}_u^{X,t}; \quad \mathcal{N} = \{N \subset \Omega, \mathbb{P}^{t,x}\text{-negligible}\}.$$

The filtration  $(\mathcal{F}_v)_{v \geq t}$  is by definition complete with respect to  $\mathbb{P}^{t,x} \forall x \in U$  and right continuous (E.Protter [32] p.36 Theorem 47). Since  $X$  is a Feller and càdlàg Markov process

with respect to  $\mathcal{G}_v^{X,t,x}$ , then it remains Markov process with respect to  $(\mathcal{F}_v)_{v \geq t}$ . We can find a version of  $X$  which satisfies the following conditions

- (i)  $\mathcal{U}$  contains all one-point sets.  $\Delta$  denotes a point not included in  $U$ .
- (ii)  $X_t = x$   $\mathbb{P}^{t,x}$ -a.s. for every  $t \in [0, \infty), x \in U$
- (iii) For every  $0 \leq t \leq s$  and  $A \in \mathcal{U}$  the function  $x \mapsto \mathbb{P}^{t,x}(X_s \in A)$  is  $\mathcal{U}$ -measurable.
- (iv) For every  $0 \leq u \leq t \leq s, A \in \mathcal{U}$ , we have  $\mathbb{P}^{u,x}(X_s \in A | \mathcal{F}_{[u,t]}) = \mathbb{P}^{t,X_t}(X_s \in A)$ ,  $\mathbb{P}^{u,x}$ -a.s.
- (v) For every  $\omega \in \Omega$ , the number of jump is finite on every bounded interval.

The class of Markov processes we will consider in this paper will be described by means of a special form of the joint law  $Q$  of the first jump time  $T_1$  and the corresponding position  $X_{T_1}$ . To avoid the complication, we first fix  $t \geq 0$  and  $x \in U$ , and define the first jump time

$$T_1(\omega) = \inf \{s > t : X_s(\omega) \neq X_t(\omega)\},$$

with the convention that  $\inf \emptyset = \infty$ . Set  $X_\infty(\omega) = \Delta \notin U$  for all  $\omega \in \Omega$ , where  $\Delta$  and let  $S := ([0, \infty), \times U) \cup \{(\infty, \Delta)\}$  with its Borel  $\sigma$ -algebra  $\mathcal{S} = \sigma(\mathcal{B}([0, \infty)) \otimes \mathcal{U}, (0, \infty))$ . Then  $(T_1, X_{T_1})$  is a random variable with values in  $(S, \mathcal{S})$ . Its law under  $\mathbb{P}^{t,x}$  will be denoted by  $Q(t, x, \cdot)$ .

The distribution  $Q$  is constructed via a transformation function  $\nu(t, x, A)$ , for all  $t \in [0, T], x \in U$ , and  $A \in \mathcal{U}$  ( we require that  $\nu$  is a positive measure ). We also assume  $(t, x) \mapsto \nu(t, x, A)$  is  $\mathcal{B}([0, \infty)) \otimes \mathcal{U}$ -measurable for all  $A \in \mathcal{U}$ , such that

$$\sup_{t \in [0, T], x \in U} \nu(t, x, U) < \infty, \quad \nu(t, x, \{x\}) = 0, \quad \text{for all } t \in [0, \infty), x \in U. \quad (2.1)$$

Define

$$\lambda(t, x) = \nu(t, x, U), \quad \pi(t, x, A) = \begin{cases} \frac{\nu(t, x, A)}{\lambda(t, x)}, & \text{if } \lambda(t, x) > 0 \\ 1_A(x), & \text{if } \lambda(t, x) = 0 \end{cases}$$

Therefore  $\lambda$  is a nonnegative bounded measurable function called *jump rate function* and  $\pi$  is a transition probability on  $U$  called *jump measure*. Note that we have  $\nu(t, x, A) = \lambda(t, x)\pi(t, x, A)$  for all  $t \in [0, T], x \in U, A \in \mathcal{U}$ .

Given  $\nu$  and  $Q$  as described above, we will require that for the Markov process  $X$ , we have for  $0 \leq t \leq a < b \leq \infty, x \in U, A \in \mathcal{U}$

$$Q(t, x, (a, b) \times A) = \int_a^b \pi(s, x, A) \lambda(s, x) \exp\left(-\int_t^s \lambda(r, x) dr\right) ds \quad (2.2)$$

Note that  $Q(t, x, \cdot)$  stated in (2.2) defines a probability measure on  $(S, \mathcal{S})$ .

The following result gives an explicit construction of the process  $X$  in terms of a discrete time Markov chain and a sequence of exponential distributed random variables. The result shows that the distributions  $\mathbb{P}^{t,x}$  are uniquely determined by the rate kernel  $\lambda(t, x)$ .

**Theorem 2.1.** (Theorem 10.19 [26]) *Let  $X$  be a pure-jump Feller process, with càdlàg path and rate kernel  $\lambda = c\nu$ . Then, there exist a Markov process  $Y$  on  $\mathbb{N}$  with transition  $\nu$  and a*

sequence of iid exponentially distributed random variables  $\gamma_1, \gamma_2 \dots$  with mean 1 such that a.s

$$X_t = Y_n \quad \text{for } t \in [\tau_n, \tau_{n+1}), \quad n \in \mathbb{N},$$

where

$$\tau_n = \sum_{k=1}^n \frac{\gamma_k}{c(t, Y_{k-1})}, \quad n \in \mathbb{N}.$$

Such that  $c(t, x) = (\mathbb{E}^{t,x} \tau_1)^{-1}$ ,  $\nu(t, x, B) = \mathbb{P}^{t,x}(X_{\tau_1} \in B)$ .

**Example 2.2.** *Pseudo-Poisson Process*, is a process of the form  $X_t = Y \circ N_t$  a.s., where  $Y$  is a discrete-time Markov chain in some measurable space  $S$  and  $N$  is an independent homogeneous Poisson process. Letting  $\nu$  be the transition kernel of  $Y$  and  $\lambda$  the constant rate of  $N$ . We may construct a kernel

$$\lambda(t, x, B) = c(t, x) \nu(t, x, B \setminus \{x\}), \quad x \in S, \quad B \in \mathcal{B}(S). \quad (2.3)$$

**Proposition 2.3.** (Proposition 10.22 [26]) *A process  $X$  in some Borel space  $S$  is pseudo-Poisson if and only if it is a pure-jump Markov with a bounded rate function. Specifically, if  $X = Y \circ N$  a.s., for some Markov chain  $Y$  with transition kernel  $\nu$  and an independent Poisson process  $N$  with constant rate  $c$ , then  $X$  has the rate kernel in (2.3).*

## 2.2 Marked point processes and the associated martingales

In the following, we fix a pair  $(t, x) \in [0, \infty) \times U$  and look at the process  $X$  under the probability  $\mathbb{P}^{t,x}$ . For every  $t \geq 0$  we denote by  $\mathbb{F}^t$  the filtration  $(\mathcal{F}_{[t,s]})_{s \in [t, \infty)}$ . The predictable  $\sigma$ -algebra (respectively, the progressive  $\sigma$ -algebra) on  $\Omega \times [t, \infty)$  will be denoted by  $\mathcal{P}^t$  (respectively, by  $Prog^t$ ). The same symbols will also denote the restriction to  $\Omega \times [t, T]$  for some  $T > t$ .

For every  $t \geq 0$  we define a sequence  $(T_n^t)_{n \geq 0}$  of random variables with values in  $[0, \infty]$  setting

$$T_0^t(\omega) = t, \quad T_{n+1}^t(\omega) = \inf \{s > T_n^t(\omega) : X_s(\omega) \neq X_{T_n^t(\omega)}(\omega)\}, \quad \text{with } \inf \emptyset = \infty. \quad (2.4)$$

Therefor, we have for every  $t \geq 0$  and  $\omega \in \Omega$

- $T_n^t(\omega) \leq T_{n+1}^t(\omega)$ , for all  $n \geq 0$ ;
- $T_n^t(\omega) < T_{n+1}^t(\omega)$  if  $T_n^t(\omega) < \infty$ , for all  $n \geq 0$ ;
- $T_n^t(\omega) \nearrow \infty$ , since  $X$  is non explosive.

Hence, the sequence  $(T_n^t, X_{T_n^t})_{n \geq 0}$  is well defines a marked point process with values in  $[0, \infty] \times U$ . For  $\omega \in \Omega$ , we associate to each marked point process the random measure  $p^t$  on  $((t, \infty) \times U, \mathcal{B}((t, \infty)) \times \mathcal{U})$ :

$$p^t(\omega, C) = \sum_{n \geq 1} \mathbb{1}_{((T_n^t(\omega), X_{T_n^t(\omega)}) \in C)}, \quad C \in \mathcal{B}((t, \infty)) \otimes \mathcal{U},$$

we also use the notation  $p^t(ds, dy)$ . Note that

$$p^t((t, s] \times A) = \sum_{n \geq 1} \mathbb{1}_{\{T_n^t \leq s\}} \mathbb{1}_{\{X_{T_n^t} \in A\}}, \quad s \geq t, A \in \mathcal{U}.$$

By general results (see [25]), it turns out that for every nonnegative  $\mathcal{P}^t \otimes \mathcal{U}$ -measurable function  $H_s(\omega, y)$  defined on  $\Omega \times [t, \infty) \times U$ , we have

$$\mathbb{E}^{t,x} \int_t^\infty \int_U H_s(y) p^t(dsdy) = \mathbb{E}^{t,x} \int_t^\infty \int_U H_s(y) \nu(s, X_s, dy) ds. \quad (2.5)$$

The random measure  $\nu(s, X_s, dy) ds$  is called the compensator, or the dual predictable projection of  $p^t(dsdy)$ .

Now fix  $T > t$ . Let  $H$  be a  $\mathcal{P}^t \otimes \mathcal{U}$ -measurable process such that

$$\int_t^T \int_U |H_s(y)| \nu(s, X_s, dy) ds < \infty, \quad \mathbb{P}^{t,x}\text{-a.s.}$$

Then we can define the following stochastic integral

$$\begin{aligned} \int_t^s \int_U H_r(y) q^t(dr dy) &:= \int_t^s \int_U H_r(y) p^t(dr dy) \\ &\quad - \int_t^s \int_U H_r(y) \nu(r, X_r, dy) dr, \quad s \in [t, T] \end{aligned}, \quad (2.6)$$

as the difference of ordinary integrals with respect to  $p^t(dsdy)$  and  $\nu(s, X_s^{t,x}, dy) ds$ . Here and in the following the symbol  $\int_a^b$  is to be understood as an integral over the interval  $(a, b]$ . We shorten this identity writing

$$q^t(dsdy) = p^t(dsdy) - \nu(s, X_{s-}^{t,x}, dy) ds.$$

Note that since  $p^t$  is a discrete random measure, the integral with respect to  $p^t$  is a sum:

$$\int_t^s \int_U H_r(y) p^t(dr dy) = \sum_{n \geq 1, T_n^t \leq s} H_{T_n^t}(X_{T_n^t}), \quad s \in [t, T],$$

and it is always well defined since  $T_n^t \nearrow \infty$ .

For  $m \geq 1$  we define  $\mathbb{L}^m(p^t)$  as the space of  $\mathcal{P}^t \otimes \mathcal{U}$ -measurable processes with real values  $H$  such that

$$\mathbb{E}^{t,x} \int_t^T \int_U |H_s(y)|^m p^t(dsdy) = \mathbb{E}^{t,x} \int_t^T \int_U |H_s(y)|^m \nu(s, X_s, dy) ds < \infty.$$

Note that the equality of integrals follows from (2.5).

Note that the  $\mathbb{F}^t$  is the filtration generated by the process  $X$  completed and right continuous, hence, we have the following martingale representation result for marked point processes which we will need to construct a solution to the reflected BSDEs.

**Theorem 2.4.** (Theorem 2 [10]) *Given  $(t, x) \in [0, T] \times U$ , let  $M$  be an  $\mathbb{F}^t$ -martingale on  $[t, T]$  with respect to  $\mathbb{P}^{t,x}$ . Then there exists a process  $H \in \mathbb{L}^1(p^t)$  such that*

$$M_s = M_t + \int_t^s \int_U H_r(y) q^t(dr dy), \quad s \in [t, T], \quad \mathbb{P}^{t,x}\text{-a.s.} \quad (2.7)$$

### 2.3 Reflected BSDE

Next we fix  $T > 0$  and a pair  $(t, x) \in [0, T] \times U$ . We look at all processes under the probability  $\mathbb{P}^{t,x}$ . Let  $T_n^t$ ,  $\mathbb{F}^t$ ,  $\mathcal{P}^t$ ,  $Prog^t$ ,  $p^t(ds, dy)$ ,  $q^t(ds, dy)$  and  $\mathbb{L}^2(p^t)$  as before, we are interested in studying the following family of reflected backward stochastic differential equations parametrized by  $(t, x)$ :  $\mathbb{P}^{t,x}$ -a.s.

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}(\cdot)) dr \\ \quad - \int_s^T \int_U Z_r^{t,x}(y) q^t(dr dy) + K_T^{t,x} - K_s^{t,x}, \quad \forall s \in [t, T] \\ Y_s^{t,x} \geq h(X_s^{t,x}), \quad \forall s \in [t, T] \\ \int_t^T (Y_{s^-}^{t,x} - h(X_{s^-}^{t,x})) dK_s^{t,x} = 0 \end{cases} \quad (2.8)$$

We introduce the space  $\mathbb{M}^{t,x}$  of the processes  $(Y, Z, K)$  on  $[t, T]$ , such that  $Y$  is real-valued and  $Prog^t$ -measurable,  $Z : \Omega \times [t, T] \times U \rightarrow \mathbb{R}$  is  $\mathcal{P}^t \otimes \mathcal{U}$ -measurable,  $K$  is càdlàg increasing  $\mathbb{F}^t$ -adapted with  $K_t = 0$  and

$$\|(Y, Z)\|_{\mathbb{M}^{t,x}}^2 := \mathbb{E}^{t,x} \int_t^T |Y_s|^2 ds + \mathbb{E}^{t,x} \int_t^T \int_U |Z_s(y)|^2 \nu(s, X_s, dy) ds + \mathbb{E}^{t,x} K_T^2 < \infty.$$

A solution to the reflected BSDE (5.3) is a triplet  $(Y^{t,x}, Z^{t,x}, K^{t,x})$  that lies in  $\mathbb{M}^{t,x}$ .

#### Assumption 2.5.

1. The final condition  $g : U \rightarrow \mathbb{R}$  is  $\mathcal{U}$ -measurable and

$$\mathbb{E}^{t,x} |g(X_T^{t,x})|^2 < \infty.$$

2. The generator  $f$  is such that

(i) for every  $s \in [0, T], x \in K, r \in \mathbb{R}, f(s, x, r, \cdot)$  is a mapping  $\mathbb{L}^2(U, \mathcal{U}, \nu(s, x, dy)) \rightarrow \mathbb{R}$ ;

(ii) for every bounded and  $\mathcal{U}$ -measurable function  $z : U \rightarrow \mathbb{R}$ , the mapping

$$(s, x, r) \mapsto f(s, x, r, z(\cdot)) \text{ is } \mathcal{B}([0, T]) \otimes \mathcal{U} \otimes \mathcal{B}(\mathbb{R})\text{-measurable}; \quad (2.9)$$

(iii) there exist  $L \geq 0, L' \geq 0$  such that for every  $s \in [0, T], x \in U, r, r' \in \mathbb{R}, z, z' \in \mathbb{L}^2(U, \mathcal{U}, \nu(s, x, dy))$ ,

$$\begin{aligned} & |f(s, x, r, z(\cdot)) - f(s, x, r', z'(\cdot))| \\ & \leq L' |r - r'| + L \left( \int_U |z(y) - z'(y)|^2 \nu(s, x, dy) \right)^{1/2}; \end{aligned} \quad (2.10)$$

(iv)

$$\mathbb{E}^{t,x} \int_t^T |f(s, X_s, 0, 0)|^2 ds < \infty$$

3. The barrier  $h : U \rightarrow \mathbb{R}$  is  $\mathcal{U}$ -measurable such that

$$\mathbb{E}^{t,x} \sup_{s \in [t, T]} h^2(X_s^{t,x}) < \infty. \quad (2.11)$$



*Remark 2.6.* We recall that the absolute continuity of measure  $\tilde{p}^t(ds, dy) = \nu(s, X_s^{t,x}, dy)ds$  is equivalent to the fact that the jumps of the marked point process  $p^t$  are totally inaccessible: (see [24]). We will often use the following consequence: since  $K^{t,x}$  is predictable, its jumps are disjoint from the jumps of  $p^t$ ; so at any jump time of  $K^{t,x}$  we also have a jump of  $Y^{t,x}$  with the same size, but of opposite sign:  $\mathbb{P}^{t,x}$ -a.s.

$$\Delta K_s^{t,x} \mathbb{1}_{\Delta K_s^{t,x} > 0} = (-\Delta Y_s^{t,x})^+ \mathbb{1}_{\Delta K_s^{t,x} > 0}, \quad s \in [t, T]. \quad (2.12)$$

**Lemma 2.7.** (Lemma 3.3 [7]) *Let  $f$  be a generator satisfying Assumption (2.5)-(i)(ii)(iii). If  $Z^{t,x} \in \mathbb{L}^2(p^t)$ , then the mapping*

$$(\omega, s, y) \mapsto f(s, X_{s^-}^{t,x}(\omega), y, Z_s^{t,x}(\omega, \cdot))$$

is  $\mathcal{P}^t \times \mathcal{B}(\mathbb{R})$ -measurable.

if, in addition,  $Y^{t,x}$  is a  $\text{Prog}^t$ -measurable process, then

$$(\omega, s) \mapsto f(s, X_{s^-}^{t,x}(\omega), Y_s^{t,x}(\omega), Z_s^{t,x}(\omega, \cdot))$$

is  $\text{Prog}^t$ -measurable.

### 3 Reflected BSDE with given generator

In this section we first study the reflected BSDE in the case when the generator  $f$  does not depend on  $(Y, Z)$  but is a given process that satisfy an integrability condition (3.2) stated below.

In this case BSDE (5.3) reduces

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f_r dr - \int_s^T \int_U Z_r^{t,x}(y) q^t(dr dy) + K_T^{t,x} - K_s^{t,x}, & \forall s \in [t, T] \\ Y_s^{t,x} \geq h(X_s^{t,x}), & \forall s \in [t, T] \\ \int_t^T (Y_{s^-} - h(X_{s^-}^{t,x})) dK_s^{t,x} = 0 \end{cases} \quad (3.1)$$

Next, using Snell envelope theory (see Appendix [16]), we show existence and uniqueness of a solution.

**Proposition 3.1.** *Let assumption (2.5)-(1)(3) and let  $f : \Omega \times [t, T] \rightarrow \mathbb{R}$  be a  $\text{Prog}^t$ -measurable process such that*

$$\mathbb{E}^{t,x} |f_s|^2 ds < \infty. \quad (3.2)$$

then there exists a unique triplet  $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \mathbb{M}^{t,x}$  solution to the BSDE (3.1).

*Proof.* Let's start with the *existence*. First we define the càdlàg process  $\eta$  by

$$\eta_s = \int_t^{s \wedge T} f_r dr + h(X_r) \mathbb{1}_{s < T} + g(X_T) \mathbb{1}_{s \geq T}.$$

Note that  $\eta$  is of class  $[D]$  since  $\mathbb{E}^{t,x} \sup_{t \leq s \leq T} |\eta_s|^2 < \infty$ . Indeed:

We have clearly that

$$\mathbb{E}^{t,x} \sup_{t \leq s \leq T} |\eta_s|^2 \leq 3\mathbb{E}^{t,x} \int_t^T |f|^2 dr + 3\mathbb{E}^{t,x} \sup_{t \leq s \leq T} |h(X_s^{t,x})|^2 + 3\mathbb{E}^{t,x} |g(X_T^{t,x})|^2 < \infty.$$

For  $t \leq s \leq T$  and a stopping time  $\theta$ , let

$$Y_s^{t,x} = \text{esssup}_{s \leq \theta \leq T} \mathbb{E}^{t,x} \left[ \int_s^{\theta \wedge T} f_r dr + h(X_\theta^{t,x}) \mathbb{1}_{\theta < T} + g(X_T^{t,x}) \mathbb{1}_{\theta \geq T} | \mathcal{F}_s \right]. \quad (3.3)$$

It follows that

$$Y_s^{t,x} + \int_t^{s \wedge T} f_r dr = \text{esssup}_{s \leq \theta \leq T} \mathbb{E}^{t,x} [\eta_\theta | \mathcal{F}_s].$$

Hence, since  $Y$  is of class  $[D]$ , it follows that the process  $R(\eta) = (Y_s^{t,x} + \int_t^{s \wedge T} f_r dr)_{t \leq s \leq T}$  is the Snell envelope of  $\eta$ . Since  $\eta$  is càdlàg, its Snell envelope  $R(\eta)$ , and hence  $Y^{t,x}$  exist and càdlàg. Furthermore, for any  $t \leq s \leq T$  we have

$$\mathbb{E}^{t,x} \left[ \sup_{s \in [t, T]} (Y_s^{t,x})^2 \right] < \infty.$$

In fact, by definition of  $Y^{t,x}$  we get

$$\begin{aligned} |Y_s^{t,x}| &\leq \mathbb{E}^{t,x} \left[ \int_t^T |f_r| dr + \sup_{t \leq r \leq T} |h(X_r^{t,x})| + |g(X_T^{t,x})| | \mathcal{F}_s \right] \\ &\leq \mathbb{E}^{t,x} \left[ (T-t) \left( \int_t^T |f_r|^2 dr \right)^{\frac{1}{2}} + \sup_{t \leq r \leq T} |h(X_r^{t,x})| + |g(X_T^{t,x})| | \mathcal{F}_s \right] = N_s. \end{aligned}$$

Under Assumption (2.5)-(1)(3) and (3.2),  $N$  is a square integrable martingale. Then by Doob's inequality we get

$$\mathbb{E}^{t,x} \left[ \sup_{s \in [t, T]} |Y_s|^2 \right] \leq \mathbb{E}^{t,x} \left[ \sup_{s \in [t, T]} N_s^2 \right] \leq C \mathbb{E}^{t,x} [N_T^2] < \infty.$$

Now according to Doob-Meyer decomposition theorem, there exist an uniformly integrable martingale  $M$  and a predictable increasing process  $A$  starting from zero, such that

$$R(\eta)_s = R(\eta)_t + M_s - A_s, \quad \text{for all } s \in [t, T] \quad \mathbb{P}^{t,x}\text{-a.s.}$$

Hence, we obtain

$$Y_s^{t,x} = Y_t^{t,x} - \int_t^s f_r dr + M_s - A_s, \quad \text{for all } s \in [t, T] \quad \mathbb{P}^{t,x}\text{-a.s.} \quad (3.4)$$

Note that  $M$  is a càdlàg  $\mathbb{F}^t$ -martingale since that the filtration  $\mathbb{F}^t$  generated by the process  $X^{t,x}$  is completed and right continuous, hence by representation theorem for square martingales, there exist  $Z^{t,x} \in \mathbb{L}^1(p^t)$  such that

$$M_s = M_t + \int_t^s \int_U Z_r^{t,x}(y) q^t(dr dy), \quad s \in [t, T], \quad \mathbb{P}^{t,x}\text{-a.s.} \quad (3.5)$$

Setting,  $\theta = s$  in (3.3), we get that  $\mathbb{P}^{t,x}$ -a.s.  $Y_s^{t,x} \geq h(X_s^{t,x})$  for all  $s \leq T$  and  $Y_T^{t,x} = g(X_T^{t,x})$ .

Let us set  $K_s^{t,x} = A_s$ ,  $t \leq s \leq T$ ; then we have that  $\mathbb{E}^{t,x} K_T^2 < \infty$ , since

$$\mathbb{E}^{t,x} \sup_{s \in [t, T]} Y_s^2 < \infty.$$

Furthermore,  $\Delta K^{t,x}$  is supported by the set  $\{t \leq s \leq T; R(\eta)_{s^-} = \eta_{s^-}\}$ , but  $R(\eta)_{s^-} = \eta_{s^-}$  is equivalent of that  $Y_s^{t,x} = h(X_{s^-}^{t,x}) \mathbb{1}_{s \leq T} + g(X_T^{t,x}) \mathbb{1}_{s > T}$ , hence

$$\Delta K_s^{t,x} = \Delta K_s^{t,x} \mathbb{1}_{\{Y_s^{t,x} = h(X_{s^-}^{t,x})\}}, \text{ for all } s \in [t, T], \quad \mathbb{P}^{t,x}\text{-a.s.}$$

Replacing now (3.5) in (3.4), we conclude that the first equality in (3.1) is verified.

In order to finish the proof of existence it remains to show that the minimal push condition (Skorohod condition) in (3.1) is satisfied, that is

$$\int_t^T (Y_{s^-}^{t,x} - h(X_{s^-}^{t,x})) dK_s^{t,x} = 0.$$

Note that the process  $K^{t,x}$  can be decomposed into  $K^c + K^d$ , where  $K^c$  (resp.  $K^d$ ) is the continuous (resp. the purely discontinuous) part of  $K$ . Hence we have

$$\int_t^T (Y_{s^-}^{t,x} - h(X_{s^-}^{t,x})) dK_s = \int_t^T (Y_{s^-}^{t,x} - h(X_{s^-}^{t,x})) dK_s^c + \sum_{t \leq s \leq T} (Y_{s^-}^{t,x} - h(X_{s^-}^{t,x})) \Delta K_s.$$

First, since jumps of  $K^{t,x}$  are supported by  $\{Y_{s^-}^{t,x} = h(X_{s^-}^{t,x})\}$ , then we get

$$\sum_{t \leq s \leq T} (Y_{s^-}^{t,x} - h(X_{s^-}^{t,x})) \Delta K_s^{t,x} = \sum_{t \leq s \leq T} (Y_{s^-}^{t,x} - h(X_{s^-}^{t,x})) \Delta K_s^{t,x} \mathbb{1}_{\{Y_{s^-}^{t,x} = h(X_{s^-}^{t,x})\}} = 0.$$

On the other hand, let  $\tilde{Y} = Y^{t,x} + K^d$  and  $\tilde{\eta} = \eta + K^d$ , hence  $\tilde{Y} + \int_t^\cdot f_r dr$  is the Snell envelope of  $\tilde{\eta}$ . Indeed,  $\tilde{Y}_s + \int_t^s f_r dr = \tilde{Y}_t + M_s - K_s^c$  is a supermartingale which dominate  $\tilde{\eta}$ . Let  $S$  be another supermartingale that dominate  $\tilde{\eta}$ , hence,  $S_s - K_s^d$  still a supermartingale such that  $S_s - K_s^d \geq \eta$ . Henceforth,  $S_s - K_s^d \geq Y_s^{t,x} + \int_t^s f_r dr$  and then  $S_s \geq \tilde{Y}_s + \int_t^s f_r dr$ , whence the desired result.

Let the stopping time defined as

$$\rho_s^* = \inf \{r \geq s : M_r \neq R(\tilde{\eta})_r\} \wedge T = \inf \{r \geq s : K_r^c > K_s^c\} \wedge T,$$

we have  $K_s^c = K_{\rho_s^*}^c$  since  $K^c$  is continuous. On the other hand  $\rho^*$  is the largest optimal stopping time since  $\tilde{Y}_s + \int_t^s f_r dr = \tilde{Y}_t + M_s - K_s^c$  is regular (see [16] Appendix). Moreover, we have  $\tilde{Y}_{\rho_s^*} + \int_t^{\rho_s^*} f_r dr = \tilde{\eta}_{\rho_s^*}$ , which yields  $\tilde{Y}_{\rho_s^*} = h(X_{\rho_s^*}^{t,x}) + K_{\rho_s^*}^d$ . Now let

$$\rho_s = \inf \{r \geq s : \tilde{Y}_r = h(X_r^{t,x}) + K_r^d\} \wedge T,$$

then we have  $\rho_s \leq \rho_s^*$  and hence  $K_s^c = K_{\rho_s}^c$  for all  $t \leq s \leq T$  which implies

$$0 = \int_t^T (\tilde{Y}_s - h(X_s^{t,x})) dK_s^c = \int_t^T (Y_s^{t,x} - h(X_s^{t,x})) dK_s^c.$$

Let us now prove the uniqueness of solution. Let  $(Y', Z', K')$  and  $(Y'', Z'', K'')$  be two solutions to RBSDE (3.1), and define

$$\bar{Y} = Y' - Y'' \quad \bar{Z} = Z' - Z'' \quad \bar{K} = K' - K'',$$

hence  $(\bar{Y}, \bar{Z}, \bar{K})$  satisfies

$$\bar{Y}_s = - \int_s^T \int_U \bar{Z}_r(y) q^t(dr dy) + \bar{K}_T - \bar{K}_s. \quad (3.6)$$

Now assuming that  $(\bar{Y}, \bar{Z}, \bar{K})$  is a solution in  $\mathbb{M}^{t,x}$ . Using Itô's formula we obtain

$$\begin{aligned} e^{\beta t} |\bar{Y}_t|^2 &= -\beta \int_t^T e^{\beta r} |\bar{Y}_r|^2 dr - 2 \int_t^T \int_U e^{\beta r} \bar{Y}_{r-} \bar{Z}_r(y) q^t(dr dy) \\ &\quad + 2 \int_t^T e^{\beta r} \bar{Y}_{r-} d\bar{K}_r - \sum_{t < r \leq T} e^{\beta r} |\Delta \bar{Y}_r|^2. \end{aligned} \quad (3.7)$$

The last term can be divided in totally inaccessible jumps from  $q(dsde)$  and predictable jumps, from the process  $\bar{K}$ , hence:

$$\begin{aligned} \sum_{t < s \leq T} e^{\beta s} |\Delta \bar{Y}_s|^2 &\geq \sum_{t < T_n \leq T} e^{\beta T_n} |\bar{Z}_{T_n}(X_{T_n}^{t,x})|^2 = \int_t^T \int_U |\bar{Z}_s(y)|^2 p^t(ds dy) \\ &= \int_t^T \int_E |\bar{Z}_s(y)|^2 q^t(ds dy) + \int_t^T \int_U |\bar{Z}_s(y)|^2 \nu(s, X_s^{t,x}, dy) dr. \end{aligned}$$

The integrals with respect to  $q^t(ds dy)$  are martingales, because the integrand processes  $e^{\beta r} \bar{Y}_{r-} \bar{Z}_r(y)$  and  $e^{\beta r} \bar{Z}_r^2(y)$  are in  $\mathbb{L}^1(p^t)$ : Indeed, from Young inequality, we get

$$\begin{aligned} \mathbb{E}^{t,x} \int_t^T \int_U e^{\beta r} |\bar{Y}_{r-}| |\bar{Z}_r(y)| \nu(r, X_r^{t,x}, dy) dr \\ \leq \frac{1}{2} \mathbb{E}^{t,x} \int_t^T \int_U e^{\beta r} |\bar{Y}_{r-}|^2 \nu(r, X_r^{t,x}, dy) dr + \frac{1}{2} \mathbb{E} \int_t^T \int_U e^{\beta r} |\bar{Z}_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr \\ \leq \sup_{t,x} \nu(t, x, U) \frac{e^{\beta T}}{2} \mathbb{E}^{t,x} \int_t^T |\bar{Y}_r|^2 dr + \frac{e^{\beta T}}{2} \mathbb{E}^{t,x} \int_t^T \int_U |\bar{Z}_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr < \infty. \end{aligned}$$

Similarly, we have

$$\mathbb{E}^{t,x} \int_t^T \int_U e^{\beta r} |\bar{Z}_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr \leq e^{\beta T} \mathbb{E}^{t,x} \int_t^T \int_U |\bar{Z}_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr < \infty.$$

By taking expectation in (3.7), we obtain

$$\begin{aligned} \beta \mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{Y}_r|^2 dr + \mathbb{E}^{t,x} \int_t^T \int_U e^{\beta s} |\bar{Z}_s(y)|^2 \nu(s, X_s^{t,x}, dy) ds \\ \leq 2e^{\beta T} \mathbb{E}^{t,x} \int_t^T \bar{Y}_{r-} d\bar{K}_r. \end{aligned}$$

Now, since the solutions satisfy the minimal push condition, we have

$$\begin{aligned}
\int_t^T \bar{Y}_{s-} d\bar{K}_s &= \underbrace{\int_t^T (Y'_{s-} - h(X_{s-}^{t,x})) dK'_s}_{=0} - \underbrace{\int_t^T (Y'_{s-} - h(X_{s-}^{t,x})) dK''_s}_{\geq 0} \\
&\quad - \underbrace{\int_t^T (Y''_{s-} - h(X_{s-}^{t,x})) dK'_s}_{\geq 0} + \underbrace{\int_t^T (Y''_{s-} - h(X_{s-}^{t,x})) dK''_s}_{=0} \\
&\leq 0,
\end{aligned}$$

it follows that

$$\beta \mathbb{E}^{t,x} \int_s^T e^{\beta r} |\bar{Y}_r|^2 dr + \mathbb{E}^{t,x} \int_s^T \int_U e^{\beta r} |\bar{Z}_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr = 0,$$

hence the uniqueness of  $Y$  and  $Z$ . From equation (3.6) we get  $\bar{K}_T = \bar{K}_s$ , for all  $s \in [t, T]$ , then  $\bar{K}_T = 0$  since  $\bar{K}_t = 0$ , then  $\bar{K}_s = 0$  for all  $s \in [t, T]$ , which gives the uniqueness of  $K$ .

We conclude the proof showing that  $(Y, Z, K) \in \mathbb{M}^{t,x}$ . We have already noticed that  $\mathbb{E}[K_T^2] < \infty$ . Next we define the sequence of stopping times:

$$S_n = \inf\{s \in [t, T] : \int_t^s |Y_r|^2 dr + \int_t^s \int_U |Z_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr > n\} \wedge T,$$

and we consider the Itô's formula applied to  $e^{\beta r} |Y_r|^2$  between  $t$  and  $S_n$ , we obtain

$$\begin{aligned}
e^{\beta S_n} |Y_{S_n}|^2 &= e^{\beta t} |Y_t|^2 + \beta \int_t^{S_n} e^{\beta r} |Y_r|^2 dr + 2 \int_t^{S_n} \int_U e^{\beta r} Y_r Z_r(y) q^t(dr dy) - 2 \int_t^{S_n} e^{\beta r} Y_r f_r dr \\
&\quad - 2 \int_t^{S_n} e^{\beta r} Y_r dK_r + \sum_{t < r \leq S_n} e^{\beta r} \Delta K_r^2 + \int_t^{S_n} \int_U e^{\beta r} |Z_r(y)|^2 p^t(dr dy),
\end{aligned}$$

we have that for all  $s \in [t, T]$

$$\int_t^s e^{\beta r} Y_r dK_r = \underbrace{\int_t^s e^{\beta r} (Y_r - h(X_r^{t,x})) dK_r}_{=0} + \int_t^s e^{\beta r} h(X_r^{t,x}) dK_r$$

Neglecting the positive terms  $e^{\beta t} |Y_t|^2$  and  $\sum_{t < r \leq S_n} e^{\beta r} \Delta K_r^2$  the previous equation becomes

$$\begin{aligned}
e^{\beta S_n} |Y_{S_n}|^2 &\geq \beta \int_t^{S_n} e^{\beta r} |Y_r|^2 dr + 2 \int_t^{S_n} \int_U e^{\beta r} Y_r Z_r(y) q^t(dr dy) - 2 \int_t^{S_n} e^{\beta r} Y_r f_r dr \\
&\quad - 2 \int_t^{S_n} e^{\beta r} h(X_r^{t,x}) dK_r + \int_t^{S_n} \int_U e^{\beta r} |Z_r(y)|^2 q^t(dr dy) + \int_t^{S_n} \int_U e^{\beta r} |Z_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr,
\end{aligned}$$

by definition of  $(S_n)^s$  we have

$$\begin{aligned}
2\mathbb{E}^{t,x} \int_t^{S_n} \int_U e^{\beta r} Y_r Z_r(y) \nu(r, X_r^{t,x}, dy) dr &\leq e^{\beta T} \sup_{t,x} \nu(t, x, U) \mathbb{E}^{t,x} \int_t^{S_n} |Y_r|^2 dr \\
e^{\beta T} \mathbb{E}^{t,x} \int_t^{S_n} \int_U |Z_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr &\leq n e^{\beta T} (1 + \sup_{t,x} \nu(t, x, U)) < \infty.
\end{aligned}$$

We obtain then that the terms with respect  $q^t(dsdy)$  are martingales. Taking expectation in both sides, we get

$$\begin{aligned}
& \beta \mathbb{E}^{t,x} \int_t^{S_n} e^{\beta r} |Y_r|^2 dr + \mathbb{E}^{t,x} \int_U e^{\beta r} |Z_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr \\
& \leq \mathbb{E}^{t,x} e^{\beta S_n} |Y_{S_n}|^2 + 2 \mathbb{E}^{t,x} \int_t^{S_n} e^{\beta r} Y_r f_r dr + 2 \mathbb{E}^{t,x} \int_t^{S_n} e^{\beta r} h(X_{r-}^{t,x}) dK_r \\
& \leq \mathbb{E}^{t,x} \sup_{t \leq s \leq T} e^{\beta s} |Y_s|^2 + \frac{1}{\beta} \mathbb{E}^{t,x} \int_t^{S_n} e^{\beta r} |Y_r|^2 + \beta e^{\beta T} \mathbb{E}^{t,x} \int_t^T |f_r|^2 dr \\
& \quad + e^{\beta T} \mathbb{E}^{t,x} \sup_{t \leq s \leq T} h^2(X_s^{t,x}) + e^{\beta T} \mathbb{E}^{t,x} K_T^2
\end{aligned}$$

define a positive constant by  $c^{-1}(\beta) = \min(\beta - \frac{1}{\beta}, 1)$ , hence we get

$$\begin{aligned}
& \mathbb{E}^{t,x} \int_t^{S_n} e^{\beta r} |Y_r|^2 dr + \mathbb{E}^{t,x} \int_U e^{\beta r} |Z_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr \\
& \leq c(\beta) e^{\beta T} \mathbb{E}^{t,x} \left[ \sup_{t \leq s \leq T} |Y_s|^2 + \beta \int_t^T |f_r|^2 dr + \sup_{t \leq s \leq T} h^2(X_s^{t,x}) + K_T^2 \right] < \infty,
\end{aligned}$$

Now let  $S = \lim_n S_n$ , by definition of  $(S_n)_{n \geq 0}$ , we have  $S = T$ , whence, the last estimate follows that  $(Y, Z, K) \in \mathbb{M}^{t,x}$ .  $\square$

## 4 Reflected BSDE

Now instead of the case where the generator  $f$  is just a  $Prog^t$ -measurable process, we will consider a generator  $f$  which depends on solution, that is the reflected BSDE (5.3).

We use a fixed point argument to show the existence and uniqueness of solution, to this end we define the space  $\mathbb{M}_\beta^{t,x}$  of processes  $(Y, Z)$  on  $[t, T]$ , such that  $Y$  is a real-valued and  $Prog^t$ -measurable and  $Z : \Omega \times [t, T] \times U \rightarrow \mathbb{R}$ , endowed with the norm

$$\|(Y, Z)\|_{\mathbb{M}_\beta^{t,x}}^2 := C \|Y\|^{2,\beta} + \|Z\|_\nu^{2,\beta},$$

where

$$\|Y\|^{2,\beta} := \mathbb{E}^{t,x} \int_t^T e^{\beta s} |Y_s|^2 ds, \quad \|Z\|_\nu^{2,\beta} := \mathbb{E}^{t,x} \int_t^T \int_U e^{\beta s} |Z_s(y)|^2 \nu(s, X_s^{t,x}, dy) ds,$$

for some constants  $C > 0$  and  $\beta > 0$ .

**Theorem 4.1.** *Suppose that Assumption (2.5) holds for some  $(t, x) \in [0, T] \times U$ . Then there exists a unique solution to the reflected BSDE (5.3).*

*Proof.* Define the map

$$\Gamma : \mathbb{M}_\beta^{t,x} \rightarrow \mathbb{M}_\beta^{t,x},$$

such that for  $(\varphi, \psi) \in \mathbb{M}_\beta^{t,x}$ ,  $(Y, Z) = \Gamma(\varphi, \psi)$  is the solution of the following reflected BSDE

$$\begin{cases}
Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, \varphi_r, \psi_r(\cdot)) dr \\
\quad - \int_s^T \int_U Z_r^{t,x}(y) q^t(dr dy) + K_T^{t,x} - K_s^{t,x}, \quad \forall s \in [t, T] \\
Y_s^{t,x} \geq h(X_s^{t,x}), \quad \forall s \in [t, T] \\
\int_t^T (Y_{s-}^{t,x} - h(X_{s-}^{t,x})) dK_s^{t,x} = 0
\end{cases} \quad (4.1)$$

The map  $\Gamma$  is well defined: Indeed, for a fixed pair  $(\varphi, \psi) \in \mathbb{M}_\beta^{t,x}$ , thanks to Assumption (2.5)-(2), we have

$$\begin{aligned} \mathbb{E}^{t,x} \int_t^T |f(r, X_r^{t,x}, \varphi_r, \psi_r(\cdot))|^2 dr &\leq 3\mathbb{E}^{t,x} \left[ \int_t^T |f(r, X_r^{t,x}, 0, 0)|^2 dr + L' \int_t^T |\varphi_r|^2 dr \right] \\ &\quad + 3L\mathbb{E}^{t,x} \left[ \int_t^T \int_U |\psi_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr \right] < \infty, \end{aligned}$$

so by Proposition (3.1), there exist a unique triplet  $(Y, Z, K)$  solution to (4.1).

We show that  $\Gamma$  is a contraction on  $\mathbb{M}_\beta^{t,x}$ . Consider  $(\varphi', \psi')$  and  $(\varphi'', \psi'')$  be two elements in  $\mathbb{M}_\beta^{t,x}$  and let  $(Y', Z') = \Gamma(\varphi', \psi')$  and  $(Y'', Z'') = \Gamma(\varphi'', \psi'')$  be the associated solutions. Denote  $\bar{Y} = Y' - Y''$ ,  $\bar{Z} = Z' - Z''$ ,  $\bar{K} = K' - K''$ ,  $\bar{\varphi} = \varphi' - \varphi''$ ,  $\bar{\psi} = \psi' - \psi''$ , and  $\bar{f} = f(r, X_r^{t,x}, \varphi'_r, \psi'_r(\cdot)) - f(r, X_r^{t,x}, \varphi''_r, \psi''_r(\cdot))$ , hence  $(\bar{Y}, \bar{Z}, \bar{K})$  satisfies

$$\bar{Y}_s = \int_s^T \bar{f}_r dr - \int_s^T \int_U \bar{Z}_r(y) q^t(dr dy) + \bar{K}_T - \bar{K}_s.$$

From Itô's formula applied on  $e^{\beta s} |\bar{Y}_s|^2$ , we get

$$\begin{aligned} -e^{\beta t} |\bar{Y}_t|^2 &= \beta \int_t^T e^{\beta r} |\bar{Y}_r|^2 dr - 2 \int_t^T e^{\beta r} \bar{Y}_r \bar{f}_r dr + 2 \int_t^T \int_U e^{\beta r} \bar{Y}_r \bar{Z}_r(y) q^t(dr dy) \\ &\quad - 2 \underbrace{\int_t^T e^{\beta r} \bar{Y}_r d\bar{K}_r}_{\leq 0} + \underbrace{\sum_{t < r \leq T} e^{\beta r} |\Delta \bar{K}_r|^2}_{\geq 0} + \int_t^T \int_U e^{\beta r} |Z_r(y)|^2 q^t(dr, dy) \\ &\quad + \int_t^T \int_U e^{\beta r} |Z_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr. \end{aligned}$$

From the Lipschitz conditions of  $\bar{f}$  and the fact that the integrals with respect to  $q^t$  are martingales, we obtain

$$\begin{aligned} &\beta \mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{Y}_r|^2 dr + \mathbb{E}^{t,x} \int_t^T \int_U e^{\beta r} |Z_r(y)|^2 \nu(r, X_r^{t,x}, dy) dr \\ &\leq 2\mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{Y}_r| |\bar{f}_r| dr \\ &\leq 2L\mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{Y}_r| \left( \int_U |\bar{\psi}(y)|^2 \right)^{\frac{1}{2}} \nu(r, X_r^{t,x}, dy) dr + 2L'\mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{Y}_r| |\bar{\varphi}_r| dr \\ &\leq \delta \mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{\psi}(y)|^2 \nu(r, X_r^{t,x}, dy) dr + \frac{L^2}{\delta} \mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{Y}_r|^2 dr \\ &\quad + \lambda L' \mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{Y}_r|^2 dr + \frac{L'}{\lambda} \mathbb{E}^{t,x} \int_t^T e^{\beta r} |\bar{\varphi}_r|^2 dr \end{aligned}$$

for some constants  $\delta > 0$  and  $\lambda > 0$ , then it follows that

$$\left( \beta - \frac{L^2}{\delta} - \lambda L' \right) \|\bar{Y}\|^{2,\beta} + \|\bar{Z}\|_{\nu}^{2,\beta} \leq \delta \|\bar{\psi}\|_{\nu}^{2,\beta} + \frac{L'}{\lambda} \|\bar{\varphi}\|^{2,\beta},$$

by choosing  $\beta > L^2 + 2L'$ , it is possible to find  $\delta \in (0, 1)$  such that

$$\beta > \frac{L^2}{\delta} + \frac{2L'}{\sqrt{\delta}}.$$

If  $L' = 0$ , we see that  $\Gamma$  is  $\delta$ -contraction on  $\mathbb{M}_\beta^{t,x}$ , for  $C = \beta - \frac{L^2}{\delta}$ .

If  $L' > 0$ , we choose  $\lambda = \frac{1}{\delta}$ , hence we get

$$\frac{L'}{\sqrt{\delta}} \|\bar{Y}\|^{2,\beta} + \|\bar{Z}\|_{\nu}^{2,\beta} \leq \delta \left( \frac{L'}{\sqrt{\delta}} \|\bar{\varphi}\|^{2,\beta} + \|\bar{\psi}\|_{\nu}^{2,\beta} \right),$$

hence,  $\Gamma$  is a  $\delta$ -contraction on  $\mathbb{M}_\beta^{t,x}$ , for  $C = \frac{L'}{\sqrt{\delta}}$ . Therefore, using the contraction theorem, there exists a unique fixed point  $(Y, Z) \in \mathbb{M}_\beta^{t,x}$ , such that  $(Y, Z) = \Gamma(Y, Z)$  with the associated  $K$  is the solution to the reflected BSDE (5.3).  $\square$

## 5 Relation with quasi-variational inequalities

In this section, we present some quasi-variational inequalities associated to the Markov process  $X$  and we show that their solution can be represented probabilistically by means of an appropriate reflected BSDE of the type studied above.

Next, we suppose that the data  $(g, f, h)$  of the RBSDE satisfies Hypothesis (2.5) and the additional conditions

1.  $g : U \rightarrow \mathbb{R}$  is continuous and has at most polynomial growth at infinity.
2.  $f : [0, T] \times U \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is jointly continuous in  $t$  and  $x$  and for some  $C > 0$ ,  $p \in \mathbb{N}$ , satisfies

$$|f(t, x, 0, 0)| \leq (1 + |x|^p). \quad t \in [0, T], \quad x \in U. \quad (5.1)$$

3.  $h : [0, T] \times U \rightarrow \mathbb{R}$  is jointly continuous in  $t$  and  $x$  and for some  $C > 0$ ,  $p \in \mathbb{N}$ , satisfies

$$|h(t, x)| \leq (1 + |x|^p). \quad t \in [0, T], \quad x \in U. \quad (5.2)$$

We assume moreover that  $h(T, x) \leq g(x)$ , for all  $x \in U$ .

It follows from the results of the above sections that for each  $(t, x) \in [0, T] \times U$ , there exists a unique triplet  $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \mathbb{M}^{t,x}$  which solves the following reflected BSDE

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}(\cdot)) dr \\ \quad - \int_s^T \int_U Z_r^{t,x}(y) q^t(dr dy) + K_T^{t,x} - K_s^{t,x}, \quad \forall s \in [t, T] \\ Y_s^{t,x} \geq h(X_s^{t,x}), \quad \forall s \in [t, T] \\ \int_t^T (Y_{s^-}^{t,x} - h(X_{s^-}^{t,x})) dK_s^{t,x} = 0 \end{cases}. \quad (5.3)$$

We now consider the related problem of quasi-variational inequalities.

**Definition 5.1.** A solution of the quasi-variational inequalities problem is a function  $v : [0, T] \times U \rightarrow \mathbb{R}$  which satisfies

$$\begin{cases} \min \{v(t, x) - h(t, x), -\partial_t v(t, x) - \mathcal{L}_t v(t, x) - f(t, x, v(t, x), v(t, \cdot)) - v(t, x)\} = 0, \\ \quad t \in [0, T], x \in U, \\ v(T, x) = g(x), \end{cases} \quad (5.4)$$



where  $\mathcal{L}$  is the generator of the Markov process  $X$  defined by

$$\mathcal{L}_t \phi(x) = \int_U (\phi(y) - \phi(x)) \nu(t, x, dy), \quad t \in [0, T], \quad x \in U.$$

More precisely, we shall consider solutions of (5.4) in the viscosity sens.

**Definition 5.2.** a) A continuous function  $v$  is said to be a viscosity subsolution of (5.4) if  $v(T, x) \leq g(x)$ ,  $x \in U$  and if for any point  $(t_0, x_0) \in [0, T] \times U$  and for any  $\varphi \in C^{1,2}([0, T] \times U)$  such that  $\varphi(t_0, x_0) = v(t_0, x_0)$  and  $\varphi - v$  attains its minimum at  $(t_0, x_0)$ , then

$$\min \{v(t, x) - h(t, x), -\partial_t v(t, x) - \mathcal{L}_t v(t, x) - f(t, x, v(t, x), v(t, \cdot) - v(t, x))\} \leq 0.$$

b) A continuous function  $v$  is said to be a viscosity supersolution of (5.4) if  $v(T, x) \leq g(x)$ ,  $x \in U$  and if for any point  $(t_0, x_0) \in [0, T] \times U$  and for any  $\varphi \in C^{1,2}([0, T] \times U)$  such that  $\varphi(t_0, x_0) = v(t_0, x_0)$  and  $\varphi - v$  attains its maximum at  $(t_0, x_0)$ , then

$$\min \{v(t, x) - h(t, x), -\partial_t v(t, x) - \mathcal{L}_t v(t, x) - f(t, x, v(t, x), v(t, \cdot) - v(t, x))\} \geq 0.$$

c) A continuous function  $v$  is said to be a viscosity solution if it is both a viscosity subsolution and supersolution.

We now define

$$v(t, x) = Y_t^{t,x}, \quad (t, x) \in [0, T] \times U. \quad (5.5)$$

**Lemma 5.3.**  $(t, x) \mapsto v(t, x)$  is continuous.

*Proof.* We can define the solution  $Y_s^{t,x}$  for all  $s \in [0, T]$  by choosing

$$Y_s^{t,x} = Y_t^{t,x}, \quad \text{for all } 0 \leq s \leq t.$$

It suffice to show that whenever  $(t_n, x_n) \rightarrow (t, x)$

$$\mathbb{E}^{t,x} \sup_{0 \leq s \leq T} |Y_s^{t_n, x_n} - Y_s^{t,x}|^2 \rightarrow 0.$$

Let  $\bar{X} = X^{t_n, x_n} - X^{t,x}$ ,  $\bar{Y} = Y^{t_n, x_n} - Y^{t,x}$ ,  $\bar{Z} = Z^{t_n, x_n} - Z^{t,x}$ ,  $\bar{K} = K^{t_n, x_n} - K^{t,x}$  and  $\bar{f} = f(\cdot, X^{t_n, x_n}, Y^{t_n, x_n}, Z^{t_n, x_n}) - f(\cdot, X^{t,x}, Y^{t,x}, Z^{t,x})$ . Then by Itô's formula and Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned} & \mathbb{E}^{t,x} \sup_{0 \leq s \leq T} |Y_s^{t_n, x_n} - Y_s^{t,x}|^2 + \mathbb{E}^{t,x} \left( \int_s^T |Z_r^{t_n, x_n}(y) - Z_r^{t,x}(y)|^2 \nu(r, X_r^{t,x}, dy) dr \right) \\ & \leq \mathbb{E}^{t,x} |g(X_T^{t,x}) - g(X_T^{t_n, x_n})|^2 + \frac{1}{2} \mathbb{E}^{t,x} \sup_{0 \leq s \leq T} |Y_s^{t_n, x_n} - Y_s^{t,x}|^2 \\ & \quad + 4(T-t) \mathbb{E}^{t,x} \left( \int_t^T (f(r, X_r^{t_n, x_n}, Y_r^{t_n, x_n}, Z_r^{t_n, x_n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}))^2 dr \right) \\ & \quad + 4C \mathbb{E}^{t,x} \left( \int_s^T |Z_r^{t_n, x_n}(y) - Z_r^{t,x}(y)|^2 \nu(r, X_r^{t,x}, dy) dr \right) \\ & \quad + \mathbb{E}^{t,x} \left( \sup_{t \leq s \leq T} |h(s, X_s^{t_n, x_n}) - h(s, X_s^{t,x})|^2 \right) \mathbb{E}^{t,x} (K_T^{t_n, x_n})^2 \end{aligned}$$

hence for  $C < \frac{1}{4}$ , we get

$$\begin{aligned} \mathbb{E}^{t,x} \sup_{0 \leq s \leq T} |Y_s^{t_n, x_n} - Y_s^{t,x}|^2 &\leq \mathbb{E}^{t,x} |g(X_T^{t,x}) - g(X_T^{t_n, x_n})|^2 \\ &+ 4(T-t) \mathbb{E}^{t,x} \left( \int_t^T (f(r, X_r^{t_n, x_n}, Y_r^{t_n, x_n}, Z_r^{t_n, x_n}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}))^2 dr \right) \\ &+ \mathbb{E}^{t,x} \left( \sup_{t \leq s \leq T} |h(s, X_s^{t_n, x_n}) - h(s, X_s^{t,x})|^2 \right) \mathbb{E}^{t,x} (K_T^{t_n, x_n})^2 \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ , which follow from the continuity assumptions and the polynomial growth of  $f$ ,  $g$  and  $h$ .  $\square$

In the order to show the existence of solution to the equation (5.4), we are going to use the approximation of the reflected BSDE (5.3) by penalization. For each  $(t, x) \in [0, T] \times U$ ,  $n \in \mathbb{N}^*$ , let  $(Y_s^{n,t,x}, Z_s^{n,t,x})_{t \leq s \leq T}$  denote the solution of the BSDE

$$\begin{aligned} Y_s^{n,t,x} &= g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}(\cdot)) dr \\ &+ n \int_s^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dr - \int_s^T \int_U Z_r^{n,t,x}(y) q^t(dr dy). \end{aligned} \quad (5.6)$$

It is known from ([7] Theorem 4.4) that

$$v^n(t, x) = Y_t^{n,t,x}, \quad (t, x) \in [0, T] \times U,$$

is the solution of the nonlinear variation Kolmogorov equation

$$\begin{cases} \partial_t v^n(t, x) + \mathcal{L}_t v^n(t, x) + f^n(t, x, v^n(t, x), v^n(t, \cdot)) - v^n(t, x) = 0 \\ v^n(T, x) = g(x) \end{cases}$$

Where  $f^n(t, x, y, z) = f(t, x, y, z) + n(y - h(t, x))^-$ . Moreover, we suppose that

$$\sup_{t \in [0, T], x \in U} (|g(x)| + |f(t, x, 0, 0)|) < \infty. \quad (5.7)$$

Then, we have  $Y_s^{n,t,x} = v^n(s, X_s^{t,x})$  and  $Z_s^{n,t,x}(y) = v(s, y) - v^n(s, X_s^{t,x})$ .

**Theorem 5.4.** *For each  $(t, x) \in [0, T] \times U$ , we have*

$$\sup_{(t,x) \in [0, T] \times U} |v(t, x) - v^n(t, x)|^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

*Proof.* First, let  $(Y^{n,t,x}, Z^{n,t,x})$  be a solution of equation (5.6), then from ([7], Corollary 3.6), there exists a positive constant  $C$  depending only on  $T$ ,  $L$  and  $L'$ , such that

$$\mathbb{E}^{t,x} \int_t^T |Y_s^{n,t,x}|^2 ds + \mathbb{E}^{t,x} \int_t^T \int_K |Z_s^{n,t,x}(y)|^2 \nu(s, X_s^{t,x}, dy) ds \leq C. \quad (5.8)$$

Setting

$$K_s^{n,t,x} = n \int_t^s (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dr.$$

From now on, the proof will be divided into five steps and in the following  $C$  will be denoted as a constant whose value can vary from line to line.

Step 1: There exists an  $\mathcal{F}_s$ -adapted process  $(Y_s^{n,t,x})_{t \leq s \leq T}$  such that

$$\lim_{n \rightarrow \infty} E^{t,x} \int_t^T |Y_s^{n,t,x} - Y_s^{t,x}|^2 ds = 0.$$

Notice that for all  $n \geq 0$  and  $(s, x, y, z) \in [0, T] \times U \times \mathbb{R} \times \mathbb{R}$

$$f^n(s, x, y, z) \leq f^{n+1}(s, x, y, z),$$

Therefore, by the comparison theorem (A.1), we have

$$Y_s^{n,t,x} \leq Y_s^{n+1,t,x}, \quad \text{for all } s \in [t, T], \quad \mathbb{P}^{t,x}\text{-a.s.}$$

Hence,  $Y^{n,t,x} \nearrow Y^{t,x}$ .

In view of (5.8) and Fataou's Lemma, we have

$$E^{t,x} \sup_{t \leq s \leq T} |Y_s^{t,x}|^2 \leq C,$$

Then, it follows by the dominated convergence theorem

$$\lim_{n \rightarrow \infty} E^{t,x} \int_t^T |Y_s^{n,t,x} - Y_s^{t,x}|^2 ds = 0. \quad (5.9)$$

Step 2:  $\lim_{n \rightarrow \infty} \mathbb{E}^{t,x} \left[ \sup_{t \leq s < T} |(Y_s^{n,t,x} - h(s, X_s^{t,x}))^-|^2 \right] = 0$ .

Let  $(\bar{Y}^{n,t,x}, \bar{Z}^{n,t,x})_{t \leq s \leq T}$  be the solution of the following BSDE

$$\begin{aligned} \bar{Y}_s^{n,t,x} &= g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, \bar{Y}_r^{n,t,x}, \bar{Z}_r^{n,t,x}(\cdot)) dr \\ &\quad + n \int_s^T (h(r, X_r^{t,x}) - \bar{Y}_r^{n,t,x}) dr - \int_s^T \int_U \bar{Z}_r^{n,t,x}(y) q^t(dr dy), \end{aligned} \quad (5.10)$$

by the comparison theorem, we have for all  $t \leq s \leq T$  and  $n \geq 0$ ,  $Y_s^{n,t,x} \geq \bar{Y}_s^{n,t,x}$   $\mathbb{P}^{t,x}$ -a.s.

Now let  $\theta$  be a stopping time such that  $t \leq \theta \leq T$ . Then, applying Itô's formula to  $\bar{Y}_s^{n,t,x} e^{-n(s-\theta)}$ , we have

$$\begin{aligned} \bar{Y}_\theta^{n,t,x} &= e^{-n(T-\theta)} g(X_T^{t,x}) + n \int_\theta^T e^{-n(s-\theta)} \bar{Y}_s^{n,t,x} ds \\ &\quad + \int_\theta^T e^{-n(s-\theta)} f(s, X_s^{t,x}, \bar{Y}_s^{n,t,x}, \bar{Z}_s^{n,t,x}(\cdot)) ds \\ &\quad + n \int_s^T e^{-n(s-\theta)} (h(s, X_s^{t,x}) - \bar{Y}_s^{n,t,x}) ds - \int_s^T \int_U e^{-n(s-\theta)} \bar{Z}_s^{n,t,x}(y) q^t(ds dy), \end{aligned}$$

since the integral with respect to  $q^t(ds, sy)$  is a martingale, we get by taking the conditional expectation.

$$\begin{aligned} \bar{Y}_\theta^{n,t,x} &= \mathbb{E}^{t,x} \left[ e^{-n(T-\theta)} g(X_T^{t,x}) + \int_\theta^T e^{-n(s-\theta)} f(s, X_s^{t,x}, \bar{Y}_s^{n,t,x}, \bar{Z}_s^{n,t,x}(\cdot)) ds \right. \\ &\quad \left. + n \int_s^T e^{-n(s-\theta)} h(s, X_s^{t,x}) ds \mid \mathcal{F}_\theta \right]. \end{aligned}$$

It is easily seen that

$$e^{-n(T-\theta)}g(X_T^{t,x}) \rightarrow g(X_T^{t,x})\mathbb{1}_{\theta=T}$$

$\mathbb{P}^{t,x}$ -a.s and in  $\mathbb{L}^2(\Omega)$ . Moreover, using Itô's formula, we have

$$\begin{aligned} n \int_s^T e^{-n(s-\theta)}h(s, X_s^{t,x})ds &= h(\theta, X_\theta^{t,x})\mathbb{1}_{\theta < T} + \int_s^T e^{-n(r-\theta)}(\partial_r h(r, X_r^{t,x}) + \mathcal{L}_r h(r, X_r^{t,x}))dr \\ &\quad + \int_s^T \int_U e^{-n(r-\theta)}(h(r, y) - h(r, X_r^{t,x}))q^t(dr dy) \\ &\rightarrow h(\theta, X_\theta^{t,x})\mathbb{1}_{\theta < T}. \end{aligned}$$

$\mathbb{P}^{t,x}$ -a.s and in  $\mathbb{L}^2(\Omega)$ . Moreover,

$$\left| \int_\tau^T e^{-n(s-\theta)}f(s, \bar{Y}_s^{n,t,x}, \bar{Z}_s^{n,t,x})ds \right| \leq \frac{1}{\sqrt{2n}} \left( \int_0^T f^2(s, \bar{Y}_s^{n,t,x}, \bar{Z}_s^{n,t,x})ds \right)^{\frac{1}{2}},$$

hence

$$\mathbb{E}^{t,x} \left[ \int_\theta^T e^{-n(s-\theta)}f(s, X_s^{t,x}, \bar{Y}_s^{n,t,x}, \bar{Z}_s^{n,t,x}(\cdot))ds \mid \mathcal{F}_\theta \right] \rightarrow 0,$$

in  $\mathbb{L}^2(\Omega)$  as  $n \rightarrow \infty$ . Consequently

$$\bar{Y}_\theta^{n,t,x} \rightarrow g(X_T^{t,x})\mathbb{1}_{\theta=T} + h(\theta, X_\theta^{t,x})\mathbb{1}_{\theta < T} \quad \text{in } \mathbb{L}^2(\Omega) \text{ as } n \rightarrow \infty.$$

Therefore,  $Y_\theta^{t,x} \geq h(\theta, X_\theta^{t,x})$ . Hence, from this and the Section Theorem in [11], it follows that  $\mathbb{P}^{t,x}$ -a.s.  $Y_s^{t,x} \geq h(s, X_s^{t,x})$ ,  $t \leq s \leq T$  and then

$$(Y_s^{n,t,x} - h(s, X - s^{t,x}))^- \searrow 0, \quad t \leq s \leq T,$$

since  $(Y_s^{n,t,x} - h(s, X_s^{t,x}))^- \leq (h(s, X - s^{t,x}) - Y_s^{n,t,x})^+ \leq |h(s, X - s^{t,x})| + |\bar{Y}_s^{0,t,x}|$ , then, it follows from the dominated convergence theorem

$$\lim_{n \rightarrow \infty} E \left[ \sup_{t \leq s < T} \left| (Y_s^{n,t,x} - h(s, X_s^{t,x}))^- \right|^2 \right] = 0.$$

Step 3:  $\lim_{n \rightarrow \infty} \mathbb{E}^{t,x} \left[ \sup_{t \leq s < T} |Y_s^{n,t,x} - Y_s^{t,x}|^2 \right] = 0$  and there exist,  $\mathbb{F}$ -adapted processes  $Z^{t,x}$  and  $K^{t,x}$  such that

$$\lim_{n \rightarrow \infty} \mathbb{E}^{t,x} \left[ \int_t^T \int_U |Z_s^{n,t,x} - Z_s^{t,x}|^2 \nu(s, X_s^{t,x}, dy) ds + \sup_{t \leq s \leq T} |K_s^{n,t,x} - K_s^{t,x}|^2 \right] = 0.$$

Indeed, by Itô's formula and taking expectation in both sides, we have for all  $n \geq p \geq 0$  and

$t \leq s \leq T$

$$\begin{aligned}
& \mathbb{E}^{t,x} |Y_s^{n,t,x} - Y_s^{p,t,x}|^2 + \mathbb{E}^{t,x} \int_s^T \int_U |Z_r^{n,t,x} - Z_r^{p,t,x}|^2 \nu(s, X_s^{t,x}, dy) ds \\
& \leq 2\mathbb{E}^{t,x} \int_s^T (f(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}) - f(r, X_r^{t,x}, Y_r^{p,t,x}, Z_r^{p,t,x})) (Y_r^{n,t,x} - Y_r^{p,t,x}) dr \\
& \quad + 2\mathbb{E}^{t,x} \int_s^T (Y_r^{n,t,x} - Y_r^{p,t,x}) d(K_r^{n,t,x} - K_r^{p,t,x}) \\
& \leq 2L\mathbb{E}^{t,x} \int_s^T |Y_r^{n,t,x} - Y_r^{p,t,x}|^2 dr + 2L'\mathbb{E}^{t,x} \int_s^T \int_U |Y_s^{n,t,x} - Y_s^{p,t,x}| |Z_r^{n,t,x} - Z_r^{p,t,x}| \nu(r, X_r^{t,x}, dy) dr \\
& \quad + 2\mathbb{E}^{t,x} \int_s^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dK_r^{p,t,x} + 2\mathbb{E}^{t,x} \int_s^T (Y_r^{p,t,x} - h(r, X_r^{t,x}))^- dK_r^{n,t,x} \\
& \leq 2(L + L'^2 \sup_{t,x} \nu(t, x, U)) \mathbb{E}^{t,x} \int_s^T |Y_r^{n,t,x} - Y_r^{p,t,x}|^2 dr \\
& \quad + \frac{1}{2} \mathbb{E}^{t,x} \int_s^T \int_U |Z_r^{n,t,x}(y) - Z_r^{p,t,x}(y)| \nu(r, X_r^{t,x}, dy) dr \\
& \quad + 2 \int_s^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dK_r^{p,t,x} + 2\mathbb{E}^{t,x} \int_s^T (Y_r^{p,t,x} - h(r, X_r^{t,x}))^- dK_r^{n,t,x}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \mathbb{E}^{t,x} \int_s^T \int_U |Z_r^{n,t,x} - Z_r^{p,t,x}|^2 \nu(r, X_r^{t,x}, dy) dr \\
& \leq 2(L + L'^2 \sup_{t,x} \nu(t, x, U)) \mathbb{E}^{t,x} \int_s^T |Y_r^{n,t,x} - Y_r^{p,t,x}|^2 dr \\
& \quad + 2 \int_s^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dK_r^{p,t,x} \\
& \quad + 2\mathbb{E}^{t,x} \int_s^T (Y_r^{p,t,x} - h(r, X_r^{t,x}))^- dK_r^{n,t,x}.
\end{aligned}$$

Using the results of step 2, we get

$$\begin{aligned}
& \mathbb{E}^{t,x} \int_s^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dK_r^{p,t,x} \\
& \leq \left( \mathbb{E}^{t,x} \sup_{t \leq s \leq T} |(Y_r^{p,t,x} - h(r, X_r^{t,x}))^-|^2 \right)^{\frac{1}{2}} (\mathbb{E}^{t,x} (K_T^{n,t,x})^2)^{\frac{1}{2}} \rightarrow 0,
\end{aligned}$$

as  $n, p \rightarrow \infty$ .

$$\begin{aligned}
& \mathbb{E}^{t,x} \int_s^T (Y_r^{p,t,x} - h(r, X_r^{t,x}))^- dK_r^{n,t,x} \\
& \leq \left( \mathbb{E}^{t,x} \sup_{t \leq s \leq T} |(Y_r^{n,t,x} - h(r, X_r^{t,x}))^-|^2 \right)^{\frac{1}{2}} (\mathbb{E}^{t,x} (K_T^{p,t,x})^2)^{\frac{1}{2}} \rightarrow 0,
\end{aligned}$$

as  $n, p \rightarrow \infty$ .

Hence, it follows from that and (5.9)

$$\lim_{n \rightarrow \infty} \mathbb{E}^{t,x} \left[ \int_t^T \int_U |Z_s^{n,t,x} - Z_s^{t,x}|^2 \nu(s, X_s^{t,x}, dy) ds \right].$$

Moreover, using Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\mathbb{E}^{t,x} \left[ \sup_{t \leq s < T} |Y_s^{n,t,x} - Y_s^{t,x}|^2 \right] &\leq (2L + L'^2 \sup_{t,x} \nu(t,x,U)) \mathbb{E}^{t,x} \int_s^T |Y_r^{n,t,x} - Y_r^{p,t,x}|^2 dr \\
&+ L' \mathbb{E}^{t,x} \left[ \int_t^T \int_U |Z_s^{n,t,x} - Z_s^{p,t,x}|^2 \nu(s, X_s^{t,x}, dy) ds \right] + 2 \int_t^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dK_r^{p,t,x} \\
&+ 2 \mathbb{E}^{t,x} \int_t^T (Y_r^{p,t,x} - h(r, X_r^{t,x}))^- dK_r^{n,t,x} \\
&+ 2 \mathbb{E}^{t,x} \sup_{t \leq s < T} \left| \int_s^T \int_U |(Y_r^{n,t,x} - Y_r^{p,t,x})(Z_s^{n,t,x} - Z_s^{p,t,x})q(dr dy)| \right| \\
&\leq (2L + L'^2 \sup_{t,x} \nu(t,x,U)) \mathbb{E}^{t,x} \int_s^T |Y_r^{n,t,x} - Y_r^{p,t,x}|^2 dr \\
&+ L' \mathbb{E}^{t,x} \left[ \int_t^T \int_U |Z_s^{n,t,x} - Z_s^{p,t,x}|^2 \nu(s, X_s^{t,x}, dy) ds \right] + 2 \int_s^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dK_r^{p,t,x} \\
&+ 2 \mathbb{E}^{t,x} \int_s^T (Y_r^{p,t,x} - h(r, X_r^{t,x}))^- dK_r^{n,t,x} + \frac{1}{2} \mathbb{E}^{t,x} \sup_{t \leq s < T} |Y_s^{n,t,x} - Y_s^{t,x}|^2 \\
&\quad + C \mathbb{E}^{t,x} \int_t^T \int_U |Z_s^{n,t,x} - Z_s^{p,t,x}|^2 \nu(s, X_s^{t,x}, dy) ds,
\end{aligned}$$

hence

$$\mathbb{E}^{t,x} \left[ \sup_{t \leq s < T} |Y_s^{n,t,x} - Y_s^{t,x}|^2 \right] \longrightarrow 0$$

as  $n, p \rightarrow \infty$ , and then

$$\lim_{n \rightarrow \infty} \mathbb{E}^{t,x} \left[ \sup_{t \leq s < T} |Y_s^{n,t,x} - Y_s^{t,x}|^2 \right] = 0.$$

Finally, since for all  $n \leq 0$ ,  $t \leq s \leq T$

$$\begin{aligned}
K_s^{n,t,x} - K_s^{p,t,x} &= Y_s^{n,t,x} - Y_s^{p,t,x} - (Y_t^{n,t,x} - Y_t^{p,t,x}) \\
&- \int_t^s f(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}) - f(r, X_r^{t,x}, Y_r^{p,t,x}, Z_r^{p,t,x}) dr \\
&+ \int_t^s \int_U Z_s^{n,t,x}(y) - Z_s^{p,t,x}(y) q(dr dy),
\end{aligned}$$

Hence, it follows

$$\begin{aligned}
\sup_{t \leq s \leq T} |K_s^{n,t,x} - K_s^{p,t,x}|^2 &\leq 4|Y_s^{n,t,x} - Y_s^{p,t,x}|^2 + 4|(Y_t^{n,t,x} - Y_t^{p,t,x})|^2 \\
&+ 8L \sup_{(t,x)} \nu(t,x,U) \int_t^T |Y_r^{n,t,x} - Y_r^{p,t,x}|^2 dr \\
&+ 8L'(T-t) \int_t^T \int_U |Z_r^{n,t,x} - Z_r^{p,t,x}|^2 \nu(r, X_r^{t,x}, dy) dr \\
&+ 4C \left( \int_t^T \int_U (Z_r^{n,t,x} - Z_r^{p,t,x})^{\frac{1}{2}} \nu(r, X_r^{t,x}, dy) dr \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\mathbb{E}^{t,x} \sup_{t \leq s \leq T} |K_s^{n,t,x} - K_s^{p,t,x}|^2 \rightarrow 0, \quad \text{as } n, p \rightarrow \infty.$$

Consequently, there exists a pair of ( $\mathbb{F}$ -adapted processes  $(Z^{t,x}, K^{t,x})$  such that

$$\lim_{n \rightarrow \infty} \mathbb{E}^{t,x} \left[ \int_t^T \int_U |Z_s^{n,t,x} - Z_s^{t,x}|^2 \nu(s, X_s^{t,x}, dy) ds + \sup_{t \leq s \leq T} |K_s^{n,t,x} - K_s^{t,x}|^2 \right] = 0.$$

Step 4: The limit process  $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$  is the solution to the reflected BSDE (5.3). Obviously, the process  $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$  satisfies

$$\begin{aligned} Y_s^{t,x} &= g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}(\cdot)) dr \\ &\quad - \int_s^T \int_U Z_r^{t,x}(y) q^t(dr dy) + K_T^{t,x} - K_s^{t,x}, \quad \forall s \in [t, T] \end{aligned}$$

Also using step 1, we get

$$\mathbb{E}^{t,x} (h(s, X_s^{t,x}) - Y_s^{t,x})^2 \leq \liminf_{n \rightarrow \infty} \mathbb{E}^{t,x} \left( \sup_{t \leq s \leq T} |(Y_s^{n,t,x} - h(s, X_s^{t,x}))^-|^2 \right) \rightarrow 0,$$

Hence

$$Y_s^{t,x} \leq h(s, X_s^{t,x}), \quad t \leq s \leq T, \quad \mathbb{P}^{t,x}\text{-a.s.} \quad (5.11)$$

Clearly,  $K^{t,x}$  is increasing. Moreover, we have just seen  $(Y_s^{t,x}, K_s^{t,x})$  tends to  $(Y_s^{t,x}, K_s^{t,x})$  uniformly in  $s$  in probability. Then the measure  $dK^{n,t,x}$  tends to  $dK^{t,x}$  weakly in probability and

$$\int_t^T Y_s^{n,t,x} - h(s, X_s^{t,x}) dK_s^{n,t,x} \rightarrow \int_t^T Y_s^{t,x} - h(s, X_s^{t,x}) dK_s^{t,x}.$$

Using (5.11), we get obviously

$$\int_t^T Y_s^{t,x} - h(s, X_s^{t,x}) dK_s^{t,x} \geq 0.$$

On the other hand

$$\int_t^T Y_s^{n,t,x} - h(s, X_s^{t,x}) dK_s^{n,t,x} \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

Hence

$$\int_t^T Y_s^{t,x} - h(s, X_s^{t,x}) dK_s^{t,x} = 0.$$

Consequently, we have

$$\begin{aligned} |v^n(t, x) - v(t, x)|^2 &= |Y_t^{n,t,x} - Y_t^{t,x}|^2 \\ &\leq \mathbb{E}^{t,x} \left[ \sup_{t \leq s \leq T} |Y_s^{n,t,x} - Y_s^{t,x}|^2 \right] \rightarrow 0. \end{aligned}$$

□

**Theorem 5.5.** *The function  $v$  defined by the equation (5.5) is a viscosity solution of the equation (5.4).*

*Proof.* First, let us show that  $u$  is a viscosity subsolution of equation (5.4). Let  $(t_0, x_0) \in [0, T] \times U$  and  $\varphi \in C^{1,2}([0, T] \times U)$  be such that  $\varphi(t_0, x_0) = v(t_0, x_0)$ . Suppose that  $h(t_0, x_0) < v(t_0, x_0)$  and that

$$-\partial_s \varphi(t_0, x_0) - \mathcal{L}_s \varphi(t_0, x_0) - f(t_0, x_0, v(t_0, x_0), \varphi(t_0, \cdot) - \varphi(t_0, x_0)) > 0,$$

and we will find a contradiction.

By continuity, we can suppose that there exists  $\epsilon > 0$  and  $\eta_\epsilon > 0$  such that for each  $(t, x)$ ,  $t_0 \leq t \leq t_0 + \eta_\epsilon$  and  $|x - x_0| \leq \epsilon$ , we have  $v(t, x) \geq h(t, x) + \epsilon$  and

$$-\partial_t \varphi(t, x) - \mathcal{L}_t \varphi(t, x) - f(t, x, v(t, x), \varphi(t, \cdot) - \varphi(t, x)) > \epsilon. \quad (5.12)$$

We define the stopping time

$$\tau = \inf\{s \geq t_0 : |X_s^{t_0, x_0} - x_0| > \eta_\epsilon\} \wedge (t_0 + \eta_\epsilon). \quad (5.13)$$

By definition of stopping time, we have for  $t_0 \leq s \leq \tau$

$$v(s, X_s^{t_0, x_0}) \geq h(s, X_s^{t_0, x_0}) + \epsilon.$$

Consequently, the process  $(X_s^{t_0, x_0})$  is constant on  $[t_0, \tau]$  and, hence,

$$Y_s^{t_0, x_0} = Y_\tau^{t_0, x_0} + \int_s^\tau f(r, X_r^{t_0, x_0}, Y_r^{t_0, x_0}, Z_r^{t_0, x_0}) dr - \int_s^\tau \int_U Z_r^{t_0, x_0}(y) q^{t_0}(dr, dy), \quad t_0 \leq s \leq \tau.$$

On the other hand, applying Itô's formula to  $\varphi(s, X_s^{t_0, x_0})$  gives  $\mathbb{P}^{t_0, x_0}$ -a.s.

$$\begin{aligned} \varphi(s, X_s^{t_0, x_0}) &= \varphi(\tau, X_\tau^{t_0, x_0}) - \int_s^\tau (\partial_r \varphi(r, X_r^{t_0, x_0}) + \mathcal{L}_r \varphi(r, X_r^{t_0, x_0})) dr \\ &\quad - \int_s^\tau \int_U (\varphi(r, y) - \varphi(r, X_{r-}^{t_0, x_0})) q^{t_0}(dr dy), \quad s \in [t_0, T]. \end{aligned}$$

We can see that  $(\varphi(s, X_s^{t_0, x_0}), \varphi(s, \cdot) - \varphi(s, X_s^{t_0, x_0}))$  is the solution of the BSDE corresponding to the terminal  $\varphi(\tau, X_\tau^{t_0, x_0})$  condition and the generator  $-\partial_s \varphi(s, X_s^{t_0, x_0}) - \mathcal{L}_s \varphi(s, X_s^{t_0, x_0})$ . Now by assumption (5.12)

$$-[\partial_s \varphi(s, X_s^{t_0, x_0}) + \mathcal{L}_s \varphi(s, X_s^{t_0, x_0})] - f(s, X_s^{t_0, x_0}, v(s, X_s^{t_0, x_0}), \varphi(s, \cdot) - \varphi(s, X_s^{t_0, x_0})) > \epsilon$$

$$-\partial_t \varphi(t, x) - \mathcal{L}_t \varphi(t, x) - f(t, x, v(t, x), \varphi(t, \cdot) - \varphi(t, x)) > \epsilon, \quad t_0 \leq s \leq \tau.$$

Also

$$\varphi(\tau, X_\tau^{t_0, x_0}) \geq v(\tau, X_\tau^{t_0, x_0}) = Y_\tau^{t_0, x_0}.$$

Consequently, using comparison theorem (A.1), we get

$$\varphi(t_0, x_0) > \varphi(t_0, x_0) - \epsilon(\tau - t_0) \geq v(t_0, x_0),$$

which leads to a contradiction.

Next, let us show that  $v$  is a viscosity supersolution of equation (5.4).

Let  $(t_0, x_0) \in [0, T] \times U$  and  $\varphi \in C^{1,2}([0, T] \times U)$  be such that  $\varphi(t_0, x_0) = v(t_0, x_0)$ .

Since the solution  $Y_s^{t_0, x_0}$  stays above the obstacle  $h$ , hence,  $v(t_0, x_0) \geq h(t_0, x_0)$ . We have to show that

$$-\partial_s \varphi(t_0, x_0) - \mathcal{L}_s \varphi(t_0, x_0) - f(t_0, x_0, v(t_0, x_0), \varphi(t_0, \cdot) - \varphi(t_0, x_0)) \geq 0.$$



We suppose that

$$-\partial_s \varphi(t_0, x_0) - \mathcal{L}_s \varphi(t_0, x_0) - f(t_0, x_0, v(t_0, x_0), \varphi(t_0, \cdot) - \varphi(t_0, x_0)) < 0.$$

By continuity, we can suppose that there exists  $\epsilon > 0$  and  $\eta_\epsilon > 0$  such that for each  $(t, x)$ ,  $t_0 \leq t \leq t_0 + \eta_\epsilon$  and  $|x - x_0| \leq \epsilon$ , we have

$$-\partial_t \varphi(t, x) - \mathcal{L}_t \varphi(t, x) - f(t, x, v(t, x), \varphi(t, \cdot) - \varphi(t, x)) \leq -\epsilon. \quad (5.14)$$

We define the stopping time

$$\tau = \inf\{s \geq t_0 : |X_s^{t_0, x_0} - x_0| > \eta_\epsilon\} \wedge (t_0 + \eta_\epsilon). \quad (5.15)$$

By definition of stopping time, we have for  $t_0 \leq s \leq \tau$

$$Y_s^{t_0, x_0} = Y_\tau^{t_0, x_0} + \int_s^\tau f(r, X_r^{t_0, x_0}, Y_r^{t_0, x_0}, Z_r^{t_0, x_0}) dr - \int_s^\tau \int_U Z_r^{t_0, x_0}(y) q^{t_0}(dr, dy) + K_\tau^{t_0, x_0} - K_s^{t_0, x_0}.$$

On the other hand, as in the case of subsolution, by applying Itô's formula to  $\varphi(s, X_s^{t_0, x_0})$ , we get  $\mathbb{P}^{t_0, x_0}$ -a.s.

$$\begin{aligned} \varphi(s, X_s^{t_0, x_0}) &= \varphi(\tau, X_\tau^{t_0, x_0}) - \int_s^\tau (\partial_r \varphi(r, X_r^{t_0, x_0}) + \mathcal{L}_r \varphi(r, X_r^{t_0, x_0})) dr \\ &\quad - \int_s^\tau \int_U (\varphi(r, y) - \varphi(r, X_{r-}^{t_0, x_0})) q^{t_0}(dr dy), \quad s \in [t_0, T]. \end{aligned}$$

Note that the couple  $(\varphi(s, X_s^{t_0, x_0}), \varphi(s, \cdot) - \varphi(s, X_s^{t_0, x_0}))$  is the solution of the BSDE corresponding to the terminal  $\varphi(\tau, X_\tau^{t_0, x_0})$  condition and the generator  $-\partial_s \varphi(s, X_s^{t_0, x_0}) - \mathcal{L}_s \varphi(s, X_s^{t_0, x_0})$ . Now by assumption (5.14)

$$-[\partial_s \varphi(s, X_s^{t_0, x_0}) + \mathcal{L}_s \varphi(s, X_s^{t_0, x_0})] - [f(s, X_s^{t_0, x_0}, v(s, X_s^{t_0, x_0}), \varphi(s, \cdot) - \varphi(s, X_s^{t_0, x_0})) + dK_s^{t_0, x_0}] \leq -\epsilon$$

Also

$$\varphi(\tau, X_\tau^{t_0, x_0}) \geq v(\tau, X_\tau^{t_0, x_0}) = Y_\tau^{t_0, x_0}.$$

Consequently, using comparison theorem (A.1), we get

$$\varphi(t_0, x_0) < \varphi(t_0, x_0) + \epsilon(\tau - t_0) \leq Y_{t_0}^{t_0, x_0} = v(t_0, x_0),$$

which leads to a contradiction.  $\square$

## A Comparison theorem for BSDE associated to jump Markov processes

In this section we establish a comparison theorem for BSDE associated to jump Markov processes. Consider the following BSDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}(\cdot)) dr - \int_s^T \int_U Z_r^{t,x}(y) q^t(dr dy). \quad (A.1)$$

The solution is a couple  $(Y^{t,x}, Z^{t,x})$ , which exist and unique from [7].

In the following  $\mathcal{E}(H)$  denotes the Doleans-Dade exponential of the process

$$\int_t^\cdot \int_E (H_s(e) - 1) q(dsde),$$

that is

$$\mathcal{E}(H) = \prod_{n \geq 1} H_{T_n}(X_{T_n}) e^{\int_t^s \int_U (1 - H_r(y)) \nu(r, X_r^{t,x}, dy) dr}.$$

We now introduce for a  $\mathcal{P}^t \otimes \mathcal{U}$ -measurable and positive function  $\lambda$  the supermartingale

$$L_s^\lambda = \mathcal{E}(\lambda) = \prod_{n \geq 1} \lambda_{T_n}(X_{T_n}) e^{\int_t^s \int_U (1 - \lambda_s(y)) \nu(r, X_r^{t,x}, dy) dr}.$$

When  $L^\lambda$  is a martingale, we define the absolutely continuous probability  $\mathbb{Q}^{t,x} \ll \mathbb{P}^{t,x}$  as

$$\frac{\mathbb{Q}^{t,x}}{\mathbb{P}^{t,x}} = L_T^\lambda, \quad \text{for all } (t, x) \in [0, T] \times U.$$

In this case, under  $\mathbb{Q}^{t,x}$  the compensator of  $p$  becomes

$$\lambda_s(y) \nu(r, X_r^{t,x}, dy) dr.$$

Assume that for all  $(s, y) \in [t, T] \times U$ ,  $0 \leq \lambda_s(y) \leq M$ . Hence, from ([17], Proposition 2.2),  $L^\lambda$  is a  $\mathbb{P}^{t,x}$ -martingale, and for any process  $H \in \mathbb{L}^1(p)$ , the process

$$\left( \int_t^s \int_U H_r(y) (p(dr, dy) - \lambda_r(dy) \nu(r, X_r^{t,x}, dy)) \right)_{t \leq s \leq T}$$

is a  $\mathbb{Q}^{t,x}$ -martingale.

**Theorem A.1.** *Let  $(g^i, f^i)_{i=1,2}$  be two sets of data for which hypotheses (2.5)-(1)(2) hold. Let  $(Y^{t,x,i}, Z^{t,x,i})_{i=1,2}$  be the corresponding solutions. Assume that*

$$g^2(X_T^{t,x}) \leq g^1(X_T^{t,x}) \text{ and } f^2(s, X_s^{t,x}, Y_s^{t,x,1}, Z_s^{t,x,1}) \leq f^1(s, X_s^{t,x}, Y_s^{t,x,1}, Z_s^{t,x,1}) \quad \mathbb{P}^{t,x}\text{-a.s.}$$

for all  $s \in [t, T]$ . Assume there is a  $\mathcal{P}^t \otimes \mathcal{U}$ -measurable function  $\lambda$  such that  $0 \leq \lambda_s(y) \leq M$  and for all  $(s, x, y) \in [t, T] \times U \times \mathbb{R}$  and  $z^2, z^1 \in \mathcal{L}^2(U, \mathcal{U}, \nu(s, X_s^{t,x}, dy))$

$$f^2(s, x, y, z^2) - f^2(s, x, y, z^1) \leq \int_U \lambda(e) (z^2(e) - z^1(e)) \nu(s, x, de).$$

Then we have that  $Y_s^{t,x,2} \leq Y_s^{t,x,1}$   $\mathbb{P}^{t,x}$ -a.s. for all  $s \in [t, T]$ .

*Proof.* Define

$$\begin{aligned} \bar{g} &= g^2(X_T^{t,x}) - g^1(X_T^{t,x}), \quad \bar{f}_s = f^2(s, X_s^{t,x}, Y_s^{t,x,1}, Z_s^{t,x,1}) - f^1(s, X_s^{t,x}, Y_s^{t,x,1}, Z_s^{t,x,1}), \\ \bar{Y} &= Y^{t,x,2} - Y^{t,x,1} \text{ and } \bar{Z} = Z^{t,x,2} - Z^{t,x,1}. \end{aligned}$$

$\bar{Y}$  satisfies

$$\begin{aligned} \bar{Y}_s &= \bar{g} + \int_s^T (f^2(s, X_r^{t,x}, Y_r^{t,x,2}, Z_r^{t,x,2}) - f^1(r, X_r^{t,x}, Y_r^{t,x,1}, Z_r^{t,x,1})) dr \\ &\quad - \int_s^T \int_U \bar{Z}_r(y) q^t(dr, dy). \end{aligned}$$

Setting

$$\phi_s = \frac{f^2(s, X_s^{t,x}, Y_s^{t,x,2}, Z_s^{t,x,2}) - f^2(s, X_s^{t,x}, Y_s^{t,x,1}, Z_s^{t,x,1})}{\bar{Y}_s} \mathbb{1}_{\{\bar{Y}_s \neq 0\}} \leq L$$

Thus, the equation can be written as

$$\begin{aligned} \bar{Y}_s = \bar{g} + \int_s^T \phi_r \bar{Y}_r dr + \int_s^T (f^2(s, X_r^{t,x}, Y_r^{t,x,1}, Z_r^{t,x,1}) - f^2(r, X_r^{t,x}, Y_r^{t,x,1}, Z_r^{t,x,2})) dr \\ + \int_s^T \bar{f}_r dr - \int_s^T \int_U \bar{Z}_r(y) q^t(dr, dy). \end{aligned}$$

Consider now the positive process  $(\Gamma_s)_{t \leq s \leq T}$  defined by

$$\Gamma_s = e^{\int_t^s \phi_r dr}, \quad \Gamma_t = 1.$$

We now apply Itô's formula to  $\Gamma \bar{Y}$  between  $s$  and  $T$ , we get

$$\begin{aligned} \Gamma_T \bar{g} = \Gamma_s \bar{Y}_s - \int_s^T \Gamma_r (f^2(s, X_r^{t,x}, Y_r^{t,x,2}, Z_r^{t,x,1}) - f^2(r, X_r^{t,x}, Y_r^{t,x,1}, Z_r^{t,x,2})) dr \\ - \int_s^T \Gamma_r \bar{f}_r dr + \int_s^T \int_U \Gamma_r \bar{Z}_r(y) q^t(dr, dy). \end{aligned}$$

Since  $\bar{g}$  and  $\bar{f}$  are non-positive, we obtain

$$\begin{aligned} \Gamma_s \bar{Y}_s &\leq \int_s^T \Gamma_r (f^2(s, X_r^{t,x}, Y_r^{t,x,2}, Z_r^{t,x,1}) - f^2(r, X_r^{t,x}, Y_r^{t,x,1}, Z_r^{t,x,2})) dr \\ &\quad - \int_s^T \int_U \Gamma_r \bar{Z}_r(y) q^t(dr, dy) \\ &\leq \int_s^T \Gamma_r \lambda_r(y) \bar{Z}_r(y) \nu(r, X_r^{t,x}, dy) dr - \int_s^T \int_U \Gamma_r \bar{Z}_r(y) q^t(dr, dy). \end{aligned}$$

Consider  $\eta > 0$ . Then for  $\epsilon < \eta$  we add to both sides of the previous inequality the term

$$\epsilon \int_s^T \int_U \Gamma_r \bar{Z}_r(y) \nu(r, X_r^{t,x}, dy) dr.$$

Obtaining

$$\begin{aligned} \Gamma_s \bar{Y}_s + \epsilon \int_s^T \int_U \Gamma_r \bar{Z}_r(y) \nu(r, X_r^{t,x}, dy) dr \\ \leq - \int_s^T \int_U \Gamma_r \bar{Z}_r(y) [p(dr, dy) - (\lambda_r(y) + \epsilon + 1) \nu(r, X_r^{t,x}, dy) dr]. \end{aligned}$$

Now we can consider  $\lambda_r(y) + \epsilon + 1$  as Girsanov kernel and then we can introduce the equivalent probability

$$\mathbb{Q}^{t,x,\epsilon} = L_T^{\lambda+\epsilon+1} \mathbb{P}^{t,x}, \quad \text{for all } (t,x) \in [0, T] \times U.$$

Therefore, any  $\mathbb{P}^{t,x}$ -martingale is a  $\mathbb{Q}^{t,x,\epsilon}$ -martingale, thus by taking  $\mathbb{Q}^{t,x,\epsilon}$  conditional expectation on  $\mathcal{F}_s$  in the previous inequality, we get for any  $\epsilon < \eta$

$$\Gamma_s \bar{Y}_s + \epsilon \mathbb{E}^{\mathbb{Q}^{t,x,\epsilon}} \left[ \int_s^T \int_U \Gamma_r \bar{Z}_r(y) \nu(r, X_r^{t,x}, dy) dr \mid \mathcal{F}_s \right] \leq 0, \quad \mathbb{Q}^{t,x,\epsilon}\text{-a.s.}$$

and then

$$\Gamma_s \bar{Y}_s + \epsilon \mathbb{E}^{\mathbb{Q}^{t,x,\epsilon}} \left[ \int_s^T \int_U \Gamma_r \bar{Z}_r(y) \nu(r, X_r^{t,x}, dy) dr \mid \mathcal{F}_s \right] \leq 0, \quad \mathbb{P}^{t,x}\text{-a.s.}$$

In the order to conclude the proof, we have to show that

$$\epsilon \mathbb{E}^{\mathbb{Q}^{t,x,\epsilon}} \left[ \int_s^T \int_U \Gamma_r \bar{Z}_r(y) \nu(r, X_r^{t,x}, dy) dr \mid \mathcal{F}_s \right] \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

To this end, we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}^{t,x,\epsilon}} \left[ \int_s^T \int_U \Gamma_r \bar{Z}_r(y) \nu(r, X_r^{t,x}, dy) dr \mid \mathcal{F}_s \right] \\ &= \mathbb{E}^{t,x} \left[ \frac{L_T^{\lambda+\epsilon+1}}{L_s^{\lambda+\epsilon+1}} \int_s^T \int_U \Gamma_r \bar{Z}_r(y) \nu(r, X_r^{t,x}, dy) dr \mid \mathcal{F}_s \right] \\ &\leq T \mathbb{E}^{t,x} \left[ \left( \frac{L_T^{\lambda+\epsilon+1}}{L_s^{\lambda+\epsilon+1}} \right)^2 \mid \mathcal{F}_s \right]^{\frac{1}{2}} \mathbb{E}^{t,x} \left[ \int_s^T \int_U \Gamma_r^2 \bar{Z}_r^2(y) \nu(r, X_r^{t,x}, dy) dr \mid \mathcal{F}_s \right]^{\frac{1}{2}}. \end{aligned}$$

From ([17], Proposition 2.2), we have

$$\mathbb{E}^{t,x} \left[ \left( \frac{L_T^{\lambda+\epsilon+1}}{L_s^{\lambda+\epsilon+1}} \right)^2 \mid \mathcal{F}_s \right] \leq e^{3+T(M+\eta+1)^4 \sup_{t,x} \nu(t,x,U)}, \quad \mathbb{P}^{t,x}\text{-a.s.}$$

On the other hand, we have

$$\begin{aligned} & \mathbb{E}^{t,x} \left[ \int_s^T \int_U \Gamma_r^2 \bar{Z}_r^2(y) \nu(r, X_r^{t,x}, dy) dr \mid \mathcal{F}_s \right] \\ &\leq e^{2(T-t)L} \mathbb{E}^{t,x} \left[ \int_s^T \int_U \bar{Z}_r^2(y) \nu(r, X_r^{t,x}, dy) dr \mid \mathcal{F}_s \right], \quad \mathbb{P}^{t,x}\text{-a.s.} \end{aligned}$$

Hence the theorem is proven.  $\square$

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