Upscaling of a double porosity problem with jumps in thin porous media

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Abstract

We study the homogenization of a double porosity diffusion problem in a thin highly heterogeneous composite medium formed by two materials separated by an imperfect interface, where the solution and its flux exhibit jumps. By applying homogenization techniques specific to the thin periodic domain under study, one derives the limit problem.

Key words: thin domain, porous medium, imperfect interface, asymptotic methods.

AMS subject classification: 76A20, 76S05, 74A50, 76M45.

1 Introduction

Our goal in this paper is to study a diffusion problem in a specific thin porous medium, namely a composite material formed by two constituents occupying a thin three-dimensional domain of height ε denoted Ω^ε. The result is still valid in $\mathbb{R}^d\,(d \geq 2)$, but the three-dimensional case is physically more relevant, if we think, for instance, to applications in filtering type problems. The domain Ω^ε is divided in two open subdomains, denoted by Ω_1^ε and Ω_2^ε and separated by an imperfect interface Γ^ε. The subdomain Ω_2^ε, assumed to be disconnected, is formed by inclusions which are ε- periodically distributed in an horizontal layer and do not touch the boundary of Ω^ε (see Figure 1). The domain described above is a mathematical model corresponding to the physical situation of a filtering material (for instance, textile, paper, biological tissue, soil) constituted of three thin horizontal layers of total height ε. Two of the layers - the top and the bottom one, respectively - are identical (material 1) and the third one in between is formed by two materials with high contrasting permeability properties. This last layer, which plays the role of a filter for the structure, is made up of the two materials 1 and 2, the material with high porosity (corresponding to the subdomain Ω_2^ε) being fully included in this layer.

In such a geometry, we analyze the asymptotic behavior, as the small parameter ε tends to zero, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of problem (2.2) stated below. We first remark that, as already mentioned, the order of magnitude of the permeability of the material occupying the domain Ω_2^ε is very low (so the porosity is high), of order ε^2, while the permeability of the material occupying the domain Ω_1^ε is supposed to be of order one. We next remark that the flux of the solution is discontinuous across the interface separating the two materials (see Remark 2.1) and depends linearly on the jump of the solution. This problem presents various sources of singularities, all described in terms of the small positive parameter ε: the geometric one related to the interspersed periodic distribution of the components, the material one related to the permeabilities and the ones generated by the presence of an imperfect interface between the two materials.

A similar problem, in the same geometry, was addressed in [18] and in [[4], Section 7]. The main difference for our study is the presence of jumps, both in the solution and in its flux, while in the above mentioned papers the solution of the initial problem does not have any jump. Problems with jumps were already encountered in classical porous media, but, as far as we know, they are new in the context of the thin porous media we consider.

Our aim is to study the behavior of problem (2.2) when ε tends to zero. We apply the periodic unfolding method (see [16] and the references therein), with operators suited to the present geometry. We study two

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representative cases for the function G^ε describing the flux jump, stated explicitly in Section 2, relations (2.3) and (2.4), which lead in the limit to different modified regularized models of diffusion. Both models, presented in Theorem 4.2 and Theorem 4.5, respectively, are special cases of double porosity models, in which the dimension reduction phenonomenon occurs. The effect of the small height of the domain Ω^ε is reflected in the fact that in the limit the diffusion occurs only in the horizontal bi-dimensional domain ω, where the homogenized problems are stated (see (4.9) and (4.23), respectively). Nevertheless, the solution of these homogenized problems keeps track of the local vertical variable y_3 via the values of the constant homogenized coefficients (see Remark 4.3). The combined effects of the small height, the disconnected geometry of the second material Ω_2^ε and the low permeability in Ω_2^ε are reflected in the fact that the limit function (4.11) depends on the third variable. As far as the flux jump is concerned, in the first case, a new global source term, macroscopically distributed over the bi-dimensional domain ω, appears in the right-hand side of the homogenized equation (4.9). In the second case, we note the emergence of the non-homogeneous Neumann cell problem (4.25) and the presence of its solution in the corrector (4.24).

For mathematical studies of similar diffusion problems in thin periodic media, we refer, for instance, to [12], [18], [28], [31], [34], [30], [22], [24], [25] and the references therein. For elasticity problems in related thin periodic domains, we refer to [13], [17], [36], [14], [15], [26], [27]. For flow problems in thin porous media, we refer, for example, to [4], [21], [5], [6]. Double porosity phenomena are treated in [7], [1], [39], [40], [4], [28], [31], [38], [20], [10] (see, also, the references therein). The first study of diffusion problems with jumps in solution is due to [8] and since then there is a wide mathematical literature on the topic; it is impossible and out of the scope of this paper to mention all the existing relevant results in this area (see, for instance, [33], [39], [40], [41], [19] and the references therein). The study of problems with jumps in flux is much more recent in the mathematical literature; we refer the reader to [23], [29], [9], [10], [11], [2], [3].

The paper is organized as follows: in Section 2, we introduce the microscopic problem and we fix the notation. In Section 3, we give the appropriate unfolding operators and their properties. In Section 4, we state and prove the main homogenization results of this paper.

2 Setting of the problem

We start by describing the geometry of Ω^ε, which represents a two-phase thin porous medium (see Figure 1). Let ω be a smooth and bounded domain in $\mathbb{R}^2$. We denote the independent variable $x \in \mathbb{R}^3$ as $x = (x_1, x_2, x_3) = (\bar{x}, x_3)$ and we define

$$\Omega^\varepsilon = \omega \times (0, \varepsilon) = \{x = (\bar{x}, x_3) \in \mathbb{R}^3 |\, \bar{x} \in \omega,\, 0 < x_3 < \varepsilon\}. \tag{2.1}$$

Here, $\varepsilon \in (0,1)$ is a sequence of strictly positive numbers such that $\dfrac{1}{\varepsilon} \in \mathbb{N}^*$. This small parameter is related to the characteristic dimension of our domain. Thus, Ω^ε is a thin heterogeneous layer, both its thickness and the periodicity of its heterogeneities being of order ε. The domain ω is chosen so that Ω^ε is the union of a finite number (depending on ε) of replicated unit cells $Y = (0,1)^3$, rescaled with ε. One has $Y = Y_1 \cup \overline{Y}_2$, where Y_1 and Y_2 are two non-empty disjoint connected open subsets of Y such that $\overline{Y}_2 \subset Y$. We assume that $\Gamma = \partial Y_2$ is Lipschitz continuous. For each $\kappa \in \mathbb{Z}^3$, we denote $Y_\alpha^\kappa = \kappa + Y_\alpha$, for $\alpha \in \{1,2\}$. We also define, for each ε, $\mathbb{Z}_\varepsilon = \left\{\kappa \in \mathbb{Z}^3 |\, \varepsilon \overline{Y}_2^k \subset \Omega^\varepsilon\right\}$; we set $\Omega_2^\varepsilon = \bigcup_{\kappa \in \mathbb{Z}_\varepsilon} \left(\varepsilon Y_2^k\right)$ and $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega}_2^\varepsilon$. The boundary of Ω_2^ε is denoted by Γ^ε and n^ε is the unit outward normal to Ω_2^ε. The boundary of the domain Ω^ε is split into three parts: Σ_D^ε, the lateral boundary of the domain Ω^ε, $\Sigma_D^\varepsilon = \{x \in \mathbb{R}^3 |\, \bar{x} \in \partial\omega,\, 0 < x_3 < \varepsilon\}$, and $\Sigma_+^{\varepsilon,N}$, $\Sigma_-^{\varepsilon,N}$, the top and the bottom boundaries, respectively, $\Sigma_+^{\varepsilon,N} = \{x \in \mathbb{R}^3 |\, \bar{x} \in \omega,\, x_3 = \varepsilon\}$, $\Sigma_-^{\varepsilon,N} = \{x \in \mathbb{R}^3 |\, \bar{x} \in \omega,\, x_3 = 0\}$.

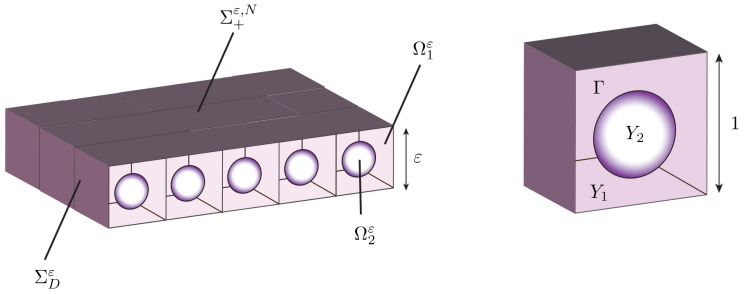

Figure 1: Domain $\Omega^\varepsilon = \omega \times (0, \varepsilon)$ and unit cell Y.

Our goal is to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the following problem:

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_1^\varepsilon) = f \quad \text{in } \Omega_1^\varepsilon, \\ -\text{div}\,(\varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon) = f \quad \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon\,(u_1^\varepsilon - u_2^\varepsilon) - G^\varepsilon \quad \text{on } \Gamma^\varepsilon, \\ \varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon\,(u_1^\varepsilon - u_2^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot \nu_\pm^\varepsilon = \varepsilon k_\pm \quad \text{on } \Sigma_\pm^{\varepsilon,N}, \\ u_1^\varepsilon = 0 \quad \text{on } \Sigma_D^\varepsilon. \end{cases} \tag{2.2}$$

Remark 2.1 *We remark that the flux of the solution is discontinuous across* Γ^ε*. Indeed,*

$$A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon - \varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = -G^\varepsilon.$$

All the assumptions made on the data are listed below.

(H1) For $\lambda, \mu \in \mathbb{R}$, with $0 < \lambda \le \mu$, let $\mathcal{M}(\lambda, \mu, Y)$ be the set of all the matrices $A = (a_{ij}) \in (L^\infty(Y))^{3\times 3}$ such that for any $\xi \in \mathbb{R}^3$, $\lambda|\xi|^2 \le (A(y)\xi, \xi) \le \mu|\xi|^2$, almost everywhere in Y. For a symmetric matrix $A \in \mathcal{M}(\lambda, \mu, Y)$, 1–periodic in the first two variables y_1 and y_2, we set

$$A^\varepsilon(x) = A^\varepsilon(\bar{x}, x_3) = A\left(\frac{\bar{x}}{\varepsilon}, \frac{x_3}{\varepsilon}\right) = A\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega^\varepsilon,$$

which is well-defined since, due to (2.1), one has $\dfrac{x_3}{\varepsilon} = y_3 \in (0, 1)$.

(H2) The function $f \in L^2(\omega)$ is given.

(H3) Let h be a function 1–periodic in the first two variables y_1 and y_2 such that $h \in L^\infty(\Gamma)$ and there exists $h_0 \in \mathbb{R}$ with $0 < h_0 < h(y)$ a.e. on Γ. We set

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \Gamma^\varepsilon.$$

(H4) Let g be a function 1–periodic in the first two variables y_1 and y_2 that belongs to $L^2(\Gamma)$. We define

$$g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \text{ a.e. on } \Gamma^\varepsilon.$$

For the given function G^ε in (2.2), we consider the following two relevant situations (see [16] and the references therein):

$$\textbf{Case 1}: G^\varepsilon(x) = \varepsilon g\Big(\frac{x}{\varepsilon}\Big), \text{ if } \mathcal{M}_\Gamma(g) \neq 0, \tag{2.3}$$

$$\textbf{Case 2}: G^\varepsilon(x) = g\Big(\frac{x}{\varepsilon}\Big), \text{ if } \mathcal{M}_\Gamma(g) = 0. \tag{2.4}$$

Here, $\mathcal{M}_\Gamma(g) = \dfrac{1}{|\Gamma|}\displaystyle\int_\Gamma g(y)\,\mathrm{d}y$ denotes the mean value of the function g on Γ.

(H5) The functions $k_+ \in L^2(\omega)$ and $k_- \in L^2(\omega)$ are given.

In order to write the variational formulation of problem (2.2), we introduce, for every positive $\varepsilon < 1$, the Hilbert space

$$H^\varepsilon = V^\varepsilon \times H^1(\Omega_2^\varepsilon).$$

The space $V^\varepsilon = \left\{ v \in H^1(\Omega_1^\varepsilon) |\, v = 0 \text{ on } \Sigma_D^\varepsilon \right\}$ is endowed with the norm $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}$, for any $v \in V^\varepsilon$, and the space $H^1(\Omega_2^\varepsilon)$ is equipped with the standard norm. On the space H^ε, we consider the scalar product

$$(u,v)_{H^\varepsilon} = \int_{\Omega_1^\varepsilon} \nabla u_1 \nabla v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \varepsilon^2 \nabla u_2 \nabla v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} (u_1 - u_2)(v_1 - v_2) \,\mathrm{d}\sigma_x, \tag{2.5}$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ belong to H^ε. The norm generated by the scalar product (2.5) is given by

$$\|v\|^2_{H^\varepsilon} = \|\nabla v_1\|^2_{L^2(\Omega_1^\varepsilon)} + \varepsilon^2 \|\nabla v_2\|^2_{L^2(\Omega_2^\varepsilon)} + \varepsilon \|v_1 - v_2\|^2_{L^2(\Gamma^\varepsilon)}. \tag{2.6}$$

The variational formulation of problem (2.2) is the following one: find $u^\varepsilon \in H^\varepsilon$ such that

$$a(u^\varepsilon, v) = l(v), \quad \forall v \in H^\varepsilon, \tag{2.7}$$

where the bilinear form $a : H^\varepsilon \times H^\varepsilon \to \mathbb{R}$ and the linear form $l : H^\varepsilon \to \mathbb{R}$ are given by

$$a(u,v) = \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1 \nabla v_1 \,\mathrm{d}x + \varepsilon^2 \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2 \nabla v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)(v_1 - v_2) \,\mathrm{d}\sigma_x$$

and

$$l(v) = \int_{\Omega_1^\varepsilon} f v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f v_2 \,\mathrm{d}x + \int_{\Gamma^\varepsilon} G^\varepsilon v_1 \,\mathrm{d}\sigma_x + \varepsilon \int_{\Sigma_+^{\varepsilon,N}} k_+ v_1 \,\mathrm{d}\sigma_x^+ + \varepsilon \int_{\Sigma_-^{\varepsilon,N}} k_- v_1 \,\mathrm{d}\sigma_x^-,$$

respectively.

We give in the next lemma a result which is a key argument allowing us to prove existence, uniqueness and *a priori* estimates for the solution of the variational problem (2.7). In the sequel, unless otherwise mentioned, by C we denote a positive constant which is independent of ε and whose value can change from line to line.

Lemma 2.2 *For every v given in the space H^ε, the following inequalities hold true:*

$$\|v_1\|_{L^2(\Omega_1^\varepsilon)} \le C\|v\|_{H^\varepsilon}, \qquad \|v_2\|_{L^2(\Omega_2^\varepsilon)} \le C\|v\|_{H^\varepsilon}.$$

Proof. The first inequality is a direct consequence of the definition (2.6), together with the Poincaré inequality applied to functions from the space V^ε, namely

$$\|v_1\|_{L^2(\Omega_1^\varepsilon)} \le C\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}. \tag{2.8}$$

In order to prove the second inequality, we make use of the following relations:

$$\|v_2\|_{L^2(\Omega_2^\varepsilon)} \le C(\varepsilon\|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)} + \sqrt{\varepsilon}\|v_2\|_{L^2(\Gamma^\varepsilon)}) \tag{2.9}$$

and

$$\sqrt{\varepsilon}\|v_1\|_{L^2(\Gamma^\varepsilon)} \le C(\varepsilon\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} + \|v_1\|_{L^2(\Omega_1^\varepsilon)}). \tag{2.10}$$

Inequality (2.9) is an adaptation of a result in [39] to the present geometry. For the derivation of (2.10), we refer the reader to [32]. The triangular inequality applied in (2.9), together with (2.10) and (2.8), imply

$$\|v_2\|_{L^2(\Omega_2^\varepsilon)} \le C(\varepsilon\|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)} + \sqrt{\varepsilon}\|v_2 - v_1\|_{L^2(\Gamma^\varepsilon)} + \sqrt{\varepsilon}\|v_1\|_{L^2(\Gamma^\varepsilon)}) \le$$

$$C(\varepsilon\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} + \varepsilon\|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)} + \sqrt{\varepsilon}\|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)} + \|v_1\|_{L^2(\Omega_1^\varepsilon)}) \le$$

$$C(\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} + \varepsilon\|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)} + \sqrt{\varepsilon}\|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}),$$

and the second inequality then follows, by using the definition (2.6).

□

Lemma 2.3 *For G^ε satisfying hypothesis* **(H4)** *and $v_1 \in V^\varepsilon$, the following estimate holds:*

$$I = \left| \int_{\Gamma^\varepsilon} G^\varepsilon(x) v_1(x) \,\mathrm{d}\sigma_x \right| \le \sqrt{\varepsilon} C \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}.$$

Proof. In order to prove this result, we follow the ideas in [16], Proposition 4.50. By using Definition 3.1 and Proposition 3.3 below, one has

$$\int_{\Gamma^\varepsilon} G^\varepsilon(x) v_1(x)\,\mathrm{d}\sigma_x = \int_{\omega\times\Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{T}_1^\varepsilon(v_1)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y =$$

$$\int_{\omega\times\Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon)\left[\mathcal{T}_1^\varepsilon(v_1) - \mathcal{M}_\varepsilon^1(v_1)\right]\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y + \int_{\omega\times\Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{M}_\varepsilon^1(v_1)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y = I_1 + I_2,$$

where the operator $\mathcal{M}_\varepsilon^1 : L^2(\omega) \longrightarrow L^2(\omega)$ is defined by

$$\mathcal{M}_\varepsilon^1(v_1)(\bar{x}) = \frac{1}{|Y_1|}\int_{Y_1} \mathcal{T}_1^\varepsilon(v_1)(\bar{x}, y)\,\mathrm{d}y.$$

For the integral I_1, one has

$$|I_1| \le \|\mathcal{T}_b^\varepsilon(G^\varepsilon)\|_{L^2(\omega\times\Gamma)}\|\mathcal{T}_1^\varepsilon(v_1) - \mathcal{M}_\varepsilon^1(v_1)\|_{L^2(\omega\times\Gamma)} \le$$

$$C\|\mathcal{T}_b^\varepsilon(G^\varepsilon)\|_{L^2(\omega\times\Gamma)}\|\nabla_y\mathcal{T}_1^\varepsilon(v_1)\|_{L^2(\omega\times Y_1)} \le \sqrt{\varepsilon}C\|\mathcal{T}_b^\varepsilon(G^\varepsilon)\|_{L^2(\omega\times\Gamma)}\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)},$$

where we use Poincaré-Wirtinger inequality and Proposition 3.3, (iii) and (iv).

For the integral I_2, one has

$$|I_2| \le \left|\int_{\omega\times\Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{M}_\varepsilon^1(v_1)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y\right| \le |\Gamma|\left|\int_\omega \mathcal{M}_\varepsilon^b(G^\varepsilon)\mathcal{M}_\varepsilon^1(v_1)\,\mathrm{d}\bar{x}\right| \le$$

$$C\|\mathcal{M}_\varepsilon^b(G^\varepsilon)\|_{L^2(\omega)}\|\mathcal{M}_\varepsilon^1(v_1)\|_{L^2(\omega)} \le \frac{C}{\sqrt{\varepsilon}}\|\mathcal{M}_\varepsilon^b(G^\varepsilon)\|_{L^2(\omega)}\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)},$$

where $\mathcal{M}_\varepsilon^b(G^\varepsilon) = \dfrac{1}{|\Gamma|}\displaystyle\int_\Gamma \mathcal{T}_b^\varepsilon(G^\varepsilon)(\cdot, y)\,\mathrm{d}\sigma_y$ and we used the definition of $\mathcal{M}_\varepsilon^1$, Proposition 3.3, (iii) and Poincaré inequality. Combining the previous estimates on I_1 et I_2, we obtain:

$$I \le \sqrt{\varepsilon}C\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}\left(\|\mathcal{T}_b^\varepsilon(G^\varepsilon)\|_{L^2(\omega\times\Gamma)} + \frac{1}{\varepsilon}\|\mathcal{M}_\varepsilon^b(G^\varepsilon)\|_{L^2(\omega)}\right).$$

According to hypothesis **(H4)**, one gets the desired result.

□

Theorem 2.4 *For any $\varepsilon \in (0,1)$, the variational problem (2.7) has a unique solution $u^\varepsilon \in H^\varepsilon$. Moreover, there exists a constant $C > 0$, independent of ε, such that*

$$\frac{1}{\sqrt{\varepsilon}}\|u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \le C, \quad \frac{1}{\sqrt{\varepsilon}}\|u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \le C \tag{2.11}$$

and

$$\frac{1}{\sqrt{\varepsilon}}\|\nabla u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \le C, \quad \sqrt{\varepsilon}\|\nabla u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \le C, \quad \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \le C. \tag{2.12}$$

Proof. In order to prove the existence and the uniqueness of the solution for problem (2.7), we apply the Lax-Milgram theorem for the space H^ε endowed with the norm (2.6). Due to the hypotheses (**H1**) and (**H2**), we easily get that the bilinear form a is coercive and continuous. Indeed, we have

$$a(v,v) \ge C\|v\|_{H^\varepsilon}^2, \quad \forall v \in H^\varepsilon,$$

and

$$a(u,v) \le C\|u\|_{H^\varepsilon}\|v\|_{H^\varepsilon}, \quad \forall u, v \in H^\varepsilon.$$

Let us prove now that the linear form l is continuous. More precisely, we shall prove

$$l(v) \le \sqrt{\varepsilon}C\|v\|_{H^\varepsilon}, \quad \forall v \in H^\varepsilon.$$

One obviously has

$$|l(v)| \leq \|f\|_{L^2(\Omega_1^\varepsilon)}\|v_1\|_{L^2(\Omega_1^\varepsilon)} + \|f\|_{L^2(\Omega_2^\varepsilon)}\|v_2\|_{L^2(\Omega_2^\varepsilon)} + \left|\int_{\Gamma^\varepsilon} G^\varepsilon(x)v_1(x)\,\mathrm{d}\sigma_x\right| +$$

$$\left|\varepsilon\int_{\Sigma_+^{\varepsilon,N}} k_+ v_1\,\mathrm{d}\sigma_x^+\right| + \left|\varepsilon\int_{\Sigma_-^{\varepsilon,N}} k_- v_1\,\mathrm{d}\sigma_x^-\right|. \tag{2.13}$$

By using the hypotheses (**H2**), (**H4**), (**H5**) and Lemma 2.3, we get

$$\|f\|_{L^2(\Omega_1^\varepsilon)}\|v_1\|_{L^2(\Omega_1^\varepsilon)} \leq \sqrt{\varepsilon}C\|v_1\|_{L^2(\Omega_1^\varepsilon)}; \quad \|f\|_{L^2(\Omega_2^\varepsilon)}\|v_2\|_{L^2(\Omega_2^\varepsilon)} \leq \sqrt{\varepsilon}C\|v_2\|_{L^2(\Omega_2^\varepsilon)},$$

$$\left|\int_{\Gamma^\varepsilon} G^\varepsilon(x)v_1(x)\,\mathrm{d}\sigma_x\right| \leq \sqrt{\varepsilon}C\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}; \quad \left|\varepsilon\int_{\Sigma_\pm^{\varepsilon,N}} k_\pm v_1\,\mathrm{d}\sigma_x^\pm\right| \leq \sqrt{\varepsilon}C\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}.$$

Coming back to (2.13), we obtain:

$$|l(v)| \leq \sqrt{\varepsilon}C(\|v_1\|_{L^2(\Omega_1^\varepsilon)} + \|v_2\|_{L^2(\Omega_2^\varepsilon)} + \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}).$$

By using Lemma 2.2 and the definition (2.6), we get the continuity of l. Thus, the Lax-Milgram theorem applies. In order to obtain the *a priori* estimates (2.11) and (2.12), we take $v = u^\varepsilon$ in the variational formulation (2.7). By using the coerciveness of a and the continuity of l, we get

$$\|u^\varepsilon\|_{H^\varepsilon} \leq \sqrt{\varepsilon}C,$$

which obviously implies (2.12). Estimates (2.11) are then obtained by applying Lemma 2.2.

□

3 Unfolding operators for the thin domain and compactness results

We recall here the definition and the main properties of the unfolding operators specific to the geometry of our thin periodic domain (see [35], [34], [16], [26], [19]). One of the main features of these operators is that they allow us to simultaneously perform homogenization and dimension reduction. For $x \in \mathbb{R}^3$, we denote by $[x]_Y$ its integer part $\kappa \in \mathbb{Z}^3$, such that $x - [x]_Y \in Y$, and we set $\{x\}_Y = x - [x]_Y$ for $x \in \mathbb{R}^3$. So, for every $x \in \mathbb{R}^3$, we have $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right)$. In particular, if $x \in \Omega^\varepsilon$ (see (2.1)), one has $x = \varepsilon\left(\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right)$.

According, for instance, to Section 3.1 in [26], one has:

Definition 3.1 *For any Lebesgue measurable function φ on $\Omega_\alpha^\varepsilon$, $\alpha \in \{1,2\}$, we define the periodic unfolding operators by the formula*

$$\mathcal{T}_\alpha^\varepsilon(\varphi)(\bar{x},y) = \varphi\left(\varepsilon\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \varepsilon y\right), \quad \text{for a.e. } (\bar{x},y) \in \omega \times Y_\alpha.$$

If φ is a function defined in Ω^ε, for simplicity, we write $\mathcal{T}_\alpha^\varepsilon(\varphi)$ instead of $\mathcal{T}_\alpha^\varepsilon(\varphi|_{\Omega_\alpha^\varepsilon})$.

For any function φ which is Lebesgue-measurable on Γ^ε, the periodic boundary unfolding operator $\mathcal{T}_b^\varepsilon$ is defined by

$$\mathcal{T}_b^\varepsilon(\varphi)(\bar{x},y) = \varphi\left(\varepsilon\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \varepsilon y\right), \quad \text{for a.e. } (\bar{x},y) \in \omega \times \Gamma.$$

Remark 3.2 *We notice that if $\varphi \in W^{1,p}(\Omega_\alpha^\varepsilon)$ ($p \in [1,+\infty]$), then $\mathcal{T}_b^\varepsilon(\varphi)$ is, in fact, the trace of $\mathcal{T}_\alpha^\varepsilon(\varphi)$ on $\omega \times \Gamma$.*

Proposition 3.3 *The operators $\mathcal{T}_\alpha^\varepsilon$ are linear and continuous from $L^2(\Omega_\alpha^\varepsilon)$ to $L^2(\omega \times Y_\alpha)$ and*

(i) if φ and ψ are two Lebesgue measurable functions on $\Omega_\alpha^\varepsilon$, one has $\mathcal{T}_\alpha^\varepsilon(\varphi\psi) = \mathcal{T}_\alpha^\varepsilon(\varphi)\mathcal{T}_\alpha^\varepsilon(\psi)$;

(ii) for every $\varphi \in L^1(\Omega_\alpha^\varepsilon)$*, one has*

$$\int_{\Omega_\alpha^\varepsilon} \varphi(x)\,\mathrm{d}x = \varepsilon \int_{\omega\times Y_\alpha} \mathcal{T}_\alpha^\varepsilon(\varphi)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}y;$$

(iii) for every $\varphi \in L^2(\Omega_\alpha^\varepsilon)$*, one has*

$$\|\mathcal{T}_\alpha^\varepsilon(\varphi)\|_{L^2(\omega\times Y_\alpha)} = \frac{1}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\Omega_\alpha^\varepsilon)};$$

(iv) for every $\varphi \in H^1(\Omega_\alpha^\varepsilon)$*, one has* $\nabla_y\left(\mathcal{T}_\alpha^\varepsilon(\varphi)\right) = \varepsilon\mathcal{T}_\alpha^\varepsilon(\nabla\varphi)$ *a.e. in* $\omega\times Y_\alpha$*;*

(v) for every $\varphi \in L^1(\Gamma^\varepsilon)$*, one has*

$$\int_{\Gamma^\varepsilon} \varphi(x)\,\mathrm{d}\sigma_x = \int_{\omega\times\Gamma} \mathcal{T}_b^\varepsilon(\varphi)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y;$$

(vi) for every $\varphi \in L^2(\Gamma^\varepsilon)$*, one has*

$$\|\mathcal{T}_b^\varepsilon(\varphi)\|_{L^2(\omega\times\Gamma)} = \|\varphi\|_{L^2(\Gamma^\varepsilon)}.$$

Remark 3.4 *Throughout this paper, we are allowed to not write explicitly the measure of* $\mathcal{Y}_1 = (0,1)^2$*, since it equals 1.*

One has the following compactness result:

Proposition 3.5 *Let* $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon) \in H^\varepsilon$ *be such that*

$$\|v^\varepsilon\|_{H^\varepsilon} \le \sqrt{\varepsilon}C. \tag{3.1}$$

Then, up to a subsequence, still denoted by ε*, there exist* $v_1 \in H_0^1(\omega)$*,* $\widehat{v}_1 \in L^2\left(\omega, H^1_{\overline{per}}(Y_1)\right)$ *and* $\widehat{v}_2 \in L^2\left(\omega, H^1(Y_2)\right)$ *such that*

$$\mathcal{T}_1^\varepsilon(v_1^\varepsilon) \rightharpoonup v_1 \text{ weakly in } L^2\left(\omega, H^1(Y_1)\right),$$

$$\mathcal{T}_1^\varepsilon(\nabla_{\bar{x}} v_1^\varepsilon) \rightharpoonup \nabla_{\bar{x}} v_1 + \nabla_{\bar{y}}\widehat{v}_1 \text{ weakly in } L^2(\omega\times Y_1),$$

$$\mathcal{T}_1^\varepsilon(\partial_{x_3} v_1^\varepsilon) \rightharpoonup \partial_{y_3}\widehat{v}_1 \text{ weakly in } L^2(\omega\times Y_1),$$

$$\mathcal{T}_2^\varepsilon(v_2^\varepsilon) \rightharpoonup \widehat{v}_2 \text{ weakly in } L^2(\omega, H^1(Y_2)),$$

$$\varepsilon\mathcal{T}_2^\varepsilon(\nabla v_2^\varepsilon) \rightharpoonup \nabla_y\widehat{v}_2 \text{ weakly in } L^2\left(\omega\times Y_2\right),$$

where $\mathcal{M}_\Gamma(\widehat{v}_1) = 0$ *for almost every* $\bar{x}\in\omega$ *and the space* $H^1_{\overline{per}}(Y_1)$ *is defined by*

$$H^1_{\overline{per}}(Y_1) = \{v \in H^1(Y_1)\,|\,v \text{ is 1-periodic in } y_1 \text{ and } y_2\}.$$

Proof. Relation (3.1) and the definition of the norm in H^ε imply

$$\frac{1}{\sqrt{\varepsilon}}\|v_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} + \frac{1}{\sqrt{\varepsilon}}\|\nabla v_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \le C,$$

$$\frac{1}{\sqrt{\varepsilon}}\|v_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} + \sqrt{\varepsilon}\|\nabla v_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \le C.$$

Then, according to Definition 3.1 and Proposition 3.3, (iii), (iv), there exists a constant $C>0$, independent of ε, such that

$$\|\mathcal{T}_1^\varepsilon(v_1^\varepsilon)\|_{L^2(\omega\times Y_1)} \le C, \qquad \|\mathcal{T}_1^\varepsilon(\nabla v_1^\varepsilon)\|_{L^2(\omega\times Y_1)} \le C,$$

$$\|\mathcal{T}_2^\varepsilon(v_2^\varepsilon)\|_{L^2(\omega\times Y_2)} \le C, \qquad \varepsilon\|\mathcal{T}_2^\varepsilon(\nabla v_2^\varepsilon)\|_{L^2(\omega\times Y_2)} \le C.$$

The convergences follow by applying results in [34], namely Proposition 4.7, Proposition 4.4 (i) for the sequences (v_1^ε) and (∇v_1^ε), and, respectively, Proposition 4.4 (ii) for the sequences (v_2^ε) and (∇v_2^ε) (see, also, [30], [26], [16]).

4 Homogenization results

Our goal in this section is to pass to the limit, with $\varepsilon \to 0$, in the variational formulation (2.7) of problem (2.2). To this end, we make use of the periodic unfolding operators and the general compactness results given in Section 3. More precisely, using the *a priori* estimates (2.11)-(2.12) and the general compactness results from Proposition 3.5, it follows that there exist $u_1 \in H_0^1(\omega)$, $\widehat{u}_1 \in L^2(\omega, H^1_{\overline{\rm per}}(Y_1))$, $\widehat{u}_2 \in L^2(\omega, H^1(Y_2))$, with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$, and such that, up to a subsequence, for $\varepsilon \to 0$, we get:

$$\begin{array}{l} \mathcal{T}_1^\varepsilon(u_1^\varepsilon) \rightharpoonup u_1 \quad \text{weakly in } L^2(\omega, H^1(Y_1)), \\ \mathcal{T}_1^\varepsilon(\nabla_{\bar{x}} u_1^\varepsilon) \rightharpoonup \nabla_{\bar{x}}\, u_1 + \nabla_{\bar{y}} \widehat{u}_1 \quad \text{weakly in } L^2(\omega \times Y_1), \\ \mathcal{T}_1^\varepsilon(\partial_{x_3} u_1^\varepsilon) \rightharpoonup \partial_{y_3} \widehat{u}_1 \quad \text{weakly in } L^2(\omega \times Y_1), \\ \mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup \widehat{u}_2 \quad \text{weakly in } L^2(\omega, H^1(Y_2)), \\ \varepsilon \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\omega \times Y_2). \end{array} \tag{4.1}$$

The special form of the limits in convergences $(4.1)_2$ and $(4.1)_3$ suggests us to introduce the following notation: to every $w = w(\bar{x}) \in H^1(\omega)$, whose gradient $\nabla_{\bar{x}} w(\bar{x})$ has two components, we associate the tridimensional vector $\overline{\nabla} w(\bar{x})$ defined by

$$\overline{\nabla} w(\bar{x}) = (\nabla_{\bar{x}} w(\bar{x}), 0).$$

Let $W_{\overline{\rm per}}(Y_1) = \{v \in H^1_{\overline{\rm per}}(Y_1) \,|\, \mathcal{M}_\Gamma(v) = 0\}$. We introduce the space

$$\mathcal{V} = H_0^1(\omega) \times L^2\left(\omega; W_{\overline{\rm per}}(Y_1)\right) \times L^2\left(\omega, H^1(Y_2)\right)$$

and for all $V = (v, \widehat{v}_1, \widehat{v}_2) \in \mathcal{V}$ we define

$$\|V\|_{\mathcal{V}}^2 = \left\|\overline{\nabla} v + \nabla_y \widehat{v}_1\right\|^2_{L^2(\omega \times Y_1)} + \|\nabla_y \widehat{v}_2\|^2_{L^2(\omega \times Y_2)} + \|v - \widehat{v}_2\|^2_{L^2(\omega \times \Gamma)},$$

which is a norm (see, for instance, Lemma 5.4 in [16]).

In order to pass to the limit in (2.7), we will distinguish between two cases, depending on the form of the function G^ε.

Case 1: $G^\varepsilon = \varepsilon g\left(\dfrac{x}{\varepsilon}\right)$, if $\mathcal{M}_\Gamma(g) \neq 0$.

Theorem 4.1 *Under the assumption* (2.3)*, the unique solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (2.7) converges, in the sense of (4.1), to the unique solution $(u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{V}$ of the following unfolded limit problem:*

$$\begin{aligned} &\int_{\omega \times Y_1} A(y)(\overline{\nabla} u_1 + \nabla_y \widehat{u}_1)(\overline{\nabla}\varphi + \nabla_y \Phi_1)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \int_{\omega \times Y_2} A(y) \nabla_y \widehat{u}_2 \nabla_y \Phi_2\, \mathrm{d}\bar{x}\, \mathrm{d}y + \\ &\int_{\omega \times \Gamma} h(y)(u_1 - \widehat{u}_2)(\varphi - \Phi_2)\, \mathrm{d}\bar{x}\, \mathrm{d}\sigma_y = |Y_1| \int_\omega f(\bar{x})\varphi(\bar{x})\, \mathrm{d}\bar{x} + \int_{\omega \times Y_2} f(\bar{x})\Phi_2(\bar{x}, y)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \\ &|\Gamma| \mathcal{M}_\Gamma(g) \int_\omega \varphi(\bar{x})\, \mathrm{d}\bar{x} + \int_\omega k_+(\bar{x})\varphi(\bar{x})\, \mathrm{d}\bar{x} + \int_\omega k_-(\bar{x})\varphi(\bar{x})\, \mathrm{d}\bar{x}, \end{aligned} \tag{4.2}$$

for all $\varphi \in H_0^1(\omega)$, $\Phi_1 \in L^2(\omega, H^1_{\overline{per}}(Y_1))$ and $\Phi_2 \in L^2(\omega, H^1(Y_2))$.

Proof. In order to get the limit problem (4.2), we unfold the variational formulation (2.7). By using the properties of the unfolding operators, we obtain

$$\begin{aligned} &\varepsilon \int_{\omega \times Y_1} \mathcal{T}_1^\varepsilon(A^\varepsilon)\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon)\mathcal{T}_1^\varepsilon(\nabla v_1)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \varepsilon \int_{\omega \times Y_2} \mathcal{T}_2^\varepsilon(A^\varepsilon)\mathcal{T}_2^\varepsilon(\varepsilon \nabla u_2^\varepsilon)\mathcal{T}_2^\varepsilon(\varepsilon \nabla v_2)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \\ &\varepsilon \int_{\omega \times \Gamma} h(y)(\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon))(\mathcal{T}_1^\varepsilon(v_1) - \mathcal{T}_2^\varepsilon(v_2))\, \mathrm{d}\bar{x}\, \mathrm{d}\sigma_y = \\ &\varepsilon \int_{\omega \times Y_1} \mathcal{T}_1^\varepsilon(f)\mathcal{T}_1^\varepsilon(v_1)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \varepsilon \int_{\omega \times Y_2} \mathcal{T}_2^\varepsilon(f)\mathcal{T}_2^\varepsilon(v_2)\, \mathrm{d}\bar{x}\, \mathrm{d}y + \int_{\omega \times \Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{T}_1^\varepsilon(v_1)\, \mathrm{d}\bar{x}\, \mathrm{d}\sigma_y + \end{aligned}$$

$$\varepsilon\int_{\omega\times\mathcal{Y}_1}\mathcal{T}^\varepsilon(k_+)(\bar{x},\bar{y})\mathcal{T}_1^\varepsilon(v_1)(\bar{x},\bar{y},\varepsilon)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}+\varepsilon\int_{\omega\times\mathcal{Y}_1}\mathcal{T}^\varepsilon(k_-)(\bar{x},\bar{y})\mathcal{T}_1^\varepsilon(v_1)(\bar{x},\bar{y},0)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y},\tag{4.3}$$

where $\mathcal{Y}_1=(0,1)^2$, $\bar{y}=(y_1,y_2)$ and $\mathcal{T}^\varepsilon$ is the classical unfolding operator from [16], Definition 1.2, considered here as an operator from $L^2(\omega)$ to $L^2(\omega\times\mathcal{Y}_1)$.

Recalling that $x=(\bar{x},x_3)$, we choose in the unfolded problem (4.3) the test functions

$$v_1(\bar{x},x_3)=\varphi(\bar{x})+\varepsilon\omega_1(\bar{x})\,\psi_1\left(\left\{\frac{\bar{x}}{\varepsilon}\right\},\frac{x_3}{\varepsilon}\right),\quad v_2(\bar{x},x_3)=\omega_2(\bar{x})\,\psi_2\left(\left\{\frac{\bar{x}}{\varepsilon}\right\},\frac{x_3}{\varepsilon}\right),\tag{4.4}$$

with $\varphi,\omega_1,\omega_2\in\mathcal{D}(\omega)$, $\psi_1\in H^1_{\overline{\mathrm{per}}}(Y_1)$, $\psi_2\in H^1(Y_2)$. One has the following convergences:

$$\mathcal{T}_1^\varepsilon(v_1)\to\varphi(\bar{x})\quad\text{strongly in } L^2(\omega\times Y_1),\tag{4.5}$$

$$\mathcal{T}_1^\varepsilon(\nabla v_1)\to\overline{\nabla}\varphi(\bar{x})+\nabla_y\Phi_1\quad\text{strongly in } L^2(\omega\times Y_1),\tag{4.6}$$

$$\mathcal{T}_2^\varepsilon(v_2)\to\Phi_2(\bar{x},y)\quad\text{strongly in } L^2(\omega\times Y_2),\tag{4.7}$$

$$\mathcal{T}_2^\varepsilon(\varepsilon\nabla v_2)\to\nabla_y\Phi_2\quad\text{strongly in } L^2(\omega\times Y_2),\tag{4.8}$$

where $\Phi_1(\bar{x},y)=\omega_1(\bar{x})\,\psi_1(y)$ and $\Phi_2(\bar{x},y)=\omega_2(\bar{x})\,\psi_2(y)$.

After dividing equation (4.3) by ε, we pass to the limit with $\varepsilon\to0$, by using convergences (4.1) and (4.5)-(4.8). The passage to the limit for the terms which need more attention is detailed in what follows:

$$\frac{1}{\varepsilon}\int_{\omega\times\Gamma}\mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{T}_1^\varepsilon(v_1)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y=\int_{\omega\times\Gamma}\mathcal{T}_b^\varepsilon\left(g\left(\left\{\frac{\bar{x}}{\varepsilon}\right\},\frac{x_3}{\varepsilon}\right)\right)\mathcal{T}_1^\varepsilon\left(\varphi(\bar{x})+\varepsilon\omega_1(\bar{x})\psi_1\left(\left\{\frac{\bar{x}}{\varepsilon}\right\},\frac{x_3}{\varepsilon}\right)\right)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y=$$

$$\int_{\omega\times\Gamma}g(y)\mathcal{T}_1^\varepsilon(\varphi)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y+\varepsilon\int_{\omega\times\Gamma}g(y)\mathcal{T}_1^\varepsilon(\omega_1)(\bar{x},y)\mathcal{T}_1^\varepsilon(\psi_1)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y\to|\Gamma|\mathcal{M}_\Gamma(g)\int_\omega\varphi(\bar{x})\,\mathrm{d}\bar{x},$$

$$\int_{\omega\times\mathcal{Y}_1}\mathcal{T}^\varepsilon(k_+)(\bar{x},\bar{y})\mathcal{T}_1^\varepsilon(v_1)(\bar{x},\bar{y},\varepsilon)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\to\int_{\omega\times\mathcal{Y}_1}k_+(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}=\int_\omega k_+(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x},$$

$$\int_{\omega\times\mathcal{Y}_1}\mathcal{T}^\varepsilon(k_-)(\bar{x},\bar{y})\mathcal{T}_1^\varepsilon(v_1)(\bar{x},\bar{y},0)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\to\int_\omega k_-(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x},\ \text{by using }\mathbf{(H5)},(4.4)\text{ and }|\mathcal{Y}_1|=1.$$

By the density of $\mathcal{D}(\omega)\otimes H^1_{\overline{\mathrm{per}}}(Y_1)$ in $L^2(\omega,H^1_{\overline{\mathrm{per}}}(Y_1))$ and of $\mathcal{D}(\omega)\otimes H^1(Y_2)$ in $L^2(\omega,H^1(Y_2))$, we obtain (4.2). The existence and the uniqueness for the solution of problem (4.2) is a consequence of the Lax-Milgram theorem. Due to the uniqueness of $(u_1,\widehat{u}_1,\widehat{u}_2)\in\mathcal{V}$, all the above convergences hold true for the whole sequence. □

Theorem 4.2 *Under the assumption* (2.3)*, the unique solution $u^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon)$ of the variational problem (2.7) converges, in the sense of (4.1), to $(u_1,\widehat{u}_1,\widehat{u}_2)\in\mathcal{V}$, where u_1 is the unique solution of the homogenized problem*

$$\begin{cases}-div_{\bar{x}}\left(A^{hom}\nabla_{\bar{x}}u_1(\bar{x})\right)=f(\bar{x})+|\Gamma|\mathcal{M}_\Gamma(g)+k_+(\bar{x})+k_-(\bar{x})\quad\text{in }\omega,\\ u_1=0\quad\text{on }\partial\omega\end{cases}\tag{4.9}$$

and

$$\widehat{u}_1(\bar{x},y)=-\sum_{j=1}^{2}\frac{\partial u_1}{\partial x_j}(\bar{x})\chi_1^j(y)\quad\text{in }\omega\times Y_1,\tag{4.10}$$

$$\widehat{u}_2(\bar{x},y)=u_1(\bar{x})+f(\bar{x})\chi_2(y)\quad\text{in }\omega\times Y_2.\tag{4.11}$$

Here, A^{hom} is the constant homogenized 2×2 matrix whose entries are defined, for $i,j\in\{1,2\}$, by

$$A_{ij}^{hom}=\int_{Y_1}\left(a_{ij}-\sum_{k=1}^{3}a_{ik}\frac{\partial\chi_1^j}{\partial y_k}\right)\mathrm{d}y.\tag{4.12}$$

The function $\chi_1=(\chi_1^1,\chi_1^2)\in(H^1_{\overline{per}}(Y_1))^2$ and the scalar function $\chi_2\in H^1(Y_2)$ are the weak solutions of the following cell problems:

$$\begin{cases}-div_y(A(y)(\nabla_y\chi_1^j-e_j))=0\quad\text{in }Y_1,\\ (A(y)(\nabla_y\chi_1^j-e_j))\cdot n=0\qquad\text{on }\Gamma,\\ (A(y)(\nabla_y\chi_1^j-e_j))\cdot\nu_\pm=0\qquad\text{on }\Sigma_\pm^1,\\ \mathcal{M}_\Gamma(\chi_1^j)=0\end{cases}\tag{4.13}$$

and

$$\begin{cases} -div_y(A(y)\nabla_y\chi_2) = 1 \quad in\ Y_2, \\ A(y)\nabla_y\chi_2 \cdot n + h\chi_2 = 0 \quad on\ \Gamma, \end{cases} \tag{4.14}$$

where n denotes the unit outward normal to Y_2 and $\nu_\pm = (0,0,\pm 1)$.

Proof. By choosing $\varphi = 0$ in the unfolded limit problem (4.2), we obtain:

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla}u_1 + \nabla_y\widehat{u}_1)\nabla_y\Phi_1 \,\mathrm{d}\bar{x}\,\mathrm{d}y + \int_{\omega\times Y_2} A(y)\nabla_y\widehat{u}_2\nabla_y\Phi_2 \,\mathrm{d}\bar{x}\,\mathrm{d}y -$$

$$\int_{\omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\Phi_2 \,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y = \int_{\omega\times Y_2} f(\bar{x})\Phi_2(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}y, \tag{4.15}$$

for all $\Phi_1 \in L^2(\omega, H^1_{\overline{\mathrm{per}}}(Y_1))$ and $\Phi_2 \in L^2(\omega, H^1(Y_2))$. Then, taking $\Phi_2 = 0$ in (4.15), we get:

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla}u_1 + \nabla_y\widehat{u}_1)\nabla_y\Phi_1 \,\mathrm{d}\bar{x}\,\mathrm{d}y = 0. \tag{4.16}$$

Choosing $\Phi_1 \in L^2(\omega, H^1_{\overline{\mathrm{per}}}(Y_1))$ such that $\Phi_1 = 0$ on $\omega \times \partial Y_1$ and integrating by parts with respect to y, we are formally led to

$$-\mathrm{div}_y(A(y)\nabla_y\widehat{u}_1) = \mathrm{div}_y(A(y)\overline{\nabla}u_1) \quad \text{in } \omega\times Y_1.$$

Taking now in (4.16) a test function $\Phi_1 \in L^2(\omega, H^1_{\overline{\mathrm{per}}}(Y_1))$ which is zero on $\Sigma^1_\pm$, we obtain

$$(A(y)\nabla_y\widehat{u}_1)\cdot n = -(A(y)\overline{\nabla}u_1)\cdot n \quad \text{on } \omega\times\Gamma.$$

Taking in (4.16) a test function $\Phi_1 \in L^2(\omega, H^1_{\overline{\mathrm{per}}}(Y_1))$ which is zero on Σ^1_- (and which is zero on Σ^1_+), we get

$$(A(y)\nabla_y\widehat{u}_1)\cdot \nu_+ = -(A(y)\overline{\nabla}u_1)\cdot\nu_+ \quad \text{on } \omega\times\Sigma^1_+$$

and

$$(A(y)\nabla_y\widehat{u}_1)\cdot \nu_- = -(A(y)\overline{\nabla}u_1)\cdot\nu_- \quad \text{on } \omega\times\Sigma^1_-.$$

The linearity of the problem suggests us to search $\widehat{u}_1(\bar{x},y) = -\overline{\nabla}u_1(\bar{x})\cdot\chi_1(y)$, where the vector $\chi_1(y) = (\chi^1_1(y),\chi^2_1(y),\chi^3_1(y))$ belonging to $(H^1_{\overline{\mathrm{per}}}(Y_1))^3$ has to be determined. Recalling that $\overline{\nabla}u_1(\bar{x}) = (\nabla_{\bar{x}}u_1(\bar{x}),0)$, we notice that only the first two components of χ_1 will play a role in our analysis. Inserting this factorization into the equation, we therefore obtain the two local problems (4.13).

By choosing now $\Phi_1 = 0$ in (4.15), we get:

$$\int_{\omega\times Y_2} A(y)\nabla_y\widehat{u}_2\nabla_y\Phi_2 \,\mathrm{d}\bar{x}\,\mathrm{d}y - \int_{\omega\times\Gamma} h(y)(u_1-\widehat{u}_2)\Phi_2\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y = \int_{\omega\times Y_2} f(\bar{x})\Phi_2(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}y,$$

for all $\Phi_2 \in L^2(\omega, H^1(Y_2))$. This formally implies

$$-\mathrm{div}_y(A(y)\nabla_y\widehat{u}_2) = f \quad \text{in } \omega\times Y_2$$

and

$$A(y)\nabla_y\widehat{u}_2\cdot n = h(y)(u_1-\widehat{u}_2) \quad \text{on } \omega\times\Gamma,$$

which suggests us to look for the function $\widehat{u}_2$ of the form

$$\widehat{u}_2(\bar{x},y) = u_1(\bar{x}) + f(\bar{x})\chi_2(y) \quad \text{in } \omega\times Y_2.$$

By replacing this form of $\widehat{u}_2$ in the two previous equations, we obtain

$$-\mathrm{div}_y\left(A(y)(f(\bar{x})\nabla_y\chi_2)\right) = f(\bar{x}) \quad \text{in } \omega\times Y_2$$

and

$$A(y)\left(f(\bar{x})\nabla_y\chi_2\right)\cdot n = -h(y)f(\bar{x})\chi_2(y) \quad \text{on } \omega\times\Gamma,$$

which imply that the scalar function χ_2 is the solution of the Robin cell problem (4.14).

By choosing now $\Phi_1 = \Phi_2 = 0$ in (4.2), we obtain:

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla}u_1 + \nabla_y\widehat{u}_1)\overline{\nabla}\varphi\,\mathrm{d}\bar{x}\,\mathrm{d}y + \int_{\omega\times\Gamma} h(y)(u_1-\widehat{u}_2)\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y =$$

$$|Y_1|\int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + |\Gamma|\mathcal{M}_\Gamma(g)\int_\omega \varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_\omega k_+(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_\omega k_-(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x}. \tag{4.17}$$

We notice that

$$\int_{\omega\times\Gamma} h(y)(u_1-\widehat{u}_2)\varphi\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y = -\int_{\omega\times\Gamma} h(y)f(\bar{x})\chi_2(y)\varphi(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y =$$

$$\left(-\int_\Gamma h(y)\chi_2(y)\,\mathrm{d}\sigma_y\right)\int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} = \left(\int_\Gamma A(y)\nabla_y\chi_2(y)\cdot n\,\mathrm{d}\sigma_y\right)\int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} =$$

$$\left(\int_{Y_2}\mathrm{div}_y(A(y)\nabla_y\chi_2(y))\,\mathrm{d}y\right)\int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} =$$

$$\int_{Y_2}(-1)\,\mathrm{d}y\int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} = -|Y_2|\int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x},$$

and then relation (4.17) implies

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla}u_1 + \nabla_y\widehat{u}_1)\overline{\nabla}\varphi\,\mathrm{d}\bar{x}\,\mathrm{d}y = \int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + |\Gamma|\mathcal{M}_\Gamma(g)\int_\omega\varphi(\bar{x})\,\mathrm{d}\bar{x} +$$

$$\int_\omega k_+(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_\omega k_-(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x}.$$

We integrate this last equality by parts with respect to $\bar{x}$ and, by using (4.10) and (4.13), we are led to the homogenized problem (4.9) with the homogenized coefficients given by (4.12).

□

Remark 4.3 *The 2×2 homogenized matrix A^{hom} is lower dimensional with respect to the initial matrix A, but information coming from the vertical direction of the initial problem is preserved. Indeed, the value of the constant coefficients* (4.12) *is influenced by the third local variable y_3, through the solution χ_1 of the cell problem* (4.13)*. Consequently, the solution u_1 of the homogenized problem* (4.9) *is implicitly influenced by the third local variable y_3. The solution u_1 also depends on the third direction via the vertical fluxes k_+ and k_-, respectively. As far as the limit solutions $\widehat{u}_1$ and $\widehat{u}_2$ are concerned, their dependence on the third local variable y_3 is explicit, via the solutions χ_1 and χ_2 of the three-dimensional local problems* (4.13) *and* (4.14)*, respectively.*

We analyze now the second relevant situation for the jump function G^ε.

Case 2: $G^\varepsilon(x) = g\left(\dfrac{x}{\varepsilon}\right)$, if $\mathcal{M}_\Gamma(g) = 0$.

Theorem 4.4 *Under the assumption* (2.4)*, the unique solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (2.7) converges, in the sense of (4.1), to the unique solution $(u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{V}$ of the following unfolded limit problem:*

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla}u_1 + \nabla_y\widehat{u}_1)(\overline{\nabla}\varphi + \nabla_y\Phi_1)\,\mathrm{d}\bar{x}\,\mathrm{d}y + \int_{\omega\times Y_2} A(y)\nabla_y\widehat{u}_2\nabla_y\Phi_2\,\mathrm{d}\bar{x}\,\mathrm{d}y +$$

$$\int_{\omega\times\Gamma} h(y)(u_1-\widehat{u}_2)(\varphi-\Phi_2)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y = |Y_1|\int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_{\omega\times Y_2} f(\bar{x})\Phi_2(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}y +$$

$$\int_{\omega\times\Gamma} g(y)\Phi_1(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y + \int_\omega k_+(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_\omega k_-(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x}, \tag{4.18}$$

for all $\varphi \in H_0^1(\omega)$, $\Phi_1 \in L^2(\omega, H^1_{\overline{per}}(Y_1))$, $\Phi_2 \in L^2(\omega, H^1(Y_2))$.

Proof. To obtain the problem (4.18), we pass to the limit in the unfolded form of the variational formulation (2.7) with the test functions (4.4), which satisfy (4.5)-(4.8). The only difference with respect to the proof of Theorem 4.1 is the passage to the limit in the term involving the function G^ε. More precisely, we have now:

$$\frac{1}{\varepsilon}\int_{\omega\times\Gamma}\mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{T}_1^\varepsilon(v_1)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y=\frac{1}{\varepsilon}\int_{\omega\times\Gamma}\mathcal{T}_b^\varepsilon\left(g\left(\left\{\frac{\bar{x}}{\varepsilon}\right\},\frac{x_3}{\varepsilon}\right)\right)\mathcal{T}_1^\varepsilon\left(\varphi(\bar{x})+\varepsilon\omega_1(\bar{x})\psi_1\left(\left\{\frac{\bar{x}}{\varepsilon}\right\},\frac{x_3}{\varepsilon}\right)\right)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y=$$

$$\frac{1}{\varepsilon}\int_{\omega\times\Gamma}g(y)\mathcal{T}_1^\varepsilon(\varphi)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y+\int_{\omega\times\Gamma}g(y)\mathcal{T}_1^\varepsilon(\omega_1)(\bar{x},y)\mathcal{T}_1^\varepsilon(\psi_1)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y\to\int_{\omega\times\Gamma}g(y)\Phi_1(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y,$$

since

$$\frac{1}{\varepsilon}\int_{\omega\times\Gamma}g(y)\mathcal{T}_b^\varepsilon(\varphi)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y\to 0. \tag{4.19}$$

Indeed, let us denote

$$J^\varepsilon=\frac{1}{\varepsilon}\int_{\omega\times\Gamma}g(y)\mathcal{T}_b^\varepsilon(\varphi)(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y.$$

Then,

$$J^\varepsilon=\frac{1}{\varepsilon}\int_{\omega\times\Gamma}g(y)\left[\mathcal{T}_1^\varepsilon(\varphi)-\mathcal{M}_\Gamma(\mathcal{T}_1^\varepsilon(\varphi))\right]\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y+\frac{1}{\varepsilon}\int_{\omega\times\Gamma}g(y)\mathcal{M}_\Gamma(\mathcal{T}_1^\varepsilon(\varphi))\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y=J_1^\varepsilon+J_2^\varepsilon.$$

Set $y_\Gamma=y-\mathcal{M}_\Gamma(y)$. For the first integral J_1^ε, we have

$$J_1^\varepsilon\to\int_{\omega\times\Gamma}g(y)y_\Gamma\cdot\overline{\nabla}\varphi\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y, \tag{4.20}$$

since, following the same ideas as in [16], [19], and [11], one can prove that

$$\frac{\mathcal{T}_1^\varepsilon(\varphi)-\mathcal{M}_\Gamma(\mathcal{T}_1^\varepsilon(\varphi))}{\varepsilon}\to y_\Gamma\cdot\overline{\nabla}\varphi\quad\text{strongly in } L^2(\omega\times\Gamma).$$

Using the equality

$$\int_{\omega\times\Gamma}g(y)y_\Gamma\cdot\overline{\nabla}\varphi(\bar{x})\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y=\int_{\partial\omega\times\Gamma}g(y)y_\Gamma\varphi(\bar{x})\cdot\nu\,\mathrm{d}\sigma_{\bar{x}}\,\mathrm{d}\sigma_y=0,$$

which holds true since $\varphi\in\mathcal{D}(\omega)$ (here, ν is the unit normal exterior to ω), we get

$$J_1^\varepsilon\to 0. \tag{4.21}$$

The second integral J_2^ε equals zero, since $\mathcal{M}_\Gamma(g)=0$ and $\mathcal{M}_\Gamma(\mathcal{T}_1^\varepsilon(\varphi))$ independent on y imply

$$\int_{\omega\times\Gamma}g(y)\mathcal{M}_\Gamma(\mathcal{T}_1^\varepsilon(\varphi))\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y=0. \tag{4.22}$$

Thus, from (4.21) and (4.22), we obtain (4.19). Then, by using the density of $\mathcal{D}(\omega)\otimes H^1_{\overline{\mathrm{per}}}(Y_1)$ in $L^2(\omega,H^1_{\overline{\mathrm{per}}}(Y_1))$ and of $\mathcal{D}(\omega)\otimes H^1(Y_2)$ in $L^2(\omega,H^1(Y_2))$, we are led to the unfolded limit problem (4.18). Due to the uniqueness of $(u_1,\widehat{u}_1,\widehat{u}_2)\in\mathcal{V}$, which can be proven by the Lax-Milgram theorem, all the above convergences hold true for the whole sequence and our theorem is proven. □

Theorem 4.5 *Under the assumption* (2.4)*, the unique solution* $u^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon)$ *of the variational problem (2.7) converges, in the sense of (4.1), to* $(u_1,\widehat{u}_1,\widehat{u}_2)\in\mathcal{V}$*, where* u_1 *is the unique solution of the homogenized problem*

$$\begin{cases}-div_{\bar{x}}\left(A^{hom}\nabla u_1(\bar{x})\right)=f(\bar{x})+k_+(\bar{x})+k_-(\bar{x}) & \text{in } \omega,\\ u_1=0 & \text{on } \partial\omega\end{cases} \tag{4.23}$$

and

$$\widehat{u}_1(\bar{x},y)=-\sum_{j=1}^{2}\frac{\partial u_1}{\partial x_j}(\bar{x})\chi_1^j(y)+\eta(y), \tag{4.24}$$

$$\widehat{u}_2(\bar{x},y)=u_1(\bar{x})+f(\bar{x})\chi_2(y).$$

Here, A^{hom} is the homogenized matrix whose entries are given by (4.12) and the functions χ_1^j and χ_2 are defined by (4.13) and (4.14). The function η, $1-$periodic in y_1 and y_2, is the unique solution of the following non-homogeneous Neumann cell problem:

$$\begin{cases} -div_y(A(y)\nabla_y\eta) = 0 \quad in\ Y_1, \\ A(y)\nabla_y\eta\cdot n = -g(y) \quad on\ \Gamma, \\ A(y)\nabla_y\eta\cdot \nu_\pm = 0 \quad on\ \Sigma_\pm^1, \\ \mathcal{M}_\Gamma(\eta) = 0, \end{cases} \tag{4.25}$$

where n denotes the unit outward normal to Y_2 and $\nu_\pm = (0,0,\pm 1)$.

Proof. By taking $\varphi = 0$ in the unfolded limit problem (4.18), we obtain:

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla}u_1 + \nabla_y\widehat{u}_1)\nabla_y\Phi_1\,\mathrm{d}\bar{x}\,\mathrm{d}y + \int_{\omega\times Y_2} A(y)\nabla_y\widehat{u}_2\nabla_y\Phi_2\,\mathrm{d}\bar{x}\,\mathrm{d}y -$$

$$\int_{\omega\times\Gamma} h(y)(u_1-\widehat{u}_2)\Phi_2\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y = \int_{\omega\times Y_2} f(\bar{x})\Phi_2(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}y + \int_{\omega\times\Gamma} g(y)\Phi_1(\bar{x},y)\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y, \tag{4.26}$$

for all $\Phi_1 \in L^2(\omega, H^1_{\overline{\mathrm{per}}}(Y_1))$, $\Phi_2 \in L^2(\omega, H^1(Y_2))$.

By choosing $\Phi_1 = 0$ in (4.26), we have

$$\int_{\omega\times Y_2} A(y)\nabla_y\widehat{u}_2\nabla_y\Phi_2\,\mathrm{d}\bar{x}\,\mathrm{d}y - \int_{\omega\times\Gamma} h(y)(u_1-\widehat{u}_2)\Phi_2\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y = \int_{\omega\times Y_2} f(\bar{x})\Phi_2\,\mathrm{d}\bar{x}\,\mathrm{d}y.$$

By taking now suitable test functions Φ_2, we formally obtain

$$-\mathrm{div}_y(A(y)\nabla_y\widehat{u}_2) = f \quad \text{in } \omega\times Y_2$$

and

$$A(y)\nabla_y\widehat{u}_2\cdot n = h(y)(u_1-\widehat{u}_2) \quad \text{on } \omega\times\Gamma. \tag{4.27}$$

We then find the functions $\widehat{u}_2$ and χ_2 exactly like in the proof of Theorem 4.2.

Now, let us take $\Phi_2 = 0$ in (4.26). We formally obtain

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla}u_1 + \nabla_y\widehat{u}_1)\nabla_y\Phi_1\,\mathrm{d}\bar{x}\,\mathrm{d}y = \int_{\omega\times\Gamma} g(y)\Phi_1\,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y.$$

By taking suitable test functions Φ_1, we obtain

$$-\mathrm{div}_y(A(y)\nabla_y\widehat{u}_1) = \mathrm{div}_y(A(y)\overline{\nabla}u_1) \quad \text{in } \omega\times Y_1, \tag{4.28}$$

$$(A(y)\nabla_y\widehat{u}_1)\cdot n = -(A(y)\overline{\nabla}u_1)\cdot n - g(y) \quad \text{on } \omega\times\Gamma \tag{4.29}$$

$$(A(y)\nabla_y\widehat{u}_1)\cdot \nu_+ = -(A(y)\overline{\nabla}u_1)\cdot \nu_+ \quad \text{on } \omega\times\Sigma_+^1,$$

and

$$(A(y)\nabla_y\widehat{u}_1)\cdot \nu_- = -(A(y)\overline{\nabla}u_1)\cdot \nu_- \quad \text{on } \omega\times\Sigma_-^1.$$

The linearity of the problem (4.28) and the presence of the function g in relation (4.29) suggest us to search

$$\widehat{u}_1(x,y) = -\overline{\nabla}u_1(\bar{x})\cdot\chi_1(y) + \eta(y), \tag{4.30}$$

where the vector $\chi_1(y) = (\chi_1^1(y), \chi_1^2(y), \chi_1^3(y))$ belonging to $(H^1_{\overline{\mathrm{per}}}(Y_1))^3$ and the function η in $H^1(Y_1)$ have to be determined. Recalling that $\overline{\nabla}u_1(\bar{x}) = (\nabla_{\bar{x}}u_1(\bar{x}), 0)$, we notice that only the first two components of χ_1 will play a role in our analysis. Actually, the functions χ_1^j are defined by (4.13) and the function η remains to be found. To this end, we replace $\widehat{u}_1$ given by (4.30) in (4.28)-(4.29). By using (4.13), we deduce that the scalar function η is the unique solution of the cell problem (4.25), which is a non-homogeneous Neumann problem. The compatibility condition for this problem is satisfied, thanks to the hypothesis (2.4).

By choosing now $\Phi_1 = \Phi_2 = 0$ in (4.18), we get:

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla} u_1 + \nabla_y \widehat{u}_1)\overline{\nabla}\varphi \,\mathrm{d}\bar{x}\,\mathrm{d}y + \int_{\omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\varphi \,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y =$$

$$|Y_1| \int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x}. \tag{4.31}$$

Also, since

$$u_1(\bar{x}) - \widehat{u}_2(\bar{x}, y) = -f(\bar{x})\chi_2(y) \quad \text{in } \omega \times Y_2,$$

we have, as in the proof of Theorem 4.2, the equality

$$\int_{\omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\varphi \,\mathrm{d}\bar{x}\,\mathrm{d}\sigma_y = -|Y_2| \int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x}$$

and relation (4.31) then becomes:

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla} u_1 + \nabla_y \widehat{u}_1)\overline{\nabla}\varphi \,\mathrm{d}\bar{x}\,\mathrm{d}y = \int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_\omega k_+(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_\omega k_+(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x}.$$

We integrate this last equality by parts with respect to $\bar{x}$ and, using (4.30) and the definition (4.12) of the matrix A^{hom}, we obtain

$$-\mathrm{div}_{\,\bar{x}}\,(A^{\mathrm{hom}}\nabla_{\bar{x}} u_1) = f + k_+ + k_- + \mathrm{div}_{\bar{x}}\left(\int_{Y_1} A(y)\nabla\eta(y)\,\mathrm{d}y\right) \quad \text{in } \omega,$$

which leads immediately to the homogenized problem (4.23). □

Remark 4.6 *We remark that the homogenized matrix A^{hom} and, so, the solution u_1, are independent of the function h, while the limit $\widehat{u}_2$ depends on h, via the function χ_2. As far as the contribution of the jump function g is concerned, its effects are analogous to the ones observed in [9], [10] and [11] for similar problems studied in classical porous media: in case one, the function g is recovered as an additional source term in the homogenized problem* (4.9)*, while in case two the function g is recovered at a finer scale, in the expression of the corrector* (4.24)*, via the solution of the Neumman-type problem* (4.25)*.*

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