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Antonin Monteil, Paul Pegon

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# MASS CONCENTRATION IN RESCALED FIRST ORDER INTEGRAL FUNCTIONALS

ANTONIN MONTEIL

*Université Paris-Est Créteil Val-de-Marne, LAMA, France*

PAUL PEGON

*Université Paris-Dauphine, Ceremade & INRIA, Project team Mokaplan, France*

ABSTRACT. We consider first order local minimization problems of the form  $\min \int_{\mathbb{R}^N} f(u, \nabla u)$  under a mass constraint  $\int_{\mathbb{R}^N} u = m \in \mathbb{R}$ . We prove that the minimal energy function  $H(m)$  is always concave on  $(-\infty, 0)$  and  $(0, +\infty)$ , and that relevant rescalings of the energy, depending on a small parameter  $\varepsilon$ ,  $\Gamma$ -converge in the weak topology of measures towards the  $H$ -mass, defined for atomic measures  $\sum_i m_i \delta_{x_i}$  as  $\sum_i H(m_i)$ . We also consider space dependent Lagrangians  $f(x, u, \nabla u)$ , which cover the case of space dependent  $H$ -masses  $\sum_i H(x_i, m_i)$ , and also the case of a family of Lagrangians  $(f_\varepsilon)_\varepsilon$  converging as  $\varepsilon \rightarrow 0$ . The  $\Gamma$ -convergence result holds under mild assumptions on  $f$ , and covers several situations including homogeneous  $H$ -masses in any dimension  $N \geq 2$  for exponents above a critical threshold, and all concave  $H$ -masses in dimension  $N = 1$ . Our result yields in particular the concentration of Cahn-Hilliard fluids into droplets, and is related to the approximation of branched transport by elliptic energies.

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## NOTATION

$B_r(x)$	open ball of radius $r$ centered at $x$ ;
$B_r$	open ball $B_r(0)$ ;
$\mathcal{M}(\mathbb{R}^N)$	set of finite signed Borel measures on $\mathbb{R}^N$ ;
$\mathcal{M}_+(\mathbb{R}^N)$	set of finite positive Borel measures on $\mathbb{R}^N$ ;
$\tau_x \mu$	Borel measure $A \mapsto \mu(A - x)$ if $\mu \in \mathcal{M}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ ;
$c_B \mu$	Borel measure $\tau_{-x}(\mu \llcorner B)$ if $B$ is the ball $B_r(x)$ ;
$\mu_\ell \xrightarrow{\mathcal{C}'_0} \mu$	weak convergence of measures, i.e. weak- $\star$ convergence in duality with the space $\mathcal{C}'_0(\mathbb{R}^N)$ of continuous functions vanishing at infinity;
$\mu_\ell \xrightarrow{\mathcal{C}'_b} \mu$	narrow convergence of measures, i.e. weak- $\star$ convergence in duality with the space of continuous and bounded function $\mathcal{C}'_b(\mathbb{R}^N)$ ;
$\Sigma$	set of increasing maps $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ ;
$\sigma_1 \preceq \sigma_2$	$\sigma_1, \sigma_2 \in \Sigma$ are such that $\sigma_1(\llbracket n, +\infty \rrbracket) \subseteq \sigma_2(\mathbb{N})$ for some $n \in \mathbb{N}$ ;
$\pm$	fixed to either $+$ or $-$ in the whole statement or proof, and $\mp = -(\pm)$ .

## 1. INTRODUCTION

1.1. **Setting.** Let  $N \in \mathbb{N}^*$  and let  $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  be a Borel function. Consider the following energy functional, defined for any fixed  $x \in \mathbb{R}^N$  on the set of finite Borel measures  $\mathcal{M}(\mathbb{R}^N)$  on  $\mathbb{R}^N$  by

$$(1.1) \quad \mathcal{E}_f^x(u) = \begin{cases} \int_{\mathbb{R}^N} f(x, u(y), \nabla u(y)) \, dy & \text{if } u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N), \\ +\infty & \text{otherwise.} \end{cases}$$

The minimization of this energy under a mass constraint gives rise to the notion of minimal cost function, defined by

$$(1.2) \quad H_f(x, m) := \inf \left\{ \mathcal{E}_f^x(u) : u \in W_{\text{loc}}^{1,1} \cap L^1(\mathbb{R}^N) \text{ such that } \int_{\mathbb{R}^N} u = m \right\} \in [0, +\infty].$$

Note that the minimization can be restricted to non-negative functions when  $m \geq 0$  and non-positive functions when  $m \leq 0$ , as explained in [Remark 2.2](#).

As a preliminary result, which deserves interest on its own, we will establish the following:

**Theorem 1.1.** *Let  $x \in \mathbb{R}^N$ . The map  $m \mapsto H_f(x, m)$  is concave non-decreasing on  $(0, +\infty)$ . If we further assume that  $f(x, 0, 0) = 0$  then  $H_f(x, 0) = 0$  and  $H_f(x, \cdot)$  is either identically  $+\infty$  on  $(0, +\infty)$  or continuous on  $[0, +\infty)$ ; it is in particular lower semicontinuous on  $\mathbb{R}_+$ . A similar statement holds on  $(-\infty, 0]$ .*

The proof is very simple and works with no further assumptions on  $f$ , as a consequence of [Theorem 2.1](#) and the following [Remark 2.2](#).

Our main purpose is to prove that under some conditions, if  $(f_\varepsilon)_{\varepsilon>0}$  is a family of functions  $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  converging pointwise to  $f$  as  $\varepsilon \rightarrow 0$ , then the rescaled energy functionals  $\mathcal{E}_\varepsilon$ , defined for each  $\varepsilon > 0$  on  $\mathcal{M}(\mathbb{R}^N)$  by

$$(1.3) \quad \mathcal{E}_\varepsilon(u) = \begin{cases} \int_{\mathbb{R}^N} f_\varepsilon(x, \varepsilon^N u(x), \varepsilon^{N+1} \nabla u(x)) \varepsilon^{-N} dx & \text{if } u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases}$$

$\Gamma$ -converge as  $\varepsilon \rightarrow 0$ , for the narrow or weak convergence of measures, to the  $H_f$ -mass, defined on  $\mathcal{M}(\mathbb{R}^N)$  by (see [Definition 2.3](#)):

$$\begin{aligned} \mathbf{M}^{H_f}(u) &:= \int_{\mathbb{R}^N} H_f(x, u(\{x\})) d\mathcal{H}^0(x) \\ &\quad + \int_{\mathbb{R}^N} H'_f(x, 0^+) du_+^d + \int_{\mathbb{R}^N} H'_f(x, 0^-) du_-^d(x). \end{aligned}$$

where  $u = u^a + u^d$  is the decomposition of  $u$  into its atomic part  $u^a$  and its diffuse part  $u^d$ ,  $u^d = u_+^d - u_-^d$  is the Jordan decomposition of  $u^d$  into positive and negative parts, and  $H'_f(x, 0^\pm) = \lim_{m \rightarrow 0^\pm} \frac{H_f(x, m)}{|m|} \in [0, +\infty]$ .

This kind of singular limit of integral functionals is reminiscent of several variational models with physical relevance which have been the object of intensive mathematical analysis, such as Cahn-Hilliard fluids with concentration on droplets [[BDS96](#)] or on singular interfaces [[MM77](#)], toy models for micromagnetism and liquid crystals like Aviles-Giga [[AG99](#)] and Landau-de Gennes [[BPP12](#)], or Ginzburg-Landau theory of superconductivity [[BBH17](#)].

The fact that  $\mathbf{M}^{H_f}$  is expected to be the  $\Gamma$ -limit of  $\mathcal{E}_\varepsilon$  is due to the following observation: if  $B_r(x_0) \subseteq \mathbb{R}^N$  and  $u_\varepsilon(x) =: \varepsilon^{-N} v_\varepsilon(\varepsilon^{-1}(x - x_0))$ , then  $\int_{B_r(x_0)} u_\varepsilon = \int_{B_{r/\varepsilon}} v_\varepsilon$  and

$$\int_{B_r(x_0)} f_\varepsilon(x, \varepsilon^N u_\varepsilon(x), \varepsilon^{N+1} \nabla u_\varepsilon(x)) \varepsilon^{-N} dx = \int_{B_{r/\varepsilon}} f_\varepsilon(x_0 + \varepsilon y, v_\varepsilon(y), \nabla v_\varepsilon(y)) dy,$$

so that the energy contribution of a mass  $m \geq 0$  contained in a ball  $B_r(x_0)$  should be of the order of  $H_f(x_0, m)$ .

Nevertheless, it is not true in general that  $\mathbf{M}^{H_f}$  is the  $\Gamma$ -limit of the functionals  $\mathcal{E}_\varepsilon$  (see [Section 1.3](#) below). We will need a couple of assumptions on  $f$  and  $f_\varepsilon$  detailed in the next section.

**1.2. Assumptions and main result.** Our first two assumptions are rather standard and guarantee the sequential lower semicontinuity of the functionals  $\mathcal{E}_f^x$ ,

(H<sub>1</sub>)  $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  is lower semicontinuous,

(H<sub>2</sub>)  $f(x, u, \cdot)$  is convex for every  $x \in \mathbb{R}^N$ ,  $u \in \mathbb{R}$ .

In order for vanishing parts to have no energetic contribution, we will impose

(H<sub>3</sub>)  $f(x, 0, 0) = 0$  for every  $x \in \mathbb{R}^N$ .

We also need continuity in the spatial variable,

(H<sub>4</sub>)  $f(\cdot, u, \xi)$  is continuous for every  $u \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^N$ .

Next, we need a compactness assumption which ensures relative compactness in the weak topology of  $W_{\text{loc}}^{1,p}(\mathbb{R}^N)$  for sequences of bounded energy  $\mathcal{E}_f^x$  and bounded mass; it will also be needed in obtaining lower bounds for the energy (see [Proposition 3.8](#)):

(H<sub>5</sub>) there exist  $\alpha, \beta \in (0, +\infty)$ ,  $p \in (1, +\infty)$  such that for all  $(x, u, \xi) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ ,

$$f(x, u, \xi) \geq \alpha|\xi|^p - \beta u.$$

We also impose a condition on the slope of  $f(x, \cdot, \xi)$  at the origin which will be needed in order to identify the initial slope of  $H_f(x, \cdot)$  (see [Section 2.3](#)), and rules out some non-trivial scale invariant Lagrangians for which the expected  $\Gamma$ -convergence result fails (see [Section 1.3](#)),

(H<sub>6</sub>) for every  $x_0 \in \mathbb{R}^N$ ,

$$(1.4) \quad f'_-(x_0, 0^\pm, 0) := \liminf_{(x, u, \xi) \rightarrow (x_0, 0^\pm, 0)} \frac{f(x, u, \xi)}{|u|} \geq \limsup_{u \rightarrow 0^\pm} \sup_{|\xi|=1} \frac{f(x_0, u, \rho(|u|)\xi)}{|u|},$$

with  $\rho \equiv 0$  if  $N = 1$  and for some  $\rho \in \mathcal{C}((0, 1], (0, +\infty))$  satisfying

$$\int_0^1 \left( \int_y^1 \frac{dt}{\rho(t)} \right)^N dy < +\infty \quad \text{if } N \geq 2.$$

Since our aim is not to care much about the dependence on  $x$ , we shall impose a spatial quasi-homogeneity condition:

(H<sub>7</sub>) there exists  $C < +\infty$  such that for every  $x, y \in \mathbb{R}^N$ ,  $u \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^N$ ,

$$f(y, u, \xi) \leq C(f(x, u, \xi) + u).$$

Last of all, we need the family of functions  $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  to converge towards  $f$  in a suitable sense, namely, we assume

(H<sub>8</sub>)  $f_\varepsilon \uparrow f$  and  $f'_{\varepsilon,-}(\cdot, 0^\pm, 0) \uparrow f'_-(\cdot, 0^\pm, 0)$  as  $\varepsilon \rightarrow 0$ .

Notice that this assumption is empty if  $f_\varepsilon$  does not depend on  $\varepsilon$ .

Our main result is the following:

**Theorem 1.2.** *If  $(f_\varepsilon)_{\varepsilon>0}$  satisfies (H<sub>8</sub>) with each  $f_\varepsilon$  satisfying (H<sub>1</sub>)–(H<sub>5</sub>) and the limit  $f$  satisfying (H<sub>6</sub>)–(H<sub>7</sub>), then  $\mathbf{M}^{H_f}$  is the  $\Gamma$ -limit as  $\varepsilon \rightarrow 0$  of the functionals  $\mathcal{E}_\varepsilon$ , defined in (1.3), for both the weak convergence and the narrow convergence of measures.*

In particular, as a  $\Gamma$ -limit, the functional  $\mathbf{M}^{H_f}$  must be lower semicontinuous for the weak convergence of measures (and so for the narrow convergence as well). This implies that  $H_f$  is lower semicontinuous on  $\mathbb{R}^N \times \mathbb{R}$  (see [Proposition 2.5](#)).

We point out that for the  $\Gamma$ -lim sup, we need weaker assumptions on  $f_\varepsilon$  and  $f$  (see [Proposition 4.2](#)), which will be useful for some applications (see [Section 5.5](#)).

We will allow ourselves slight abuses of notation. We will sometimes consider Lagrangians defined on  $\mathbb{R} \times \mathbb{R}^N$  which do not depend on  $x$  and still refer to hypotheses [\(H<sub>1</sub>\)](#)–[\(H<sub>8</sub>\)](#); we will use the notation  $\mathcal{E}_f$  instead of  $\mathcal{E}_f^x$  in [\(1.1\)](#) and consider  $H_f$  as a function of  $m$  only in [\(1.2\)](#). Besides, we will also consider Lagrangians defined only for  $u \in \mathbb{R}_+$ , which may be thought as defined for  $u \in \mathbb{R}$ , set to  $+\infty$  when  $u$  is negative<sup>1</sup>. The resulting minimal cost function  $H_f$  and its associated  $H_f$ -mass may be thought as defined on  $\mathbb{R}_+$  and  $\mathcal{M}_+(\mathbb{R}^N)$  respectively, as they will be infinite on the respective complements.

**1.3. Remarks and applications.** We start with two situations where the expected  $\Gamma$ -convergence fails and which justify the importance of [\(H<sub>3</sub>\)](#) and [\(H<sub>6</sub>\)](#), then we provide examples and applications of our result, as a short summary of [Section 5](#), where full details are provided. We restrict our attention to positive measures and Lagrangians defined for  $u \geq 0$ .

*Lagrangians giving energy to vanishing parts.* By assumption [\(H<sub>3</sub>\)](#) no energy is given to any set where a function  $u$  vanishes. It is a necessary condition for  $\mathbf{M}^{H_f}$  to be lower semicontinuous (a necessary condition to be a  $\Gamma$ -limit) and not identically  $+\infty$ . Indeed if  $\mathbf{M}^{H_f}$  is lower semicontinuous and finite for some measure  $u \in \mathcal{M}(\mathbb{R}^N)$  then thanks to [Theorem 2.1](#) we know that  $\mathbf{M}^{H_f}(tu) \leq \mathbf{M}^{H_f}(u)$  for every  $t \in (0, 1)$  and thus  $\mathbf{M}^{H_f}(0) \leq \liminf_{t \rightarrow 0^+} \mathbf{M}^{H_f}(tu) \leq \mathbf{M}^{H_f}(u) < +\infty$ . Thus  $\mathbf{M}^{H_f}$  is not identically  $+\infty$  if and only if  $\mathbf{M}^{H_f}(0) < +\infty$ , i.e.  $\int_{\mathbb{R}^N} H_f(x, 0) d\mathcal{H}^0(x) < +\infty$ . But since  $H_f(x, 0) = (+\infty) \times f(x, 0, 0)$  this can only happen if  $f(\cdot, 0, 0) \equiv 0$ . This justifies imposing [\(H<sub>3</sub>\)](#).

*Scale invariant Lagrangians.* In the particular case where  $f(x, u, \xi) = u^{-p(1-\frac{1}{p^*})} |\xi|^p$  and  $f_\varepsilon \equiv f$ , with  $p \in (1, N)$  and  $p^* = \frac{pN}{N-p}$ , we find that

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(x, \varepsilon^N u, \varepsilon^{N+1} \nabla u) \varepsilon^{-N} = \int_{\mathbb{R}^N} u^{-p(1-\frac{1}{p^*})} |\nabla u|^p = \mathcal{E}_f(u),$$

i.e. the rescaled energies  $\mathcal{E}_\varepsilon$  do not depend on  $\varepsilon > 0$ . A scaling analysis also shows that the associated minimal cost function satisfies  $H_f(m) = m^{1-\frac{N}{p^*}} H_f(1)$ . Moreover, it can be seen (see [Section 5.1](#)) that  $0 < H_f(1) < +\infty$ , which implies that the  $\Gamma$ -limit of  $\mathcal{E}_\varepsilon$ , which is nothing but the lower semicontinuous relaxation of  $\mathcal{E}_f$ , does not coincide with  $\mathbf{M}_{H_f}$ . Considering the perturbation of  $f$  given by  $\tilde{f}(x, u, \xi) = f(x, u, \xi) + |\xi|^p$ , we find a Lagrangian satisfying all our assumptions except [\(H<sub>6</sub>\)](#) (note that  $|\xi|^p$  is needed in [\(H<sub>5</sub>\)](#)), and such that the associated rescaled energies do not  $\Gamma$ -converge to  $\mathbf{M}_{H_f}$ . Hence, an assumption like [\(H<sub>6</sub>\)](#) is required in our  $\Gamma$ -convergence result. We will even see that the lower semicontinuity of  $H_f$  and  $\mathbf{M}_{H_f}$  is not guaranteed without [\(H<sub>6</sub>\)](#).

<sup>1</sup>Notice that if any of our assumptions is satisfied for a Lagrangian defined for  $u \in \mathbb{R}_+$ , then it holds also for the Lagrangian extended to  $\mathbb{R}$  in this way.

*Concave  $H$ -masses in dimension one.* Consider the energy

$$\mathcal{E}_f(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + c(u) \quad \text{with Lagrangian} \quad f(x, u, \xi) = |\xi|^2 + c(u).$$

In dimension  $N = 1$ , it is shown in [Wir19] that for any concave continuous function  $H$  with  $H(0) = 0$ , there exists a suitable  $c \geq 0$  such that  $H_f = H$ . As explained in Section 5.2, Theorem 1.2 implies that the rescaled energies

$$(1.5) \quad \mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(\varepsilon^N u, \varepsilon^{N+1} \nabla u) \varepsilon^{-N} = \int_{\mathbb{R}^N} \varepsilon^{N+2} |\nabla u|^2 + \varepsilon^{-N} c(\varepsilon^N u).$$

$\Gamma$ -converge to  $\mathbf{M}^H$ , leading to an elliptic approximation of any concave  $H$ -mass in dimension one. In dimension  $N \geq 2$ , we will show that  $H_f$  must be concave on  $(0, +\infty)$ , and strictly concave after the possible initial interval where it is linear (see Proposition 2.8); however, we have no solution to the inverse problem, consisting in characterizing the class of attainable minimal cost functions  $H = H_f$  for Lagrangians  $f$  satisfying our assumptions.

*Homogeneous  $H$ -masses in any dimension.* We consider the functional

$$(1.6) \quad \mathcal{E}_f(u) = \int_{\mathbb{R}^N} f(u, \nabla u) = \int_{\mathbb{R}^N} |\nabla u|^p + u^s, \quad p > 1, \quad s \in (-p', 1].$$

Then, the rescaled energies

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(\varepsilon^N u, \varepsilon^{N+1} \nabla u) \varepsilon^{-N} = \int_{\mathbb{R}^N} \varepsilon^{pN+p-N} |\nabla u|^p + \varepsilon^{-(1-s)N} u^s$$

$\Gamma$ -converge to a non-trivial multiple of some  $\alpha$ -mass  $\mathbf{M}^\alpha := \mathbf{M}^{t \rightarrow t^\alpha}$  where the exponent  $\alpha = (1 - \frac{s}{p} + \frac{s}{N})(1 - \frac{s}{p} + \frac{1}{N})^{-1}$  ranges over  $(1 - \frac{2}{N+1}, 1]$  when  $(s, p)$  varies in its range and  $N \geq 1$ . More details are given in Section 5.3.

*Cahn-Hilliard approximations of droplets models.* Following the works of [BDS96; Dub98], we consider the functionals

$$(1.7) \quad \mathcal{W}_\varepsilon(u) = \int_{\mathbb{R}^N} \varepsilon^{-\rho} (W(u) + \varepsilon |\nabla u|^2),$$

where  $W(t) \sim_{t \rightarrow +\infty} t^s$  for some exponent  $s \in (-2, 1)$ . As shown in Section 5.5, we may rewrite these functionals to fit our general framework, and recover known  $\Gamma$ -convergence results, under slightly more general assumptions, as stated in Theorem 5.1. The  $\Gamma$ -limit is a non-trivial multiple of the  $\alpha$ -mass with  $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$ .

*Elliptic approximations of Branched Transport.* The energy of Branched Transport (see [BCM09] for an account of the theory), in its Eulerian formulation, is an  $H$ -mass defined this time on vector measures  $w$  whose divergence is also a measure,

$$(1.8) \quad \mathbf{M}_1^H(w) := \int_{\Sigma} H(x, \theta(x)) \, d\mathcal{H}^1(x) + \int_{\mathbb{R}^d} H'(x, 0^+) \, d|w^\perp|,$$

where  $w = \theta\xi \cdot \mathcal{H}^1 \llcorner \Sigma + w^\perp$  is the decomposition of  $w$  into its 1-rectifiable and 1-diffuse parts (see Section 5.4 for more details). An elliptic approximation of Modica-Mortola type has been introduced in [OS11] for  $H(m) = m^\alpha$ ,  $\alpha \in (0, 1)$ , and their  $\Gamma$ -convergence result in dimension  $d = 2$  has been extended to any dimension in [Mon15] by a slicing method which relates the energy of  $w$  to the energy of its

slicings. The same slicing method, together with [Theorem 1.2](#), would allow to prove the  $\Gamma$ -convergence of the functionals

$$(1.9) \quad \mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} f_\varepsilon(x, \varepsilon^{d-1}|w|(x), \varepsilon^d|\nabla w|(x))\varepsilon^{1-d} dx & \text{if } w \in W_{\text{loc}}^{1,1}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

toward  $\mathbf{M}_1^{H_f}$  for Lagrangians  $f_\varepsilon \rightarrow f$  satisfying [\(H<sub>1</sub>\)](#)–[\(H<sub>8</sub>\)](#), thus covering a wide range of concave  $H$ -masses over vector measures with divergence.

**1.4. Structure of the paper.** In [Section 2](#), we prove the concavity of the minimal cost function  $H_f$  with respect to the mass variable  $m$  in full generality ([Theorem 2.1](#)), we establish useful properties of general  $H$ -masses, and we identify the slope at the origin of  $H_f$  in terms of  $f$  under our assumption ([Proposition 2.6](#) and [Proposition 2.7](#)). In [Section 3](#), we apply a concentration-compactness principle to provide a profile decomposition theorem for sequences of positive measures ([Theorem 3.2](#)), which is used to obtain our main lower bound for the energy  $\mathcal{E}_f$  ([Proposition 3.10](#)) and also yields an existence criterion for profiles with minimal energy under a mass constraint ([Proposition 3.12](#)). [Section 4](#) is dedicated to proving lower and upper bounds on the rescaled energies  $\mathcal{E}_\varepsilon$  ([Proposition 4.1](#) and [Proposition 4.2](#)) that imply in particular our main  $\Gamma$ -convergence result ([Theorem 1.2](#)). Last of all, in [Section 5](#), we provide counterexamples and several examples of energy functionals that fall into our framework, as summarized in the previous section.

## 2. MINIMAL COST FUNCTION AND $H$ -MASS

In this section, we study the properties of general  $H$ -masses, of costs  $H_f$  associated with general Lagrangians  $f$ , and we relate the slope of  $H_f$  at  $m = 0$  to that of  $f$  at  $(u, \xi) = (0, 0)$  in the variable  $u$ , under particular conditions.

### 2.1. Concavity and lower semicontinuity of the minimal cost function.

Our concavity result stated in [Theorem 1.1](#) is a consequence of:

**Theorem 2.1.** *Let  $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$  be Borel measurable. The function*

$$(2.1) \quad H_f(m) = \inf \left\{ \int_{\mathbb{R}^N} f(u, \nabla u) : u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+), \int_{\mathbb{R}^N} u = m \right\}, \quad m \geq 0,$$

*is either identically  $+\infty$  on  $(0, +\infty)$ , or it is everywhere finite, continuous, concave and non-decreasing on  $(0, +\infty)$ . In the latter case, if moreover  $f(0, 0) = 0$  then  $\lim_{m \rightarrow 0^+} H_f(m) = H_f(0) = 0$ .*

The fact that [Theorem 2.1](#) implies [Theorem 1.1](#) is due to the following remark.

*Remark 2.2.* The minimization in [\(1.2\)](#) can be restricted to non-negative (resp. non-positive) functions when  $m \geq 0$  (resp.  $m \leq 0$ ). For instance fix  $x \in \mathbb{R}^N$ ,  $m \geq 0$ , and take  $u \in W_{\text{loc}}^{1,1} \cap L^1(\mathbb{R}^N)$  such that  $\int_{\mathbb{R}^N} u = m$ . If  $u = u_+ - u_-$  is the decomposition of  $u \in W_{\text{loc}}^{1,1} \cap L^1(\mathbb{R}^N)$  into positive and negative parts, then  $\mathcal{E}_f^x(u) = \mathcal{E}_g(u_+) + \mathcal{E}_{\tilde{g}}(u_-)$ , with  $g, \tilde{g}$  defined on  $\mathbb{R}_+ \times \mathbb{R}^N$  by  $g(u, \xi) = f(x, u, \xi)$  for every  $(u, \xi)$  and  $\tilde{g}(u, \xi) = f(x, -u, -\xi)$  for  $(u, \xi) \neq (0, 0)$ ,  $\tilde{g}(0, 0) = 0$ . In particular  $\mathcal{E}_f^x(u) \geq H_g(m_+) + H_{\tilde{g}}(m_-)$  where  $m_\pm = \int_{\mathbb{R}^N} u_\pm$ . Since  $H_g$  is non-decreasing on  $(0, +\infty)$  and  $H_{\tilde{g}}$  is non-negative, we therefore have  $\mathcal{E}_f^x(u) \geq H_g(m)$  since  $m_+ \geq m$ . A similar reasoning holds for non-positive functions when  $m \leq 0$ .



Note that, considering Lagrangians  $f$  taking infinite values, [Theorem 2.1](#) covers the case where we have a constraint  $(u, \nabla u) \in A$ , where  $A \subseteq \mathbb{R} \times \mathbb{R}^N$  is Borel measurable.

*Proof.* We first prove that  $H_f$  is concave on  $(0, +\infty)$ . Let  $m > 0$  and  $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+)$  such that  $\int_{\mathbb{R}^N} u = m$ . We pick a non-zero vector  $v \in \mathbb{R}^N$  and for every  $t \in \mathbb{R}$ , we set  $u^t(\cdot) = u(\cdot + tv)$  and

$$u \wedge u^t(\cdot) = \min\{u(\cdot), u^t(\cdot)\}, \quad u \vee u^t(\cdot) = \max\{u(\cdot), u^t(\cdot)\}.$$

We have  $u \wedge u^t + u \vee u^t = u + u^t$ . Hence

$$(2.2) \quad \int_{\mathbb{R}^N} u \wedge u^t + \int_{\mathbb{R}^N} u \vee u^t = 2 \int_{\mathbb{R}^N} u = 2m.$$

Moreover, it is standard that  $u \wedge u^t = u - (u^t - u)_- \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$  with  $\nabla(u \wedge u^t) = \nabla u$  a.e. in  $\{u \leq u^t\}$  and  $\nabla(u \wedge u^t) = \nabla u^t$  a.e. in  $\{u > u^t\}$ . Since  $u \vee u^t = u + u^t - u \wedge u^t$ , we have similar identities for  $u \vee u^t$ , and we obtain

$$(2.3) \quad \mathcal{E}_f(u \wedge u^t) + \mathcal{E}_f(u \vee u^t) = \mathcal{E}_f(u) + \mathcal{E}_f(u^t) = 2\mathcal{E}_f(u).$$

Now, let  $M : t \mapsto \int_{\mathbb{R}^N} u \wedge u^t$ . In view of [\(2.2\)](#), [\(2.3\)](#), and by definition of  $H$ , we have proved

$$(2.4) \quad H_f(M(t)) + H_f(2m - M(t)) \leq 2\mathcal{E}_f(u).$$

Now, by continuity of translations in  $L^1$  and since the map  $(x, y) \mapsto x \wedge y$  is Lipschitz on  $\mathbb{R}^2$ , we have that  $M$  is continuous on  $\mathbb{R}$  with  $M(0) = m$ . Moreover  $\lim_{t \rightarrow +\infty} M(t) = 0$  by dominated convergence. So, by the intermediate value theorem  $M(\mathbb{R}) \supseteq (0, m]$ . Hence, we have proved  $H_f(\theta) + H_f(2m - \theta) \leq 2\mathcal{E}_f(u)$  for every  $\theta \in (0, m]$ . Taking the infimum over  $u$  such that  $\int_{\mathbb{R}^N} u = m$ , we obtain

$$\frac{H_f(\theta) + H_f(2m - \theta)}{2} \leq H_f(m), \quad \forall \theta \in (0, m],$$

that is,  $H_f$  is midpoint concave on  $(0, +\infty)$ . Since  $H_f$  is also bounded below (by 0), we can deduce that  $H_f$  is concave  $(0, +\infty)$  (see [\[RV73, Section 72\]](#)), and since  $H_f \geq 0$ , either  $H_f$  is identically  $+\infty$  on  $(0, +\infty)$ , or it is finite everywhere, continuous, concave and non-decreasing on  $(0, +\infty)$ .

We now justify that if  $H_f(m) < +\infty$  for some  $m > 0$  and  $f(0, 0) = 0$ , then  $\lim_{m \rightarrow 0^+} H_f(m) = 0 = H_f(0)$ . Let  $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+)$  such that  $\int_{\mathbb{R}^N} u = m > 0$  and  $\mathcal{E}_f(u) < +\infty$ , and set

$$t_* := \sup\{t \geq 0 : M(t) > 0\} \in [0, +\infty], \quad \text{where} \quad M(t) = \int_{\mathbb{R}^N} u \wedge u^t.$$

Since  $M$  is continuous with  $M(0) = \int_{\mathbb{R}^N} u > 0$  and  $\lim_{t \rightarrow +\infty} M(t) = 0$  as seen above, we have that  $t_* \in (0, +\infty]$ ,  $\lim_{t \rightarrow t_*} M(t) = 0$  and  $M(t)$  does not vanish identically near  $t_*$ . Moreover, if  $t_* = +\infty$ , since  $u^t \rightarrow 0$  locally in measure, by dominated convergence,

$$\begin{aligned} \limsup_{m \rightarrow 0^+} H_f(m) &\leq \limsup_{t \rightarrow +\infty} \mathcal{E}_f(u \wedge u^t) = \limsup_{t \rightarrow +\infty} \int_{\{u < u^t\}} f(u, \nabla u) + \int_{\{u^{-t} \geq u\}} f(u, \nabla u) \\ &\leq 2f(0, 0)|\{u = 0\}| = 0. \end{aligned}$$

If  $t_* < +\infty$ , we have  $u \wedge u^{t_*} = 0$  a.e. and  $u^t \rightarrow u^{t_*}$  locally in measure as  $t \rightarrow t_*$  by continuity of translation in  $L^1$ . Thus using dominated convergence again,

$$\begin{aligned} \limsup_{m \rightarrow 0^+} H_f(m) &\leq \limsup_{t \rightarrow (t_*)^-} \mathcal{E}_f(u \wedge u^t) = \limsup_{t \rightarrow (t_*)^-} \int_{\{u < u^t\}} f(u, \nabla u) + \int_{\{u^{-t} \geq u\}} f(u, \nabla u) \\ &= \int_{\{u < u^{t_*}\}} f(u, \nabla u) + \int_{\{u^{-t_*} \geq u\}} f(u, \nabla u) \\ &= \mathcal{E}_f(u \wedge u^{t_*}) = f(0, 0) \times (+\infty) = 0. \quad \square \end{aligned}$$

## 2.2. $H$ -transform and $H$ -mass.

**Definition 2.3.** Let  $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$  be a Borel measurable function having left/right slopes at the origin defined for each  $x \in \mathbb{R}^N$  by

$$(2.5) \quad H'(x, 0^\pm) := \lim_{m \rightarrow 0^\pm} \frac{H(x, m)}{|m|} \in [0, +\infty].$$

We define the  $H$ -transform of a finite signed Borel measure  $u \in \mathcal{M}(\mathbb{R}^N)$  by:

$$H(u) := H(\cdot, u(\{\cdot\})) \mathcal{H}^0 + H'(\cdot, 0^+) u_+^d + H'(\cdot, 0^-) u_-^d,$$

where  $u = u^a + u^d$  is the decomposition of  $u$  into its atomic part  $u^a$  and its diffuse (or non-atomic) part  $u^d$ ,  $u^d = u_+^d - u_-^d$  is the Jordan decomposition of  $u^d$  into positive and negative parts.

The  $H$ -mass of  $u$  is then defined as the total variation of  $H(u)$ , that is:

$$\begin{aligned} \mathbf{M}^H(u) &:= \int_{\mathbb{R}^N} H(x, u(\{x\})) \, d\mathcal{H}^0(x) \\ &\quad + \int_{\mathbb{R}^N} H'(x, 0^+) \, du_+^d(x) + \int_{\mathbb{R}^N} H'(x, 0^-) \, du_-^d(x). \end{aligned}$$

$\mathbf{M}^H(u)$  is a natural spatially non-homogeneous extension (depending on the position  $x$ ) of the  $H$ -mass of  $k$ -dimensional flat currents<sup>2</sup> from Geometric Measure Theory, introduced by [Fle66] (see also the more recent works [DH03; Col+17]).

We say that  $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$  is *mass-concave* if  $m \mapsto H(x, m)$  is concave on  $(0, +\infty)$  and  $(-\infty, 0)$  for every  $x \in \mathbb{R}^N$ . From [BB90], we have the following result<sup>3</sup>:

**Proposition 2.4** ([BB90, Theorem 3.3]). *Assume that  $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$  is lower semicontinuous, mass-concave and  $H(\cdot, 0) \equiv 0$ . Then  $\mathbf{M}^H$  is sequentially lower semicontinuous on  $\mathcal{M}(\mathbb{R}^N)$  for the weak topology.*

From another work from the same authors [BB93, Theorem 3.2], we know that under some further assumptions on  $H$ ,  $\mathbf{M}^H$  is the relaxation for the weak topology of the functional

$$\mathbf{M}_{\text{atom}}^H(u) = \begin{cases} \sum_{i=1}^k H(x_i, m_i) & \text{if } u = \sum_{i=1}^k m_i \delta_{x_i} \text{ with } k \in \mathbb{N}^*, x_i \in \mathbb{R}^N, m_i \in \mathbb{R}, \\ +\infty & \text{otherwise.} \end{cases}$$

<sup>2</sup>In the case  $k = 0$ , since signed measures are merely 0-currents with finite mass.

<sup>3</sup>In the notation of this paper, we take  $\mu = 0$  and  $f(x, s) = |s|^2$ ; we have  $\varphi_f(x, 0) = 0$  and  $\varphi_f(x, s) = +\infty$  if  $s \neq 0$ .

We need a slightly different result<sup>4</sup>, namely that for any function  $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$  satisfying all the assumptions of [Proposition 2.4](#) except the lower semicontinuity, the relaxation of  $\mathbf{M}_{\text{atom}}^H$  for the narrow sequential convergence is  $\mathbf{M}^{H_{\text{lsc}}}$ , where  $H_{\text{lsc}}$  is the lower semicontinuous envelope of  $H$ , which can be expressed as

$$(2.6) \quad H_{\text{lsc}}(x, m) = \sup\{G(x, m) : G \leq H \text{ with } G \text{ lower semicontinuous}\}.$$

It is worth noticing that if  $H(\cdot, 0) \equiv 0$  and  $H$  is mass-concave, then these properties hold also for  $H_{\text{lsc}}$ .

**Proposition 2.5.** *Let  $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$  be a function which is mass-concave and such that  $H(\cdot, 0) \equiv 0$ . Then, the sequentially lower semicontinuous envelope of  $\mathbf{M}_{\text{atom}}^H$  in the narrow topology of  $\mathcal{M}(\mathbb{R}^N)$  is given by  $\mathbf{M}^{H_{\text{lsc}}}$ , namely we have:*

$$(2.7) \quad \mathbf{M}^{H_{\text{lsc}}} = \sup\left\{F : F \leq \mathbf{M}_{\text{atom}}^H, F \text{ sequentially narrowly l.s.c. on } \mathcal{M}(\mathbb{R}^N)\right\}.$$

We point out that for a general  $H$ , for  $\mathbf{M}^H$  to be sequentially lower semicontinuous (for the narrow topology) it is necessary that  $H$  is lsc on  $\mathbb{R}^N \times (0, +\infty)$ . However, neither the subadditivity of  $H$  in  $m$  nor its lower semicontinuity on  $\mathbb{R}^N \times \mathbb{R}_+$  are necessary. Indeed,  $\mathbf{M}^H$  is sequentially lower semicontinuous if for instance  $H(x, m) = +\infty$  when  $x \neq 0, m > 0$ ,  $H(x, 0) = 0$  when  $x \neq 0$  and  $H(0, \cdot)$  is any lower semicontinuous function. Nevertheless the subadditivity in the mass  $m$  and the lower semicontinuity would be necessary if  $H$  did not depend on  $x$ .

*Proof of [Proposition 2.5](#).* Since  $H_{\text{lsc}}$  is lower semicontinuous and mass-concave, we know from [Proposition 2.4](#) that  $\mathbf{M}^{H_{\text{lsc}}}$  is sequentially lower semicontinuous in the weak topology hence also in the narrow topology of  $\mathcal{M}(\mathbb{R}^N)$ . Since  $\mathbf{M}^{H_{\text{lsc}}} \leq \mathbf{M}_{\text{atom}}^H$ , we deduce that  $\mathbf{M}^{H_{\text{lsc}}}$  is lower or equal than the sequentially lower semicontinuous envelope of  $\mathbf{M}_{\text{atom}}^H$  in the narrow topology, i.e. the right hand side in [\(2.7\)](#), which we denote by  $F : \mathcal{M}(\mathbb{R}^N) \rightarrow \mathbb{R}_+$ . We shall see that  $F \leq \mathbf{M}^{H_{\text{lsc}}}$ .

We first prove that

$$(2.8) \quad F \leq \mathbf{M}_{\text{atom}}^{H_{\text{lsc}}}.$$

For this, we let  $u = \sum_{i=1}^k m_i \delta_{x_i}$  be a finitely atomic positive measure and we let  $u_n := \sum_{i=1}^k m_{i,n} \delta_{x_{i,n}}$  where for each  $i \in \{1, \dots, k\}$ ,  $(x_{i,n})_{n \in \mathbb{N}}$  is a sequence of points converging to  $x_i$  and  $(m_{i,n})_{n \in \mathbb{N}}$  is a sequence converging to  $m_i$  such that  $H_{\text{lsc}}(x_i, m_i) = \lim_{n \rightarrow \infty} H(x_{i,n}, m_{i,n})$ . Then  $(u_n)_{n \in \mathbb{N}}$  converges narrowly to  $u$  and, by lower semicontinuity,

$$\begin{aligned} F(u) &\leq \liminf_{n \rightarrow \infty} F(u_n) \leq \liminf_{n \rightarrow \infty} \mathbf{M}_{\text{atom}}^H(u_n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^k H(x_{i,n}, m_{i,n}) = \sum_{i=1}^k H_{\text{lsc}}(x_i, m_i), \end{aligned}$$

so that  $F(u) \leq \mathbf{M}_{\text{atom}}^{H_{\text{lsc}}}(u)$  as wanted.

We now prove that  $F \leq \mathbf{M}^{H_{\text{lsc}}}$ . Let  $u \in \mathcal{M}(\mathbb{R}^N)$  and  $u = u^a + u^d$  be the decomposition of  $u$  into its atomic part  $u^a = \sum_{i=1}^k m_i \delta_{x_i}$ , with  $k \in \mathbb{N} \cup \{+\infty\}$  (here,  $k = 0$  if there is no atom), and its diffuse part  $u^d$ , and let  $u^d = u^d_+ - u^d_-$  be the Jordan decomposition of  $u^d$  into positive and negative parts. We then discretize

<sup>4</sup>In [[BB93](#), Theorem 3.2],  $H$  is assumed to be lower semicontinuous and the authors make a further coercivity assumption (assumption (3.5) in the paper) that we want to avoid.

$u_{\pm}^d$  by taking  $n \in \mathbb{N}^*$ , a partition  $(Q_i^n)_{i \in \{1, \dots, (n2^n)^N\}}$  of  $[-n, n]^N$  by means of cubes of the form  $Q_i^n = c_i^n + 2^{-n}[-1, 1]^N$  with  $c_i^n \in \mathbb{R}^N$ , and we define

$$u_n := \sum_{i=1}^{n \wedge k} m_i \delta_{x_i} + \sum_{i=1}^{(n2^n)^N} u_+^d(Q_i^n) \delta_{x_i^n} - \sum_{i=1}^{(n2^n)^N} u_-^d(Q_i^n) \delta_{y_i^n},$$

where for each  $i \in \{1, \dots, (n2^n)^N\}$ ,  $x_i^n, y_i^n \in \bar{Q}_i^n$  are some points such that

$$(2.9) \quad H_{\text{lsc}}'(x_i^n, 0^+) = \inf_{x \in \bar{Q}_i^n} H_{\text{lsc}}'(x, 0^+), \quad H_{\text{lsc}}'(y_i^n, 0^-) = \inf_{x \in \bar{Q}_i^n} H_{\text{lsc}}'(x, 0^-).$$

Such points exist since  $\bar{Q}_i^n$  is compact and since by concavity,

$$(2.10) \quad H_{\text{lsc}}'(x, 0^{\pm}) = \sup_{\pm m > 0} \frac{H_{\text{lsc}}(x, m)}{|m|},$$

so that  $H_{\text{lsc}}'(\cdot, 0^{\pm})$  are lower semicontinuous as suprema of lower semicontinuous functions.

The sequence  $(u_n)_{n \in \mathbb{N}}$  converges narrowly to  $u$ . We deduce by lower semicontinuity of  $F$ ,

$$\begin{aligned} F(u) &\leq \liminf_{n \rightarrow \infty} F(u_n) \stackrel{(2.8)}{\leq} \liminf_{n \rightarrow \infty} \mathbf{M}_{\text{atom}}^{\text{lsc}}(u_n) \\ &= \liminf_{n \rightarrow \infty} \sum_{i=1}^{n \wedge k} H_{\text{lsc}}(x_i, m_i) + \sum_{i=1}^{(n2^n)^N} \left( H_{\text{lsc}}(x_i^n, u_+^d(Q_i^n)) + H_{\text{lsc}}(y_i^n, -u_-^d(Q_i^n)) \right), \\ &\stackrel{(2.10)}{\leq} \liminf_{n \rightarrow \infty} \sum_{i=1}^{n \wedge k} H_{\text{lsc}}(x_i, m_i) + \sum_{i=1}^{(n2^n)^N} H_{\text{lsc}}'(x_i^n, 0^+) u_+^d(Q_i^n) \\ &\quad + \sum_{i=1}^{(n2^n)^N} H_{\text{lsc}}'(y_i^n, 0^-) u_-^d(Q_i^n) \\ &\stackrel{(2.9)}{\leq} \liminf_{n \rightarrow \infty} \sum_{i=1}^{n \wedge k} H_{\text{lsc}}(x_i, m_i) + \sum_{i=1}^{(n2^n)^N} \int_{Q_i^n} H_{\text{lsc}}'(\cdot, 0^+) du_+^d + \int_{Q_i^n} H_{\text{lsc}}'(\cdot, 0^-) du_-^d \\ &= \sum_{i=1}^k H_{\text{lsc}}(x_i, m_i) + \int_{\mathbb{R}^N} H_{\text{lsc}}'(\cdot, 0^+) du_+^d + \int_{\mathbb{R}^N} H_{\text{lsc}}'(\cdot, 0^-) du_-^d \\ &= \mathbf{M}^{H_{\text{lsc}}}(u), \end{aligned}$$

where we have used monotone convergence in the last but one equality.  $\square$

### 2.3. Slope at the origin of the minimal cost function.

**Proposition 2.6.** *Let  $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$  be a lower semicontinuous function such that  $f(0, 0) = 0$ , with  $N \geq 2$ . For every function  $\rho \in \mathcal{C}((0, 1], (0, +\infty))$  such that*

$$(2.11) \quad \int_0^1 \left( \int_y^1 \frac{dt}{\rho(t)} \right)^N dy < +\infty,$$

the function  $H_f$  defined in (2.1) satisfies

$$(2.12) \quad \lim_{m \rightarrow 0^+} \frac{H_f(m)}{m} \leq \limsup_{u \rightarrow 0^+} \sup_{\xi \in \mathbb{S}^{N-1}} \frac{f(u, \rho(u)\xi)}{u}.$$

*Proof.* For every  $y \geq 0$ , we let

$$F(y) = \int_y^1 \frac{dt}{\rho(t)} \in [0, +\infty].$$

The function  $F$  is decreasing, and belongs to  $\mathcal{C}^1((0, 1])$  and  $L^N((0, 1])$  by assumption. We now consider the solution of the ODE  $v'_\varepsilon = -\rho(v_\varepsilon)$ , with  $v_\varepsilon(0) = \varepsilon$ , given by

$$v_\varepsilon(r) = \begin{cases} F^{-1}(F(\varepsilon) + r), & \text{if } 0 \leq r < F(0) - F(\varepsilon), \\ 0 & \text{if } r \geq F(0) - F(\varepsilon), \end{cases}$$

with  $F(0)$  possibly equal to  $+\infty$ . Notice that  $v_\varepsilon \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$  because it is non-increasing and bounded, hence it has finite total variation, and because it is of class  $\mathcal{C}^1$  except possibly at  $r_\varepsilon := F(0) - F(\varepsilon)$ , where it has no jump. As a consequence the radial profile defined by  $u_\varepsilon(x) := v_\varepsilon(|x|)$  belongs to  $W_{\text{loc}}^{1,1}(\mathbb{R}^N)$  and we compute, using the change of variables  $s = v_\varepsilon(r)$  (i.e.  $r = F(s) - F(\varepsilon)$ ) and an integration by parts combined with monotone convergence.

$$\begin{aligned} m_\varepsilon &:= \int_{\mathbb{R}^N} u_\varepsilon = |\mathbb{S}^{N-1}| \int_0^\infty v_\varepsilon(r) r^{N-1} dr \\ &= -|\mathbb{S}^{N-1}| \int_0^\varepsilon s(F(s) - F(\varepsilon))^{N-1} F'(s) ds \\ &= |\mathbb{S}^{N-1}| \lim_{t \downarrow 0} \left( \int_t^\varepsilon \frac{(F(s) - F(\varepsilon))^N}{N} ds - \left[ s \frac{(F(s) - F(\varepsilon))^N}{N} \right]_t^\varepsilon \right) \\ &= |\mathbb{S}^{N-1}| \int_0^\varepsilon \frac{(F(s) - F(\varepsilon))^N}{N} ds \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

The equality on the last line holds because  $\lim_{t \rightarrow 0^+} \int_t^\varepsilon (F - F(\varepsilon))^N < +\infty$  (since  $F \in L^N((0, 1])$ ), hence  $\lim_{t \rightarrow 0} t(F(t) - F(\varepsilon))^N$  exists by existence of the limit in the previous line, and it must be zero (again, because  $F \in L^N((0, 1])$ ).

Moreover, since  $\sup_{(0, +\infty)} v_\varepsilon = \varepsilon$ ,

$$\mathcal{E}(u_\varepsilon) = \int_0^\infty \int_{\mathbb{S}^{N-1}} f(v_\varepsilon(r), v'_\varepsilon(r)\xi) r^{N-1} d\mathcal{H}^{N-1}(\xi) dr \leq m_\varepsilon \sup_{u \leq \varepsilon, |\xi|=1} \frac{f(u, \rho(u)\xi)}{u}.$$

By assumption, we deduce that

$$\limsup_{m \rightarrow 0^+} \frac{H_f(m)}{m} \leq \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathcal{E}(u_\varepsilon)}{m_\varepsilon} \leq \limsup_{u \rightarrow 0^+} \sup_{\xi \in \mathbb{S}^{N-1}} \frac{f(u, \rho(u)\xi)}{u}. \quad \square$$

In dimension  $N = 1$ , we need no other assumption than  $H_f < +\infty$ , as stated below.

**Proposition 2.7.** *Let  $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$  be Borel measurable with  $N = 1$ . The minimal cost function  $H_f$  is either identically infinite on  $(0, +\infty)$ , or it satisfies (2.12) with  $\rho \equiv 0$ , i.e.*

$$\lim_{m \rightarrow 0^+} \frac{H_f(m)}{m} \leq \limsup_{u \rightarrow 0^+} \frac{f(u, 0)}{u}.$$

*Proof.* One can assume that there exists  $u \in W_{\text{loc}}^{1,1}(\mathbb{R}, \mathbb{R}_+)$  with  $0 < \int_{\mathbb{R}} u < +\infty$  and  $\mathcal{E}(u) < +\infty$ . In particular, up to changing the value of  $u$  on a negligible set,  $u$  is

continuous on  $\mathbb{R}$ . Let  $\varepsilon \in (0, \sup_{\mathbb{R}} u)$ , set  $A_\varepsilon := \{x : u(x) = \varepsilon\}$  which is non-empty by the intermediate value theorem and integrability of  $u$ , and define

$$a_\varepsilon = \begin{cases} \inf A_\varepsilon & \text{if } \inf A_\varepsilon > -\infty, \\ \text{any point in } (-\infty, -\varepsilon^{-1}) \cap A_\varepsilon & \text{otherwise,} \end{cases}$$

$$b_\varepsilon = \begin{cases} \sup A_\varepsilon & \text{if } \sup A_\varepsilon < +\infty, \\ \text{any point in } (\varepsilon^{-1}, +\infty) \cap A_\varepsilon & \text{otherwise.} \end{cases}$$

By continuity and integrability of  $u$ ,  $u(a_\varepsilon) = u(b_\varepsilon) = \varepsilon$  and

$$\sup_{x \in \mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u(x) \leq \varepsilon \vee \sup_{|x| > \varepsilon^{-1}} u(x) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Moreover  $a_\varepsilon, b_\varepsilon$  converge to points  $-\infty \leq a \leq b \leq +\infty$ , hence  $u = 0$  on  $\mathbb{R} \setminus (a, b)$  and by dominated convergence, since  $\nabla u = 0$  a.e. on  $\{u = 0\}$ ,

$$+\infty > \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u + f(u, \nabla u) = f(0, 0) \mathcal{L}(\mathbb{R} \setminus (a, b)).$$

Notice that this limit is necessary zero. Let  $m > 0$ . If  $\varepsilon$  is small enough, then  $\int_{\mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u < m$  so that we can take  $R_\varepsilon > 0$  such that  $\varepsilon R_\varepsilon = m - \int_{\mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u$ . We then define

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if } x \leq a_\varepsilon, \\ \varepsilon & \text{if } a_\varepsilon < x < a_\varepsilon + R_\varepsilon, \\ u(b_\varepsilon + x - (a_\varepsilon + R_\varepsilon)) & \text{if } x \geq a_\varepsilon + R_\varepsilon, \end{cases}$$

so that  $\int_{\mathbb{R}} v_\varepsilon = m$ . Moreover,

$$\mathcal{E}(v_\varepsilon) = \mathcal{E}(u, \mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]) + R_\varepsilon f(\varepsilon, 0).$$

Hence, as  $R_\varepsilon = \frac{m+o(1)}{\varepsilon}$  as  $\varepsilon \rightarrow 0$ ,

$$H_f(m) \leq \limsup_{\varepsilon \rightarrow 0^+} \mathcal{E}(v_\varepsilon) = m \limsup_{\varepsilon \rightarrow 0^+} \frac{f(\varepsilon, 0)}{\varepsilon}. \quad \square$$

#### 2.4. Strict concavity of the minimal cost function in dimension $N \geq 2$ .

We show that in dimension  $N \geq 2$ , the minimal cost function must be strictly concave away from the possible initial interval where it is linear:

**Proposition 2.8.** *Assume that  $N \geq 2$  and that  $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  satisfies (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>5</sub>) and (H<sub>6</sub>). Let*

$$m_* = \sup\{m \geq 0 : H_f(x_0, \cdot) \text{ is linear on } [0, m]\},$$

where  $x_0 \in \mathbb{R}^N$  is fixed. Then,  $H_f(x_0, \cdot)$  is strictly concave on  $(m_*, +\infty)$ . A similar statement holds on  $\mathbb{R}_-$ .

A similar result does not hold in dimension 1 since any continuous concave function  $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $H(0) = 0$  can be written as  $H = H_f$  with  $f$  satisfying all our assumptions (H<sub>1</sub>)–(H<sub>8</sub>) (see Section 5.2).

In the rest of this section, we will systematically omit the dependence on  $x_0$  of  $H_f(x_0, m)$  since  $x_0$  is fixed.

We denote by  $\mathcal{M}_m^f$  the set of non-negative minimizers (see Remark 2.2) of mass  $m \in \mathbb{R}_+$ :

$$(2.13) \quad \mathcal{M}_m^f := \left\{ u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+) : \mathcal{E}_f(u) = H_f(m) \text{ and } \int_{\mathbb{R}^N} u = m \right\}.$$

The proof of [Proposition 2.8](#) is based on the following observation:

**Lemma 2.9.** *Let  $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$  be Borel measurable and let  $u_i \in \mathcal{M}_{m_i}^f$  with  $m_i \in \mathbb{R}_+$  for  $i = 1, 2$ . Let also  $u_* := \min\{u_1, u_2\}$ ,  $u^* := \max\{u_1, u_2\}$ ,  $m_* := \int_{\mathbb{R}^N} u_*$  and  $m^* := \int_{\mathbb{R}^N} u^*$ . If  $H_f$  is affine on  $[m_*, m^*]$  then  $u_* \in \mathcal{M}_{m_*}^f$  and  $u^* \in \mathcal{M}_{m^*}^f$ .*

*Proof of Lemma 2.9.* We use the same observations as in the proof of [Theorem 2.1](#). In particular, we have  $m_* + m^* = m_1 + m_2$ ; since  $H_f$  is affine on  $[m_*, m^*]$ , it yields

$$H_f(m_*) + H_f(m^*) = H_f(m_1) + H_f(m_2).$$

But we have also

$$H_f(m_*) + H_f(m^*) \leq \mathcal{E}_f(u_*) + \mathcal{E}_f(u^*) = \mathcal{E}_f(u_1) + \mathcal{E}_f(u_2) = H_f(m_1) + H_f(m_2),$$

so that the inequalities we used, i.e.  $H_f(m_*) \leq \mathcal{E}_f(u_*)$  and  $H_f(m^*) \leq \mathcal{E}_f(u^*)$ , are actually equalities.  $\square$

We also use an elementary Sobolev type inequality:

**Lemma 2.10.** *Let  $N \geq 2$ ,  $p \in (1, +\infty)$  and  $\omega \subset \mathbb{R}^{N-1}$  be a bounded open set. For every  $u \in W_{\text{loc}}^{1,p}(\mathbb{R} \times \omega)$ ,*

$$\int_{\omega} \|u(\cdot, x')\|_{L^\infty(\mathbb{R})} dx' \leq \|u\|_{L^1(\mathbb{R} \times \omega)} + |\omega|^{\frac{p-1}{p}} \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\mathbb{R} \times \omega)}.$$

*Proof of Lemma 2.10.* We prove the lemma when  $u \in \mathcal{C}^1(\mathbb{R} \times \omega)$ ; the general case follows by approximation. For every  $x_1, y_1 \in \mathbb{R}$ ,  $x' \in \omega$ , we have

$$u(x_1, x') = u(y_1, x') + \int_{y_1}^{x_1} \frac{\partial u}{\partial x_1}(t, x') dt.$$

By averaging in the variable  $y_1$ , we deduce

$$|u(x_1, x')| \leq \int_{x_1 - \frac{1}{2}}^{x_1 + \frac{1}{2}} |u(y_1, x')| dy_1 + \int_{x_1 - \frac{1}{2}}^{x_1 + \frac{1}{2}} \left| \frac{\partial u}{\partial x_1}(t, x') \right| dt.$$

The result follows from Hölder inequality after integrating over  $\omega$ .  $\square$

*Proof of Proposition 2.8.* Assume by contradiction that the concave function  $H_f$  is not strictly concave on  $(m_*, +\infty)$  which means that there exists  $m \in (m_*, +\infty)$  and  $\eta > 0$  such that  $H_f$  is affine on  $[m - \eta, m + \eta]$ . (Note that  $\eta \leq m - m_*$  by definition of  $m_*$ .) Moreover, we will see in [Proposition 3.12](#) that  $\mathcal{M}_m^f$  is not empty because  $H_f$  is not linear on  $[0, m]$ . We let  $u \in \mathcal{M}_m^f$ .

As before, we shall use the notations  $\wedge$  and  $\vee$  for the minimum and maximum; we also let  $(e_1, \dots, e_N)$  be the canonical basis of  $\mathbb{R}^N$ . Knowing that  $\tau \mapsto u(\cdot + \tau)$  is continuous in  $\mathbb{R}^N$  for every  $u \in L^1(\mathbb{R}^N)$ , that  $u \mapsto u(\cdot + \tau)$  is isometric in  $L^1(\mathbb{R}^N)$  for every  $\tau \in \mathbb{R}^N$ , that the map  $(x, y) \mapsto x \wedge y$  is Lipschitz on  $\mathbb{R}^N \times \mathbb{R}^N$ , and since the set  $\mathcal{M}_{m+\frac{\eta}{2}}$  is compact in  $L^1$  up to translations in view of [Remark 3.13](#), we deduce that there exists  $\delta_0 > 0$  such that

$$(2.14) \quad \|u \wedge u(\cdot + \delta e_2)\|_{L^1(\mathbb{R}^N)} > m \quad \text{for all } \delta \in (0, \delta_0) \text{ and } u \in \mathcal{M}_{m+\frac{\eta}{2}}^f.$$

We now construct by induction a sequence  $(t_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}_+$  and a sequence  $(u_n)_{n \in \mathbb{N}}$  in  $\mathcal{M}_m^f$  such that

$$(2.15) \quad t_{n+1} \geq t_n + \delta_0 \quad \text{and} \quad u_n(x) \leq U(x) \wedge U(x + t_n e_2) \quad \forall x \in \mathbb{R}^N,$$

where we have set

$$U(x) := \operatorname{ess\,sup}_{t \in \mathbb{R}} u(x + te_1).$$

To this aim, we first set  $u_0 := u$  and  $t_0 = 0$ . Then, if we assume that  $t_n$  and  $u_n$  are constructed as before, we first pick an  $\delta_n^1 \in \mathbb{R}_+$  such that  $v_n := u_n \vee u_n(\cdot + \delta_n^1 e_1)$  satisfies  $\int_{\mathbb{R}^N} v_n = m + \frac{\eta}{2}$ , which is possible since  $\eta \leq m$ , as we argued in the proof of [Theorem 2.1](#). Similarly, we pick a  $\delta_n^2 \in \mathbb{R}_+$  such that  $u_{n+1} := v_n \wedge v_n(\cdot + \delta_n^2 e_2)$  satisfies  $\int_{\mathbb{R}^N} u_{n+1} = m$ , and we set  $t_{n+1} = t_n + \delta_n^2$ . By [Lemma 2.9](#),  $v_n \in \mathcal{M}_{m+\frac{\eta}{2}}^f$  and  $u_n \in \mathcal{M}_m^f$ . By [\(2.14\)](#), we have  $\delta_n^2 \geq \delta_0$ , thus insuring the first condition in [\(2.15\)](#). For the second condition, we observe that for all  $x = (x_1, x') \in \mathbb{R}^N$ ,

$$u_{n+1}(x) \leq \left( \sup_{t \in \mathbb{R}} u_n(x + te_1) \right) \wedge \left( \sup_{t \in \mathbb{R}} u_n(x + te_1 + \delta_n^2 e_2) \right) \leq U(x) \wedge U(x + t_{n+1} e_2),$$

where in the last inequality we have used the induction hypothesis [\(2.15\)](#).

We now show that the sequence  $(u_n \mathcal{L}^N)_{n \in \mathbb{N}}$  is vanishing which will contradict the compactness of  $\mathcal{M}_m^f$  in  $L^1$  up to translations.

For this, we let  $(x_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathbb{R}^N$  and  $(u_{n_k})_{k \in \mathbb{N}}$  be a subsequence of  $(u_n)_{n \in \mathbb{N}}$  such that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \int_{x+[0,1]^N} u_n = \lim_{k \rightarrow \infty} \int_{x_k+[0,1]^N} u_{n_k}.$$

By [\(H5\)](#), we have  $\frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^N)$ . Using this fact, estimate [\(2.15\)](#), and [Lemma 2.10](#) with  $\omega$  a unit cube in  $\mathbb{R}^{N-1}$ , we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{x_k+[0,1]^N} u_{n_k} &\leq \liminf_{k \rightarrow \infty} \int_{x_k+[0,1]^N} U \wedge \int_{x_k+t_{n_k}e_2+[0,1]^N} U \\ &\leq \liminf_{k \rightarrow \infty} \left( \|u\|_{L^1(\{0 \leq (x-x_k) \cdot e_2 \leq 1\})} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\{0 \leq (x-x_k) \cdot e_2 \leq 1\})} \right) \\ &\wedge \liminf_{k \rightarrow \infty} \left( \|u\|_{L^1(\{t_{n_k} \leq (x-x_k) \cdot e_2 \leq t_{n_k} + 1\})} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\{t_{n_k} \leq (x-x_k) \cdot e_2 \leq t_{n_k} + 1\})} \right), \end{aligned}$$

and the conclusion follows since the sequences  $(x_k \cdot e_2)_{k \in \mathbb{N}}$  and  $(t_{n_k} + x_k \cdot e_2)_{k \in \mathbb{N}}$  cannot be both bounded as  $\lim_{k \rightarrow \infty} t_{n_k} = \infty$ .  $\square$

### 3. LOWER BOUND FOR THE ENERGY AND EXISTENCE OF OPTIMAL PROFILES

Our main tool to localize the energy and obtain a lower bound relies on a profile decomposition for bounded sequences of positive measures, which is reminiscent of the concentration-compactness principle of P.-L. Lions. This differs from classical strategies to localize the energy which are based on suitable cut-offs. Naturally, this concentration-compactness result also provides a criterion for the existence of optimal profiles in [\(1.2\)](#).

**3.1. Profile decomposition by concentration-compactness.** We prove a profile decomposition theorem for bounded sequences of positive measures over  $\mathbb{R}^N$ , which is essentially equivalent to [[Mar14](#), Theorem 1.5] in the Euclidean case. We have added an extra information on mass conservation that will be useful, and provide a self-contained simple proof. We start with a definition.



**Definition 3.1.** A sequence of positive measures  $(\mu_n)_{n \in \mathbb{N}} \in \mathcal{M}_+(\mathbb{R}^N)$  is *vanishing* if

$$\sup_{x \in \mathbb{R}^N} \mu_n(B_1(x)) \xrightarrow{n \rightarrow \infty} 0.$$

Any bounded sequence of positive measures over  $\mathbb{R}^N$  may be decomposed (up to subsequence) into a countable collection of narrowly converging “bubbles” and a vanishing part, accounting for the total mass of the sequence, as stated in the following theorem.

**Theorem 3.2.** *For every bounded sequence  $(\mu_n)_{n \in \mathbb{N}}$  of positive Borel measures on  $\mathbb{R}^N$ , there exists a subsequence  $(\mu_n)_{n \in \sigma(\mathbb{N})}$ ,  $\sigma \in \Sigma$ , a non-decreasing sequence of integers  $(k_n)_{n \in \sigma(\mathbb{N})}$  converging to some  $k \in \mathbb{N} \cup \{+\infty\}$ , a sequence of non-trivial positive Borel measures  $(\mu^i)_{0 \leq i < k}$ , and for every  $n \in \sigma(\mathbb{N})$ , a collection of balls  $(B_n^i)_{0 \leq i < k_n}$  centered at points of  $\text{supp } \mu_n$  such that, writing for all  $n \in \sigma(\mathbb{N})$ ,*

$$(3.1) \quad \mu_n = \mu_n^b + \mu_n^v, \quad \text{where } \mu_n^b = \sum_{0 \leq i < k_n} \mu_n \llcorner B_n^i,$$

(A) *bubbles emerge:*  $(c_{B_n^i} \mu_n)_{n \in \sigma(\mathbb{N})} \xrightarrow{n \rightarrow \infty} \mu^i$  for every  $i < k$ ,<sup>5</sup>

(B) *bubbles split:*  $\min_{0 \leq i < j < k_n} \text{dist}(B_n^i, B_n^j) \xrightarrow{n \rightarrow \infty} +\infty$ ,

(C) *bubbles diverge:*  $\min_{0 \leq i < k_n} \text{diam}(B_n^i) \xrightarrow{n \rightarrow \infty} +\infty$ ,

(D) *the bubbling mass is conserved:*  $\|\mu_n^b\| \xrightarrow{\ell \rightarrow \infty} \sum_{0 \leq i < k} \|\mu^i\|$ ,

(E) *the remaining part is vanishing:*  $\sup_{x \in \mathbb{R}^N} \mu_n^v(B_1(x)) \xrightarrow{n \rightarrow \infty} 0$ .

Before proving **Theorem 3.2**, we introduce the “bubbling” function of a sequence of finite signed measures  $(\mu_n)_{n \in \mathbb{N}}$ :

$$(3.2) \quad m((\mu_n)_{n \in \mathbb{N}}) := \sup \left\{ \|\mu\| : (\tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)})_{\ell \in \mathbb{N}} \xrightarrow{\mathcal{C}'_0} \mu, \sigma \in \Sigma, x_{\sigma(\ell)} \in \mathbb{R}^N (\forall \ell) \right\}.$$

Although we will use this function on signed measures, we will start from a sequence of positive measures and use the following characterization of vanishing sequences, which holds only in the case of positive measures:

**Lemma 3.3.** *A sequence  $(\mu_n)_{n \in \mathbb{N}}$  of finite positive measures over  $\mathbb{R}^N$  is vanishing if and only if  $m((\mu_n)_{n \in \mathbb{N}}) = 0$ .*

*Proof.* Assume that  $(\mu_n)_{n \in \mathbb{N}}$  is vanishing and that  $(\tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)})_{\ell \in \mathbb{N}} \xrightarrow{\mathcal{C}'_0} \mu$  for some  $\sigma \in \Sigma$  and some sequence of points  $(x_{\sigma(\ell)})_{\ell \in \mathbb{N}}$ . Then, for every  $x \in \mathbb{R}^N$ ,

$$\mu(B_1(x)) \leq \liminf_{\ell \rightarrow \infty} \tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_1(x)) = \liminf_{\ell \rightarrow \infty} \mu_{\sigma(\ell)}(B_1(x + x_{\sigma(\ell)})) = 0,$$

i.e.  $\mu = 0$  and thus  $m((\mu_\ell)_{\ell \in \mathbb{N}}) = 0$ .

Conversely, if  $(\mu_n)_{n \in \mathbb{N}}$  is not vanishing, then there exists  $\varepsilon > 0$ ,  $\sigma \in \Sigma$  and a sequence of points  $(x_n)_{n \in \sigma(\mathbb{N})}$  in  $\mathbb{R}^N$  such that  $\mu_n(B_1(x_n)) \geq \varepsilon$  for every  $n \in \sigma(\mathbb{N})$ .

Up to further extraction, one can assume that  $(\tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)})_{\ell \in \mathbb{N}} \xrightarrow{\mathcal{C}'_0} \mu \in \mathcal{M}(\mathbb{R}^N)$ . We have

$$\mu(\bar{B}_1(0)) \geq \limsup_{\ell \rightarrow \infty} \tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)}(\bar{B}_1(0)) = \limsup_{\ell \rightarrow \infty} \mu_{\sigma(\ell)}(\bar{B}_1(x_{\sigma(\ell)})) \geq \varepsilon > 0,$$

<sup>5</sup>Recall that  $c_B \mu = (x \mapsto x - y)_\# (\mu \llcorner B)$  if  $B = B_r(y)$  and  $\mu \in \mathcal{M}(\mathbb{R}^N)$ .

which entails  $m((\mu_\ell)_{\ell \in \mathbb{N}}) \geq \varepsilon > 0$ .  $\square$

*Proof of Theorem 3.2.* If  $(\mu_n)_{n \in \mathbb{N}}$  is vanishing, then we take  $\sigma = \text{Id}$  and  $k = 0$ , so that  $\mu_{\sigma(\ell)} = \mu_\ell = \mu_\ell^v$ , (A) to (D) are empty statements and (E) is satisfied since  $(\mu_n)_{n \in \mathbb{N}}$  is vanishing. Assume on the contrary that  $(\mu_n)_{n \in \mathbb{N}}$  is not vanishing. We shall construct the bubbles by induction and prove their properties in several steps.

*Step 1: construction of bubbles centers.* At first step (step 0), since  $m((\mu_n)_{n \in \mathbb{N}}) > 0$ , there exists  $\sigma_0 \in \Sigma$  and a sequence of points  $(x_n^0)_{n \in \sigma_0(\mathbb{N})}$ , such that

$$(3.3) \quad (\tau_{-x_n^0} \mu_n)_{n \in \sigma_0(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^0 \in \mathcal{M}(\mathbb{R}^N) \quad \text{with} \quad \|\mu^0\| \geq \frac{1}{2} m((\mu_n)_{n \in \mathbb{N}}).$$

We then set  $\mu_n^0 := \mu_n - \tau_{x_n^0} \mu^0$  and we continue by induction, starting from the sequence  $(\mu_n^0)_{n \in \sigma_0(\mathbb{N})}$ . More precisely, assume that for a fixed step  $k-1 \in \mathbb{N}$ , for every  $i \in \mathbb{N}$  such that  $0 \leq i \leq k-1$ , we have built  $\mu^i \in \mathcal{M}(\mathbb{R}^N)$ ,  $\sigma_i \in \Sigma$ , points  $(x_n^i)_{n \in \sigma_i(\mathbb{N})}$  and sequences  $(\mu_n^i)_{n \in \sigma_i(\mathbb{N})} \in \mathcal{M}(\mathbb{R}^N)$  such that for every  $i$ ,

$$(3.4) \quad \sigma_i \preceq \sigma_{i-1},$$

$$(3.5) \quad \mu_n^i = \mu_n - \sum_{0 \leq j \leq i} \tau_{x_n^j} \mu^j, \quad (\forall n \in \sigma_i(\mathbb{N})),$$

$$(3.6) \quad (\tau_{-x_n^i} \mu_n^{i-1})_{n \in \sigma_i(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^i,$$

$$(3.7) \quad \|\mu^i\| \geq \frac{1}{2} m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}) > 0,$$

where  $\sigma_{-1} := \text{Id}$ ,  $(\mu_n^{-1}) := (\mu_n)$ . If  $m((\mu_n^{k-1})_{n \in \sigma_{k-1}(\mathbb{N})}) = 0$ , we stop; otherwise, we proceed to the next step  $k$  to build  $\sigma_k, \mu^k, (x_n^k)_{n \in \sigma_k(\mathbb{N})}, (\mu_n^k)$  as we did at step  $k=0$ , starting with  $(\mu_n^{k-1})_{n \in \sigma_{k-1}(\mathbb{N})}$ . Either the induction stops at some step  $k-1 \in \mathbb{N}$  for which  $m((\mu_n^{k-1})_{n \in \sigma_{k-1}(\mathbb{N})}) = 0$  or the previous objects are defined for every  $i \in \mathbb{N}$ , in which case we let  $k := +\infty$ .

*Step 2: splitting of bubbles centers.* We prove that

$$(3.8) \quad \lim_{\sigma_i(\mathbb{N}) \ni n \rightarrow \infty} \text{dist}(x_n^i, x_n^j) = +\infty \quad \text{for every } i, j \in \mathbb{N} \text{ with } 0 \leq j < i < k.$$

Indeed, assume by contradiction that there is a first index  $i < k$  such that for some  $j_0 < i$ ,  $(\text{dist}(x_n^i, x_n^{j_0}))_{n \in \sigma_i(\mathbb{N})}$  is not divergent. In particular, there exists  $\sigma \preceq \sigma_i$  such that  $(x_n^i - x_n^{j_0})_{n \in \sigma(\mathbb{N})} \rightarrow x \in \mathbb{R}^N$ . Moreover,  $(\text{dist}(x_n^i, x_n^j))_{n \in \sigma_i(\mathbb{N})} \rightarrow \infty$ , for every  $j < i$ ,  $j \neq j_0$  by minimality of  $i$  and the triangle inequality  $\text{dist}(x_n^j, x_n^{j_0}) \leq \text{dist}(x_n^j, x_n^i) + \text{dist}(x_n^i, x_n^{j_0})$ . Notice by (3.5) that for every  $n \in \sigma(\mathbb{N})$ ,

$$\mu_n^{i-1} = \mu_n^{j_0-1} - \tau_{x_n^{j_0}} \mu^{j_0} - \sum_{j_0 < j < i} \tau_{x_n^j} \mu^j,$$

hence taking the translation  $\tau_{-x_n^i}$ ,

$$\tau_{-x_n^i} \mu_n^{i-1} = \tau_{x_n^{j_0} - x_n^i} (\tau_{-x_n^{j_0}} \mu_n^{j_0-1} - \mu^{j_0}) - \sum_{j_0 < j < i} \tau_{x_n^j - x_n^i} \mu^j,$$

and passing to the weak limit, knowing that  $x_n^{j_0} - x_n^i \rightarrow -x$  and  $\text{dist}(x_n^j, x_n^i) \rightarrow +\infty$  for  $j_0 < j < i$ ,

$$\mu^i = \tau_{-x} (\mu^{j_0} - \mu^{j_0}) - \sum_{j_0 < j < i} 0 = 0.$$

This contradicts the fact that  $(\tau_{-x_n^i} \mu_n^{i-1})_{n \in \sigma(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^i \neq 0$  and proves (3.8).

*Step 3: weak convergence of bubbles.* From (3.5) we get

$$(3.9) \quad \tau_{-x_n^i} \mu_n^{i-1} = \tau_{-x_n^i} \mu_n - \sum_{0 \leq j < i} \tau_{-x_n^i + x_n^j} \mu^j,$$

and by (3.8), the sum converges weakly to 0, and so

$$(3.10) \quad (\tau_{-x_n^i} \mu_n)_{n \in \sigma_i(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^i \quad \text{for every } i \in \mathbb{N} \text{ with } i < k.$$

*Step 4: construction of the bubbles with mass conservation.* We now construct the extraction  $\sigma \in \Sigma$  that we need by induction: we set  $\sigma(0) = 0$  and, assuming that  $\sigma(0) < \dots < \sigma(\ell - 1)$ , with  $\ell \in \mathbb{N}^*$ , have been constructed, we set  $\sigma(\ell) := n$  with  $n \in \sigma_{\ell \wedge k - 1}(\mathbb{N})$  large enough so that  $n > \sigma(\ell - 1)$  and for every  $i < \ell \wedge k$ ,

$$(3.11) \quad \mu_n(B_\ell(x_n^i)) \leq \|\mu^i\| + 2^{-\ell},$$

and

$$(3.12) \quad \min_{0 \leq j < i} \text{dist}(x_n^i, x_n^j) \geq 4\ell.$$

Such an  $n$  exists by (3.8) and (3.10), noticing that  $\mu_n(B_\ell(x_n^i)) = (\tau_{-x_n^i} \mu_n)(B_\ell)$ . Then for each  $n = \sigma(\ell)$ ,  $\ell \in \mathbb{N}$ , we set  $k_n = \ell \wedge k$ , and for each  $i \in \{0, \dots, k_n - 1\}$ ,

$$B_n^i := B_\ell(x_n^i).$$

Finally, for every  $n \in \sigma(\mathbb{N})$ , we decompose  $\mu_n$  as expected:

$$\mu_n = \mu_n^b + \mu_n^v, \quad \text{where } \mu_n^b = \sum_{0 \leq i < k_n} \mu_n \llcorner B_n^i.$$

Let us check the four first items (A)–(D). Notice that (C) is fulfilled because  $\text{diam}(B_{\sigma(\ell)}^i) = \ell \rightarrow +\infty$  as  $\ell \rightarrow \infty$ , and (B) because of (3.12). Since for every  $i < k$ ,  $\lim_{\sigma(\mathbb{N}) \ni n \rightarrow \infty} \text{diam}(B_n^i) = +\infty$  and  $c_{B_n^i} \mu_n = (\tau_{-x_n^i} (\mu_n \llcorner B_n^i))$  for every  $n \in \sigma_i(\mathbb{N})$ ,  $(c_{B_n^i} \mu_n)_{n \in \sigma(\mathbb{N})}$  converges weakly to  $\mu^i$  by (3.10), and together with (3.11) it implies that

$$(c_{B_n^i} \mu_n)_{n \in \sigma(\mathbb{N})} \xrightarrow{\mathcal{C}'_b} \mu^i,$$

i.e. (A) is satisfied. Moreover, by (3.11) again,

$$\limsup_{\ell \rightarrow \infty} \sum_{0 \leq i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i) \leq \sum_{0 \leq i < k} \|\mu^i\| + \limsup_{\ell \rightarrow \infty} (\ell \wedge k) 2^{-\ell} = \sum_{0 \leq i < k} \|\mu^i\|,$$

and since  $k_n \rightarrow k$ , by Fatou's lemma we have,

$$\sum_{0 \leq i < k} \|\mu^i\| \leq \liminf_{\ell \rightarrow \infty} \sum_{0 \leq i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i),$$

which proves (D) because  $\sum_{0 \leq i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i) = \|\mu_{\sigma(\ell)}^b\|$ .

*Step 5: vanishing of the remaining part, proof of (E).* By Lemma 3.3, it suffices to prove that  $m((\mu_n^v)_{n \in \sigma(\mathbb{N})}) = 0$ . We claim that:

$$(3.13) \quad m((\mu_n^v)_{n \in \sigma(\mathbb{N})}) \leq m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}), \quad \text{for every } i \in \mathbb{N} \text{ with } i < k,$$

which concludes since  $m((\mu_n^k)_{n \in \sigma_{k-1}(\mathbb{N})}) = 0$  if  $k < \infty$ , and  $m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}) \rightarrow 0$  as  $i \rightarrow \infty$  if  $k = \infty$ . Indeed, if  $k = \infty$ , we have by (3.7) and (D),

$$\frac{1}{2} \sum_{i \in \mathbb{N}} m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}) \leq \sum_{i \in \mathbb{N}} \|\mu^i\| = \lim_{\ell \rightarrow \infty} \|\mu_{\sigma(\ell)}^b\| \leq \liminf_{\ell \rightarrow \infty} \|\mu_{\sigma(\ell)}\| < \infty.$$

Let us show (3.13). Let  $\bar{\sigma} \preceq \sigma$  and  $(x_n)_{n \in \bar{\sigma}(\mathbb{N})}$  be a sequence of points such that

$$(\tau_{-x_n} \mu_n^v)_{n \in \bar{\sigma}(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu \in \mathcal{M}(\mathbb{R}^N).$$

We need to prove that  $\|\mu\| \leq m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})})$  for every  $i < k$ . Assume without loss of generality that  $\|\mu\| > 0$ . Then for every  $i < k$ ,

$$(3.14) \quad (\text{dist}(x_n, x_n^i))_{n \in \bar{\sigma}(\mathbb{N})} \rightarrow \infty.$$

Otherwise, up to subsequence,  $(\text{dist}(x_n, x_n^i))_n$  would be bounded by some constant  $M$ , and for every  $r > 0$ ,

$$(\tau_{-x_n} \mu_n^v)(B_r) \leq \mu_n^v(B_{r+M}(x_n^i)) \xrightarrow{n \rightarrow \infty} 0,$$

because  $\mu_n^v$  is supported on  $\mathbb{R}^N \setminus \cup_{0 \leq i < k_n} B_n^i$  and  $B_{r+M}(x_n^i) \subseteq B_n^i$  for  $n$  large enough by (C). Hence  $\mu$  would be 0, a contradiction. Up to further extraction, one can assume that  $(\tau_{-x_n} \mu_n)_{n \in \bar{\sigma}(\mathbb{N})}$  converges weakly to a measure  $\bar{\mu} \in \mathcal{M}(\mathbb{R}^N)$ . Since  $\mu_n^v \leq \mu_n$ , we have  $\mu \leq \bar{\mu}$ . Moreover by (3.5), for every  $i < k$  and  $n \in \bar{\sigma}(\mathbb{N})$  large enough,

$$\tau_{-x_n} \mu_n^i = \tau_{-x_n} \mu_n - \sum_{0 \leq j \leq i} \tau_{x_n^j - x_n} \mu_n^j,$$

and because of (3.14) the sum converges weakly to 0, so that  $\tau_{-x_n} \mu_n^i \xrightarrow{\mathcal{C}'_0} \bar{\mu}$ , and consequently,

$$\|\mu\| \leq \|\bar{\mu}\| \leq m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}),$$

which is what had to be proved.

*Step 6: re-centering of the bubbles at points of  $\text{supp } \mu_n$ .* By (3.10),  $(\tau_{-x_n^i} \mu_n)_{n \in \sigma(\mathbb{N})}$  converges weakly to the non-trivial measure  $\mu_i$  for every  $i < k$ , thus

$$(3.15) \quad R_i/2 := \limsup_{\sigma(\mathbb{N}) \ni n \rightarrow +\infty} \text{dist}(\text{supp } \mu_n, x_n^i) < +\infty.$$

Therefore, for every  $n$  large enough, there is a point  $\tilde{x}_n^i$  such that  $|x_n^i - \tilde{x}_n^i| < R_i$  and  $\tilde{x}_n^i \in \text{supp } \mu_n$ . After a further extraction, one may assume that for every  $i$ ,  $|x_n^i - \tilde{x}_n^i| < R_i < r_i^n$  where  $\text{diam } B_n^i = 2r_i^n$  for every  $n$ , and  $(x_n^i - \tilde{x}_n^i)_{n \in \sigma(\mathbb{N})}$  converges to some  $p_i \in \mathbb{R}^N$ . Finally, we set  $\tilde{r}_i^n := r_i^n - R_i$  and  $\tilde{B}_n^i := B(\tilde{x}_n^i, \tilde{r}_i^n) \subseteq B_n^i$ . After replacing the balls  $B_n^i$  by  $\tilde{B}_n^i$ , (B) and (C) are satisfied by definition. Notice that  $(\tau_{-\tilde{x}_n^i} \mu_n)_{n \in \sigma(\mathbb{N})}$  converges weakly to  $\tilde{\mu}^i := \tau_{p_i} \mu^i$  with  $\|\tilde{\mu}^i\| = \|\mu^i\|$ , and  $\limsup_n \|c_{B_n^i} \mu_n\| = \limsup_n \mu_n(\tilde{B}_n^i) \leq \limsup_n \mu_n(B_n^i) = \|\mu^i\|$  hence (A) holds. Besides, using Fatou's lemma,

$$\begin{aligned} \limsup_n \sum_{i < k_n} \mu_n(\tilde{B}_n^i) &\leq \limsup_n \sum_{i < k_n} \mu_n(B_n^i) \\ &= \sum_{i < k} \|\mu^i\| \leq \sum_{i < k} \liminf_n \mu_n(\tilde{B}_n^i) \leq \liminf_n \sum_{i < k_n} \mu_n(\tilde{B}_n^i) \end{aligned}$$

so that  $\lim_n \sum_{i < k_n} \mu_n(\tilde{B}_n^i) = \sum_i \|\mu_i\|$  and (D) is satisfied. In particular,

$$\lim_n \sum_{i < k_n} \mu_n(B_n^i \setminus \tilde{B}_n^i) = \lim_n \sum_{i < k_n} \mu_n(B_n^i) - \lim_n \sum_{i < k_n} \mu_n(\tilde{B}_n^i) = 0,$$

and (E) holds as well.  $\square$

*Remark 3.4.* If the sequence of families of balls  $(B_n^i)_{0 \leq i < k_n}$  satisfies the conclusion of the theorem, i.e. (A)–(E), then it is also the case for any family of balls  $(\tilde{B}_n^i)_{0 \leq i < k_n}$  with the same centers as those of  $B_n^i$  and with smaller but still divergent radii (i.e. satisfying (C)). It can be easily seen following the arguments at Step 6 of the proof.

**3.2. Lower bound by concentration-compactness.** We will first establish a lower bound for the minimal energy along vanishing sequences defined on varying subsets of  $\mathbb{R}^N$ . We say that a sequence of Borel functions  $(u_n)_{n \in \mathbb{N}}$ , each defined on some open set  $\Omega_n \subseteq \mathbb{R}^N$ , is vanishing if the sequence of measures  $(|u_n| \mathcal{L}^N \llcorner \Omega_n)_{n \in \mathbb{N}}$  is vanishing in the sense of [Definition 3.1](#), namely if  $\|u_n\|_{L^1_{\text{uloc}}(\Omega_n)} \rightarrow 0$  as  $n \rightarrow \infty$ , where  $L^1_{\text{uloc}}(\Omega)$  is the set of uniformly locally integrable functions on the open set  $\Omega$ , i.e. Borel functions  $u$  on  $\Omega$  such that

$$(3.16) \quad \|u\|_{L^1_{\text{uloc}}(\Omega)} := \sup_{x \in \mathbb{R}^N} \int_{\Omega \cap (x + [0,1]^N)} |u| < +\infty.$$

It will be convenient to first extend our Sobolev functions to a neighbourhood  $\Omega_\delta$  of  $\Omega$  where for every  $\delta > 0$  and every set  $X \subseteq \mathbb{R}^N$ , we have set

$$X_\delta := \{x \in \mathbb{R}^N : \text{dist}(x, X) < \delta\}.$$

We will need to consider sufficiently regular domains for which we have an extension operator  $W^{1,p} \cap L^1_{\text{uloc}}(\Omega) \rightarrow W^{1,p} \cap L^1_{\text{uloc}}(\Omega_\delta)$ . We will only apply it to domains with smooth boundary, in which case we can use a reflexion technique. Since we want quantitative estimates, we will use the notion of *reach* of a set  $X \subseteq \mathbb{R}^N$  (see [\[Fed59\]](#)). We say that  $X$  has positive reach if there exists  $\delta > 0$  such that every  $x \in X_\delta$  has a unique nearest point  $\pi(x)$  on  $X$ . The greatest  $\delta$  for which this holds is denoted by  $\text{reach}(X)$  and the map  $x \in X_{\text{reach}(X)} \mapsto \pi(x) \in X$  is called the nearest point retraction.

*Example 3.5.* Assume that  $\Omega$  is a perforated domain  $B^0 \setminus \bigcup_{i=1}^k B^i$  where the  $B^i$  are disjoint closed balls included in some open ball  $B^0$  (possibly  $B^0 = \mathbb{R}^N$ ). Then,

$$\text{reach}(\partial\Omega) = \inf\{\text{radius}(B^i) : i = 0, \dots, k\} \cup \{\text{dist}(\partial B^i, \partial B^j) : i \neq j\}.$$

By [\[Fed59, Theorem 4.8\]](#), we have

- (i) if  $x, y \in X_\delta$  with  $0 < \delta < \delta_0 := \text{reach}(X)$ , then  $|\pi(x) - \pi(y)| \leq \frac{\delta_0}{\delta_0 - \delta} |x - y|$ ,
- (ii) if  $x \in X$  and  $D_x$  is the intersection of  $X_{\text{reach}(X)}$  with the straight line crossing  $\partial\Omega$  orthogonally at  $x$ , then  $\pi(y) = x$  for every  $y \in D_x$ .

**Lemma 3.6** (Extension). *Let  $\Omega \subseteq \mathbb{R}^N$  be an open set such that its boundary  $\partial\Omega$  is  $\mathcal{C}^1$  with positive reach<sup>6</sup>. Then, for every  $\delta \in (0, \text{reach}(\partial\Omega))$ , every  $p \in [1, +\infty)$  and every  $u \in L^1 \cap W^{1,p}(\Omega)$ , there exists  $\bar{u} \in L^1 \cap W^{1,p}(\Omega_\delta)$  such that  $\bar{u} = u$  a.e. on  $\Omega$ , and*

$$\|\bar{u}\|_{L^1(\Omega_\delta)} \leq A \|u\|_{L^1(\Omega)}, \quad \|\bar{u}\|_{L^1_{\text{uloc}}(\Omega_\delta)} \leq A \|u\|_{L^1_{\text{uloc}}(\Omega)}, \quad \|\nabla \bar{u}\|_{L^p(\Omega_\delta)} \leq A \|\nabla u\|_{L^p(\Omega)},$$

with a constant  $A < +\infty$  depending only on  $N, \delta$  and  $\text{reach}(\partial\Omega)$ .

<sup>6</sup>Thanks to [\[Fed59, Remark 4.20\]](#),  $\partial\Omega$  is actually of class  $\mathcal{C}^{1,1}$ .

*Proof.* Let  $\sigma : (\partial\Omega)_\delta \rightarrow (\partial\Omega)_\delta$  be the reflection through  $\partial\Omega$ , defined by  $\sigma(x) = 2\pi(x) - x$ . By the properties (i) and (ii) of the nearest point retraction, we have that  $\sigma = \sigma^{-1}$  (simply because  $\pi(\sigma(x)) = \pi(x)$ ) and  $\sigma$  is  $L$ -Lipschitz with a constant  $L < +\infty$  depending on  $\delta$  and  $\text{reach}(\partial\Omega)$  only.

We define<sup>7</sup>  $\bar{u}$  by  $\bar{u} = u$  on  $\Omega$  and  $\bar{u} = u \circ \sigma$  on  $\Omega_\delta \setminus \Omega$ . This map is well-defined since  $\sigma(\Omega_\delta \setminus \Omega) \subseteq \Omega$ . Indeed, if we had  $x, \sigma(x) \in \Omega_\delta \setminus \Omega$ , then the line segment  $[x, \sigma(x)]$  would meet  $\partial\Omega$  orthogonally at its center  $\pi(x)$ , and would remain out of  $\Omega$ , because otherwise there would exist a point  $y$  belonging either to  $\partial\Omega \cap (x, \pi(x))$  or  $\partial\Omega \cap (\pi(x), \sigma(x))$  thus contradicting the definition of  $\pi(x)$ . Such a situation is not possible for a  $\mathcal{C}^1$  boundary.

Moreover, by the change of variable formula and the chain rule,  $\bar{u}$  satisfies the desired estimates since  $\sigma$  is bi-Lipschitz with its Lipschitz constants controlled in terms of  $\delta$  and  $\text{reach}(\partial\Omega)$ .  $\square$

We will need a localized version of the Gagliardo–Nirenberg–Sobolev inequality in a particular case:

**Lemma 3.7.** *Let  $\Omega \subseteq \mathbb{R}^N$  be an open set such that  $\partial\Omega$  is  $\mathcal{C}^1$  with positive reach, let  $p \in [1, +\infty)$ , let  $r \geq p(1 + \frac{1}{N})$ , and assume that  $r \leq \frac{pN}{N-p}$  when  $p < N$ . Then for every  $u \in L^1 \cap W^{1,p}(\Omega)$ ,*

$$\|u\|_{L^r(\Omega)} \leq C(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)})^\alpha \|u\|_{L^1_{\text{loc}}(\Omega)}^{1-\alpha},$$

where  $\alpha \in (0, 1]$  is the unique parameter such that  $\frac{1}{r} = \alpha(\frac{1}{p} - \frac{1}{N}) + (1 - \alpha)$ , and the constant  $C < +\infty$  depends on  $N, r, p$  and  $\text{reach}(\partial\Omega)$ .

*Proof of Lemma 3.7.* We let  $u \in L^1 \cap W^{1,p}(\Omega)$  and we extend  $u$  to  $\bar{u} \in L^1 \cap W^{1,p}(\Omega_\delta)$  as in Lemma 3.6, with  $\delta := \text{reach}(\Omega)/2$ . By the Gagliardo–Nirenberg–Sobolev inequality (see [Nir59]) on the hypercube  $Q_\delta = [-\frac{\delta}{\sqrt{N}}, \frac{\delta}{\sqrt{N}}]^N \subseteq \bar{B}_\delta$ , we have for some  $C$  depending on  $N, \delta$ ,

$$\|\bar{u}\|_{L^r(Q_\delta)} \leq C\|\nabla \bar{u}\|_{L^p(Q_\delta)}^\alpha \|\bar{u}\|_{L^1(Q_\delta)}^{1-\alpha} + C\|\bar{u}\|_{L^1(Q_\delta)}.$$

We then cover  $\Omega$  with the disjoint hypercubes  $Q_\delta(c) = c + Q_\delta \subseteq \Omega_\delta$  centered at points  $c$  on the grid  $\mathcal{C} := \Omega \cap \frac{2\delta}{\sqrt{N}}\mathbb{Z}^N$ . Since  $r \geq p(1 + 1/N)$ , we can check that

$$(3.17) \quad r\alpha = \frac{r-1}{1 + \frac{1}{N} - \frac{1}{p}} \geq p.$$

By superadditivity of  $s \mapsto s^{\frac{r\alpha}{p}}$  and of  $s \mapsto s^{r\alpha}$ , we obtain

$$\begin{aligned} \|u\|_{L^r(\Omega)}^r &\leq \sum_{c \in \mathcal{C}} \|\bar{u}\|_{L^r(Q_\delta(c))}^r \\ &\leq C' \sum_{c \in \mathcal{C}} \|\nabla \bar{u}\|_{L^p(Q_\delta(c))}^{p\frac{r\alpha}{p}} \|\bar{u}\|_{L^1(Q_\delta(c))}^{r(1-\alpha)} + C' \|\bar{u}\|_{L^1(Q_\delta(c))}^r \\ &\leq C' \|\nabla \bar{u}\|_{L^p(\Omega_\delta)}^{r\alpha} \|\bar{u}\|_{L^1_{\text{loc}}(\Omega_\delta)}^{r(1-\alpha)} + C' \|\bar{u}\|_{L^1(\Omega_\delta)}^{r\alpha} \|\bar{u}\|_{L^1_{\text{loc}}(\Omega_\delta)}^{r(1-\alpha)} \\ &\leq C'' (\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)})^{r\alpha} \|u\|_{L^1_{\text{loc}}(\Omega)}^{r(1-\alpha)}. \end{aligned} \quad \square$$

<sup>7</sup>Note that  $\bar{u}$  is not defined on  $\partial\Omega$ , but this set is negligible.

**Proposition 3.8.** *Assume that  $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  satisfies **(H<sub>1</sub>)** and **(H<sub>5</sub>)** for some  $p \in (1, +\infty)$ . Consider a vanishing sequence  $(u_n)_{n \in \mathbb{N}}$  in  $W_{\text{loc}}^{1,1}(\Omega_n, \mathbb{R}_{\pm})$ , where the  $\Omega_n \subseteq \mathbb{R}^N$  are open sets with  $\mathcal{C}^1$  boundary such that  $\inf_{n \in \mathbb{N}} \text{reach}(\partial\Omega_n) > 0$ , and a sequence  $(\Phi_n)_{n \in \mathbb{N}}$  of Borel maps  $\Phi_n : \Omega_n \rightarrow \mathbb{R}^N$  such that  $\sup_{y \in \Omega_n} |\Phi_n(y) - x_0| \rightarrow 0$  as  $n \rightarrow +\infty$  for some  $x_0 \in \mathbb{R}^N$ . If  $\theta_n := \int_{\Omega_n} u_n \neq 0$  for every  $n$  and  $(\theta_n)_{n \in \mathbb{N}}$  is bounded, then:*

$$\liminf_{n \rightarrow +\infty} \frac{1}{|\theta_n|} \int_{\Omega_n} f(\Phi_n(y), u_n(y), \nabla u_n(y)) \, dy \geq f'_-(x_0, 0^\pm, 0),$$

where  $f'_-(x_0, 0^\pm, 0)$  was defined in (1.4).

*Proof of Proposition 3.8.* Suppose for example that  $u_n \geq 0$  a.e. for every  $n$ . Without loss of generality, we may assume after extracting a subsequence that:

$$(3.18) \quad K := \sup_n \frac{1}{\theta_n} \int_{\Omega_n} f(\Phi_n(y), u_n(y), \nabla u_n(y)) \, dy + \theta_n < +\infty.$$

We consider the sequence of measures  $(\nu_n)_{n \in \mathbb{N}} \in \mathcal{M}_+(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$  defined by

$$\nu_n := \frac{1}{\theta_n} (\Phi_n, u_n, \nabla u_n)_\# (u_n \mathcal{L}^N \llcorner \Omega_n), \quad n \in \mathbb{N}.$$

We are going to show in several steps that  $\nu_n \xrightarrow{\mathcal{C}'_b} \delta_{(x_0, 0, 0)}$  and deduce the result. It suffices to show that the three projections  $\nu_n^i := (\pi^i)_\# \nu_n$ ,  $i \in \{1, 2, 3\}$  converge narrowly to  $\delta_{x_0}$ ,  $\delta_0$  and  $\delta_0$  respectively. Indeed, this would imply that  $(\nu_n)$  converges narrowly to a measure concentrated on  $(x_0, 0, 0)$ , hence to  $\delta_{(x_0, 0, 0)}$  since the  $\nu_n$  are probability measures. First of all, since  $(\nu_n)$  has bounded mass and  $(\theta_n)$  is bounded, we may take a subsequence (not relabeled) such that  $\nu_n \xrightarrow{\mathcal{C}'_0} \nu$  and  $\theta_n \rightarrow \theta$  as  $n \rightarrow \infty$  for some  $\nu \in \mathcal{M}_+(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$  and  $\theta \geq 0$ .

*Step 1:*  $\nu_n^1 \xrightarrow{\mathcal{C}'_b} \delta_{x_0}$ . This is a direct consequence of the fact that  $\nu_n^1$  is concentrated on  $\Phi_n(\mathbb{R}^N)$  for every  $n$  and  $\text{dist}(\Phi_n(\mathbb{R}^N), x_0) \rightarrow 0$  as  $n \rightarrow \infty$ .

*Step 2:*  $\nu_n^2 \xrightarrow{\mathcal{C}'_b} \delta_0$ . By (3.18) and our assumption **(H<sub>5</sub>)**, there is a constant  $K_1 > 0$  with

$$(3.19) \quad \int_{\Omega_n} |\nabla u_n|^p \leq K_1 \int_{\Omega_n} u_n, \quad n \in \mathbb{N}.$$

We deduce from Markov's inequality, and Lemma 3.7 applied with  $r = p(1 + \frac{1}{N})$ , corresponding to  $\alpha = \frac{N}{N+1}$ , that

$$\begin{aligned} \nu_n^2([\eta, +\infty)) &= \frac{1}{\theta_n} \int_{\{u_n \geq \eta\}} u_n \\ &= \frac{1}{\theta_n} \int_{\{u_n \geq \eta\}} u_n^{1-r} u_n^r \\ &\leq \frac{1}{\theta_n \eta^{r-1}} \int_{\Omega_n} u_n^r \\ &\leq \frac{C}{\theta_n \eta^{r-1}} (\|\nabla u_n\|_{L^p(\Omega_n)} + \|u_n\|_{L^1(\Omega_n)})^{r\alpha} \|u_n\|_{L^1_{\text{uloc}}(\Omega_n)}^{r(1-\alpha)} \\ &\leq \frac{C'}{\eta^{r-1}} (1 + \theta_n^{p-1}) \|u_n\|_{L^1_{\text{uloc}}(\Omega_n)}^{r(1-\alpha)}, \end{aligned}$$

where in the last inequality, we have used the identity  $\alpha r = p$  and (3.19), and  $C, C'$  depend only on  $N, r, p$  and  $\inf_n \text{reach}(\partial\Omega_n)$ .

Since  $(u_n)_{n \in \mathbb{N}}$  is vanishing and  $(\theta_n)_{n \in \mathbb{N}}$  is bounded, the last term in the previous inequality goes to zero as  $n \rightarrow \infty$  and it follows that  $\nu_n^2 \xrightarrow{\mathcal{C}'_b} \delta_0$ .

*Step 3:*  $\nu_n^3 \xrightarrow{\mathcal{C}'_b} \delta_0$ . Fix  $M > 0$  and  $\eta > 0$ . One has by (3.19),

$$\begin{aligned} \nu_n^3([M, +\infty)) &= \frac{1}{\theta_n} \int_{\{|\nabla u_n| \geq M\}} u_n \leq \frac{1}{\theta_n} \int_{\{u_n < \eta\} \cap \{|\nabla u_n| \geq M\}} u_n + \frac{1}{\theta_n} \int_{\{u_n > \eta\}} u_n \\ &\leq \frac{\eta}{\theta_n} \mathcal{L}^N(\{|\nabla u_n| \geq M\}) + \nu_n^2([\eta, +\infty)) \\ &\leq \frac{\eta}{\theta_n} \frac{1}{M^p} \int_{\Omega_n} |\nabla u_n|^p + \nu_n^2([\eta, +\infty)) \\ &\leq \frac{\eta K_1}{M^p} + \nu_n^2([\eta, +\infty)). \end{aligned}$$

By the previous step, we know that  $\lim_{n \rightarrow +\infty} \nu_n^2([\eta, +\infty)) = 0$ , hence taking the superior limit as  $n \rightarrow +\infty$  then  $\eta \rightarrow 0$  we get  $\lim_{n \rightarrow +\infty} \nu_n^3([M, +\infty)) = 0$ . Since this is true for every  $M > 0$  we obtain  $\nu_n^3 \xrightarrow{\mathcal{C}'_b} \delta_0$ .

*Step 4: conclusion.* By the previous steps, we deduce that  $\nu_n \xrightarrow{\mathcal{C}'_b} \delta_{(x_0, 0, 0)}$  as  $n \rightarrow +\infty$ . We define  $g : \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$  as the lower semicontinuous envelope of  $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \ni (x, u, \xi) \mapsto \frac{1}{u} f(x, u, \xi)$ . By (H<sub>1</sub>), we have  $g(x, u, \xi) = \frac{1}{u} f(x, u, \xi)$  if  $u > 0$ , and by definition of  $f'_-$  (see (1.4)), we have  $g(x, 0, 0) = f'_-(x, 0^+, 0)$  for every  $x \in \mathbb{R}^N$ . Hence, by lower semicontinuity of  $g$  and weak convergence of  $(\nu_n)$ , we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega_n} f(\Phi_n, u_n, \nabla u_n) &\geq \liminf_{n \rightarrow \infty} \int_{\{u_n > 0\}} \frac{f(\Phi_n, u_n, \nabla u_n)}{u_n} u_n \\ &= \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N} g(x, u, \xi) d\nu_n(x, u, \xi) \\ &\geq \int_{\mathbb{R}^N} g(x, u, \xi) d\delta_{(x_0, 0, 0)} = f'_-(x_0, 0^+, 0), \end{aligned}$$

which ends the proof of the lemma.  $\square$

As a corollary, we may now relate the slope at 0 of  $H_f$  to that of  $f$ .

**Corollary 3.9.** *Assume that  $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  satisfies (H<sub>1</sub>), (H<sub>5</sub>) for some  $p \in (1, +\infty)$  and (H<sub>6</sub>). Fix  $x \in \mathbb{R}^N$ . If either  $N \geq 2$  or ( $N = 1$  and  $H_f(x, \cdot) \not\equiv +\infty$  on  $\mathbb{R}_\pm^*$ ), then  $H'_f(x, 0^\pm) = f'_-(x, 0^\pm, 0)$ .*

*Proof.* The inequality  $H'_f(x, 0^\pm) \leq f'_-(x, 0^\pm, 0)$  is a consequence of (H<sub>6</sub>), Proposition 2.6 and Proposition 2.7, and the converse inequality  $H'_f(x, 0^\pm) \geq f'_-(x, 0^\pm, 0)$  comes from Proposition 3.8. Indeed, if  $(u_n)_{n \in \mathbb{N}} \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_\pm)$  is a sequence of functions of mass  $\theta_n = \int_{\mathbb{R}^N} u_n$  going to 0 and which is almost minimizing in the sense that  $\lim_{n \rightarrow \infty} \frac{\mathcal{E}_f^x(u_n)}{|\theta_n|} = \liminf_{n \rightarrow \infty} \frac{H(x, \theta_n)}{\theta_n}$  then  $(u_n)_{n \in \mathbb{N}}$  is vanishing and Proposition 3.8 yields

$$\liminf_{n \rightarrow \infty} \frac{\mathcal{E}_f^x(u_n)}{|\theta_n|} \geq f'_-(x, 0^\pm, 0).$$



□

We now establish our main energy lower bound along sequences with bounded mass (not necessarily vanishing):

**Proposition 3.10.** *Assume that  $(f_\varepsilon)_{\varepsilon>0}$  is a family of functions  $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  satisfying **(H<sub>1</sub>)**, **(H<sub>2</sub>)**, **(H<sub>5</sub>)** and **(H<sub>8</sub>)** where  $f = \lim_\varepsilon f_\varepsilon$ . Let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers going to zero,  $(R_n)_{n \in \mathbb{N}}$  and  $(r_n)_{n \in \mathbb{N}}$  be two sequences in  $(0, +\infty]$  such that  $\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} R_n - r_n = +\infty$ ,  $(u_n)_{n \in \mathbb{N}}$  be a sequence of functions  $u_n \in W_{\text{loc}}^{1,1}(B_{R_n}, \mathbb{R}_\pm)$  with finite limit mass  $m := \lim_{n \rightarrow \infty} \int_{B_{r_n}} u_n$ , and  $(\Phi_n)_{n \in \mathbb{N}}$  be a sequence of Borel maps  $\Phi_n : B_{R_n} \rightarrow \mathbb{R}^N$  such that*

$$(3.20) \quad \sup_{y \in B_{R_n}} |\Phi_n(y) - x_0| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for some } x_0 \in \mathbb{R}^N.$$

*Then there exists a family  $(u^i)_{0 \leq i < k}$  of functions in  $W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_\pm)$  with  $k \in \mathbb{N} \cup \{+\infty\}$ , such that  $m_i := \int_{\mathbb{R}^N} u^i \in \mathbb{R}_\pm^*$  for every  $i$ , and*

$$(3.21) \quad m = m_v + \sum_{0 \leq i < k} m_i \quad \text{with } \pm m_v \geq 0,$$

$$(3.22)$$

$$\liminf_{n \rightarrow \infty} \int_{B_{R_n}} f_{\varepsilon_n}(\Phi_n, u_n, \nabla u_n) \geq |m_v| f'_-(x_0, 0^\pm, 0) + \sum_{0 \leq i < k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i).$$

*Proof.* Suppose for example that  $u_n \geq 0$  a.e. for every  $n$ . We first assume, up to subsequence, that the left hand side of (3.22) is a finite limit. We apply the profile decomposition [Theorem 3.2](#) to the sequence of positive measures  $\mu_n = u_n \mathcal{L}^N \llcorner B_{r_n}$  where, we assume the extraction  $\sigma$  to be the identity for convenience, and we use the same notation as in [Theorem 3.2](#). In particular, for each bubble  $B_n^i = B_{r_n^i}(x_n^i)$ , with  $0 \leq i < k_n$ , we have  $x_n^i \in \text{supp } \mu_n \subseteq \bar{B}_{r_n}$ . By assumption, we have  $\lim_{n \rightarrow \infty} (R_n - r_n) = +\infty$ ; hence, up to reducing the radii of the balls  $B_n^i$  if necessary, in such a way that their radii still diverge (see [Remark 3.4](#)), we can assume that

$$(3.23) \quad B_n^i \subseteq B_{R_n-1}, \quad 0 \leq i < k_n.$$

For each  $0 \leq i < k_n$ , we let  $u_n^i := u_n(\cdot + x_n^i)$ . Since (3.22) is assumed to be finite, we get that the sequence  $(u_n^i)_n$  is bounded in  $W_{\text{loc}}^{1,p}(\mathbb{R}^N)$  by **(H<sub>5</sub>)**. Hence, after a further extraction if needed, we get that  $(u_n^i)_{n \in \mathbb{N}} \rightharpoonup u^i$  weakly in  $W_{\text{loc}}^{1,p}(\mathbb{R}^N)$  for some limit  $u^i$ , for every  $0 \leq i < k = \lim k_n$ . Setting  $m_i = \int_{\mathbb{R}^N} u^i$  for every  $i$ , by **(D)** in [Theorem 3.2](#), we have

$$m_v := m - \sum_{0 \leq i < k} m_i = \lim_{n \rightarrow \infty} \int_{B_{R_n} \setminus \cup_{0 \leq i < k_n} B_n^i} u_n.$$

Fix  $\varepsilon > 0$ . We decompose the energy as

$$(3.24) \quad \int_{B_{R_n}} f_\varepsilon(\Phi_n, u_n, \nabla u_n) = \int_{B_{R_n} \setminus \cup_{0 \leq i < k_n} B_n^i} f_\varepsilon(\Phi_n, u_n, \nabla u_n) \\ + \sum_{0 \leq i < k_n} \int_{B_{r_n^i}} f_\varepsilon(\Phi_n(\cdot + x_n^i), u_n^i, \nabla u_n^i).$$

Note that the domains  $\Omega_n := B_{R_n} \setminus \cup_{0 \leq i < k} B_n^i$  satisfy  $\inf_{n \in \mathbb{N}} \text{reach}(\partial\Omega_n) > 0$  as noticed in [Example 3.5](#), thanks to [\(3.23\)](#) and [\(B\)](#), [\(C\)](#) in [Theorem 3.2](#). Hence, applying [Proposition 3.8](#) to the Lagrangian  $f_\varepsilon$ , we obtain

$$(3.25) \quad \liminf_{n \rightarrow \infty} \int_{B_{R_n} \setminus \cup_{0 \leq i < k} B_n^i} f_\varepsilon(\Phi_n, u_n, \nabla u_n) \geq m_v(f_\varepsilon)'_-(x_0, 0^+, 0).$$

Moreover, by the lower semicontinuity of integral functionals (see [\[But89, Theorem 4.1.1\]](#)), in view of [\(3.20\)](#), we have for each  $i$  with  $0 \leq i < k$ ,

$$(3.26) \quad \liminf_{n \rightarrow \infty} \int_{B_n^i} f_\varepsilon(\Phi_n(\cdot + x_n^i), u_n^i, \nabla u_n^i) \geq \int_{\mathbb{R}^N} f_\varepsilon(x_0, u^i, \nabla u^i).$$

Finally, by [\(3.24\)](#), [\(3.25\)](#), [\(3.26\)](#), [\(H<sub>8</sub>\)](#) together with monotone convergence, we deduce that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{B_{R_n}} f_{\varepsilon_n}(\Phi_n, u_n, \nabla u_n) \\ & \geq \lim_{\varepsilon \rightarrow 0^+} \left( m_v(f_\varepsilon)'_-(x_0, 0^+, 0) + \sum_{0 \leq i < k} \int_{\mathbb{R}^N} f_\varepsilon(x_0, u^i, \nabla u^i) \right) \\ & = m_v f'_-(x_0, 0^+, 0) + \sum_{0 \leq i < k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i). \end{aligned}$$

The similar statement for non-positive functions is obtained in the same way.  $\square$

**3.3. Existence of optimal profiles.** For the existence of an optimal profile in [\(1.2\)](#), we need a criterion that rules out splitting and vanishing of minimizing sequences:

**Lemma 3.11.** *Let  $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a concave function. Then  $H$  is subadditive, and if for some  $0 < \theta < m$  one has  $H(m) = H(m - \theta) + H(\theta)$ , then  $H$  is linear on  $(0, m)$ .*

*Proof.* By concavity,  $t \mapsto \frac{H(t)}{t}$  is non-increasing. Hence,

$$H(m) = \theta \frac{H(m)}{m} + (m - \theta) \frac{H(m)}{m} \leq \theta \frac{H(\theta)}{\theta} + (m - \theta) \frac{H(m - \theta)}{m - \theta}.$$

But, by assumption, the last inequality is an equality which means that  $\frac{H(m)}{m} = \frac{H(\theta)}{\theta} = \frac{H(m - \theta)}{m - \theta}$ . In particular, the monotone function  $t \mapsto \frac{H(t)}{t}$  must be constant on  $[\theta, m]$ , i.e.  $H$  must be linear on  $[\theta, m]$ . By concavity this is only possible if  $H$  is linear on  $[0, m]$ .  $\square$

We can now state and prove our existence result:

**Proposition 3.12.** *Assume that  $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  satisfies [\(H<sub>1</sub>\)](#), [\(H<sub>2</sub>\)](#), [\(H<sub>5</sub>\)](#) and [\(H<sub>6</sub>\)](#). Let  $x_0 \in \mathbb{R}^N$  and  $m \in \mathbb{R}_+$  (resp.  $m \in \mathbb{R}_-$ ). If  $H_f(x_0, \cdot)$  is not linear on  $[0, m]$  (resp.  $[m, 0]$ ), then [\(1.2\)](#) admits a solution  $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ , i.e.  $\int_{\mathbb{R}^N} u = m$  and  $\int_{\mathbb{R}^N} f(x_0, u, \nabla u) = H_f(x_0, m)$ , such that  $u \geq 0$  (resp.  $u \leq 0$ ) in  $\mathbb{R}^N$ .*

*Proof.* We consider the case  $m \geq 0$ , the case  $m < 0$  can then be deduced by considering  $\tilde{f}(x, u, \xi) = f(x, -u, -\xi)$ . We assume without loss of generality that  $H_f$  is finite on  $(0, +\infty)$ , otherwise by [Theorem 2.1](#) there is nothing to prove. By [Remark 2.2](#), the admissible class in [\(1.2\)](#) can be reduced to non-negative functions. In particular, if  $m = 0$ , then  $u = 0$  is the only non-negative solution. If  $m > 0$ ,

we apply [Proposition 3.10](#) in the following situation:  $f_\varepsilon \equiv f$ ,  $R_n \equiv +\infty$ ,  $\Phi_n \equiv x_0$ ,  $(u_n)_{n \in \mathbb{N}}$  is a minimizing sequence for the minimization problem in [\(1.2\)](#), and  $(r_n)_{n \in \mathbb{N}}$  is a sequence of positive radii going to  $+\infty$  such that  $\lim_{n \rightarrow \infty} \int_{B_{r_n}} u_n = m$ . We obtain

$$H_f(x_0, m) \geq m_v f'_-(x_0, 0^+, 0) + \sum_{0 \leq i < k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i),$$

with  $k \in \mathbb{N} \cup \{+\infty\}$ ,  $u^i \in W_{\text{loc}}^{1,p}(\mathbb{R}^N, \mathbb{R}_+)$  and  $m = \sum_{0 \leq i < k} m_i + m_v$ , where  $m_i := \int_{\mathbb{R}^N} u^i$ . By [Proposition 2.6](#) and [Proposition 2.7](#), in view of our assumption [\(H<sub>6</sub>\)](#), and since  $H_f$  is assumed to be finite on  $(0, +\infty)$  (for the case  $N = 1$ ), we have  $f'_-(x_0, 0^+, 0) \geq H'_f(x_0, 0)$ . Moreover, by [Theorem 2.1](#), we have  $m_v H'_f(x_0, 0) \geq H_f(x_0, m_v)$ . Hence, by definition of  $H_f$ ,

$$H_f(x_0, m) \geq H_f(x_0, m_v) + \sum_{0 \leq i < k} H_f(x_0, m_i).$$

Since the concave function  $H_f(x_0, \cdot)$  is not linear on  $[0, m]$ , by [Lemma 3.11](#), we have either  $k = 1$  and  $m_v = 0$ , and we are done, or  $k = 0$  and  $m = m_v$ . But in the latter case, we would have  $H_f(x_0, m) = m H'_f(x_0, 0)$  which implies that the monotone function  $t \mapsto \frac{H_f(x_0, t)}{t}$  is constant on  $[0, m]$ , i.e. that  $H_f(x_0, \cdot)$  is linear on  $[0, m]$ . This contradicts our assumption.  $\square$

*Remark 3.13.* Notice that the end of the proof actually shows, under the given assumptions, that the set of minimizers for a given mass  $m$  is compact in  $L^1$  modulo translations.

#### 4. $\Gamma$ -CONVERGENCE OF THE RESCALED ENERGIES TOWARDS THE $H$ -MASS

We establish lower and upper bounds for the  $\Gamma$  – lim inf and  $\Gamma$  – lim sup respectively, from which we deduce the proof of our main  $\Gamma$ -convergence result. The upper bound on the  $\Gamma$  – lim sup holds under more general assumptions and will be needed in [Section 5.5](#).

**4.1. Lower bound for the  $\Gamma$  – lim inf.** Given a Borel function  $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ , we define for every  $(x, m) \in \mathbb{R}^N \times \mathbb{R}$ ,

$$(4.1) \quad H_f^-(x, m) := H_f(x, m) \wedge (f'_-(x, 0^\pm, 0)|m|), \quad \text{if } \pm m \geq 0,$$

recalling that  $H_f$  is defined in [\(1.2\)](#) and  $f'_-(x, 0^\pm, 0)$  in [\(1.4\)](#), with the usual convention  $(\pm\infty) \times 0 = 0$ . Notice that under [\(H<sub>6</sub>\)](#), in view of [Proposition 2.6](#) and [Proposition 2.7](#) we have  $H_f^-(x, m) = H_f(x, m)$ , provided  $H_f \not\equiv +\infty$  on  $\mathbb{R}_+^*$  and  $\mathbb{R}_-^*$ .

**Proposition 4.1.** *Assume that  $(f_\varepsilon)_{\varepsilon > 0}$  is a family of functions  $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$  satisfying [\(H<sub>1</sub>\)](#), [\(H<sub>2</sub>\)](#), [\(H<sub>5</sub>\)](#) and [\(H<sub>8</sub>\)](#) where  $f = \lim_{\varepsilon \rightarrow 0} f_\varepsilon$ . Let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers going to zero,  $(u_n)_{n \in \mathbb{N}}$  be a sequence in  $W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ , and let*

$$e_n(x) := f_{\varepsilon_n}(x, \varepsilon_n^N u_n(x), \varepsilon_n^{N+1} \nabla u_n(x)) \varepsilon_n^{-N}, \quad x \in \mathbb{R}^N,$$

be the energy density of  $u_n$ . If  $u_n \mathcal{L}^N \xrightarrow{\mathcal{C}'_0} u \in \mathcal{M}(\mathbb{R}^N)$  and  $e_n \mathcal{L}^N \xrightarrow{\mathcal{C}'_0} e \in \mathcal{M}(\mathbb{R}^N)$ , then

$$(4.2) \quad e \geq H_f^-(u).$$

In particular,  $\Gamma(\mathcal{C}'_0) - \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^{H_f^-}$ .

*Proof of Proposition 4.1.* Set  $H := H_f^-$ . Let us assume first that  $u_n \geq 0$  a.e. for every  $n$ . To obtain (4.2), it is enough to prove that for every  $x_0 \in \mathbb{R}^N$ ,

$$(4.3) \quad e(\{x_0\}) \geq H(x_0, u(\{x_0\})).$$

and that if  $x_0 \in \text{supp } u$  is not an atom of  $u$ , then

$$(4.4) \quad \limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} \geq H'(x_0, 0^+),$$

Indeed (4.3) implies that  $e \geq (H(u))^a$  (the atomic part of the measure  $H(u)$ ) while (4.4) implies that  $e \geq H'(\cdot, 0^+)u^d = (H(u))^d$ , by Radon-Nikodým theorem (see [AFP00, Theorem 2.22]); these two relations yield  $e \geq (H(u))^a + (H(u))^d = H(u)$  as required.

We fix  $x_0 \in \text{supp } u$  and proceed in several steps.

*Step 1: blow-up near  $x_0$ .* We first take two sequences of positive radii  $(R_\ell)_{\ell \in \mathbb{N}} \rightarrow 0$  and  $(r_\ell)_{\ell \in \mathbb{N}}$  such that for every  $\ell \in \mathbb{N}$ ,  $r_\ell \in (0, R_\ell)$ ,

$$(4.5) \quad e(\partial B_{R_\ell}(x_0)) = u(\partial B_{r_\ell}(x_0)) = 0,$$

and

$$(4.6) \quad \lim_{\ell \rightarrow \infty} \frac{e(B_{R_\ell}(x_0))}{u(B_{r_\ell}(x_0))} = \limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))}.$$

This last property is obtained by taking first a sequence  $(\rho_\ell)_\ell$  such that

$$\limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} = \lim_{\ell \rightarrow \infty} \frac{e(B_{\rho_\ell}(x_0))}{u(B_{\rho_\ell}(x_0))},$$

then using monotone convergence the measures to get first  $r_\ell$  then  $R_\ell$  such that  $0 < r_\ell < R_\ell < \rho_\ell$ ,  $u(B_{r_\ell}(x_0)) \geq (1 - 2^{-\ell})u(B_{\rho_\ell}(x_0))$  and  $e(B_{R_\ell}(x_0)) \geq (1 - 2^{-\ell})e(B_{\rho_\ell}(x_0))$ .

By weak convergence and (4.5), according to [AFP00, Proposition 1.62 b)], we have for every  $\ell \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} e_n(B_{R_\ell}(x_0)) = e(B_{R_\ell}(x_0)) \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{B_{r_\ell}(x_0)} u_n = u(B_{r_\ell}(x_0)).$$

Hence, there exists an extraction  $(n_\ell)_{\ell \in \mathbb{N}} \in \Sigma$  such that

$$(4.7) \quad \lim_{\ell \rightarrow \infty} \frac{r_\ell}{\varepsilon_{n_\ell}} = +\infty \quad \text{and} \quad \lim_{\ell \rightarrow \infty} \frac{R_\ell - r_\ell}{\varepsilon_{n_\ell}} = +\infty,$$

satisfying the following conditions:

$$(4.8) \quad u(\{x_0\}) = \lim_{\ell \rightarrow \infty} \int_{B_{r_\ell}(x_0)} u_{n_\ell}, \quad e(\{x_0\}) = \lim_{\ell \rightarrow \infty} e_{n_\ell}(B_{R_\ell}(x_0)),$$

and

$$(4.9) \quad \limsup_{\ell \rightarrow \infty} \frac{e(B_{R_\ell}(x_0))}{u(B_{r_\ell}(x_0))} = \lim_{\ell \rightarrow \infty} \frac{e_{n_\ell}(B_{R_\ell}(x_0))}{\int_{B_{r_\ell}(x_0)} u_{n_\ell}}.$$

We may rewrite the mass and energy in terms of the re-scaled map  $v_\ell$  defined by

$$(4.10) \quad v_\ell(y) := \varepsilon_{n_\ell}^N u_{n_\ell}(x_0 + \varepsilon_{n_\ell} y), \quad y \in \mathbb{R}^N, \ell \in \mathbb{N},$$

as follows:

$$(4.11) \quad \int_{B_{r_\ell}(x_0)} u_{n_\ell} = \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell,$$

and

$$(4.12) \quad e_{n_\ell}(B_{R_\ell}(x_0)) = \int_{B_{\varepsilon_{n_\ell}^{-1} R_\ell}} f_{\varepsilon_{n_\ell}}(x_0 + \varepsilon_{n_\ell} y, v_\ell(y), \nabla v_\ell(y)) \, dy.$$

*Step 2: proof of (4.3).* By [Proposition 3.10](#), we have

$$(4.13) \quad \begin{aligned} e(\{x_0\}) &= \lim_{\ell \rightarrow \infty} \int_{B_{\varepsilon_{n_\ell}^{-1} R_\ell}} f_{\varepsilon_{n_\ell}}(x_0 + \varepsilon_{n_\ell} y, v_\ell(y), \nabla v_\ell(y)) \, dy \\ &\geq m_v f'_-(x_0, 0^+, 0) + \sum_{0 \leq i < k} H_f(x_0, m_i). \end{aligned}$$

Here  $k \in \mathbb{N} \cup \{+\infty\}$  and  $m = m_v + \sum_{0 \leq i < k} m_i$ , with  $m_i > 0$ ,  $m_v \geq 0$  and

$$m = \lim_{\ell \rightarrow \infty} \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell = u(\{x_0\}).$$

Since the function  $H = H_f^-$ , defined in [\(4.1\)](#), is the infimum of two functions which are mass-concave, it is mass-concave hence subadditive. From [\(4.13\)](#) we thus arrive at

$$e(\{x_0\}) \geq H(x_0, m_v) + \sum_{0 \leq i < k} H(x_0, m_i) \geq H\left(x_0, m_v + \sum_{0 \leq i < k} m_i\right) = H(x_0, u(\{x_0\})).$$

*Step 3: proof of (4.4).* Fix  $\varepsilon > 0$  and assume that  $m = u(\{x_0\}) = 0$ . In that case, we apply [Proposition 3.8](#) to the sequence of functions  $(v_\ell)_{\ell \in \mathbb{N}}$  defined on the sets  $\Omega_\ell = B_{\varepsilon_{n_\ell}^{-1} r_\ell}$  and the function  $f_\varepsilon$  to get, thanks to [\(H<sub>8</sub>\)](#):

$$\begin{aligned} \limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} &= \lim_{\ell \rightarrow \infty} \frac{e_{n_\ell}(B_{R_\ell}(x_0))}{\int_{B_{r_\ell}(x_0)} u_{n_\ell}} \\ &\geq \liminf_{\ell \rightarrow \infty} \frac{1}{\int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell} \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} f_\varepsilon(x_0 + \varepsilon_{n_\ell} y, v_\ell(y), \nabla v_\ell(y)) \\ &\geq (f_\varepsilon)'_-(x_0, 0^+, 0). \end{aligned}$$

Taking the limit  $\varepsilon \rightarrow 0^+$ , we deduce by [\(H<sub>8</sub>\)](#) and [\(4.1\)](#):

$$(4.14) \quad \limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} \geq f'_-(x_0, 0^+, 0) \geq H'(x_0, 0^+).$$

In view of the discussion at the beginning of the proof, we have now proved [\(4.2\)](#).

*Step 4: proof of (4.2) for signed  $(u_n)_n$ .* Notice that the preceding reasoning for non-negative  $u_n$  applies also to the case of non-positive  $u_n$ . Let us handle the case where the  $(u_n)$ 's may change sign. We simply apply the above cases to the positive and negative parts  $((u_n)_\pm)_n$  which converge weakly (up to subsequence) to some measures  $u^\pm \in \mathcal{M}_+(\mathbb{R}^N)$  which satisfy  $u = u^+ - u^-$ , so that  $e \geq H(\pm u^\pm)$ . We know that the Jordan decomposition  $u = u_+ - u_-$  is minimal, so that  $u_\pm \leq u^\pm$

and  $u_+ \perp u_-$ . By monotonicity of the  $H$ -transform, due to [Theorem 2.1](#),  $e \geq H(\pm u^\pm) \geq H(\pm u_\pm)$ , and since  $H(u_+) \perp H(-u_-)$ , we get

$$e \geq H(u_+) + H(-u_-) = H(u).$$

*Step 5: lower bound for the  $\Gamma$ -lim inf.* We justify that [\(4.2\)](#) implies the lower bound  $\Gamma(\mathcal{C}'_0) - \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^H$ . Indeed, fix  $u \in \mathcal{M}(\mathbb{R}^N)$  and consider a family  $(u_\varepsilon)_{\varepsilon > 0}$  weakly converging to  $u$  as  $\varepsilon \rightarrow 0$ . We need to show that  $\mathbf{M}^H(u) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon)$ . Assume without loss of generality that the inferior limit is finite and take a sequence of positive numbers  $(\varepsilon_n)_{n \in \mathbb{N}} \rightarrow 0$  such that this inferior limit is equal to  $\lim_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n})$ . Now the energy density  $e_n$  associated to  $u_n = u_{\varepsilon_n}$  has bounded mass and up to extracting a subsequence one may assume that it converges weakly to some measure  $e \in \mathcal{M}_+(\mathbb{R}^N)$ . By the previous steps,  $e \geq H(u)$ , and by lower semicontinuity and monotonicity of the mass:

$$\liminf_{\varepsilon \rightarrow 0^+} \mathcal{E}_\varepsilon(u_\varepsilon) = \liminf_{n \rightarrow \infty} \|e_n\| \geq \|e\| \geq \|H(u)\| = \mathbf{M}^H(u). \quad \square$$

**4.2. Upper bound for the  $\Gamma$ -lim sup.** In this section, we introduce the following substitute for [\(H4\)](#), [\(H7\)](#) and [\(H8\)](#), where  $f, (f_\varepsilon)_{\varepsilon > 0}$  are Borel maps from  $\mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$  to  $[0, +\infty]$ :

(U) there exists  $C < +\infty$  such that for every  $x, y \in \mathbb{R}^N$ ,  $u \in \mathbb{R}$  and  $\xi \in \mathbb{R}^N$ ,

$$\limsup_{\varepsilon \rightarrow 0^+} f_\varepsilon(x + \varepsilon y, u, \xi) \leq f(x, u, \xi) \quad \text{and} \quad f_\varepsilon(y, u, \xi) \leq C(f(x, u, \xi) + u) \quad \forall \varepsilon > 0.$$

**Proposition 4.2.** *Assume that  $f, (f_\varepsilon)_{\varepsilon > 0}$  satisfy (U) and (H3). If  $u \in \mathcal{M}(\mathbb{R}^N)$ , then there exists  $(u_\varepsilon)_{\varepsilon > 0} \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$  such that  $u_\varepsilon \mathcal{L}^N \xrightarrow{\mathcal{C}'_b} u$  when  $\varepsilon \rightarrow 0$  and which satisfies*

$$\limsup_{\varepsilon \rightarrow 0^+} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathbf{M}^{H_{f,\text{lsc}}}(u),$$

where  $H_{f,\text{lsc}} \leq H_f$  stands for the lower semicontinuous envelope of  $H_f$ , defined in [\(2.6\)](#). In other words, we have  $\Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \leq \mathbf{M}^{H_{f,\text{lsc}}}$ .

*Proof of Proposition 4.2.* Let  $F = \Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon$ . As an upper  $\Gamma$ -limit,  $F$  is sequentially lower semicontinuous in the narrow topology. Hence, by [Proposition 2.5](#), it is enough to prove that  $F(u) \leq \mathbf{M}^{H_f}(u)$  whenever  $u$  is finitely atomic. Let  $u = \sum_{i=1}^k m_i \delta_{x_i}$  with  $k \in \mathbb{N}$ ,  $m_i \in \mathbb{R}$ ,  $x_i \in \mathbb{R}^N$ , and assume without loss of generality that  $x_i \neq x_j$  when  $i \neq j$  and  $\mathbf{M}^{H_f}(u) < +\infty$ . Fix  $\eta > 0$ . For each  $i = 1, \dots, k$ , there exists  $u_i \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$  such that  $\int_{\mathbb{R}^N} u_i = m_i$  and  $\int_{\mathbb{R}^N} f(x_i, u_i, \nabla u_i) \leq H(x_i, m_i) + \eta < +\infty$ . We define for every  $i = 1, \dots, k$ ,

$$(4.15) \quad u_\varepsilon^i(x) = \varepsilon^{-N} u_i(\varepsilon^{-1}(x - x_i)), \quad x \in \mathbb{R}^N,$$

and

$$(4.16) \quad u_\varepsilon = \max\{u_\varepsilon^i : i = 1, \dots, k\},$$

which converge narrowly as measures to  $u$  as  $\varepsilon \rightarrow 0$ . We have by change of variables:

$$\begin{aligned} \mathcal{E}_\varepsilon(u_\varepsilon) &\leq \sum_{i=1}^k \int_{\{u_\varepsilon = u_\varepsilon^i\}} f_\varepsilon(x, \varepsilon^N u_\varepsilon^i(x), \varepsilon^{N+1} \nabla u_\varepsilon^i(x)) \varepsilon^{-N} dx \\ &\leq \sum_{i=1}^k \mathcal{E}_\varepsilon(u_\varepsilon^i) = \sum_{i=1}^k \int_{\mathbb{R}^N} f_\varepsilon(x_i + \varepsilon x, u_i, \nabla u_i). \end{aligned}$$

Using our assumption **(U)** and the dominated convergence theorem, one gets as  $\varepsilon \rightarrow 0$ :

$$F(u) \leq \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \sum_{i=1}^k \int_{\mathbb{R}^N} f(x_i, u_i, \nabla u_i) \leq \sum_{i=1}^k H(x_i, m_i) + k\eta = \mathbf{M}^H(u) + k\eta.$$

The conclusion follows by arbitrariness of  $\eta > 0$ .  $\square$

**4.3. Proof of the main  $\Gamma$ -convergence result.** We now explain how [Theorem 1.2](#) follows from [Proposition 4.1](#) and [Proposition 4.2](#).

*Proof of [Theorem 1.2](#).* The lower bound  $\Gamma(\mathcal{C}'_0) - \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^{H_f^-}$  follows from [Proposition 4.1](#), and the upper bound  $\Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \leq \mathbf{M}^{H_{f,\text{lsc}}}$  from [Proposition 4.2](#), where the assumption **(U)** is a consequence of **(H<sub>4</sub>)**, **(H<sub>7</sub>)** and **(H<sub>8</sub>)**. In the case  $N = 2$ , by [Proposition 2.6](#) and assumption **(H<sub>6</sub>)**, we have  $H_f^- = H_f$ , and  $H_f \geq H_{f,\text{lsc}}$  by definition, so that both  $\Gamma - \liminf$  and  $\Gamma - \limsup$  (for weak and narrow topologies) coincide. In the case  $N = 1$ , by [Proposition 2.7](#) either we have again  $H_f^- = H_f$  on  $\mathbb{R}^N \times \mathbb{R}_+$  by **(H<sub>6</sub>)** and **(H<sub>3</sub>)**, or we have  $H_f(x_0, \cdot) \equiv +\infty$  on  $(0, +\infty)$  for some  $x_0 \in \mathbb{R}$ . In that case, by **(H<sub>7</sub>)** we necessarily have  $\mathcal{E}_\varepsilon(u) = +\infty$  for every  $u$  such that  $\int_{\mathbb{R}^N} u > 0$ , and  $H_f(x, \cdot) \equiv H_f(x_0, \cdot)$  on  $\mathbb{R}_+$  for every  $x \in \mathbb{R}$ . It implies that both  $\Gamma - \liminf$  and  $\Gamma - \limsup$  (for the weak and narrow topologies) coincide with<sup>8</sup>  $\chi_{\{0\}} = \mathbf{M}^{H_f}$  on  $\mathcal{M}_+(\mathbb{R}^N)$ . Similarly, they coincide on finite negative measures.  $\square$

## 5. EXAMPLES, COUNTEREXAMPLES AND APPLICATIONS

**5.1. Scale-invariant Lagrangians and necessity of assumption **(H<sub>6</sub>)**.** Our assumption **(H<sub>6</sub>)** is not very standard, but we need a condition of this type in order to get  $\Gamma$ -convergence of the rescaled energies  $\mathcal{E}_\varepsilon$  towards  $\mathbf{M}^{H_f}$ , as shown by the following class of scale-invariant Lagrangians:

$$(5.1) \quad f_\varepsilon(x, u, \xi) = f(u, \xi) \quad \text{with} \quad f(u, \xi) = \begin{cases} u^{p(\frac{1}{p^*}-1)} |\xi|^p & \text{if } u > 0, \\ 0 & \text{else,} \end{cases}$$

where  $p \in (1, N)$ ,  $N \in \mathbb{N}^*$  and  $p^* := \frac{pN}{N-p}$ . By straightforward computations,  $\mathcal{E}_\varepsilon(u) = \mathcal{E}_f(u) := \int_{\mathbb{R}^N} f(u, \nabla u)$  for every  $\varepsilon > 0$  and  $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^N)$  in that case.

Moreover, the associated minimal cost function  $H_f$  is not trivial. Indeed, applying the Gagliardo–Nirenberg–Sobolev inequality,

$$\left( \int_{\mathbb{R}^N} |v|^{p^*} \right)^{\frac{1}{p^*}} \leq C \left( \int_{\mathbb{R}^N} |\nabla v|^p \right)^{\frac{1}{p}}, \quad \forall v \in L^{p^*} \cap W_{\text{loc}}^{1,1}(\mathbb{R}^N),$$

to the function<sup>9</sup>  $v = u^{\frac{1}{p^*}}$ , we obtain that for every  $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+) \cap L^1(\mathbb{R}^N)$ ,

$$\left( \int_{\mathbb{R}^N} u \right)^{\frac{p}{p^*}} \leq \left( \frac{C}{p^*} \right)^p \int_{\{u>0\}} u^{\frac{p}{p^*}-p} |\nabla u|^p = \left( \frac{C}{p^*} \right)^p \mathcal{E}_f(u).$$

<sup>8</sup> $\chi_C(x) = 0$  if  $x \in C$  and  $+\infty$  otherwise.

<sup>9</sup>Actually, we apply it to  $v_\varepsilon = \phi_\varepsilon(u)$  where  $\phi_\varepsilon$  is a suitable approximation of  $(\cdot)^{\frac{1}{p^*}}$  and take  $\varepsilon \rightarrow 0$ .

Hence, for every  $m > 0$ , we have  $H_f(m) > 0$ , and even  $H_f(m) < +\infty$  since any function  $u = v^{p^*}$ , with  $v \in W^{1,p}(\mathbb{R}^N, \mathbb{R}_+)$ , has finite energy. Replacing  $u$  by  $mu$  in the infimum defining  $H_f$  in (1.2), we actually obtain

$$(5.2) \quad H_f(m) = m^{1-\frac{p}{p^*}} H_f(1), \quad 0 < H_f(1) < +\infty.$$

In that case, it is clear that the  $\Gamma$ -limit of  $\mathcal{E}_\varepsilon \equiv \mathcal{E}$  in the weak or narrow topology of  $\mathcal{M}_+(\mathbb{R}^N)$ , that is the lower semicontinuous relaxation of  $\mathcal{E}_f$ , does not coincide with  $\mathbf{M}^{H_f}$ ; indeed, the first functional is finite on diffuse measures whose density has finite energy, while the second functional is always infinite for non-trivial diffuse measures since  $H'_f(0) = +\infty$ .

These scaling invariant Lagrangians are ruled out by our assumption (H<sub>6</sub>). All the other assumptions are satisfied except (H<sub>5</sub>). Note that the following perturbation of  $f$ ,

$$\tilde{f}(u, \xi) = (1 + u^{p(\frac{1}{p^*}-1)})|\xi|^p$$

satisfies all the assumptions except (H<sub>6</sub>), and provides a counterexample to the  $\Gamma$ -convergence. Indeed,  $\mathbf{M}_{H_{\tilde{f}}} \geq \mathbf{M}_{H_f}$  is still infinite on diffuse measures, while (the relaxation of)  $\mathcal{E}_{\tilde{f}}$  is finite for any diffuse measure whose density has finite energy.

We stress that an assumption like (H<sub>6</sub>) is actually needed, even for the lower semicontinuity of the function  $H_f$  – recall that if  $\mathbf{M}_{H_f}$  is a  $\Gamma$ -limit, then it must be lower semicontinuous by [Bra02, Proposition 1.28], which in turn implies that the function  $H_f$  is lower semicontinuous by Proposition 2.5. Indeed, consider the Lagrangians

$$f(x, u, \xi) = (1 + u^{p(\frac{1}{p^*}-1)})|\xi|^{p(x)},$$

with  $p \in \mathcal{C}^0(\mathbb{R}^N, (1, N))$  such that  $p(0) = p \in (1, N)$  and  $p(x) > p$  when  $x \neq 0$ . Then, we have  $H_f(0, m) = m^{1-\frac{p}{p^*}} H(1)$ , but  $H_f(x, \cdot) \equiv 0$  if  $x \neq 0$  as can be easily seen via the change of function  $\varepsilon^N u(\varepsilon \cdot)$ , with  $\varepsilon > 0$  small.

**5.2. General concave costs in dimension one.** It has been proved in [Wir19] that for any continuous concave function  $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $H(0) = 0$ , there exists a function  $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $c(0) = 0$ ,  $u \mapsto \frac{c(u)}{u}$  is lower semicontinuous and non-increasing on  $(0, +\infty)$ , and for every  $m \geq 0$ ,

$$H(m) = \inf \left\{ \int_{\mathbb{R}} |u'|^2 + c(u) : u \in W_{\text{loc}}^{1,1}(\mathbb{R}, \mathbb{R}_+), \int_{\mathbb{R}} u = m \right\}.$$

The Lagrangians of the form  $f_\varepsilon(x, u, \xi) = |\xi|^2 + c(u)$ , in dimension  $N = 1$ , satisfy all our assumptions (H<sub>1</sub>)–(H<sub>8</sub>), hence our  $\Gamma$ -convergence result stated in Theorem 1.2 yields the  $\Gamma$ -convergence of the functionals

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon^3 |u'|^2 + \frac{c(\varepsilon u)}{\varepsilon}, \quad u \in W^{1,2}(\mathbb{R}, \mathbb{R}_+),$$

towards  $\mathbf{M}^H$  for both the weak and narrow convergence of measures. Therefore, we may find an elliptic approximation of any concave  $H$ -mass. Let us stress that  $c$  is determined in [Wir19] from  $H$  through several operations including a deconvolution problem, but no closed form solution is given in general; nonetheless, an explicit solution is provided if  $c$  is affine by parts.



In higher dimension  $N \geq 2$ , [Proposition 2.8](#) tells us that the class of functions  $H = H_f$  with  $f$  satisfying [\(H<sub>1</sub>\)–\(H<sub>8</sub>\)](#) is smaller, namely,  $H$  must satisfy:

$$(5.3) \quad \exists m_* \geq 0, \quad \begin{cases} H \text{ is linear on } [0, m_*], \\ H \text{ is strictly concave } (m_*, +\infty). \end{cases}$$

We have no positive or negative answer to the inverse problem, consisting in finding  $f$  satisfying our assumptions such that  $H_f = H$ , for a given continuous concave function  $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying [\(5.3\)](#).

**5.3. Homogeneous costs in any dimension.** In this section, we provide Lagrangians  $f$  to obtain the  $\alpha$ -mass  $\mathbf{M}^\alpha := \mathbf{M}^{t \rightarrow t^\alpha}$  in any dimension  $N$  for a wide range of exponents, including exponents  $\alpha \in (1 - \frac{1}{N}, 1]$ . We consider for every  $p \in [1, +\infty)$ ,  $s \in (-\infty, 1]$  and  $N \in \mathbb{N}^*$ , the energy defined for every  $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+)$  by

$$(5.4) \quad \mathcal{E}_f(u) := \int_{\mathbb{R}^N} f(u, \nabla u) := \int_{\mathbb{R}^N} |\nabla u|^p + u^s.$$

Notice that for  $p > 1$ ,  $f$  satisfies all our hypotheses [\(H<sub>1</sub>\)–\(H<sub>6</sub>\)](#) (without dependence on  $x$ ), [\(H<sub>6</sub>\)](#) holding in dimension  $N \geq 2$  with  $\rho(t) = t$  for example. Thus by [Theorem 1.2](#) the re-scaled energies  $\Gamma$ -converge to the  $H_f$ -mass.

One may compute  $H_f$  substituting  $u$  by  $v$  such that  $u = m\lambda^N v(\lambda \cdot)$  in [\(1.2\)](#), where

$$(5.5) \quad \lambda = m^{\frac{s/p-1}{1+N-sN/p}}.$$

Straightforward computations give  $\int_{\mathbb{R}^N} v = 1$  if  $\int_{\mathbb{R}^N} u = m$ , and

$$\mathcal{E}_f(u) = m^\alpha \mathcal{E}_f(v), \quad \text{where } \alpha = \frac{1 - \frac{s}{p} + \frac{s}{N}}{1 - \frac{s}{p} + \frac{1}{N}},$$

thus

$$H_f(m) = cm^\alpha, \quad \text{where } c = H_f(1).$$

We look for cases when the cost is non-trivial, i.e. neither identically zero nor infinite on  $(0, +\infty)$ . Take an auxiliary exponent  $q \in [1, +\infty)$  and  $\alpha \in [0, 1]$  such that  $1 = \alpha q + (1 - \alpha)s$ . By Hölder inequality,

$$\int_{\mathbb{R}^N} u = \int_{\mathbb{R}^N} u^{\alpha q} u^{(1-\alpha)s} \leq \left( \int_{\mathbb{R}^N} u^q \right)^\alpha \left( \int_{\mathbb{R}^N} u^s \right)^{1-\alpha}.$$

Moreover, choosing  $q \in (1, p^*)$  if  $p < N$  and any  $q \in (1, +\infty)$  if  $p \geq N$ , by the Gagliardo–Nirenberg–Sobolev inequality, for every  $u \in W_{\text{loc}}^{1,1} \cap L^1(\mathbb{R}^N, \mathbb{R}_+)$ ,

$$\left( \int_{\mathbb{R}^N} u^q \right)^{\frac{1}{q}} = \|u\|_{L^q} \leq C \|\nabla u\|_{L^p}^\beta \|u\|_{L^1}^{1-\beta},$$

with  $\beta \in (0, 1)$  such that  $\frac{1}{q} = \beta \left( \frac{1}{p} - \frac{1}{N} \right) + (1 - \beta)$ . Hence,

$$\left( \int_{\mathbb{R}^N} u \right)^{1-q\alpha(1-\beta)} \leq C \left( \int_{\mathbb{R}^N} |\nabla u|^p \right)^{\frac{q\alpha\beta}{p}} \left( \int_{\mathbb{R}^N} u^s \right)^{1-\alpha},$$

and the cost is non-zero for every  $m > 0$ .

In the case  $s \in [0, 1]$ , any  $u = v^r$  with  $v \in \mathcal{C}_c^1(\mathbb{R}^N)$  is a competitor with finite energy, thus  $\mathcal{E}_f$  is non-trivial for every  $p \in [1, +\infty)$ . In the case  $s < 0$ , consider the

competitor  $u : x \mapsto (1 - |x|)_+^\gamma$  for  $\gamma > 0$  to be fixed later. Then  $\int_{\mathbb{R}^N} |\nabla u|^p < +\infty$  if and only if  $t \mapsto (1-t)^{(\gamma-1)p}$  is integrable at  $1^-$ , i.e.  $(\gamma-1)p > -1 \iff \gamma > 1-1/p$ , and  $\int_{\{u>0\}} u^s < +\infty$  if and only if  $\gamma s > -1 \iff \gamma < -1/s$ . Therefore, one may find  $\gamma > 0$  satisfying both conditions, and ensure that  $H_f$  is non-trivial, if

$$-p' < s < 0.$$

To summarize, we have shown that  $H_f$  is non-trivial if:

$$s \in (-p', 1].$$

Since  $\alpha$  is monotone in  $s$ , one may easily compute the range of  $\alpha$ . If  $p$  and  $N$  are fixed,  $\alpha$  ranges over  $\left(\frac{N-1}{N+1-1/p}, 1\right]$  when  $s \in (-p', 1]$ . Notice that when  $N = 1$  we obtain the whole range  $\alpha \in (0, 1]$ , and at least the range  $\left[1 - \frac{1}{N}, 1\right]$  for every  $p$  in dimension  $N \geq 2$ . Finally, we obtain a range  $\alpha \in \left(1 - \frac{2}{N+1}, 1\right]$  when  $p$  ranges over  $(1 + \infty)$  in dimension  $N$ .

#### 5.4. Branched transport approximation: $H$ -masses of normal 1-currents.

Branched Transport is a variant of classical optimal transport (see [San15] and Section 4.4.2 therein for a brief presentation of branched transport, and [BCM09] for a vast exposition) where the transport energy concentrates on a network, i.e. a 1-dimensional subset of  $\mathbb{R}^d$ , which has a graph structure when optimized with prescribed source and target measures. It can be formulated as a minimal flow problem,

$$\min \left\{ \mathbf{M}_1^H(w) : \operatorname{div}(w) = \mu^- - \mu^+ \right\},$$

where  $\mu^\pm$  are probability measures on  $\mathbb{R}^d$ ,  $H : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is mass-concave, and the  $H$ -mass  $\mathbf{M}_1^H$  is this time defined for finite vector measures  $w \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$  whose distributional divergence is also a finite measure; in the language of currents, it is called a 1-dimensional normal current. Any such measure may be decomposed into a 1-rectifiable part  $\theta\xi \cdot \mathcal{H}^1 \llcorner \Sigma$  where  $\theta(x) \geq 0$  and  $\xi(x)$  is a unit tangent vector to  $\Sigma$  for  $\mathcal{H}^1$ -a.e.  $x \in \Sigma$ , and a 1-diffuse part  $w^\perp$  satisfying  $|w^\perp|(A) = 0$  for every 1-rectifiable set  $A$ :

$$w = \theta\xi \cdot \mathcal{H}^1 \llcorner M + w^\perp.$$

The  $H$ -mass is then defined by:

$$(5.6) \quad \mathbf{M}_1^H(w) := \int_{\Sigma} H(x, \theta(x)) \, d\mathcal{H}^1(x) + \int_{\mathbb{R}^d} H'(x, 0) \, d|w^\perp|.$$

In the case  $H(x, m) = m^\alpha$  with  $0 < \alpha < 1$ , a family of approximations of these functional has been introduced in [OS11]:

$$(5.7) \quad \mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} \varepsilon^{\gamma_1} |\nabla w|^2 + \varepsilon^{-\gamma_2} |w|^\beta & \text{if } w \in W_{\text{loc}}^{1,2}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

with  $\beta = \frac{2-2d+2\alpha d}{3-d+\alpha(d-1)}$ ,  $\gamma_1 = (d-1)(1-\alpha)$  and  $\gamma_2 = 3-d+\alpha(d-1)$ . It has been shown in [OS11; Mon17] that the functionals  $\mathcal{E}_\varepsilon$   $\Gamma$ -converge as  $\varepsilon \rightarrow 0^+$ , in the topology of weak convergence of  $u$  and its divergence, to a non-trivial multiple of the  $\alpha$ -mass  $\mathbf{M}_1^\alpha := \mathbf{M}_1^H$  with  $H(x, m) = m^\alpha$  in dimension  $d = 2$ . The result extends to any dimension  $d$ , by [Mon15], thanks to a slicing method that relates the energy  $\mathcal{E}_\varepsilon$  with the energy of the sliced measures  $u = (w \cdot \nu)_+$  supported on the

slices  $V_a = \{x \in \mathbb{R}^d : x \cdot \nu = a\} \simeq \mathbb{R}^N$ , for any given unit vector  $\nu \in \mathbb{R}^d$ , defined by

$$\bar{\mathcal{E}}_\varepsilon(u) = \int_{\mathbb{R}^N} \varepsilon^{\gamma_1} |\nabla u|^2 + \varepsilon^{-\gamma_2} |u|^\beta.$$

The functionals  $\bar{\mathcal{E}}_\varepsilon$   $\Gamma$ -converge as  $\varepsilon \rightarrow 0^+$ , in the narrow topology, to  $c\mathbf{M}^\alpha$  for some non-trivial  $c$ , as shown in [Section 5.3](#), and one may recover every  $\alpha$ -mass in this way for  $\alpha \in \left(\frac{2d-4}{2d+1}, 1\right]$ , and in particular every so-called super-critical exponents for Branched Transport in dimension  $d$ , that is  $\alpha \in (1 - 1/d, 1]$ .

The same slicing method would allow to extend our  $\Gamma$ -convergence result stated in [Theorem 1.2](#) to functionals defined on vector measure

$$(5.8) \quad \mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} f_\varepsilon(x, \varepsilon^{d-1}|w|(x), \varepsilon^d|\nabla w|(x))\varepsilon^{1-d} dx & \text{if } w \in W_{\text{loc}}^{1,1}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

for Lagrangians  $f_\varepsilon \rightarrow f$  fitting the framework of [Theorem 1.2](#). The expected  $\Gamma$ -limit, for the weak topology of measures and their divergence measure, would be the functional  $\mathbf{M}_1^{H_f}$ , with  $H_f$  defined in [\(1.2\)](#). Note that this approach would provide approximations of  $H$ -masses for more general continuous and concave cost functions  $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $H(0) = 0$ . By [\[Wir19\]](#), we would obtain all such  $H$ -masses when  $N = 1$  (corresponding to  $d = 2$ ).

**5.5. A Cahn-Hilliard model for droplets.** Following the works [\[BDS96\]](#) in the one-dimensional case and [\[Dub98\]](#) in higher dimension, we consider functionals on  $\mathcal{M}_+(\mathbb{R}^N)$  of the form:

$$(5.9) \quad \mathcal{W}_\varepsilon(u) = \begin{cases} \int_{\mathbb{R}^N} \varepsilon^{-\rho}(W(u) + \varepsilon|\nabla u|^2) & \text{if } u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+), \\ +\infty & \text{otherwise,} \end{cases}$$

where  $W : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a Borel function satisfying  $W(t) \sim_{t \rightarrow +\infty} u^s$  for some exponent  $s \in (-\infty, 1)$ . In [\[BDS96; Dub98\]](#), it is in particular proven, under some assumptions on the slope of  $W$  at 0 and its regularity, that the family  $(\mathcal{W}_\varepsilon)_{\varepsilon>0}$   $\Gamma$ -converges to a non-trivial multiple of the  $\alpha$ -mass,  $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$ , when  $s \in (-2, 1)$  and  $\rho = \rho(s, N) := \frac{N(1-s)}{(N+2)+N(1-s)}$ . In this section, we recover this  $\Gamma$ -convergence result using our general model.

Replacing  $\varepsilon$  with  $\bar{\varepsilon} := \varepsilon^{(N+2)+N(1-s)}$  and noticing that  $1 - \rho = \frac{N+2}{(N+2)+N(1-s)}$ , one gets for every  $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+)$ :

$$\begin{aligned} \mathcal{W}_{\bar{\varepsilon}}(u) &= \int_{\mathbb{R}^N} \varepsilon^{-N(1-s)}W(u) + \varepsilon^{N+2}|\nabla u|^2 \\ &= \int_{\mathbb{R}^N} ([\varepsilon^{Ns}W(\varepsilon^{-N}\varepsilon^N u)] + |\varepsilon^{N+1}\nabla u|^2)\varepsilon^{-N} \\ &= \int_{\mathbb{R}^N} f_\varepsilon^W(x, \varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N}, \end{aligned}$$

where  $f_\varepsilon^W$  is defined for every  $x \in \mathbb{R}^N$ ,  $u \in \mathbb{R}_+$ ,  $\xi \in \mathbb{R}^N$  by

$$f_\varepsilon^W(x, u, \xi) := W_\varepsilon(u) + |\xi|^2 \quad \text{and} \quad W_\varepsilon(u) := \varepsilon^{Ns}W(\varepsilon^{-N}u).$$

Therefore if we take  $f_\varepsilon = f_\varepsilon^W$  in our general model [\(1.3\)](#) we exactly get  $\mathcal{W}_{\bar{\varepsilon}} = \mathcal{E}_{\bar{\varepsilon}}$ . The fact that  $W(u) \sim u^s$  as  $u \rightarrow +\infty$  implies that  $W_\varepsilon$  converges pointwise to the

map  $w_s : u \mapsto u^s$  if  $u > 0$ ,  $w_s(0) = 0$ , hence  $f_\varepsilon^W$  converges to  $f_s : (x, u, \xi) \mapsto w_s(u) + |\xi|^2$ .

**Theorem 5.1.** *Assume that  $W : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfies:*

- (HW<sub>1</sub>)  $W$  is lower semicontinuous,
- (HW<sub>2</sub>)  $\{W = 0\} = \{0\}$ ,
- (HW<sub>3</sub>)  $W(u) \sim_{u \rightarrow +\infty} u^s$  for some  $s \in (-\infty, 1)$ ,
- (HW<sub>4</sub>)  $\sup_{u>0} \frac{W(u)}{u^s} < +\infty$ ,
- (HW<sub>5</sub>)  $0 < \liminf_{u \rightarrow 0^+} \frac{W(u)}{u}$ .

Then  $(W_\varepsilon)_{\varepsilon>0}$   $\Gamma$ -converges to  $\mathbf{M}^{H_{f_s}}$ , for both topologies  $\mathcal{C}'_0$  and  $\mathcal{C}'_b$ , and if  $s \in (-2, 1]$  then  $\mathbf{M}^{H_{f_s}}$  is a non-trivial multiple of  $\mathbf{M}^\alpha$  where  $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$ .

To prove this theorem, we start with a simple lemma.

**Lemma 5.2.** *Assume that  $W$  satisfies (HW<sub>1</sub>)–(HW<sub>5</sub>). Then for every  $\delta \in (0, 1)$ , there exists  $c_\delta \in (0, +\infty)$  such that for every  $\varepsilon > 0$  and every  $u \in \mathbb{R}_+$ ,*

$$(5.10) \quad \delta(u^s \wedge c_\delta \varepsilon^{-N(1-s)}u) \leq W_\varepsilon(u).$$

*Proof.* Fix  $\delta \in (0, 1)$ . There exists  $M > 0$  such that  $\delta u^s \leq W(u)$  for every  $u \geq M$ . Besides, the map  $w : u \mapsto W(u)/u$  is lower semicontinuous and positive on  $(0, M]$  by (HW<sub>1</sub>) and (HW<sub>2</sub>), and since  $\liminf_{u \rightarrow 0} w(u) > 0$  by (HW<sub>5</sub>),  $w$  is necessarily bounded from below on  $(0, M]$  by some constant  $c > 0$ . As a consequence  $W_\varepsilon(u) \geq \delta u^s$  if  $u \geq \varepsilon^N M$  and  $W_\varepsilon(u) \geq c \varepsilon^{N(s-1)}u$  if  $u \leq \varepsilon^N M$ , hence:

$$\forall u \in \mathbb{R}, \quad W_\varepsilon(u) \geq \delta(u^s \wedge c \varepsilon^{-N(1-s)}u). \quad \square$$

*Proof of Theorem 5.1.* By (HW<sub>4</sub>), there exists a constant  $C$  such that  $f_\varepsilon^W \leq C f_s$  for every  $\varepsilon$ , and since  $f_\varepsilon^W$  does not depend on the  $x$  variable and converges pointwise to  $f_s$ , (U) is satisfied and our  $\Gamma$ –lim sup result stated in Proposition 4.2 yields

$$\mathbf{M}^{H_{f_s}} \geq \Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon.$$

Fix  $\delta \in (0, 1)$ . By Lemma 5.2, there exists  $c_\delta$  such that

$$\forall x, u, \xi, \quad f_\varepsilon^W(x, u, \xi) \geq \delta(|\xi|^2 + (u^s \wedge c_\delta \varepsilon^{-N(1-s)}u)) =: f_\varepsilon^\delta(x, u, \xi).$$

It is easy to check that  $f_\varepsilon^\delta$  satisfies (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>5</sub>) for every  $\varepsilon > 0$ . Moreover  $f_\varepsilon^\delta \uparrow \delta f_s$  and  $(f_\varepsilon^\delta)'_-(\cdot, 0^+, 0) = \delta c_\delta \varepsilon^{-N(1-s)} \uparrow (+\infty) = (\delta f_s)'_-(\cdot, 0^+, 0)$  as  $\varepsilon \rightarrow 0$ , thus (H<sub>8</sub>) holds for the family  $(f_\varepsilon^\delta)_{\varepsilon>0}$ , and by applying our  $\Gamma$ –lim inf result stated in Proposition 4.1 to the energies  $\mathcal{E}_\varepsilon^\delta$  induced by  $f_\varepsilon^\delta$  we get:

$$\Gamma(\mathcal{C}'_0) - \liminf \mathcal{E}_\varepsilon \geq \Gamma(\mathcal{C}'_0) - \liminf \mathcal{E}_\varepsilon^\delta \geq \mathbf{M}^{H_{\delta f_s}}.$$

We get the result by taking the limit  $\delta \rightarrow 1$ , noticing that  $(f_s)'_-(\cdot, 0^+, 0) = +\infty$ , so that  $H_{\delta f_s}^- = H_{\delta f_s} = \delta H_{f_s}$  and  $\mathbf{M}^{H_{\delta f_s}^-} = \mathbf{M}^{\delta H_{f_s}} = \delta \mathbf{M}^{H_{f_s}}$ .  $\square$

*Remark 5.3.* We recover the  $\Gamma$ -convergence results of [BDS96] and [Dub98] when  $s \in (-2, 1)$  under slightly more general assumptions: besides (HW<sub>2</sub>) and (HW<sub>3</sub>), the authors impose the existence of a non-trivial slope  $\lim_{u \rightarrow 0} \frac{W(u)}{u} \in (0, +\infty)$  and a regularity condition (either  $W$  is of class  $\mathcal{C}^1$  or continuous and non-decreasing close to 0), which are stronger than (HW<sub>1</sub>), (HW<sub>4</sub>) and (HW<sub>5</sub>). Let us stress

however that these works also tackle the cases  $s < -2$  in any dimension, where the exponent  $\rho$  has to be fixed to  $\rho(-2, N)$ , and the case  $s = -2$  in dimension one, where a logarithmic factor must be introduced, replacing  $\varepsilon^{-\rho}$  with  $\varepsilon^{-\rho(-2,1)}|\log \varepsilon|^{-1} = \varepsilon^{-1/2}|\log \varepsilon|^{-1}$ . This implies that in our model we get a trivial  $\Gamma$ -limit when  $s \leq -2$ , namely  $H_{f_s} \equiv +\infty$  on  $(0, +\infty)$ .

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