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MASS CONCENTRATION IN RESCALED FIRST ORDER INTEGRAL FUNCTIONALS

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ABSTRACT. We consider first order local minimization problems of the form $\min \int_{\mathbb{R}^N} f(u, \nabla u)$ under a mass constraint $\int_{\mathbb{R}^N} u = m \in \mathbb{R}$. We prove that the minimal energy function $H(m)$ is always concave on $(-\infty, 0)$ and $(0, +\infty)$, and that relevant rescalings of the energy, depending on a small parameter ε , Γ -converge in the weak topology of measures towards the H -mass, defined for atomic measures $\sum_i m_i \delta_{x_i}$ as $\sum_i H(m_i)$. We also consider space dependent Lagrangians $f(x, u, \nabla u)$, which cover the case of space dependent H -masses $\sum_i H(x_i, m_i)$, and also the case of a family of Lagrangians $(f_\varepsilon)_\varepsilon$ converging as $\varepsilon \rightarrow 0$. The Γ -convergence result holds under mild assumptions on f , and covers several situations including homogeneous H -masses in any dimension $N \geq 2$ for exponents above a critical threshold, and all concave H -masses in dimension $N = 1$. Our result yields in particular the concentration of Cahn-Hilliard fluids into droplets, and is related to the approximation of branched transport by elliptic energies.

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NOTATION

$B_r(x)$	open ball of radius r centered at x ;
B_r	open ball $B_r(0)$;
$\mathcal{M}(\mathbb{R}^N)$	set of finite signed Borel measures on \mathbb{R}^N ;
$\mathcal{M}_+(\mathbb{R}^N)$	set of finite positive Borel measures on \mathbb{R}^N ;
$\tau_x \mu$	Borel measure $A \mapsto \mu(A - x)$ if $\mu \in \mathcal{M}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$;
$c_B \mu$	Borel measure $\tau_{-x}(\mu \llcorner B)$ if B is the ball $B_r(x)$;
$\mu_\ell \xrightarrow{\mathcal{C}'_0} \mu$	weak convergence of measures, i.e. weak- \star convergence in duality with the space $\mathcal{C}'_0(\mathbb{R}^N)$ of continuous functions vanishing at infinity;
$\mu_\ell \xrightarrow{\mathcal{C}'_b} \mu$	narrow convergence of measures, i.e. weak- \star convergence in duality with the space of continuous and bounded function $\mathcal{C}'_b(\mathbb{R}^N)$;
Σ	set of increasing maps $\sigma : \mathbb{N} \rightarrow \mathbb{N}$;
$\sigma_1 \preceq \sigma_2$	$\sigma_1, \sigma_2 \in \Sigma$ are such that $\sigma_1(\llbracket n, +\infty \rrbracket) \subseteq \sigma_2(\mathbb{N})$ for some $n \in \mathbb{N}$;
\pm	fixed to either $+$ or $-$ in the whole statement or proof, and $\mp = -(\pm)$.

1. INTRODUCTION

1.1. **Setting.** Let $N \in \mathbb{N}^*$ and let $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ be a Borel function. Consider the following energy functional, defined for any fixed $x \in \mathbb{R}^N$ on the set of finite Borel measures $\mathcal{M}(\mathbb{R}^N)$ on \mathbb{R}^N by

$$(1.1) \quad \mathcal{E}_f^x(u) = \begin{cases} \int_{\mathbb{R}^N} f(x, u(y), \nabla u(y)) \, dy & \text{if } u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N), \\ +\infty & \text{otherwise.} \end{cases}$$

The minimization of this energy energy under a mass constraint gives rise to the notion of minimal cost function, defined by

$$(1.2) \quad H_f(x, m) := \inf \left\{ \mathcal{E}_f^x(u) : u \in W_{\text{loc}}^{1,1} \cap L^1(\mathbb{R}^N) \text{ such that } \int_{\mathbb{R}^N} u = m \right\} \in [0, +\infty].$$

Note that the minimization can be restricted to non-negative functions when $m \geq 0$ and non-positive functions when $m \leq 0$, as explained in [Remark 2.2](#).

As a preliminary result, which deserves interest on its own, we will establish the following:

Theorem 1.1. *Let $x \in \mathbb{R}^N$. The map $m \mapsto H_f(x, m)$ is concave non-decreasing on $(0, +\infty)$. If we further assume that $f(x, 0, 0) = 0$ then $H_f(x, 0) = 0$ and $H_f(x, \cdot)$ is either identically $+\infty$ on $(0, +\infty)$ or continuous on $[0, +\infty)$; it is in particular lower semicontinuous on \mathbb{R}_+ . A similar statement holds on $(-\infty, 0]$.*

The proof is very simple and works with no further assumptions on f , as a consequence of [Theorem 2.1](#) and the following [Remark 2.2](#).

Our main purpose is to prove that under some conditions, if $(f_\varepsilon)_{\varepsilon>0}$ is a family of functions $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ converging pointwise to f as $\varepsilon \rightarrow 0$, then the rescaled energy functionals \mathcal{E}_ε , defined for each $\varepsilon > 0$ on $\mathcal{M}(\mathbb{R}^N)$ by

$$(1.3) \quad \mathcal{E}_\varepsilon(u) = \begin{cases} \int_{\mathbb{R}^N} f_\varepsilon(x, \varepsilon^N u(x), \varepsilon^{N+1} \nabla u(x)) \varepsilon^{-N} dx & \text{if } u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases}$$

Γ -converge as $\varepsilon \rightarrow 0$, for the narrow or weak convergence of measures, to the H_f -mass, defined on $\mathcal{M}(\mathbb{R}^N)$ by (see [Definition 2.3](#)):

$$\begin{aligned} \mathbf{M}^{H_f}(u) &:= \int_{\mathbb{R}^N} H_f(x, u(\{x\})) d\mathcal{H}^0(x) \\ &\quad + \int_{\mathbb{R}^N} H'_f(x, 0^+) du_+^d + \int_{\mathbb{R}^N} H'_f(x, 0^-) du_-^d(x). \end{aligned}$$

where $u = u^a + u^d$ is the decomposition of u into its atomic part u^a and its diffuse part u^d , $u^d = u_+^d - u_-^d$ is the Jordan decomposition of u^d into positive and negative parts, and $H'_f(x, 0^\pm) = \lim_{m \rightarrow 0^\pm} \frac{H_f(x, m)}{|m|} \in [0, +\infty]$.

This kind of singular limit of integral functionals is reminiscent of several variational models with physical relevance which have been the object of intensive mathematical analysis, such as Cahn-Hilliard fluids with concentration on droplets [\[BDS96\]](#) or on singular interfaces [\[MM77\]](#), toy models for micromagnetism and liquid crystals like Aviles-Giga [\[AG99\]](#) and Landau-de Gennes [\[BPP12\]](#), or Ginzburg-Landau theory of superconductivity [\[BBH17\]](#).

The fact that \mathbf{M}^{H_f} is expected to be the Γ -limit of \mathcal{E}_ε is due to the following observation: if $B_r(x_0) \subseteq \mathbb{R}^N$ and $u_\varepsilon(x) =: \varepsilon^{-N} v_\varepsilon(\varepsilon^{-1}(x - x_0))$, then $\int_{B_r(x_0)} u_\varepsilon = \int_{B_{r/\varepsilon}} v_\varepsilon$ and

$$\int_{B_r(x_0)} f_\varepsilon(x, \varepsilon^N u_\varepsilon(x), \varepsilon^{N+1} \nabla u_\varepsilon(x)) \varepsilon^{-N} dx = \int_{B_{r/\varepsilon}} f_\varepsilon(x_0 + \varepsilon y, v_\varepsilon(y), \nabla v_\varepsilon(y)) dy,$$

so that the energy contribution of a mass $m \geq 0$ contained in a ball $B_r(x_0)$ should be of the order of $H_f(x_0, m)$.

Nevertheless, it is not true in general that \mathbf{M}^{H_f} is the Γ -limit of the functionals \mathcal{E}_ε (see [Section 1.3](#) below). We will need a couple of assumptions on f and f_ε detailed in the next section.

1.2. Assumptions and main result. Our first two assumptions are rather standard and guarantee the sequential lower semicontinuity of the functionals \mathcal{E}_f^x ,

(H₁) $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ is lower semicontinuous,

(H₂) $f(x, u, \cdot)$ is convex for every $x \in \mathbb{R}^N$, $u \in \mathbb{R}$.

In order for vanishing parts to have no energetic contribution, we will impose

(H₃) $f(x, 0, 0) = 0$ for every $x \in \mathbb{R}^N$.

We also need continuity in the spatial variable,

(H₄) $f(\cdot, u, \xi)$ is continuous for every $u \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

Next, we need a compactness assumption which ensures relative compactness in the weak topology of $W_{\text{loc}}^{1,p}(\mathbb{R}^N)$ for sequences of bounded energy \mathcal{E}_f^x and bounded mass; it will also be needed in obtaining lower bounds for the energy (see [Proposition 3.8](#)):

(H₅) there exist $\alpha, \beta \in (0, +\infty)$, $p \in (1, +\infty)$ such that for all $(x, u, \xi) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$,

$$f(x, u, \xi) \geq \alpha|\xi|^p - \beta u.$$

We also impose a condition on the slope of $f(x, \cdot, \xi)$ at the origin which will be needed in order to identify the initial slope of $H_f(x, \cdot)$ (see [Section 2.3](#)), and rules out some non-trivial scale invariant Lagrangians for which the expected Γ -convergence result fails (see [Section 1.3](#)),

(H₆) for every $x_0 \in \mathbb{R}^N$,

$$(1.4) \quad f'_-(x_0, 0^\pm, 0) := \liminf_{(x, u, \xi) \rightarrow (x_0, 0^\pm, 0)} \frac{f(x, u, \xi)}{|u|} \geq \limsup_{u \rightarrow 0^\pm} \sup_{|\xi|=1} \frac{f(x_0, u, \rho(|u|)\xi)}{|u|},$$

with $\rho \equiv 0$ if $N = 1$ and for some $\rho \in \mathcal{C}((0, 1], (0, +\infty))$ satisfying

$$\int_0^1 \left(\int_y^1 \frac{dt}{\rho(t)} \right)^N dy < +\infty \quad \text{if } N \geq 2.$$

Since our aim is not to care much about the dependence on x , we shall impose a spatial quasi-homogeneity condition:

(H₇) there exists $C < +\infty$ such that for every $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$f(y, u, \xi) \leq C(f(x, u, \xi) + u).$$

Last of all, we need the family of functions $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ to converge towards f in a suitable sense, namely, we assume

(H₈) $f_\varepsilon \uparrow f$ and $f'_{\varepsilon,-}(\cdot, 0^\pm, 0) \uparrow f'_-(\cdot, 0^\pm, 0)$ as $\varepsilon \rightarrow 0$.

Notice that this assumption is empty if f_ε does not depend on ε .

Our main result is the following:

Theorem 1.2. *If $(f_\varepsilon)_{\varepsilon>0}$ satisfies (H₈) with each f_ε satisfying (H₁)–(H₅) and the limit f satisfying (H₆)–(H₇), then \mathbf{M}^{H_f} is the Γ -limit as $\varepsilon \rightarrow 0$ of the functionals \mathcal{E}_ε , defined in (1.3), for both the weak convergence and the narrow convergence of measures.*

In particular, as a Γ -limit, the functional \mathbf{M}^{H_f} must be lower semicontinuous for the weak convergence of measures (and so for the narrow convergence as well). This implies that H_f is lower semicontinuous on $\mathbb{R}^N \times \mathbb{R}$ (see [Proposition 2.5](#)).

We point out that for the Γ -lim sup, we need weaker assumptions on f_ε and f (see [Proposition 4.2](#)), which will be useful for some applications (see [Section 5.5](#)).

We will allow ourselves slight abuses of notation. We will sometimes consider Lagrangians defined on $\mathbb{R} \times \mathbb{R}^N$ which do not depend on x and still refer to hypotheses [\(H₁\)](#)–[\(H₈\)](#); we will use the notation \mathcal{E}_f instead of \mathcal{E}_f^x in [\(1.1\)](#) and consider H_f as a function of m only in [\(1.2\)](#). Besides, we will also consider Lagrangians defined only for $u \in \mathbb{R}_+$, which may be thought as defined for $u \in \mathbb{R}$, set to $+\infty$ when u is negative¹. The resulting minimal cost function H_f and its associated H_f -mass may be thought as defined on \mathbb{R}_+ and $\mathcal{M}_+(\mathbb{R}^N)$ respectively, as they will be infinite on the respective complements.

1.3. Remarks and applications. We start with two situations where the expected Γ -convergence fails and which justify the importance of [\(H₃\)](#) and [\(H₆\)](#), then we provide examples and applications of our result, as a short summary of [Section 5](#), where full details are provided. We restrict our attention to positive measures and Lagrangians defined for $u \geq 0$.

Lagrangians giving energy to vanishing parts. By assumption [\(H₃\)](#) no energy is given to any set where a function u vanishes. It is a necessary condition for \mathbf{M}^{H_f} to be lower semicontinuous (a necessary condition to be a Γ -limit) and not identically $+\infty$. Indeed if \mathbf{M}^{H_f} is lower semicontinuous and finite for some measure $u \in \mathcal{M}(\mathbb{R}^N)$ then thanks to [Theorem 2.1](#) we know that $\mathbf{M}^{H_f}(tu) \leq \mathbf{M}^{H_f}(u)$ for every $t \in (0, 1)$ and thus $\mathbf{M}^{H_f}(0) \leq \liminf_{t \rightarrow 0^+} \mathbf{M}^{H_f}(tu) \leq \mathbf{M}^{H_f}(u) < +\infty$. Thus \mathbf{M}^{H_f} is not identically $+\infty$ if and only if $\mathbf{M}^{H_f}(0) < +\infty$, i.e. $\int_{\mathbb{R}^N} H_f(x, 0) d\mathcal{H}^0(x) < +\infty$. But since $H_f(x, 0) = (+\infty) \times f(x, 0, 0)$ this can only happen if $f(\cdot, 0, 0) \equiv 0$. This justifies imposing [\(H₃\)](#).

Scale invariant Lagrangians. In the particular case where $f(x, u, \xi) = u^{-p(1-\frac{1}{p^*})} |\xi|^p$ and $f_\varepsilon \equiv f$, with $p \in (1, N)$ and $p^* = \frac{pN}{N-p}$, we find that

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(x, \varepsilon^N u, \varepsilon^{N+1} \nabla u) \varepsilon^{-N} = \int_{\mathbb{R}^N} u^{-p(1-\frac{1}{p^*})} |\nabla u|^p = \mathcal{E}_f(u),$$

i.e. the rescaled energies \mathcal{E}_ε do not depend on $\varepsilon > 0$. A scaling analysis also shows that the associated minimal cost function satisfies $H_f(m) = m^{1-\frac{N}{p^*}} H_f(1)$. Moreover, it can be seen (see [Section 5.1](#)) that $0 < H_f(1) < +\infty$, which implies that the Γ -limit of \mathcal{E}_ε , which is nothing but the lower semicontinuous relaxation of \mathcal{E}_f , does not coincide with \mathbf{M}_{H_f} . Considering the perturbation of f given by $\tilde{f}(x, u, \xi) = f(x, u, \xi) + |\xi|^p$, we find a Lagrangian satisfying all our assumptions except [\(H₆\)](#) (note that $|\xi|^p$ is needed in [\(H₅\)](#)), and such that the associated rescaled energies do not Γ -converge to \mathbf{M}_{H_f} . Hence, an assumption like [\(H₆\)](#) is required in our Γ -convergence result. We will even see that the lower semicontinuity of H_f and \mathbf{M}_{H_f} is not guaranteed without [\(H₆\)](#).

¹Notice that if any of our assumptions is satisfied for a Lagrangian defined for $u \in \mathbb{R}_+$, then it holds also for the Lagrangian extended to \mathbb{R} in this way.

Concave H -masses in dimension one. Consider the energy

$$\mathcal{E}_f(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + c(u) \quad \text{with Lagrangian} \quad f(x, u, \xi) = |\xi|^2 + c(u).$$

In dimension $N = 1$, it is shown in [Wir19] that for any concave continuous function H with $H(0) = 0$, there exists a suitable $c \geq 0$ such that $H_f = H$. As explained in Section 5.2, Theorem 1.2 implies that the rescaled energies

$$(1.5) \quad \mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(\varepsilon^N u, \varepsilon^{N+1} \nabla u) \varepsilon^{-N} = \int_{\mathbb{R}^N} \varepsilon^{N+2} |\nabla u|^2 + \varepsilon^{-N} c(\varepsilon^N u).$$

Γ -converge to \mathbf{M}^H , leading to an elliptic approximation of any concave H -mass in dimension one. In dimension $N \geq 2$, we will show that H_f must be concave on $(0, +\infty)$, and strictly concave after the possible initial interval where it is linear (see Proposition 2.8); however, we have no solution to the inverse problem, consisting in characterizing the class of attainable minimal cost functions $H = H_f$ for Lagrangians f satisfying our assumptions.

Homogeneous H -masses in any dimension. We consider the functional

$$(1.6) \quad \mathcal{E}_f(u) = \int_{\mathbb{R}^N} f(u, \nabla u) = \int_{\mathbb{R}^N} |\nabla u|^p + u^s, \quad p > 1, \quad s \in (-p', 1].$$

Then, the rescaled energies

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(\varepsilon^N u, \varepsilon^{N+1} \nabla u) \varepsilon^{-N} = \int_{\mathbb{R}^N} \varepsilon^{pN+p-N} |\nabla u|^p + \varepsilon^{-(1-s)N} u^s$$

Γ -converge to a non-trivial multiple of some α -mass $\mathbf{M}^\alpha := \mathbf{M}^{t \rightarrow t^\alpha}$ where the exponent $\alpha = (1 - \frac{s}{p} + \frac{s}{N})(1 - \frac{s}{p} + \frac{1}{N})^{-1}$ ranges over $(1 - \frac{2}{N+1}, 1]$ when (s, p) varies in its range and $N \geq 1$. More details are given in Section 5.3.

Cahn-Hilliard approximations of droplets models. Following the works of [BDS96; Dub98], we consider the functionals

$$(1.7) \quad \mathcal{W}_\varepsilon(u) = \int_{\mathbb{R}^N} \varepsilon^{-\rho} (W(u) + \varepsilon |\nabla u|^2),$$

where $W(t) \sim_{t \rightarrow +\infty} t^s$ for some exponent $s \in (-2, 1)$. As shown in Section 5.5, we may rewrite these functionals to fit our general framework, and recover known Γ -convergence results, under slightly more general assumptions, as stated in Theorem 5.1. The Γ -limit is a non-trivial multiple of the α -mass with $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$.

Elliptic approximations of Branched Transport. The energy of Branched Transport (see [BCM09] for an account of the theory), in its Eulerian formulation, is an H -mass defined this time on vector measures w whose divergence is also a measure,

$$(1.8) \quad \mathbf{M}_1^H(w) := \int_{\Sigma} H(x, \theta(x)) \, d\mathcal{H}^1(x) + \int_{\mathbb{R}^d} H'(x, 0^+) \, d|w^\perp|,$$

where $w = \theta\xi \cdot \mathcal{H}^1 \llcorner \Sigma + w^\perp$ is the decomposition of w into its 1-rectifiable and 1-diffuse parts (see Section 5.4 for more details). An elliptic approximation of Modica-Mortola type has been introduced in [OS11] for $H(m) = m^\alpha$, $\alpha \in (0, 1)$, and their Γ -convergence result in dimension $d = 2$ has been extended to any dimension in [Mon15] by a slicing method which relates the energy of w to the energy of its

slicings. The same slicing method, together with [Theorem 1.2](#), would allow to prove the Γ -convergence of the functionals

$$(1.9) \quad \mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} f_\varepsilon(x, \varepsilon^{d-1}|w|(x), \varepsilon^d|\nabla w|(x))\varepsilon^{1-d} dx & \text{if } w \in W_{\text{loc}}^{1,1}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

toward $\mathbf{M}_1^{H_f}$ for Lagrangians $f_\varepsilon \rightarrow f$ satisfying [\(H₁\)](#)–[\(H₈\)](#), thus covering a wide range of concave H -masses over vector measures with divergence.

1.4. Structure of the paper. In [Section 2](#), we prove the concavity of the minimal cost function H_f with respect to the mass variable m in full generality ([Theorem 2.1](#)), we establish useful properties of general H -masses, and we identify the slope at the origin of H_f in terms of f under our assumption ([Proposition 2.6](#) and [Proposition 2.7](#)). In [Section 3](#), we apply a concentration-compactness principle to provide a profile decomposition theorem for sequences of positive measures ([Theorem 3.2](#)), which is used to obtain our main lower bound for the energy \mathcal{E}_f ([Proposition 3.10](#)) and also yields an existence criterion for profiles with minimal energy under a mass constraint ([Proposition 3.12](#)). [Section 4](#) is dedicated to proving lower and upper bounds on the rescaled energies \mathcal{E}_ε ([Proposition 4.1](#) and [Proposition 4.2](#)) that imply in particular our main Γ -convergence result ([Theorem 1.2](#)). Last of all, in [Section 5](#), we provide counterexamples and several examples of energy functionals that fall into our framework, as summarized in the previous section.

2. MINIMAL COST FUNCTION AND H -MASS

In this section, we study the properties of general H -masses, of costs H_f associated with general Lagrangians f , and we relate the slope of H_f at $m = 0$ to that of f at $(u, \xi) = (0, 0)$ in the variable u , under particular conditions.

2.1. Concavity and lower semicontinuity of the minimal cost function.

Our concavity result stated in [Theorem 1.1](#) is a consequence of:

Theorem 2.1. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$ be Borel measurable. The function*

$$(2.1) \quad H_f(m) = \inf \left\{ \int_{\mathbb{R}^N} f(u, \nabla u) : u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+), \int_{\mathbb{R}^N} u = m \right\}, \quad m \geq 0,$$

is either identically $+\infty$ on $(0, +\infty)$, or it is everywhere finite, continuous, concave and non-decreasing on $(0, +\infty)$. In the latter case, if moreover $f(0, 0) = 0$ then $\lim_{m \rightarrow 0^+} H_f(m) = H_f(0) = 0$.

The fact that [Theorem 2.1](#) implies [Theorem 1.1](#) is due to the following remark.

Remark 2.2. The minimization in [\(1.2\)](#) can be restricted to non-negative (resp. non-positive) functions when $m \geq 0$ (resp. $m \leq 0$). For instance fix $x \in \mathbb{R}^N$, $m \geq 0$, and take $u \in W_{\text{loc}}^{1,1} \cap L^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u = m$. If $u = u_+ - u_-$ is the decomposition of $u \in W_{\text{loc}}^{1,1} \cap L^1(\mathbb{R}^N)$ into positive and negative parts, then $\mathcal{E}_f^x(u) = \mathcal{E}_g(u_+) + \mathcal{E}_{\tilde{g}}(u_-)$, with g, \tilde{g} defined on $\mathbb{R}_+ \times \mathbb{R}^N$ by $g(u, \xi) = f(x, u, \xi)$ for every (u, ξ) and $\tilde{g}(u, \xi) = f(x, -u, -\xi)$ for $(u, \xi) \neq (0, 0)$, $\tilde{g}(0, 0) = 0$. In particular $\mathcal{E}_f^x(u) \geq H_g(m_+) + H_{\tilde{g}}(m_-)$ where $m_\pm = \int_{\mathbb{R}^N} u_\pm$. Since H_g is non-decreasing on $(0, +\infty)$ and $H_{\tilde{g}}$ is non-negative, we therefore have $\mathcal{E}_f^x(u) \geq H_g(m)$ since $m_+ \geq m$. A similar reasoning holds for non-positive functions when $m \leq 0$.

Note that, considering Lagrangians f taking infinite values, [Theorem 2.1](#) covers the case where we have a constraint $(u, \nabla u) \in A$, where $A \subseteq \mathbb{R} \times \mathbb{R}^N$ is Borel measurable.

Proof. We first prove that H_f is concave on $(0, +\infty)$. Let $m > 0$ and $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+)$ such that $\int_{\mathbb{R}^N} u = m$. We pick a non-zero vector $v \in \mathbb{R}^N$ and for every $t \in \mathbb{R}$, we set $u^t(\cdot) = u(\cdot + tv)$ and

$$u \wedge u^t(\cdot) = \min\{u(\cdot), u^t(\cdot)\}, \quad u \vee u^t(\cdot) = \max\{u(\cdot), u^t(\cdot)\}.$$

We have $u \wedge u^t + u \vee u^t = u + u^t$. Hence

$$(2.2) \quad \int_{\mathbb{R}^N} u \wedge u^t + \int_{\mathbb{R}^N} u \vee u^t = 2 \int_{\mathbb{R}^N} u = 2m.$$

Moreover, it is standard that $u \wedge u^t = u - (u^t - u)_- \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ with $\nabla(u \wedge u^t) = \nabla u$ a.e. in $\{u \leq u^t\}$ and $\nabla(u \wedge u^t) = \nabla u^t$ a.e. in $\{u > u^t\}$. Since $u \vee u^t = u + u^t - u \wedge u^t$, we have similar identities for $u \vee u^t$, and we obtain

$$(2.3) \quad \mathcal{E}_f(u \wedge u^t) + \mathcal{E}_f(u \vee u^t) = \mathcal{E}_f(u) + \mathcal{E}_f(u^t) = 2\mathcal{E}_f(u).$$

Now, let $M : t \mapsto \int_{\mathbb{R}^N} u \wedge u^t$. In view of [\(2.2\)](#), [\(2.3\)](#), and by definition of H , we have proved

$$(2.4) \quad H_f(M(t)) + H_f(2m - M(t)) \leq 2\mathcal{E}_f(u).$$

Now, by continuity of translations in L^1 and since the map $(x, y) \mapsto x \wedge y$ is Lipschitz on \mathbb{R}^2 , we have that M is continuous on \mathbb{R} with $M(0) = m$. Moreover $\lim_{t \rightarrow +\infty} M(t) = 0$ by dominated convergence. So, by the intermediate value theorem $M(\mathbb{R}) \supseteq (0, m]$. Hence, we have proved $H_f(\theta) + H_f(2m - \theta) \leq 2\mathcal{E}_f(u)$ for every $\theta \in (0, m]$. Taking the infimum over u such that $\int_{\mathbb{R}^N} u = m$, we obtain

$$\frac{H_f(\theta) + H_f(2m - \theta)}{2} \leq H_f(m), \quad \forall \theta \in (0, m],$$

that is, H_f is midpoint concave on $(0, +\infty)$. Since H_f is also bounded below (by 0), we can deduce that H_f is concave $(0, +\infty)$ (see [\[RV73, Section 72\]](#)), and since $H_f \geq 0$, either H_f is identically $+\infty$ on $(0, +\infty)$, or it is finite everywhere, continuous, concave and non-decreasing on $(0, +\infty)$.

We now justify that if $H_f(m) < +\infty$ for some $m > 0$ and $f(0, 0) = 0$, then $\lim_{m \rightarrow 0^+} H_f(m) = 0 = H_f(0)$. Let $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+)$ such that $\int_{\mathbb{R}^N} u = m > 0$ and $\mathcal{E}_f(u) < +\infty$, and set

$$t_* := \sup\{t \geq 0 : M(t) > 0\} \in [0, +\infty], \quad \text{where} \quad M(t) = \int_{\mathbb{R}^N} u \wedge u^t.$$

Since M is continuous with $M(0) = \int_{\mathbb{R}^N} u > 0$ and $\lim_{t \rightarrow +\infty} M(t) = 0$ as seen above, we have that $t_* \in (0, +\infty]$, $\lim_{t \rightarrow t_*} M(t) = 0$ and $M(t)$ does not vanish identically near t_* . Moreover, if $t_* = +\infty$, since $u^t \rightarrow 0$ locally in measure, by dominated convergence,

$$\begin{aligned} \limsup_{m \rightarrow 0^+} H_f(m) &\leq \limsup_{t \rightarrow +\infty} \mathcal{E}_f(u \wedge u^t) = \limsup_{t \rightarrow +\infty} \int_{\{u < u^t\}} f(u, \nabla u) + \int_{\{u^{-t} \geq u\}} f(u, \nabla u) \\ &\leq 2f(0, 0)|\{u = 0\}| = 0. \end{aligned}$$

If $t_* < +\infty$, we have $u \wedge u^{t_*} = 0$ a.e. and $u^t \rightarrow u^{t_*}$ locally in measure as $t \rightarrow t_*$ by continuity of translation in L^1 . Thus using dominated convergence again,

$$\begin{aligned} \limsup_{m \rightarrow 0^+} H_f(m) &\leq \limsup_{t \rightarrow (t_*)^-} \mathcal{E}_f(u \wedge u^t) = \limsup_{t \rightarrow (t_*)^-} \int_{\{u < u^t\}} f(u, \nabla u) + \int_{\{u^{-t} \geq u\}} f(u, \nabla u) \\ &= \int_{\{u < u^{t_*}\}} f(u, \nabla u) + \int_{\{u^{-t_*} \geq u\}} f(u, \nabla u) \\ &= \mathcal{E}_f(u \wedge u^{t_*}) = f(0, 0) \times (+\infty) = 0. \quad \square \end{aligned}$$

2.2. H -transform and H -mass.

Definition 2.3. Let $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$ be a Borel measurable function having left/right slopes at the origin defined for each $x \in \mathbb{R}^N$ by

$$(2.5) \quad H'(x, 0^\pm) := \lim_{m \rightarrow 0^\pm} \frac{H(x, m)}{|m|} \in [0, +\infty].$$

We define the H -transform of a finite signed Borel measure $u \in \mathcal{M}(\mathbb{R}^N)$ by:

$$H(u) := H(\cdot, u(\{\cdot\})) \mathcal{H}^0 + H'(\cdot, 0^+) u_+^d + H'(\cdot, 0^-) u_-^d,$$

where $u = u^a + u^d$ is the decomposition of u into its atomic part u^a and its diffuse (or non-atomic) part u^d , $u^d = u_+^d - u_-^d$ is the Jordan decomposition of u^d into positive and negative parts.

The H -mass of u is then defined as the total variation of $H(u)$, that is:

$$\begin{aligned} \mathbf{M}^H(u) &:= \int_{\mathbb{R}^N} H(x, u(\{x\})) \, d\mathcal{H}^0(x) \\ &\quad + \int_{\mathbb{R}^N} H'(x, 0^+) \, du_+^d(x) + \int_{\mathbb{R}^N} H'(x, 0^-) \, du_-^d(x). \end{aligned}$$

$\mathbf{M}^H(u)$ is a natural spatially non-homogeneous extension (depending on the position x) of the H -mass of k -dimensional flat currents² from Geometric Measure Theory, introduced by [Fle66] (see also the more recent works [DH03; Col+17]).

We say that $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$ is *mass-concave* if $m \mapsto H(x, m)$ is concave on $(0, +\infty)$ and $(-\infty, 0)$ for every $x \in \mathbb{R}^N$. From [BB90], we have the following result³:

Proposition 2.4 ([BB90, Theorem 3.3]). *Assume that $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$ is lower semicontinuous, mass-concave and $H(\cdot, 0) \equiv 0$. Then \mathbf{M}^H is sequentially lower semicontinuous on $\mathcal{M}(\mathbb{R}^N)$ for the weak topology.*

From another work from the same authors [BB93, Theorem 3.2], we know that under some further assumptions on H , \mathbf{M}^H is the relaxation for the weak topology of the functional

$$\mathbf{M}_{\text{atom}}^H(u) = \begin{cases} \sum_{i=1}^k H(x_i, m_i) & \text{if } u = \sum_{i=1}^k m_i \delta_{x_i} \text{ with } k \in \mathbb{N}^*, x_i \in \mathbb{R}^N, m_i \in \mathbb{R}, \\ +\infty & \text{otherwise.} \end{cases}$$

²In the case $k = 0$, since signed measures are merely 0-currents with finite mass.

³In the notation of this paper, we take $\mu = 0$ and $f(x, s) = |s|^2$; we have $\varphi_f(x, 0) = 0$ and $\varphi_f(x, s) = +\infty$ if $s \neq 0$.

We need a slightly different result⁴, namely that for any function $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$ satisfying all the assumptions of [Proposition 2.4](#) except the lower semicontinuity, the relaxation of $\mathbf{M}_{\text{atom}}^H$ for the *narrow* sequential convergence is $\mathbf{M}^{H_{\text{lsc}}}$, where H_{lsc} is the lower semicontinuous envelope of H , which can be expressed as

$$(2.6) \quad H_{\text{lsc}}(x, m) = \sup\{G(x, m) : G \leq H \text{ with } G \text{ lower semicontinuous}\}.$$

It is worth noticing that if $H(\cdot, 0) \equiv 0$ and H is mass-concave, then these properties hold also for H_{lsc} .

Proposition 2.5. *Let $H : \mathbb{R}^N \times \mathbb{R} \rightarrow [0, +\infty]$ be a function which is mass-concave and such that $H(\cdot, 0) \equiv 0$. Then, the sequentially lower semicontinuous envelope of $\mathbf{M}_{\text{atom}}^H$ in the narrow topology of $\mathcal{M}(\mathbb{R}^N)$ is given by $\mathbf{M}^{H_{\text{lsc}}}$, namely we have:*

$$(2.7) \quad \mathbf{M}^{H_{\text{lsc}}} = \sup\left\{F : F \leq \mathbf{M}_{\text{atom}}^H, F \text{ sequentially narrowly l.s.c. on } \mathcal{M}(\mathbb{R}^N)\right\}.$$

We point out that for a general H , for \mathbf{M}^H to be sequentially lower semicontinuous (for the narrow topology) it is necessary that H is lsc on $\mathbb{R}^N \times (0, +\infty)$. However, neither the subadditivity of H in m nor its lower semicontinuity on $\mathbb{R}^N \times \mathbb{R}_+$ are necessary. Indeed, \mathbf{M}^H is sequentially lower semicontinuous if for instance $H(x, m) = +\infty$ when $x \neq 0, m > 0$, $H(x, 0) = 0$ when $x \neq 0$ and $H(0, \cdot)$ is any lower semicontinuous function. Nevertheless the subadditivity in the mass m and the lower semicontinuity would be necessary if H did not depend on x .

Proof of [Proposition 2.5](#). Since H_{lsc} is lower semicontinuous and mass-concave, we know from [Proposition 2.4](#) that $\mathbf{M}^{H_{\text{lsc}}}$ is sequentially lower semicontinuous in the weak topology hence also in the narrow topology of $\mathcal{M}(\mathbb{R}^N)$. Since $\mathbf{M}^{H_{\text{lsc}}} \leq \mathbf{M}_{\text{atom}}^H$, we deduce that $\mathbf{M}^{H_{\text{lsc}}}$ is lower or equal than the sequentially lower semicontinuous envelope of $\mathbf{M}_{\text{atom}}^H$ in the narrow topology, i.e. the right hand side in [\(2.7\)](#), which we denote by $F : \mathcal{M}(\mathbb{R}^N) \rightarrow \mathbb{R}_+$. We shall see that $F \leq \mathbf{M}^{H_{\text{lsc}}}$.

We first prove that

$$(2.8) \quad F \leq \mathbf{M}_{\text{atom}}^{H_{\text{lsc}}}.$$

For this, we let $u = \sum_{i=1}^k m_i \delta_{x_i}$ be a finitely atomic positive measure and we let $u_n := \sum_{i=1}^k m_{i,n} \delta_{x_{i,n}}$ where for each $i \in \{1, \dots, k\}$, $(x_{i,n})_{n \in \mathbb{N}}$ is a sequence of points converging to x_i and $(m_{i,n})_{n \in \mathbb{N}}$ is a sequence converging to m_i such that $H_{\text{lsc}}(x_i, m_i) = \lim_{n \rightarrow \infty} H(x_{i,n}, m_{i,n})$. Then $(u_n)_{n \in \mathbb{N}}$ converges narrowly to u and, by lower semicontinuity,

$$\begin{aligned} F(u) &\leq \liminf_{n \rightarrow \infty} F(u_n) \leq \liminf_{n \rightarrow \infty} \mathbf{M}_{\text{atom}}^H(u_n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^k H(x_{i,n}, m_{i,n}) = \sum_{i=1}^k H_{\text{lsc}}(x_i, m_i), \end{aligned}$$

so that $F(u) \leq \mathbf{M}_{\text{atom}}^{H_{\text{lsc}}}(u)$ as wanted.

We now prove that $F \leq \mathbf{M}^{H_{\text{lsc}}}$. Let $u \in \mathcal{M}(\mathbb{R}^N)$ and $u = u^a + u^d$ be the decomposition of u into its atomic part $u^a = \sum_{i=1}^k m_i \delta_{x_i}$, with $k \in \mathbb{N} \cup \{+\infty\}$ (here, $k = 0$ if there is no atom), and its diffuse part u^d , and let $u^d = u^d_+ - u^d_-$ be the Jordan decomposition of u^d into positive and negative parts. We then discretize

⁴In [[BB93](#), Theorem 3.2], H is assumed to be lower semicontinuous and the authors make a further coercivity assumption (assumption (3.5) in the paper) that we want to avoid.

u_{\pm}^d by taking $n \in \mathbb{N}^*$, a partition $(Q_i^n)_{i \in \{1, \dots, (n2^n)^N\}}$ of $[-n, n]^N$ by means of cubes of the form $Q_i^n = c_i^n + 2^{-n}[-1, 1]^N$ with $c_i^n \in \mathbb{R}^N$, and we define

$$u_n := \sum_{i=1}^{n \wedge k} m_i \delta_{x_i} + \sum_{i=1}^{(n2^n)^N} u_+^d(Q_i^n) \delta_{x_i^n} - \sum_{i=1}^{(n2^n)^N} u_-^d(Q_i^n) \delta_{y_i^n},$$

where for each $i \in \{1, \dots, (n2^n)^N\}$, $x_i^n, y_i^n \in \bar{Q}_i^n$ are some points such that

$$(2.9) \quad H_{\text{lsc}}'(x_i^n, 0^+) = \inf_{x \in \bar{Q}_i^n} H_{\text{lsc}}'(x, 0^+), \quad H_{\text{lsc}}'(y_i^n, 0^-) = \inf_{x \in \bar{Q}_i^n} H_{\text{lsc}}'(x, 0^-).$$

Such points exist since \bar{Q}_i^n is compact and since by concavity,

$$(2.10) \quad H_{\text{lsc}}'(x, 0^{\pm}) = \sup_{\pm m > 0} \frac{H_{\text{lsc}}(x, m)}{|m|},$$

so that $H'_{\text{lsc}}(\cdot, 0^{\pm})$ are lower semicontinuous as suprema of lower semicontinuous functions.

The sequence $(u_n)_{n \in \mathbb{N}}$ converges narrowly to u . We deduce by lower semicontinuity of F ,

$$\begin{aligned} F(u) &\leq \liminf_{n \rightarrow \infty} F(u_n) \stackrel{(2.8)}{\leq} \liminf_{n \rightarrow \infty} \mathbf{M}_{\text{atom}}^{\text{lsc}}(u_n) \\ &= \liminf_{n \rightarrow \infty} \sum_{i=1}^{n \wedge k} H_{\text{lsc}}(x_i, m_i) + \sum_{i=1}^{(n2^n)^N} \left(H_{\text{lsc}}(x_i^n, u_+^d(Q_i^n)) + H_{\text{lsc}}(y_i^n, -u_-^d(Q_i^n)) \right), \\ &\stackrel{(2.10)}{\leq} \liminf_{n \rightarrow \infty} \sum_{i=1}^{n \wedge k} H_{\text{lsc}}(x_i, m_i) + \sum_{i=1}^{(n2^n)^N} H_{\text{lsc}}'(x_i^n, 0^+) u_+^d(Q_i^n) \\ &\quad + \sum_{i=1}^{(n2^n)^N} H_{\text{lsc}}'(y_i^n, 0^-) u_-^d(Q_i^n) \\ &\stackrel{(2.9)}{\leq} \liminf_{n \rightarrow \infty} \sum_{i=1}^{n \wedge k} H_{\text{lsc}}(x_i, m_i) + \sum_{i=1}^{(n2^n)^N} \int_{Q_i^n} H_{\text{lsc}}'(\cdot, 0^+) du_+^d + \int_{Q_i^n} H_{\text{lsc}}'(\cdot, 0^-) du_-^d \\ &= \sum_{i=1}^k H_{\text{lsc}}(x_i, m_i) + \int_{\mathbb{R}^N} H_{\text{lsc}}'(\cdot, 0^+) du_+^d + \int_{\mathbb{R}^N} H_{\text{lsc}}'(\cdot, 0^-) du_-^d \\ &= \mathbf{M}^{H_{\text{lsc}}}(u), \end{aligned}$$

where we have used monotone convergence in the last but one equality. \square

2.3. Slope at the origin of the minimal cost function.

Proposition 2.6. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$ be a lower semicontinuous function such that $f(0, 0) = 0$, with $N \geq 2$. For every function $\rho \in \mathcal{C}((0, 1], (0, +\infty))$ such that*

$$(2.11) \quad \int_0^1 \left(\int_y^1 \frac{dt}{\rho(t)} \right)^N dy < +\infty,$$

the function H_f defined in (2.1) satisfies

$$(2.12) \quad \lim_{m \rightarrow 0^+} \frac{H_f(m)}{m} \leq \limsup_{u \rightarrow 0^+} \sup_{\xi \in \mathbb{S}^{N-1}} \frac{f(u, \rho(u)\xi)}{u}.$$

Proof. For every $y \geq 0$, we let

$$F(y) = \int_y^1 \frac{dt}{\rho(t)} \in [0, +\infty].$$

The function F is decreasing, and belongs to $\mathcal{C}^1((0, 1])$ and $L^N((0, 1])$ by assumption. We now consider the solution of the ODE $v'_\varepsilon = -\rho(v_\varepsilon)$, with $v_\varepsilon(0) = \varepsilon$, given by

$$v_\varepsilon(r) = \begin{cases} F^{-1}(F(\varepsilon) + r), & \text{if } 0 \leq r < F(0) - F(\varepsilon), \\ 0 & \text{if } r \geq F(0) - F(\varepsilon), \end{cases}$$

with $F(0)$ possibly equal to $+\infty$. Notice that $v_\varepsilon \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$ because it is non-increasing and bounded, hence it has finite total variation, and because it is of class \mathcal{C}^1 except possibly at $r_\varepsilon := F(0) - F(\varepsilon)$, where it has no jump. As a consequence the radial profile defined by $u_\varepsilon(x) := v_\varepsilon(|x|)$ belongs to $W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ and we compute, using the change of variables $s = v_\varepsilon(r)$ (i.e. $r = F(s) - F(\varepsilon)$) and an integration by parts combined with monotone convergence.

$$\begin{aligned} m_\varepsilon &:= \int_{\mathbb{R}^N} u_\varepsilon = |\mathbb{S}^{N-1}| \int_0^\infty v_\varepsilon(r) r^{N-1} dr \\ &= -|\mathbb{S}^{N-1}| \int_0^\varepsilon s(F(s) - F(\varepsilon))^{N-1} F'(s) ds \\ &= |\mathbb{S}^{N-1}| \lim_{t \downarrow 0} \left(\int_t^\varepsilon \frac{(F(s) - F(\varepsilon))^N}{N} ds - \left[s \frac{(F(s) - F(\varepsilon))^N}{N} \right]_t^\varepsilon \right) \\ &= |\mathbb{S}^{N-1}| \int_0^\varepsilon \frac{(F(s) - F(\varepsilon))^N}{N} ds \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

The equality on the last line holds because $\lim_{t \rightarrow 0^+} \int_t^\varepsilon (F - F(\varepsilon))^N < +\infty$ (since $F \in L^N((0, 1])$), hence $\lim_{t \rightarrow 0} t(F(t) - F(\varepsilon))^N$ exists by existence of the limit in the previous line, and it must be zero (again, because $F \in L^N((0, 1])$).

Moreover, since $\sup_{(0, +\infty)} v_\varepsilon = \varepsilon$,

$$\mathcal{E}(u_\varepsilon) = \int_0^\infty \int_{\mathbb{S}^{N-1}} f(v_\varepsilon(r), v'_\varepsilon(r)\xi) r^{N-1} d\mathcal{H}^{N-1}(\xi) dr \leq m_\varepsilon \sup_{u \leq \varepsilon, |\xi|=1} \frac{f(u, \rho(u)\xi)}{u}.$$

By assumption, we deduce that

$$\limsup_{m \rightarrow 0^+} \frac{H_f(m)}{m} \leq \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathcal{E}(u_\varepsilon)}{m_\varepsilon} \leq \limsup_{u \rightarrow 0^+} \sup_{\xi \in \mathbb{S}^{N-1}} \frac{f(u, \rho(u)\xi)}{u}. \quad \square$$

In dimension $N = 1$, we need no other assumption than $H_f < +\infty$, as stated below.

Proposition 2.7. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$ be Borel measurable with $N = 1$. The minimal cost function H_f is either identically infinite on $(0, +\infty)$, or it satisfies (2.12) with $\rho \equiv 0$, i.e.*

$$\lim_{m \rightarrow 0^+} \frac{H_f(m)}{m} \leq \limsup_{u \rightarrow 0^+} \frac{f(u, 0)}{u}.$$

Proof. One can assume that there exists $u \in W_{\text{loc}}^{1,1}(\mathbb{R}, \mathbb{R}_+)$ with $0 < \int_{\mathbb{R}} u < +\infty$ and $\mathcal{E}(u) < +\infty$. In particular, up to changing the value of u on a negligible set, u is

continuous on \mathbb{R} . Let $\varepsilon \in (0, \sup_{\mathbb{R}} u)$, set $A_\varepsilon := \{x : u(x) = \varepsilon\}$ which is non-empty by the intermediate value theorem and integrability of u , and define

$$a_\varepsilon = \begin{cases} \inf A_\varepsilon & \text{if } \inf A_\varepsilon > -\infty, \\ \text{any point in } (-\infty, -\varepsilon^{-1}) \cap A_\varepsilon & \text{otherwise,} \end{cases}$$

$$b_\varepsilon = \begin{cases} \sup A_\varepsilon & \text{if } \sup A_\varepsilon < +\infty, \\ \text{any point in } (\varepsilon^{-1}, +\infty) \cap A_\varepsilon & \text{otherwise.} \end{cases}$$

By continuity and integrability of u , $u(a_\varepsilon) = u(b_\varepsilon) = \varepsilon$ and

$$\sup_{x \in \mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u(x) \leq \varepsilon \vee \sup_{|x| > \varepsilon^{-1}} u(x) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Moreover $a_\varepsilon, b_\varepsilon$ converge to points $-\infty \leq a \leq b \leq +\infty$, hence $u = 0$ on $\mathbb{R} \setminus (a, b)$ and by dominated convergence, since $\nabla u = 0$ a.e. on $\{u = 0\}$,

$$+\infty > \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u + f(u, \nabla u) = f(0, 0) \mathcal{L}(\mathbb{R} \setminus (a, b)).$$

Notice that this limit is necessary zero. Let $m > 0$. If ε is small enough, then $\int_{\mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u < m$ so that we can take $R_\varepsilon > 0$ such that $\varepsilon R_\varepsilon = m - \int_{\mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u$. We then define

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if } x \leq a_\varepsilon, \\ \varepsilon & \text{if } a_\varepsilon < x < a_\varepsilon + R_\varepsilon, \\ u(b_\varepsilon + x - (a_\varepsilon + R_\varepsilon)) & \text{if } x \geq a_\varepsilon + R_\varepsilon, \end{cases}$$

so that $\int_{\mathbb{R}} v_\varepsilon = m$. Moreover,

$$\mathcal{E}(v_\varepsilon) = \mathcal{E}(u, \mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]) + R_\varepsilon f(\varepsilon, 0).$$

Hence, as $R_\varepsilon = \frac{m+o(1)}{\varepsilon}$ as $\varepsilon \rightarrow 0$,

$$H_f(m) \leq \limsup_{\varepsilon \rightarrow 0^+} \mathcal{E}(v_\varepsilon) = m \limsup_{\varepsilon \rightarrow 0^+} \frac{f(\varepsilon, 0)}{\varepsilon}. \quad \square$$

2.4. Strict concavity of the minimal cost function in dimension $N \geq 2$.

We show that in dimension $N \geq 2$, the minimal cost function must be strictly concave away from the possible initial interval where it is linear:

Proposition 2.8. *Assume that $N \geq 2$ and that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ satisfies (H₁), (H₂), (H₅) and (H₆). Let*

$$m_* = \sup\{m \geq 0 : H_f(x_0, \cdot) \text{ is linear on } [0, m]\},$$

where $x_0 \in \mathbb{R}^N$ is fixed. Then, $H_f(x_0, \cdot)$ is strictly concave on $(m_*, +\infty)$. A similar statement holds on \mathbb{R}_- .

A similar result does not hold in dimension 1 since any continuous concave function $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $H(0) = 0$ can be written as $H = H_f$ with f satisfying all our assumptions (H₁)–(H₈) (see Section 5.2).

In the rest of this section, we will systematically omit the dependence on x_0 of $H_f(x_0, m)$ since x_0 is fixed.

We denote by \mathcal{M}_m^f the set of non-negative minimizers (see Remark 2.2) of mass $m \in \mathbb{R}_+$:

$$(2.13) \quad \mathcal{M}_m^f := \left\{ u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+) : \mathcal{E}_f(u) = H_f(m) \text{ and } \int_{\mathbb{R}^N} u = m \right\}.$$

The proof of [Proposition 2.8](#) is based on the following observation:

Lemma 2.9. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$ be Borel measurable and let $u_i \in \mathcal{M}_{m_i}^f$ with $m_i \in \mathbb{R}_+$ for $i = 1, 2$. Let also $u_* := \min\{u_1, u_2\}$, $u^* := \max\{u_1, u_2\}$, $m_* := \int_{\mathbb{R}^N} u_*$ and $m^* := \int_{\mathbb{R}^N} u^*$. If H_f is affine on $[m_*, m^*]$ then $u_* \in \mathcal{M}_{m_*}^f$ and $u^* \in \mathcal{M}_{m^*}^f$.*

Proof of Lemma 2.9. We use the same observations as in the proof of [Theorem 2.1](#). In particular, we have $m_* + m^* = m_1 + m_2$; since H_f is affine on $[m_*, m^*]$, it yields

$$H_f(m_*) + H_f(m^*) = H_f(m_1) + H_f(m_2).$$

But we have also

$$H_f(m_*) + H_f(m^*) \leq \mathcal{E}_f(u_*) + \mathcal{E}_f(u^*) = \mathcal{E}_f(u_1) + \mathcal{E}_f(u_2) = H_f(m_1) + H_f(m_2),$$

so that the inequalities we used, i.e. $H_f(m_*) \leq \mathcal{E}_f(u_*)$ and $H_f(m^*) \leq \mathcal{E}_f(u^*)$, are actually equalities. \square

We also use an elementary Sobolev type inequality:

Lemma 2.10. *Let $N \geq 2$, $p \in (1, +\infty)$ and $\omega \subset \mathbb{R}^{N-1}$ be a bounded open set. For every $u \in W_{\text{loc}}^{1,p}(\mathbb{R} \times \omega)$,*

$$\int_{\omega} \|u(\cdot, x')\|_{L^\infty(\mathbb{R})} dx' \leq \|u\|_{L^1(\mathbb{R} \times \omega)} + |\omega|^{\frac{p-1}{p}} \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\mathbb{R} \times \omega)}.$$

Proof of Lemma 2.10. We prove the lemma when $u \in \mathcal{C}^1(\mathbb{R} \times \omega)$; the general case follows by approximation. For every $x_1, y_1 \in \mathbb{R}$, $x' \in \omega$, we have

$$u(x_1, x') = u(y_1, x') + \int_{y_1}^{x_1} \frac{\partial u}{\partial x_1}(t, x') dt.$$

By averaging in the variable y_1 , we deduce

$$|u(x_1, x')| \leq \int_{x_1 - \frac{1}{2}}^{x_1 + \frac{1}{2}} |u(y_1, x')| dy_1 + \int_{x_1 - \frac{1}{2}}^{x_1 + \frac{1}{2}} \left| \frac{\partial u}{\partial x_1}(t, x') \right| dt.$$

The result follows from Hölder inequality after integrating over ω . \square

Proof of Proposition 2.8. Assume by contradiction that the concave function H_f is not strictly concave on $(m_*, +\infty)$ which means that there exists $m \in (m_*, +\infty)$ and $\eta > 0$ such that H_f is affine on $[m - \eta, m + \eta]$. (Note that $\eta \leq m - m_*$ by definition of m_* .) Moreover, we will see in [Proposition 3.12](#) that \mathcal{M}_m^f is not empty because H_f is not linear on $[0, m]$. We let $u \in \mathcal{M}_m^f$.

As before, we shall use the notations \wedge and \vee for the minimum and maximum; we also let (e_1, \dots, e_N) be the canonical basis of \mathbb{R}^N . Knowing that $\tau \mapsto u(\cdot + \tau)$ is continuous in \mathbb{R}^N for every $u \in L^1(\mathbb{R}^N)$, that $u \mapsto u(\cdot + \tau)$ is isometric in $L^1(\mathbb{R}^N)$ for every $\tau \in \mathbb{R}^N$, that the map $(x, y) \mapsto x \wedge y$ is Lipschitz on $\mathbb{R}^N \times \mathbb{R}^N$, and since the set $\mathcal{M}_{m+\frac{\eta}{2}}$ is compact in L^1 up to translations in view of [Remark 3.13](#), we deduce that there exists $\delta_0 > 0$ such that

$$(2.14) \quad \|u \wedge u(\cdot + \delta e_2)\|_{L^1(\mathbb{R}^N)} > m \quad \text{for all } \delta \in (0, \delta_0) \text{ and } u \in \mathcal{M}_{m+\frac{\eta}{2}}^f.$$

We now construct by induction a sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}_+ and a sequence $(u_n)_{n \in \mathbb{N}}$ in \mathcal{M}_m^f such that

$$(2.15) \quad t_{n+1} \geq t_n + \delta_0 \quad \text{and} \quad u_n(x) \leq U(x) \wedge U(x + t_n e_2) \quad \forall x \in \mathbb{R}^N,$$

where we have set

$$U(x) := \operatorname{ess\,sup}_{t \in \mathbb{R}} u(x + te_1).$$

To this aim, we first set $u_0 := u$ and $t_0 = 0$. Then, if we assume that t_n and u_n are constructed as before, we first pick an $\delta_n^1 \in \mathbb{R}_+$ such that $v_n := u_n \vee u_n(\cdot + \delta_n^1 e_1)$ satisfies $\int_{\mathbb{R}^N} v_n = m + \frac{\eta}{2}$, which is possible since $\eta \leq m$, as we argued in the proof of [Theorem 2.1](#). Similarly, we pick a $\delta_n^2 \in \mathbb{R}_+$ such that $u_{n+1} := v_n \wedge v_n(\cdot + \delta_n^2 e_2)$ satisfies $\int_{\mathbb{R}^N} u_{n+1} = m$, and we set $t_{n+1} = t_n + \delta_n^2$. By [Lemma 2.9](#), $v_n \in \mathcal{M}_{m+\frac{\eta}{2}}^f$ and $u_n \in \mathcal{M}_m^f$. By [\(2.14\)](#), we have $\delta_n^2 \geq \delta_0$, thus insuring the first condition in [\(2.15\)](#). For the second condition, we observe that for all $x = (x_1, x') \in \mathbb{R}^N$,

$$u_{n+1}(x) \leq \left(\sup_{t \in \mathbb{R}} u_n(x + te_1) \right) \wedge \left(\sup_{t \in \mathbb{R}} u_n(x + te_1 + \delta_n^2 e_2) \right) \leq U(x) \wedge U(x + t_{n+1} e_2),$$

where in the last inequality we have used the induction hypothesis [\(2.15\)](#).

We now show that the sequence $(u_n \mathcal{L}^N)_{n \in \mathbb{N}}$ is vanishing which will contradict the compactness of \mathcal{M}_m^f in L^1 up to translations.

For this, we let $(x_k)_{k \in \mathbb{N}}$ be a sequence in \mathbb{R}^N and $(u_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(u_n)_{n \in \mathbb{N}}$ such that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \int_{x+[0,1]^N} u_n = \lim_{k \rightarrow \infty} \int_{x_k+[0,1]^N} u_{n_k}.$$

By [\(H5\)](#), we have $\frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^N)$. Using this fact, estimate [\(2.15\)](#), and [Lemma 2.10](#) with ω a unit cube in \mathbb{R}^{N-1} , we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{x_k+[0,1]^N} u_{n_k} &\leq \liminf_{k \rightarrow \infty} \int_{x_k+[0,1]^N} U \wedge \int_{x_k+t_{n_k}e_2+[0,1]^N} U \\ &\leq \liminf_{k \rightarrow \infty} \left(\|u\|_{L^1(\{0 \leq (x-x_k) \cdot e_2 \leq 1\})} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\{0 \leq (x-x_k) \cdot e_2 \leq 1\})} \right) \\ &\wedge \liminf_{k \rightarrow \infty} \left(\|u\|_{L^1(\{t_{n_k} \leq (x-x_k) \cdot e_2 \leq t_{n_k} + 1\})} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\{t_{n_k} \leq (x-x_k) \cdot e_2 \leq t_{n_k} + 1\})} \right), \end{aligned}$$

and the conclusion follows since the sequences $(x_k \cdot e_2)_{k \in \mathbb{N}}$ and $(t_{n_k} + x_k \cdot e_2)_{k \in \mathbb{N}}$ cannot be both bounded as $\lim_{k \rightarrow \infty} t_{n_k} = \infty$. \square

3. LOWER BOUND FOR THE ENERGY AND EXISTENCE OF OPTIMAL PROFILES

Our main tool to localize the energy and obtain a lower bound relies on a profile decomposition for bounded sequences of positive measures, which is reminiscent of the concentration-compactness principle of P.-L. Lions. This differs from classical strategies to localize the energy which are based on suitable cut-offs. Naturally, this concentration-compactness result also provides a criterion for the existence of optimal profiles in [\(1.2\)](#).

3.1. Profile decomposition by concentration-compactness. We prove a profile decomposition theorem for bounded sequences of positive measures over \mathbb{R}^N , which is essentially equivalent to [[Mar14](#), Theorem 1.5] in the Euclidean case. We have added an extra information on mass conservation that will be useful, and provide a self-contained simple proof. We start with a definition.

Definition 3.1. A sequence of positive measures $(\mu_n)_{n \in \mathbb{N}} \in \mathcal{M}_+(\mathbb{R}^N)$ is *vanishing* if

$$\sup_{x \in \mathbb{R}^N} \mu_n(B_1(x)) \xrightarrow{n \rightarrow \infty} 0.$$

Any bounded sequence of positive measures over \mathbb{R}^N may be decomposed (up to subsequence) into a countable collection of narrowly converging “bubbles” and a vanishing part, accounting for the total mass of the sequence, as stated in the following theorem.

Theorem 3.2. *For every bounded sequence $(\mu_n)_{n \in \mathbb{N}}$ of positive Borel measures on \mathbb{R}^N , there exists a subsequence $(\mu_n)_{n \in \sigma(\mathbb{N})}$, $\sigma \in \Sigma$, a non-decreasing sequence of integers $(k_n)_{n \in \sigma(\mathbb{N})}$ converging to some $k \in \mathbb{N} \cup \{+\infty\}$, a sequence of non-trivial positive Borel measures $(\mu^i)_{0 \leq i < k}$, and for every $n \in \sigma(\mathbb{N})$, a collection of balls $(B_n^i)_{0 \leq i < k_n}$ centered at points of $\text{supp } \mu_n$ such that, writing for all $n \in \sigma(\mathbb{N})$,*

$$(3.1) \quad \mu_n = \mu_n^b + \mu_n^v, \quad \text{where } \mu_n^b = \sum_{0 \leq i < k_n} \mu_n \llcorner B_n^i,$$

(A) *bubbles emerge:* $(c_{B_n^i} \mu_n)_{n \in \sigma(\mathbb{N})} \xrightarrow{n \rightarrow \infty} \mu^i$ for every $i < k$,⁵

(B) *bubbles split:* $\min_{0 \leq i < j < k_n} \text{dist}(B_n^i, B_n^j) \xrightarrow{n \rightarrow \infty} +\infty$,

(C) *bubbles diverge:* $\min_{0 \leq i < k_n} \text{diam}(B_n^i) \xrightarrow{n \rightarrow \infty} +\infty$,

(D) *the bubbling mass is conserved:* $\|\mu_n^b\| \xrightarrow{\ell \rightarrow \infty} \sum_{0 \leq i < k} \|\mu^i\|$,

(E) *the remaining part is vanishing:* $\sup_{x \in \mathbb{R}^N} \mu_n^v(B_1(x)) \xrightarrow{n \rightarrow \infty} 0$.

Before proving **Theorem 3.2**, we introduce the “bubbling” function of a sequence of finite signed measures $(\mu_n)_{n \in \mathbb{N}}$:

$$(3.2) \quad m((\mu_n)_{n \in \mathbb{N}}) := \sup \left\{ \|\mu\| : (\tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)})_{\ell \in \mathbb{N}} \xrightarrow{\mathcal{C}'_0} \mu, \sigma \in \Sigma, x_{\sigma(\ell)} \in \mathbb{R}^N (\forall \ell) \right\}.$$

Although we will use this function on signed measures, we will start from a sequence of positive measures and use the following characterization of vanishing sequences, which holds only in the case of positive measures:

Lemma 3.3. *A sequence $(\mu_n)_{n \in \mathbb{N}}$ of finite positive measures over \mathbb{R}^N is vanishing if and only if $m((\mu_n)_{n \in \mathbb{N}}) = 0$.*

Proof. Assume that $(\mu_n)_{n \in \mathbb{N}}$ is vanishing and that $(\tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)})_{\ell \in \mathbb{N}} \xrightarrow{\mathcal{C}'_0} \mu$ for some $\sigma \in \Sigma$ and some sequence of points $(x_{\sigma(\ell)})_{\ell \in \mathbb{N}}$. Then, for every $x \in \mathbb{R}^N$,

$$\mu(B_1(x)) \leq \liminf_{\ell \rightarrow \infty} \tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_1(x)) = \liminf_{\ell \rightarrow \infty} \mu_{\sigma(\ell)}(B_1(x + x_{\sigma(\ell)})) = 0,$$

i.e. $\mu = 0$ and thus $m((\mu_\ell)_{\ell \in \mathbb{N}}) = 0$.

Conversely, if $(\mu_n)_{n \in \mathbb{N}}$ is not vanishing, then there exists $\varepsilon > 0$, $\sigma \in \Sigma$ and a sequence of points $(x_n)_{n \in \sigma(\mathbb{N})}$ in \mathbb{R}^N such that $\mu_n(B_1(x_n)) \geq \varepsilon$ for every $n \in \sigma(\mathbb{N})$.

Up to further extraction, one can assume that $(\tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)})_{\ell \in \mathbb{N}} \xrightarrow{\mathcal{C}'_0} \mu \in \mathcal{M}(\mathbb{R}^N)$. We have

$$\mu(\bar{B}_1(0)) \geq \limsup_{\ell \rightarrow \infty} \tau_{-x_{\sigma(\ell)}} \mu_{\sigma(\ell)}(\bar{B}_1(0)) = \limsup_{\ell \rightarrow \infty} \mu_{\sigma(\ell)}(\bar{B}_1(x_{\sigma(\ell)})) \geq \varepsilon > 0,$$

⁵Recall that $c_B \mu = (x \mapsto x - y)_\# (\mu \llcorner B)$ if $B = B_r(y)$ and $\mu \in \mathcal{M}(\mathbb{R}^N)$.

which entails $m((\mu_\ell)_{\ell \in \mathbb{N}}) \geq \varepsilon > 0$. \square

Proof of Theorem 3.2. If $(\mu_n)_{n \in \mathbb{N}}$ is vanishing, then we take $\sigma = \text{Id}$ and $k = 0$, so that $\mu_{\sigma(\ell)} = \mu_\ell = \mu_\ell^v$, (A) to (D) are empty statements and (E) is satisfied since $(\mu_n)_{n \in \mathbb{N}}$ is vanishing. Assume on the contrary that $(\mu_n)_{n \in \mathbb{N}}$ is not vanishing. We shall construct the bubbles by induction and prove their properties in several steps.

Step 1: construction of bubbles centers. At first step (step 0), since $m((\mu_n)_{n \in \mathbb{N}}) > 0$, there exists $\sigma_0 \in \Sigma$ and a sequence of points $(x_n^0)_{n \in \sigma_0(\mathbb{N})}$, such that

$$(3.3) \quad (\tau_{-x_n^0} \mu_n)_{n \in \sigma_0(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^0 \in \mathcal{M}(\mathbb{R}^N) \quad \text{with} \quad \|\mu^0\| \geq \frac{1}{2} m((\mu_n)_{n \in \mathbb{N}}).$$

We then set $\mu_n^0 := \mu_n - \tau_{x_n^0} \mu^0$ and we continue by induction, starting from the sequence $(\mu_n^0)_{n \in \sigma_0(\mathbb{N})}$. More precisely, assume that for a fixed step $k-1 \in \mathbb{N}$, for every $i \in \mathbb{N}$ such that $0 \leq i \leq k-1$, we have built $\mu^i \in \mathcal{M}(\mathbb{R}^N)$, $\sigma_i \in \Sigma$, points $(x_n^i)_{n \in \sigma_i(\mathbb{N})}$ and sequences $(\mu_n^i)_{n \in \sigma_i(\mathbb{N})} \in \mathcal{M}(\mathbb{R}^N)$ such that for every i ,

$$(3.4) \quad \sigma_i \preceq \sigma_{i-1},$$

$$(3.5) \quad \mu_n^i = \mu_n - \sum_{0 \leq j \leq i} \tau_{x_n^j} \mu^j, \quad (\forall n \in \sigma_i(\mathbb{N})),$$

$$(3.6) \quad (\tau_{-x_n^i} \mu_n^{i-1})_{n \in \sigma_i(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^i,$$

$$(3.7) \quad \|\mu^i\| \geq \frac{1}{2} m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}) > 0,$$

where $\sigma_{-1} := \text{Id}$, $(\mu_n^{-1}) := (\mu_n)$. If $m((\mu_n^{k-1})_{n \in \sigma_{k-1}(\mathbb{N})}) = 0$, we stop; otherwise, we proceed to the next step k to build $\sigma_k, \mu^k, (x_n^k)_{n \in \sigma_k(\mathbb{N})}, (\mu_n^k)$ as we did at step $k=0$, starting with $(\mu_n^{k-1})_{n \in \sigma_{k-1}(\mathbb{N})}$. Either the induction stops at some step $k-1 \in \mathbb{N}$ for which $m((\mu_n^{k-1})_{n \in \sigma_{k-1}(\mathbb{N})}) = 0$ or the previous objects are defined for every $i \in \mathbb{N}$, in which case we let $k := +\infty$.

Step 2: splitting of bubbles centers. We prove that

$$(3.8) \quad \lim_{\sigma_i(\mathbb{N}) \ni n \rightarrow \infty} \text{dist}(x_n^i, x_n^j) = +\infty \quad \text{for every } i, j \in \mathbb{N} \text{ with } 0 \leq j < i < k.$$

Indeed, assume by contradiction that there is a first index $i < k$ such that for some $j_0 < i$, $(\text{dist}(x_n^i, x_n^{j_0}))_{n \in \sigma_i(\mathbb{N})}$ is not divergent. In particular, there exists $\sigma \preceq \sigma_i$ such that $(x_n^i - x_n^{j_0})_{n \in \sigma(\mathbb{N})} \rightarrow x \in \mathbb{R}^N$. Moreover, $(\text{dist}(x_n^i, x_n^j))_{n \in \sigma_i(\mathbb{N})} \rightarrow \infty$, for every $j < i$, $j \neq j_0$ by minimality of i and the triangle inequality $\text{dist}(x_n^j, x_n^{j_0}) \leq \text{dist}(x_n^j, x_n^i) + \text{dist}(x_n^i, x_n^{j_0})$. Notice by (3.5) that for every $n \in \sigma(\mathbb{N})$,

$$\mu_n^{i-1} = \mu_n^{j_0-1} - \tau_{x_n^{j_0}} \mu^{j_0} - \sum_{j_0 < j < i} \tau_{x_n^j} \mu^j,$$

hence taking the translation $\tau_{-x_n^i}$,

$$\tau_{-x_n^i} \mu_n^{i-1} = \tau_{x_n^{j_0} - x_n^i} (\tau_{-x_n^{j_0}} \mu_n^{j_0-1} - \mu^{j_0}) - \sum_{j_0 < j < i} \tau_{x_n^j - x_n^i} \mu^j,$$

and passing to the weak limit, knowing that $x_n^{j_0} - x_n^i \rightarrow -x$ and $\text{dist}(x_n^j, x_n^i) \rightarrow +\infty$ for $j_0 < j < i$,

$$\mu^i = \tau_{-x} (\mu^{j_0} - \mu^{j_0}) - \sum_{j_0 < j < i} 0 = 0.$$

This contradicts the fact that $(\tau_{-x_n^i} \mu_n^{i-1})_{n \in \sigma(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^i \neq 0$ and proves (3.8).

Step 3: weak convergence of bubbles. From (3.5) we get

$$(3.9) \quad \tau_{-x_n^i} \mu_n^{i-1} = \tau_{-x_n^i} \mu_n - \sum_{0 \leq j < i} \tau_{-x_n^i + x_n^j} \mu^j,$$

and by (3.8), the sum converges weakly to 0, and so

$$(3.10) \quad (\tau_{-x_n^i} \mu_n)_{n \in \sigma_i(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^i \quad \text{for every } i \in \mathbb{N} \text{ with } i < k.$$

Step 4: construction of the bubbles with mass conservation. We now construct the extraction $\sigma \in \Sigma$ that we need by induction: we set $\sigma(0) = 0$ and, assuming that $\sigma(0) < \dots < \sigma(\ell - 1)$, with $\ell \in \mathbb{N}^*$, have been constructed, we set $\sigma(\ell) := n$ with $n \in \sigma_{\ell \wedge k - 1}(\mathbb{N})$ large enough so that $n > \sigma(\ell - 1)$ and for every $i < \ell \wedge k$,

$$(3.11) \quad \mu_n(B_\ell(x_n^i)) \leq \|\mu^i\| + 2^{-\ell},$$

and

$$(3.12) \quad \min_{0 \leq j < i} \text{dist}(x_n^i, x_n^j) \geq 4\ell.$$

Such an n exists by (3.8) and (3.10), noticing that $\mu_n(B_\ell(x_n^i)) = (\tau_{-x_n^i} \mu_n)(B_\ell)$. Then for each $n = \sigma(\ell)$, $\ell \in \mathbb{N}$, we set $k_n = \ell \wedge k$, and for each $i \in \{0, \dots, k_n - 1\}$,

$$B_n^i := B_\ell(x_n^i).$$

Finally, for every $n \in \sigma(\mathbb{N})$, we decompose μ_n as expected:

$$\mu_n = \mu_n^b + \mu_n^v, \quad \text{where } \mu_n^b = \sum_{0 \leq i < k_n} \mu_n \llcorner B_n^i.$$

Let us check the four first items (A)–(D). Notice that (C) is fulfilled because $\text{diam}(B_{\sigma(\ell)}^i) = \ell \rightarrow +\infty$ as $\ell \rightarrow \infty$, and (B) because of (3.12). Since for every $i < k$, $\lim_{\sigma(\mathbb{N}) \ni n \rightarrow \infty} \text{diam}(B_n^i) = +\infty$ and $c_{B_n^i} \mu_n = (\tau_{-x_n^i} (\mu_n \llcorner B_n^i))$ for every $n \in \sigma_i(\mathbb{N})$, $(c_{B_n^i} \mu_n)_{n \in \sigma(\mathbb{N})}$ converges weakly to μ^i by (3.10), and together with (3.11) it implies that

$$(c_{B_n^i} \mu_n)_{n \in \sigma(\mathbb{N})} \xrightarrow{\mathcal{C}'_b} \mu^i,$$

i.e. (A) is satisfied. Moreover, by (3.11) again,

$$\limsup_{\ell \rightarrow \infty} \sum_{0 \leq i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i) \leq \sum_{0 \leq i < k} \|\mu^i\| + \limsup_{\ell \rightarrow \infty} (\ell \wedge k) 2^{-\ell} = \sum_{0 \leq i < k} \|\mu^i\|,$$

and since $k_n \rightarrow k$, by Fatou's lemma we have,

$$\sum_{0 \leq i < k} \|\mu^i\| \leq \liminf_{\ell \rightarrow \infty} \sum_{0 \leq i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i),$$

which proves (D) because $\sum_{0 \leq i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i) = \|\mu_{\sigma(\ell)}^b\|$.

Step 5: vanishing of the remaining part, proof of (E). By Lemma 3.3, it suffices to prove that $m((\mu_n^v)_{n \in \sigma(\mathbb{N})}) = 0$. We claim that:

$$(3.13) \quad m((\mu_n^v)_{n \in \sigma(\mathbb{N})}) \leq m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}), \quad \text{for every } i \in \mathbb{N} \text{ with } i < k,$$

which concludes since $m((\mu_n^k)_{n \in \sigma_{k-1}(\mathbb{N})}) = 0$ if $k < \infty$, and $m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}) \rightarrow 0$ as $i \rightarrow \infty$ if $k = \infty$. Indeed, if $k = \infty$, we have by (3.7) and (D),

$$\frac{1}{2} \sum_{i \in \mathbb{N}} m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}) \leq \sum_{i \in \mathbb{N}} \|\mu^i\| = \lim_{\ell \rightarrow \infty} \|\mu_{\sigma(\ell)}^b\| \leq \liminf_{\ell \rightarrow \infty} \|\mu_{\sigma(\ell)}\| < \infty.$$

Let us show (3.13). Let $\bar{\sigma} \preceq \sigma$ and $(x_n)_{n \in \bar{\sigma}(\mathbb{N})}$ be a sequence of points such that

$$(\tau_{-x_n} \mu_n^v)_{n \in \bar{\sigma}(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu \in \mathcal{M}(\mathbb{R}^N).$$

We need to prove that $\|\mu\| \leq m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})})$ for every $i < k$. Assume without loss of generality that $\|\mu\| > 0$. Then for every $i < k$,

$$(3.14) \quad (\text{dist}(x_n, x_n^i))_{n \in \bar{\sigma}(\mathbb{N})} \rightarrow \infty.$$

Otherwise, up to subsequence, $(\text{dist}(x_n, x_n^i))_n$ would be bounded by some constant M , and for every $r > 0$,

$$(\tau_{-x_n} \mu_n^v)(B_r) \leq \mu_n^v(B_{r+M}(x_n^i)) \xrightarrow{n \rightarrow \infty} 0,$$

because μ_n^v is supported on $\mathbb{R}^N \setminus \cup_{0 \leq i < k_n} B_n^i$ and $B_{r+M}(x_n^i) \subseteq B_n^i$ for n large enough by (C). Hence μ would be 0, a contradiction. Up to further extraction, one can assume that $(\tau_{-x_n} \mu_n)_{n \in \bar{\sigma}(\mathbb{N})}$ converges weakly to a measure $\bar{\mu} \in \mathcal{M}(\mathbb{R}^N)$. Since $\mu_n^v \leq \mu_n$, we have $\mu \leq \bar{\mu}$. Moreover by (3.5), for every $i < k$ and $n \in \bar{\sigma}(\mathbb{N})$ large enough,

$$\tau_{-x_n} \mu_n^i = \tau_{-x_n} \mu_n - \sum_{0 \leq j \leq i} \tau_{x_n^j - x_n} \mu_n^j,$$

and because of (3.14) the sum converges weakly to 0, so that $\tau_{-x_n} \mu_n^i \xrightarrow{\mathcal{C}'_0} \bar{\mu}$, and consequently,

$$\|\mu\| \leq \|\bar{\mu}\| \leq m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}),$$

which is what had to be proved.

Step 6: re-centering of the bubbles at points of $\text{supp } \mu_n$. By (3.10), $(\tau_{-x_n^i} \mu_n)_{n \in \sigma(\mathbb{N})}$ converges weakly to the non-trivial measure μ_i for every $i < k$, thus

$$(3.15) \quad R_i/2 := \limsup_{\sigma(\mathbb{N}) \ni n \rightarrow +\infty} \text{dist}(\text{supp } \mu_n, x_n^i) < +\infty.$$

Therefore, for every n large enough, there is a point \tilde{x}_n^i such that $|x_n^i - \tilde{x}_n^i| < R_i$ and $\tilde{x}_n^i \in \text{supp } \mu_n$. After a further extraction, one may assume that for every i , $|x_n^i - \tilde{x}_n^i| < R_i < r_i^n$ where $\text{diam } B_n^i = 2r_i^n$ for every n , and $(x_n^i - \tilde{x}_n^i)_{n \in \sigma(\mathbb{N})}$ converges to some $p_i \in \mathbb{R}^N$. Finally, we set $\tilde{r}_i^n := r_i^n - R_i$ and $\tilde{B}_n^i := B(\tilde{x}_n^i, \tilde{r}_i^n) \subseteq B_n^i$. After replacing the balls B_n^i by \tilde{B}_n^i , (B) and (C) are satisfied by definition. Notice that $(\tau_{-\tilde{x}_n^i} \mu_n)_{n \in \sigma(\mathbb{N})}$ converges weakly to $\tilde{\mu}^i := \tau_{p_i} \mu^i$ with $\|\tilde{\mu}^i\| = \|\mu^i\|$, and $\limsup_n \|c_{B_n^i} \mu_n\| = \limsup_n \mu_n(\tilde{B}_n^i) \leq \limsup_n \mu_n(B_n^i) = \|\mu^i\|$ hence (A) holds. Besides, using Fatou's lemma,

$$\begin{aligned} \limsup_n \sum_{i < k_n} \mu_n(\tilde{B}_n^i) &\leq \limsup_n \sum_{i < k_n} \mu_n(B_n^i) \\ &= \sum_{i < k} \|\mu^i\| \leq \sum_{i < k} \liminf_n \mu_n(\tilde{B}_n^i) \leq \liminf_n \sum_{i < k_n} \mu_n(\tilde{B}_n^i) \end{aligned}$$

so that $\lim_n \sum_{i < k_n} \mu_n(\tilde{B}_n^i) = \sum_i \|\mu_i\|$ and (D) is satisfied. In particular,

$$\lim_n \sum_{i < k_n} \mu_n(B_n^i \setminus \tilde{B}_n^i) = \lim_n \sum_{i < k_n} \mu_n(B_n^i) - \lim_n \sum_{i < k_n} \mu_n(\tilde{B}_n^i) = 0,$$

and (E) holds as well. \square

Remark 3.4. If the sequence of families of balls $(B_n^i)_{0 \leq i < k_n}$ satisfies the conclusion of the theorem, i.e. (A)–(E), then it is also the case for any family of balls $(\tilde{B}_n^i)_{0 \leq i < k_n}$ with the same centers as those of B_n^i and with smaller but still divergent radii (i.e. satisfying (C)). It can be easily seen following the arguments at Step 6 of the proof.

3.2. Lower bound by concentration-compactness. We will first establish a lower bound for the minimal energy along vanishing sequences defined on varying subsets of \mathbb{R}^N . We say that a sequence of Borel functions $(u_n)_{n \in \mathbb{N}}$, each defined on some open set $\Omega_n \subseteq \mathbb{R}^N$, is vanishing if the sequence of measures $(|u_n| \mathcal{L}^N \llcorner \Omega_n)_{n \in \mathbb{N}}$ is vanishing in the sense of [Definition 3.1](#), namely if $\|u_n\|_{L^1_{\text{uloc}}(\Omega_n)} \rightarrow 0$ as $n \rightarrow \infty$, where $L^1_{\text{uloc}}(\Omega)$ is the set of uniformly locally integrable functions on the open set Ω , i.e. Borel functions u on Ω such that

$$(3.16) \quad \|u\|_{L^1_{\text{uloc}}(\Omega)} := \sup_{x \in \mathbb{R}^N} \int_{\Omega \cap (x + [0,1]^N)} |u| < +\infty.$$

It will be convenient to first extend our Sobolev functions to a neighbourhood Ω_δ of Ω where for every $\delta > 0$ and every set $X \subseteq \mathbb{R}^N$, we have set

$$X_\delta := \{x \in \mathbb{R}^N : \text{dist}(x, X) < \delta\}.$$

We will need to consider sufficiently regular domains for which we have an extension operator $W^{1,p} \cap L^1_{\text{uloc}}(\Omega) \rightarrow W^{1,p} \cap L^1_{\text{uloc}}(\Omega_\delta)$. We will only apply it to domains with smooth boundary, in which case we can use a reflexion technique. Since we want quantitative estimates, we will use the notion of *reach* of a set $X \subseteq \mathbb{R}^N$ (see [\[Fed59\]](#)). We say that X has positive reach if there exists $\delta > 0$ such that every $x \in X_\delta$ has a unique nearest point $\pi(x)$ on X . The greatest δ for which this holds is denoted by $\text{reach}(X)$ and the map $x \in X_{\text{reach}(X)} \mapsto \pi(x) \in X$ is called the nearest point retraction.

Example 3.5. Assume that Ω is a perforated domain $B^0 \setminus \bigcup_{i=1}^k B^i$ where the B^i are disjoint closed balls included in some open ball B^0 (possibly $B^0 = \mathbb{R}^N$). Then,

$$\text{reach}(\partial\Omega) = \inf\{\text{radius}(B^i) : i = 0, \dots, k\} \cup \{\text{dist}(\partial B^i, \partial B^j) : i \neq j\}.$$

By [\[Fed59, Theorem 4.8\]](#), we have

- (i) if $x, y \in X_\delta$ with $0 < \delta < \delta_0 := \text{reach}(X)$, then $|\pi(x) - \pi(y)| \leq \frac{\delta_0}{\delta_0 - \delta} |x - y|$,
- (ii) if $x \in X$ and D_x is the intersection of $X_{\text{reach}(X)}$ with the straight line crossing $\partial\Omega$ orthogonally at x , then $\pi(y) = x$ for every $y \in D_x$.

Lemma 3.6 (Extension). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that its boundary $\partial\Omega$ is \mathcal{C}^1 with positive reach⁶. Then, for every $\delta \in (0, \text{reach}(\partial\Omega))$, every $p \in [1, +\infty)$ and every $u \in L^1 \cap W^{1,p}(\Omega)$, there exists $\bar{u} \in L^1 \cap W^{1,p}(\Omega_\delta)$ such that $\bar{u} = u$ a.e. on Ω , and*

$$\|\bar{u}\|_{L^1(\Omega_\delta)} \leq A \|u\|_{L^1(\Omega)}, \quad \|\bar{u}\|_{L^1_{\text{uloc}}(\Omega_\delta)} \leq A \|u\|_{L^1_{\text{uloc}}(\Omega)}, \quad \|\nabla \bar{u}\|_{L^p(\Omega_\delta)} \leq A \|\nabla u\|_{L^p(\Omega)},$$

with a constant $A < +\infty$ depending only on N, δ and $\text{reach}(\partial\Omega)$.

⁶Thanks to [\[Fed59, Remark 4.20\]](#), $\partial\Omega$ is actually of class $\mathcal{C}^{1,1}$.

Proof. Let $\sigma : (\partial\Omega)_\delta \rightarrow (\partial\Omega)_\delta$ be the reflection through $\partial\Omega$, defined by $\sigma(x) = 2\pi(x) - x$. By the properties (i) and (ii) of the nearest point retraction, we have that $\sigma = \sigma^{-1}$ (simply because $\pi(\sigma(x)) = \pi(x)$) and σ is L -Lipschitz with a constant $L < +\infty$ depending on δ and $\text{reach}(\partial\Omega)$ only.

We define⁷ \bar{u} by $\bar{u} = u$ on Ω and $\bar{u} = u \circ \sigma$ on $\Omega_\delta \setminus \Omega$. This map is well-defined since $\sigma(\Omega_\delta \setminus \Omega) \subseteq \Omega$. Indeed, if we had $x, \sigma(x) \in \Omega_\delta \setminus \Omega$, then the line segment $[x, \sigma(x)]$ would meet $\partial\Omega$ orthogonally at its center $\pi(x)$, and would remain out of Ω , because otherwise there would exist a point y belonging either to $\partial\Omega \cap (x, \pi(x))$ or $\partial\Omega \cap (\pi(x), \sigma(x))$ thus contradicting the definition of $\pi(x)$. Such a situation is not possible for a \mathcal{C}^1 boundary.

Moreover, by the change of variable formula and the chain rule, \bar{u} satisfies the desired estimates since σ is bi-Lipschitz with its Lipschitz constants controlled in terms of δ and $\text{reach}(\partial\Omega)$. \square

We will need a localized version of the Gagliardo–Nirenberg–Sobolev inequality in a particular case:

Lemma 3.7. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\partial\Omega$ is \mathcal{C}^1 with positive reach, let $p \in [1, +\infty)$, let $r \geq p(1 + \frac{1}{N})$, and assume that $r \leq \frac{pN}{N-p}$ when $p < N$. Then for every $u \in L^1 \cap W^{1,p}(\Omega)$,*

$$\|u\|_{L^r(\Omega)} \leq C(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)})^\alpha \|u\|_{L^1_{\text{loc}}(\Omega)}^{1-\alpha},$$

where $\alpha \in (0, 1]$ is the unique parameter such that $\frac{1}{r} = \alpha(\frac{1}{p} - \frac{1}{N}) + (1 - \alpha)$, and the constant $C < +\infty$ depends on N, r, p and $\text{reach}(\partial\Omega)$.

Proof of Lemma 3.7. We let $u \in L^1 \cap W^{1,p}(\Omega)$ and we extend u to $\bar{u} \in L^1 \cap W^{1,p}(\Omega_\delta)$ as in Lemma 3.6, with $\delta := \text{reach}(\Omega)/2$. By the Gagliardo–Nirenberg–Sobolev inequality (see [Nir59]) on the hypercube $Q_\delta = [-\frac{\delta}{\sqrt{N}}, \frac{\delta}{\sqrt{N}}]^N \subseteq \bar{B}_\delta$, we have for some C depending on N, δ ,

$$\|\bar{u}\|_{L^r(Q_\delta)} \leq C\|\nabla \bar{u}\|_{L^p(Q_\delta)}^\alpha \|\bar{u}\|_{L^1(Q_\delta)}^{1-\alpha} + C\|\bar{u}\|_{L^1(Q_\delta)}.$$

We then cover Ω with the disjoint hypercubes $Q_\delta(c) = c + Q_\delta \subseteq \Omega_\delta$ centered at points c on the grid $\mathcal{C} := \Omega \cap \frac{2\delta}{\sqrt{N}}\mathbb{Z}^N$. Since $r \geq p(1 + 1/N)$, we can check that

$$(3.17) \quad r\alpha = \frac{r-1}{1 + \frac{1}{N} - \frac{1}{p}} \geq p.$$

By superadditivity of $s \mapsto s^{\frac{r\alpha}{p}}$ and of $s \mapsto s^{r\alpha}$, we obtain

$$\begin{aligned} \|u\|_{L^r(\Omega)}^r &\leq \sum_{c \in \mathcal{C}} \|\bar{u}\|_{L^r(Q_\delta(c))}^r \\ &\leq C' \sum_{c \in \mathcal{C}} \|\nabla \bar{u}\|_{L^p(Q_\delta(c))}^{p\frac{r\alpha}{p}} \|\bar{u}\|_{L^1(Q_\delta(c))}^{r(1-\alpha)} + C' \|\bar{u}\|_{L^1(Q_\delta(c))}^r \\ &\leq C' \|\nabla \bar{u}\|_{L^p(\Omega_\delta)}^{r\alpha} \|\bar{u}\|_{L^1_{\text{loc}}(\Omega_\delta)}^{r(1-\alpha)} + C' \|\bar{u}\|_{L^1(\Omega_\delta)}^{r\alpha} \|\bar{u}\|_{L^1_{\text{loc}}(\Omega_\delta)}^{r(1-\alpha)} \\ &\leq C'' (\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)})^{r\alpha} \|u\|_{L^1_{\text{loc}}(\Omega)}^{r(1-\alpha)}. \end{aligned} \quad \square$$

⁷Note that \bar{u} is not defined on $\partial\Omega$, but this set is negligible.

Proposition 3.8. *Assume that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ satisfies **(H₁)** and **(H₅)** for some $p \in (1, +\infty)$. Consider a vanishing sequence $(u_n)_{n \in \mathbb{N}}$ in $W_{\text{loc}}^{1,1}(\Omega_n, \mathbb{R}_{\pm})$, where the $\Omega_n \subseteq \mathbb{R}^N$ are open sets with \mathcal{C}^1 boundary such that $\inf_{n \in \mathbb{N}} \text{reach}(\partial\Omega_n) > 0$, and a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of Borel maps $\Phi_n : \Omega_n \rightarrow \mathbb{R}^N$ such that $\sup_{y \in \Omega_n} |\Phi_n(y) - x_0| \rightarrow 0$ as $n \rightarrow +\infty$ for some $x_0 \in \mathbb{R}^N$. If $\theta_n := \int_{\Omega_n} u_n \neq 0$ for every n and $(\theta_n)_{n \in \mathbb{N}}$ is bounded, then:*

$$\liminf_{n \rightarrow +\infty} \frac{1}{|\theta_n|} \int_{\Omega_n} f(\Phi_n(y), u_n(y), \nabla u_n(y)) \, dy \geq f'_-(x_0, 0^\pm, 0),$$

where $f'_-(x_0, 0^\pm, 0)$ was defined in (1.4).

Proof of Proposition 3.8. Suppose for example that $u_n \geq 0$ a.e. for every n . Without loss of generality, we may assume after extracting a subsequence that:

$$(3.18) \quad K := \sup_n \frac{1}{\theta_n} \int_{\Omega_n} f(\Phi_n(y), u_n(y), \nabla u_n(y)) \, dy + \theta_n < +\infty.$$

We consider the sequence of measures $(\nu_n)_{n \in \mathbb{N}} \in \mathcal{M}_+(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ defined by

$$\nu_n := \frac{1}{\theta_n} (\Phi_n, u_n, \nabla u_n)_\# (u_n \mathcal{L}^N \llcorner \Omega_n), \quad n \in \mathbb{N}.$$

We are going to show in several steps that $\nu_n \xrightarrow{\mathcal{C}'_b} \delta_{(x_0, 0, 0)}$ and deduce the result. It suffices to show that the three projections $\nu_n^i := (\pi^i)_\# \nu_n$, $i \in \{1, 2, 3\}$ converge narrowly to δ_{x_0} , δ_0 and δ_0 respectively. Indeed, this would imply that (ν_n) converges narrowly to a measure concentrated on $(x_0, 0, 0)$, hence to $\delta_{(x_0, 0, 0)}$ since the ν_n are probability measures. First of all, since (ν_n) has bounded mass and (θ_n) is bounded, we may take a subsequence (not relabeled) such that $\nu_n \xrightarrow{\mathcal{C}'_0} \nu$ and $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$ for some $\nu \in \mathcal{M}_+(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ and $\theta \geq 0$.

Step 1: $\nu_n^1 \xrightarrow{\mathcal{C}'_b} \delta_{x_0}$. This is a direct consequence of the fact that ν_n^1 is concentrated on $\Phi_n(\mathbb{R}^N)$ for every n and $\text{dist}(\Phi_n(\mathbb{R}^N), x_0) \rightarrow 0$ as $n \rightarrow \infty$.

Step 2: $\nu_n^2 \xrightarrow{\mathcal{C}'_b} \delta_0$. By (3.18) and our assumption **(H₅)**, there is a constant $K_1 > 0$ with

$$(3.19) \quad \int_{\Omega_n} |\nabla u_n|^p \leq K_1 \int_{\Omega_n} u_n, \quad n \in \mathbb{N}.$$

We deduce from Markov's inequality, and Lemma 3.7 applied with $r = p(1 + \frac{1}{N})$, corresponding to $\alpha = \frac{N}{N+1}$, that

$$\begin{aligned} \nu_n^2([\eta, +\infty)) &= \frac{1}{\theta_n} \int_{\{u_n \geq \eta\}} u_n \\ &= \frac{1}{\theta_n} \int_{\{u_n \geq \eta\}} u_n^{1-r} u_n^r \\ &\leq \frac{1}{\theta_n \eta^{r-1}} \int_{\Omega_n} u_n^r \\ &\leq \frac{C}{\theta_n \eta^{r-1}} (\|\nabla u_n\|_{L^p(\Omega_n)} + \|u_n\|_{L^1(\Omega_n)})^{r\alpha} \|u_n\|_{L^1_{\text{uloc}}(\Omega_n)}^{r(1-\alpha)} \\ &\leq \frac{C'}{\eta^{r-1}} (1 + \theta_n^{p-1}) \|u_n\|_{L^1_{\text{uloc}}(\Omega_n)}^{r(1-\alpha)}, \end{aligned}$$

where in the last inequality, we have used the identity $\alpha r = p$ and (3.19), and C, C' depend only on N, r, p and $\inf_n \text{reach}(\partial\Omega_n)$.

Since $(u_n)_{n \in \mathbb{N}}$ is vanishing and $(\theta_n)_{n \in \mathbb{N}}$ is bounded, the last term in the previous inequality goes to zero as $n \rightarrow \infty$ and it follows that $\nu_n^2 \xrightarrow{\mathcal{C}'_b} \delta_0$.

Step 3: $\nu_n^3 \xrightarrow{\mathcal{C}'_b} \delta_0$. Fix $M > 0$ and $\eta > 0$. One has by (3.19),

$$\begin{aligned} \nu_n^3([M, +\infty)) &= \frac{1}{\theta_n} \int_{\{|\nabla u_n| \geq M\}} u_n \leq \frac{1}{\theta_n} \int_{\{u_n < \eta\} \cap \{|\nabla u_n| \geq M\}} u_n + \frac{1}{\theta_n} \int_{\{u_n > \eta\}} u_n \\ &\leq \frac{\eta}{\theta_n} \mathcal{L}^N(\{|\nabla u_n| \geq M\}) + \nu_n^2([\eta, +\infty)) \\ &\leq \frac{\eta}{\theta_n} \frac{1}{M^p} \int_{\Omega_n} |\nabla u_n|^p + \nu_n^2([\eta, +\infty)) \\ &\leq \frac{\eta K_1}{M^p} + \nu_n^2([\eta, +\infty)). \end{aligned}$$

By the previous step, we know that $\lim_{n \rightarrow +\infty} \nu_n^2([\eta, +\infty)) = 0$, hence taking the superior limit as $n \rightarrow +\infty$ then $\eta \rightarrow 0$ we get $\lim_{n \rightarrow +\infty} \nu_n^3([M, +\infty)) = 0$. Since this is true for every $M > 0$ we obtain $\nu_n^3 \xrightarrow{\mathcal{C}'_b} \delta_0$.

Step 4: conclusion. By the previous steps, we deduce that $\nu_n \xrightarrow{\mathcal{C}'_b} \delta_{(x_0, 0, 0)}$ as $n \rightarrow +\infty$. We define $g : \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \rightarrow [0, +\infty]$ as the lower semicontinuous envelope of $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \ni (x, u, \xi) \mapsto \frac{1}{u} f(x, u, \xi)$. By (H₁), we have $g(x, u, \xi) = \frac{1}{u} f(x, u, \xi)$ if $u > 0$, and by definition of f'_- (see (1.4)), we have $g(x, 0, 0) = f'_-(x, 0^+, 0)$ for every $x \in \mathbb{R}^N$. Hence, by lower semicontinuity of g and weak convergence of (ν_n) , we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega_n} f(\Phi_n, u_n, \nabla u_n) &\geq \liminf_{n \rightarrow \infty} \int_{\{u_n > 0\}} \frac{f(\Phi_n, u_n, \nabla u_n)}{u_n} u_n \\ &= \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N} g(x, u, \xi) d\nu_n(x, u, \xi) \\ &\geq \int_{\mathbb{R}^N} g(x, u, \xi) d\delta_{(x_0, 0, 0)} = f'_-(x_0, 0^+, 0), \end{aligned}$$

which ends the proof of the lemma. \square

As a corollary, we may now relate the slope at 0 of H_f to that of f .

Corollary 3.9. *Assume that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ satisfies (H₁), (H₅) for some $p \in (1, +\infty)$ and (H₆). Fix $x \in \mathbb{R}^N$. If either $N \geq 2$ or ($N = 1$ and $H_f(x, \cdot) \not\equiv +\infty$ on \mathbb{R}_\pm^*), then $H'_f(x, 0^\pm) = f'_-(x, 0^\pm, 0)$.*

Proof. The inequality $H'_f(x, 0^\pm) \leq f'_-(x, 0^\pm, 0)$ is a consequence of (H₆), Proposition 2.6 and Proposition 2.7, and the converse inequality $H'_f(x, 0^\pm) \geq f'_-(x, 0^\pm, 0)$ comes from Proposition 3.8. Indeed, if $(u_n)_{n \in \mathbb{N}} \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_\pm)$ is a sequence of functions of mass $\theta_n = \int_{\mathbb{R}^N} u_n$ going to 0 and which is almost minimizing in the sense that $\lim_{n \rightarrow \infty} \frac{\mathcal{E}_f^x(u_n)}{|\theta_n|} = \liminf_{n \rightarrow \infty} \frac{H(x, \theta_n)}{\theta_n}$ then $(u_n)_{n \in \mathbb{N}}$ is vanishing and Proposition 3.8 yields

$$\liminf_{n \rightarrow \infty} \frac{\mathcal{E}_f^x(u_n)}{|\theta_n|} \geq f'_-(x, 0^\pm, 0).$$

□

We now establish our main energy lower bound along sequences with bounded mass (not necessarily vanishing):

Proposition 3.10. *Assume that $(f_\varepsilon)_{\varepsilon>0}$ is a family of functions $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ satisfying **(H₁)**, **(H₂)**, **(H₅)** and **(H₈)** where $f = \lim_\varepsilon f_\varepsilon$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers going to zero, $(R_n)_{n \in \mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}}$ be two sequences in $(0, +\infty]$ such that $\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} R_n - r_n = +\infty$, $(u_n)_{n \in \mathbb{N}}$ be a sequence of functions $u_n \in W_{\text{loc}}^{1,1}(B_{R_n}, \mathbb{R}_\pm)$ with finite limit mass $m := \lim_{n \rightarrow \infty} \int_{B_{r_n}} u_n$, and $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence of Borel maps $\Phi_n : B_{R_n} \rightarrow \mathbb{R}^N$ such that*

$$(3.20) \quad \sup_{y \in B_{R_n}} |\Phi_n(y) - x_0| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for some } x_0 \in \mathbb{R}^N.$$

Then there exists a family $(u^i)_{0 \leq i < k}$ of functions in $W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_\pm)$ with $k \in \mathbb{N} \cup \{+\infty\}$, such that $m_i := \int_{\mathbb{R}^N} u^i \in \mathbb{R}_\pm^*$ for every i , and

$$(3.21) \quad m = m_v + \sum_{0 \leq i < k} m_i \quad \text{with } \pm m_v \geq 0,$$

$$(3.22)$$

$$\liminf_{n \rightarrow \infty} \int_{B_{R_n}} f_{\varepsilon_n}(\Phi_n, u_n, \nabla u_n) \geq |m_v| f'_-(x_0, 0^\pm, 0) + \sum_{0 \leq i < k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i).$$

Proof. Suppose for example that $u_n \geq 0$ a.e. for every n . We first assume, up to subsequence, that the left hand side of (3.22) is a finite limit. We apply the profile decomposition **Theorem 3.2** to the sequence of positive measures $\mu_n = u_n \mathcal{L}^N \llcorner B_{r_n}$ where, we assume the extraction σ to be the identity for convenience, and we use the same notation as in **Theorem 3.2**. In particular, for each bubble $B_n^i = B_{r_n^i}(x_n^i)$, with $0 \leq i < k_n$, we have $x_n^i \in \text{supp } \mu_n \subseteq \bar{B}_{r_n}$. By assumption, we have $\lim_{n \rightarrow \infty} (R_n - r_n) = +\infty$; hence, up to reducing the radii of the balls B_n^i if necessary, in such a way that their radii still diverge (see **Remark 3.4**), we can assume that

$$(3.23) \quad B_n^i \subseteq B_{R_n-1}, \quad 0 \leq i < k_n.$$

For each $0 \leq i < k_n$, we let $u_n^i := u_n(\cdot + x_n^i)$. Since (3.22) is assumed to be finite, we get that the sequence $(u_n^i)_n$ is bounded in $W_{\text{loc}}^{1,p}(\mathbb{R}^N)$ by **(H₅)**. Hence, after a further extraction if needed, we get that $(u_n^i)_{n \in \mathbb{N}} \rightharpoonup u^i$ weakly in $W_{\text{loc}}^{1,p}(\mathbb{R}^N)$ for some limit u^i , for every $0 \leq i < k = \lim k_n$. Setting $m_i = \int_{\mathbb{R}^N} u^i$ for every i , by **(D)** in **Theorem 3.2**, we have

$$m_v := m - \sum_{0 \leq i < k} m_i = \lim_{n \rightarrow \infty} \int_{B_{R_n} \setminus \cup_{0 \leq i < k_n} B_n^i} u_n.$$

Fix $\varepsilon > 0$. We decompose the energy as

$$(3.24) \quad \int_{B_{R_n}} f_\varepsilon(\Phi_n, u_n, \nabla u_n) = \int_{B_{R_n} \setminus \cup_{0 \leq i < k_n} B_n^i} f_\varepsilon(\Phi_n, u_n, \nabla u_n) \\ + \sum_{0 \leq i < k_n} \int_{B_{r_n^i}} f_\varepsilon(\Phi_n(\cdot + x_n^i), u_n^i, \nabla u_n^i).$$

Note that the domains $\Omega_n := B_{R_n} \setminus \cup_{0 \leq i < k} B_n^i$ satisfy $\inf_{n \in \mathbb{N}} \text{reach}(\partial\Omega_n) > 0$ as noticed in [Example 3.5](#), thanks to [\(3.23\)](#) and [\(B\)](#), [\(C\)](#) in [Theorem 3.2](#). Hence, applying [Proposition 3.8](#) to the Lagrangian f_ε , we obtain

$$(3.25) \quad \liminf_{n \rightarrow \infty} \int_{B_{R_n} \setminus \cup_{0 \leq i < k} B_n^i} f_\varepsilon(\Phi_n, u_n, \nabla u_n) \geq m_v(f_\varepsilon)'_-(x_0, 0^+, 0).$$

Moreover, by the lower semicontinuity of integral functionals (see [\[But89, Theorem 4.1.1\]](#)), in view of [\(3.20\)](#), we have for each i with $0 \leq i < k$,

$$(3.26) \quad \liminf_{n \rightarrow \infty} \int_{B_n^i} f_\varepsilon(\Phi_n(\cdot + x_n^i), u_n^i, \nabla u_n^i) \geq \int_{\mathbb{R}^N} f_\varepsilon(x_0, u^i, \nabla u^i).$$

Finally, by [\(3.24\)](#), [\(3.25\)](#), [\(3.26\)](#), [\(H₈\)](#) together with monotone convergence, we deduce that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{B_{R_n}} f_{\varepsilon_n}(\Phi_n, u_n, \nabla u_n) \\ & \geq \lim_{\varepsilon \rightarrow 0^+} \left(m_v(f_\varepsilon)'_-(x_0, 0^+, 0) + \sum_{0 \leq i < k} \int_{\mathbb{R}^N} f_\varepsilon(x_0, u^i, \nabla u^i) \right) \\ & = m_v f'_-(x_0, 0^+, 0) + \sum_{0 \leq i < k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i). \end{aligned}$$

The similar statement for non-positive functions is obtained in the same way. \square

3.3. Existence of optimal profiles. For the existence of an optimal profile in [\(1.2\)](#), we need a criterion that rules out splitting and vanishing of minimizing sequences:

Lemma 3.11. *Let $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a concave function. Then H is subadditive, and if for some $0 < \theta < m$ one has $H(m) = H(m - \theta) + H(\theta)$, then H is linear on $(0, m)$.*

Proof. By concavity, $t \mapsto \frac{H(t)}{t}$ is non-increasing. Hence,

$$H(m) = \theta \frac{H(m)}{m} + (m - \theta) \frac{H(m)}{m} \leq \theta \frac{H(\theta)}{\theta} + (m - \theta) \frac{H(m - \theta)}{m - \theta}.$$

But, by assumption, the last inequality is an equality which means that $\frac{H(m)}{m} = \frac{H(\theta)}{\theta} = \frac{H(m - \theta)}{m - \theta}$. In particular, the monotone function $t \mapsto \frac{H(t)}{t}$ must be constant on $[\theta, m]$, i.e. H must be linear on $[\theta, m]$. By concavity this is only possible if H is linear on $[0, m]$. \square

We can now state and prove our existence result:

Proposition 3.12. *Assume that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ satisfies [\(H₁\)](#), [\(H₂\)](#), [\(H₅\)](#) and [\(H₆\)](#). Let $x_0 \in \mathbb{R}^N$ and $m \in \mathbb{R}_+$ (resp. $m \in \mathbb{R}_-$). If $H_f(x_0, \cdot)$ is not linear on $[0, m]$ (resp. $[m, 0]$), then [\(1.2\)](#) admits a solution $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$, i.e. $\int_{\mathbb{R}^N} u = m$ and $\int_{\mathbb{R}^N} f(x_0, u, \nabla u) = H_f(x_0, m)$, such that $u \geq 0$ (resp. $u \leq 0$) in \mathbb{R}^N .*

Proof. We consider the case $m \geq 0$, the case $m < 0$ can then be deduced by considering $\tilde{f}(x, u, \xi) = f(x, -u, -\xi)$. We assume without loss of generality that H_f is finite on $(0, +\infty)$, otherwise by [Theorem 2.1](#) there is nothing to prove. By [Remark 2.2](#), the admissible class in [\(1.2\)](#) can be reduced to non-negative functions. In particular, if $m = 0$, then $u = 0$ is the only non-negative solution. If $m > 0$,

we apply [Proposition 3.10](#) in the following situation: $f_\varepsilon \equiv f$, $R_n \equiv +\infty$, $\Phi_n \equiv x_0$, $(u_n)_{n \in \mathbb{N}}$ is a minimizing sequence for the minimization problem in [\(1.2\)](#), and $(r_n)_{n \in \mathbb{N}}$ is a sequence of positive radii going to $+\infty$ such that $\lim_{n \rightarrow \infty} \int_{B_{r_n}} u_n = m$. We obtain

$$H_f(x_0, m) \geq m_v f'_-(x_0, 0^+, 0) + \sum_{0 \leq i < k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i),$$

with $k \in \mathbb{N} \cup \{+\infty\}$, $u^i \in W_{\text{loc}}^{1,p}(\mathbb{R}^N, \mathbb{R}_+)$ and $m = \sum_{0 \leq i < k} m_i + m_v$, where $m_i := \int_{\mathbb{R}^N} u^i$. By [Proposition 2.6](#) and [Proposition 2.7](#), in view of our assumption [\(H₆\)](#), and since H_f is assumed to be finite on $(0, +\infty)$ (for the case $N = 1$), we have $f'_-(x_0, 0^+, 0) \geq H'_f(x_0, 0)$. Moreover, by [Theorem 2.1](#), we have $m_v H'_f(x_0, 0) \geq H_f(x_0, m_v)$. Hence, by definition of H_f ,

$$H_f(x_0, m) \geq H_f(x_0, m_v) + \sum_{0 \leq i < k} H_f(x_0, m_i).$$

Since the concave function $H_f(x_0, \cdot)$ is not linear on $[0, m]$, by [Lemma 3.11](#), we have either $k = 1$ and $m_v = 0$, and we are done, or $k = 0$ and $m = m_v$. But in the latter case, we would have $H_f(x_0, m) = m H'_f(x_0, 0)$ which implies that the monotone function $t \mapsto \frac{H_f(x_0, t)}{t}$ is constant on $[0, m]$, i.e. that $H_f(x_0, \cdot)$ is linear on $[0, m]$. This contradicts our assumption. \square

Remark 3.13. Notice that the end of the proof actually shows, under the given assumptions, that the set of minimizers for a given mass m is compact in L^1 modulo translations.

4. Γ -CONVERGENCE OF THE RESCALED ENERGIES TOWARDS THE H -MASS

We establish lower and upper bounds for the Γ – lim inf and Γ – lim sup respectively, from which we deduce the proof of our main Γ -convergence result. The upper bound on the Γ – lim sup holds under more general assumptions and will be needed in [Section 5.5](#).

4.1. Lower bound for the Γ – lim inf. Given a Borel function $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$, we define for every $(x, m) \in \mathbb{R}^N \times \mathbb{R}$,

$$(4.1) \quad H_f^-(x, m) := H_f(x, m) \wedge (f'_-(x, 0^\pm, 0)|m|), \quad \text{if } \pm m \geq 0,$$

recalling that H_f is defined in [\(1.2\)](#) and $f'_-(x, 0^\pm, 0)$ in [\(1.4\)](#), with the usual convention $(\pm\infty) \times 0 = 0$. Notice that under [\(H₆\)](#), in view of [Proposition 2.6](#) and [Proposition 2.7](#) we have $H_f^-(x, m) = H_f(x, m)$, provided $H_f \not\equiv +\infty$ on \mathbb{R}_+^* and \mathbb{R}_-^* .

Proposition 4.1. *Assume that $(f_\varepsilon)_{\varepsilon > 0}$ is a family of functions $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty]$ satisfying [\(H₁\)](#), [\(H₂\)](#), [\(H₅\)](#) and [\(H₈\)](#) where $f = \lim_{\varepsilon \rightarrow 0} f_\varepsilon$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers going to zero, $(u_n)_{n \in \mathbb{N}}$ be a sequence in $W_{\text{loc}}^{1,1}(\mathbb{R}^N)$, and let*

$$e_n(x) := f_{\varepsilon_n}(x, \varepsilon_n^N u_n(x), \varepsilon_n^{N+1} \nabla u_n(x)) \varepsilon_n^{-N}, \quad x \in \mathbb{R}^N,$$

be the energy density of u_n . If $u_n \mathcal{L}^N \xrightarrow{\mathcal{C}'_0} u \in \mathcal{M}(\mathbb{R}^N)$ and $e_n \mathcal{L}^N \xrightarrow{\mathcal{C}'_0} e \in \mathcal{M}(\mathbb{R}^N)$, then

$$(4.2) \quad e \geq H_f^-(u).$$

In particular, $\Gamma(\mathcal{C}'_0) - \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^{H_f^-}$.

Proof of Proposition 4.1. Set $H := H_f^-$. Let us assume first that $u_n \geq 0$ a.e. for every n . To obtain (4.2), it is enough to prove that for every $x_0 \in \mathbb{R}^N$,

$$(4.3) \quad e(\{x_0\}) \geq H(x_0, u(\{x_0\})).$$

and that if $x_0 \in \text{supp } u$ is not an atom of u , then

$$(4.4) \quad \limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} \geq H'(x_0, 0^+),$$

Indeed (4.3) implies that $e \geq (H(u))^a$ (the atomic part of the measure $H(u)$) while (4.4) implies that $e \geq H'(\cdot, 0^+)u^d = (H(u))^d$, by Radon-Nikodým theorem (see [AFP00, Theorem 2.22]); these two relations yield $e \geq (H(u))^a + (H(u))^d = H(u)$ as required.

We fix $x_0 \in \text{supp } u$ and proceed in several steps.

Step 1: blow-up near x_0 . We first take two sequences of positive radii $(R_\ell)_{\ell \in \mathbb{N}} \rightarrow 0$ and $(r_\ell)_{\ell \in \mathbb{N}}$ such that for every $\ell \in \mathbb{N}$, $r_\ell \in (0, R_\ell)$,

$$(4.5) \quad e(\partial B_{R_\ell}(x_0)) = u(\partial B_{r_\ell}(x_0)) = 0,$$

and

$$(4.6) \quad \lim_{\ell \rightarrow \infty} \frac{e(B_{R_\ell}(x_0))}{u(B_{r_\ell}(x_0))} = \limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))}.$$

This last property is obtained by taking first a sequence $(\rho_\ell)_\ell$ such that

$$\limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} = \lim_{\ell \rightarrow \infty} \frac{e(B_{\rho_\ell}(x_0))}{u(B_{\rho_\ell}(x_0))},$$

then using monotone convergence the measures to get first r_ℓ then R_ℓ such that $0 < r_\ell < R_\ell < \rho_\ell$, $u(B_{r_\ell}(x_0)) \geq (1 - 2^{-\ell})u(B_{\rho_\ell}(x_0))$ and $e(B_{R_\ell}(x_0)) \geq (1 - 2^{-\ell})e(B_{\rho_\ell}(x_0))$.

By weak convergence and (4.5), according to [AFP00, Proposition 1.62 b)], we have for every $\ell \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} e_n(B_{R_\ell}(x_0)) = e(B_{R_\ell}(x_0)) \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{B_{r_\ell}(x_0)} u_n = u(B_{r_\ell}(x_0)).$$

Hence, there exists an extraction $(n_\ell)_{\ell \in \mathbb{N}} \in \Sigma$ such that

$$(4.7) \quad \lim_{\ell \rightarrow \infty} \frac{r_\ell}{\varepsilon_{n_\ell}} = +\infty \quad \text{and} \quad \lim_{\ell \rightarrow \infty} \frac{R_\ell - r_\ell}{\varepsilon_{n_\ell}} = +\infty,$$

satisfying the following conditions:

$$(4.8) \quad u(\{x_0\}) = \lim_{\ell \rightarrow \infty} \int_{B_{r_\ell}(x_0)} u_{n_\ell}, \quad e(\{x_0\}) = \lim_{\ell \rightarrow \infty} e_{n_\ell}(B_{R_\ell}(x_0)),$$

and

$$(4.9) \quad \limsup_{\ell \rightarrow \infty} \frac{e(B_{R_\ell}(x_0))}{u(B_{r_\ell}(x_0))} = \lim_{\ell \rightarrow \infty} \frac{e_{n_\ell}(B_{R_\ell}(x_0))}{\int_{B_{r_\ell}(x_0)} u_{n_\ell}}.$$

We may rewrite the mass and energy in terms of the re-scaled map v_ℓ defined by

$$(4.10) \quad v_\ell(y) := \varepsilon_{n_\ell}^N u_{n_\ell}(x_0 + \varepsilon_{n_\ell} y), \quad y \in \mathbb{R}^N, \ell \in \mathbb{N},$$

as follows:

$$(4.11) \quad \int_{B_{r_\ell}(x_0)} u_{n_\ell} = \int_{B_{\varepsilon_{n_\ell}^{-1}r_\ell}} v_\ell,$$

and

$$(4.12) \quad e_{n_\ell}(B_{R_\ell}(x_0)) = \int_{B_{\varepsilon_{n_\ell}^{-1}R_\ell}} f_{\varepsilon_{n_\ell}}(x_0 + \varepsilon_{n_\ell}y, v_\ell(y), \nabla v_\ell(y)) \, dy.$$

Step 2: proof of (4.3). By [Proposition 3.10](#), we have

$$(4.13) \quad \begin{aligned} e(\{x_0\}) &= \lim_{\ell \rightarrow \infty} \int_{B_{\varepsilon_{n_\ell}^{-1}R_\ell}} f_{\varepsilon_{n_\ell}}(x_0 + \varepsilon_{n_\ell}y, v_\ell(y), \nabla v_\ell(y)) \, dy \\ &\geq m_v f'_-(x_0, 0^+, 0) + \sum_{0 \leq i < k} H_f(x_0, m_i). \end{aligned}$$

Here $k \in \mathbb{N} \cup \{+\infty\}$ and $m = m_v + \sum_{0 \leq i < k} m_i$, with $m_i > 0$, $m_v \geq 0$ and

$$m = \lim_{\ell \rightarrow \infty} \int_{B_{\varepsilon_{n_\ell}^{-1}r_\ell}} v_\ell = u(\{x_0\}).$$

Since the function $H = H_f^-$, defined in [\(4.1\)](#), is the infimum of two functions which are mass-concave, it is mass-concave hence subadditive. From [\(4.13\)](#) we thus arrive at

$$e(\{x_0\}) \geq H(x_0, m_v) + \sum_{0 \leq i < k} H(x_0, m_i) \geq H\left(x_0, m_v + \sum_{0 \leq i < k} m_i\right) = H(x_0, u(\{x_0\})).$$

Step 3: proof of (4.4). Fix $\varepsilon > 0$ and assume that $m = u(\{x_0\}) = 0$. In that case, we apply [Proposition 3.8](#) to the sequence of functions $(v_\ell)_{\ell \in \mathbb{N}}$ defined on the sets $\Omega_\ell = B_{\varepsilon_{n_\ell}^{-1}r_\ell}$ and the function f_ε to get, thanks to [\(H₈\)](#):

$$\begin{aligned} \limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} &= \lim_{\ell \rightarrow \infty} \frac{e_{n_\ell}(B_{R_\ell}(x_0))}{\int_{B_{r_\ell}(x_0)} u_{n_\ell}} \\ &\geq \liminf_{\ell \rightarrow \infty} \frac{1}{\int_{B_{\varepsilon_{n_\ell}^{-1}r_\ell}} v_\ell} \int_{B_{\varepsilon_{n_\ell}^{-1}r_\ell}} f_\varepsilon(x_0 + \varepsilon_{n_\ell}y, v_\ell(y), \nabla v_\ell(y)) \\ &\geq (f_\varepsilon)'_-(x_0, 0^+, 0). \end{aligned}$$

Taking the limit $\varepsilon \rightarrow 0^+$, we deduce by [\(H₈\)](#) and [\(4.1\)](#):

$$(4.14) \quad \limsup_{R \rightarrow 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} \geq f'_-(x_0, 0^+, 0) \geq H'(x_0, 0^+).$$

In view of the discussion at the beginning of the proof, we have now proved [\(4.2\)](#).

Step 4: proof of (4.2) for signed $(u_n)_n$. Notice that the preceding reasoning for non-negative u_n applies also to the case of non-positive u_n . Let us handle the case where the (u_n) 's may change sign. We simply apply the above cases to the positive and negative parts $((u_n)_\pm)_n$ which converge weakly (up to subsequence) to some measures $u^\pm \in \mathcal{M}_+(\mathbb{R}^N)$ which satisfy $u = u^+ - u^-$, so that $e \geq H(\pm u^\pm)$. We know that the Jordan decomposition $u = u_+ - u_-$ is minimal, so that $u_\pm \leq u^\pm$

and $u_+ \perp u_-$. By monotonicity of the H -transform, due to [Theorem 2.1](#), $e \geq H(\pm u^\pm) \geq H(\pm u_\pm)$, and since $H(u_+) \perp H(-u_-)$, we get

$$e \geq H(u_+) + H(-u_-) = H(u).$$

Step 5: lower bound for the Γ -lim inf. We justify that [\(4.2\)](#) implies the lower bound $\Gamma(\mathcal{C}'_0) - \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^H$. Indeed, fix $u \in \mathcal{M}(\mathbb{R}^N)$ and consider a family $(u_\varepsilon)_{\varepsilon > 0}$ weakly converging to u as $\varepsilon \rightarrow 0$. We need to show that $\mathbf{M}^H(u) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon)$. Assume without loss of generality that the inferior limit is finite and take a sequence of positive numbers $(\varepsilon_n)_{n \in \mathbb{N}} \rightarrow 0$ such that this inferior limit is equal to $\lim_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n})$. Now the energy density e_n associated to $u_n = u_{\varepsilon_n}$ has bounded mass and up to extracting a subsequence one may assume that it converges weakly to some measure $e \in \mathcal{M}_+(\mathbb{R}^N)$. By the previous steps, $e \geq H(u)$, and by lower semicontinuity and monotonicity of the mass:

$$\liminf_{\varepsilon \rightarrow 0^+} \mathcal{E}_\varepsilon(u_\varepsilon) = \liminf_{n \rightarrow \infty} \|e_n\| \geq \|e\| \geq \|H(u)\| = \mathbf{M}^H(u). \quad \square$$

4.2. Upper bound for the Γ -lim sup. In this section, we introduce the following substitute for [\(H4\)](#), [\(H7\)](#) and [\(H8\)](#), where $f, (f_\varepsilon)_{\varepsilon > 0}$ are Borel maps from $\mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$ to $[0, +\infty]$:

(U) there exists $C < +\infty$ such that for every $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$\limsup_{\varepsilon \rightarrow 0^+} f_\varepsilon(x + \varepsilon y, u, \xi) \leq f(x, u, \xi) \quad \text{and} \quad f_\varepsilon(y, u, \xi) \leq C(f(x, u, \xi) + u) \quad \forall \varepsilon > 0.$$

Proposition 4.2. *Assume that $f, (f_\varepsilon)_{\varepsilon > 0}$ satisfy (U) and (H3). If $u \in \mathcal{M}(\mathbb{R}^N)$, then there exists $(u_\varepsilon)_{\varepsilon > 0} \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ such that $u_\varepsilon \mathcal{L}^N \xrightarrow{\mathcal{C}'_b} u$ when $\varepsilon \rightarrow 0$ and which satisfies*

$$\limsup_{\varepsilon \rightarrow 0^+} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathbf{M}^{H_{f,\text{lsc}}}(u),$$

where $H_{f,\text{lsc}} \leq H_f$ stands for the lower semicontinuous envelope of H_f , defined in [\(2.6\)](#). In other words, we have $\Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \leq \mathbf{M}^{H_{f,\text{lsc}}}$.

Proof of Proposition 4.2. Let $F = \Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon$. As an upper Γ -limit, F is sequentially lower semicontinuous in the narrow topology. Hence, by [Proposition 2.5](#), it is enough to prove that $F(u) \leq \mathbf{M}^{H_f}(u)$ whenever u is finitely atomic. Let $u = \sum_{i=1}^k m_i \delta_{x_i}$ with $k \in \mathbb{N}$, $m_i \in \mathbb{R}$, $x_i \in \mathbb{R}^N$, and assume without loss of generality that $x_i \neq x_j$ when $i \neq j$ and $\mathbf{M}^{H_f}(u) < +\infty$. Fix $\eta > 0$. For each $i = 1, \dots, k$, there exists $u_i \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u_i = m_i$ and $\int_{\mathbb{R}^N} f(x_i, u_i, \nabla u_i) \leq H(x_i, m_i) + \eta < +\infty$. We define for every $i = 1, \dots, k$,

$$(4.15) \quad u_\varepsilon^i(x) = \varepsilon^{-N} u_i(\varepsilon^{-1}(x - x_i)), \quad x \in \mathbb{R}^N,$$

and

$$(4.16) \quad u_\varepsilon = \max\{u_\varepsilon^i : i = 1, \dots, k\},$$

which converge narrowly as measures to u as $\varepsilon \rightarrow 0$. We have by change of variables:

$$\begin{aligned} \mathcal{E}_\varepsilon(u_\varepsilon) &\leq \sum_{i=1}^k \int_{\{u_\varepsilon = u_\varepsilon^i\}} f_\varepsilon(x, \varepsilon^N u_\varepsilon^i(x), \varepsilon^{N+1} \nabla u_\varepsilon^i(x)) \varepsilon^{-N} dx \\ &\leq \sum_{i=1}^k \mathcal{E}_\varepsilon(u_\varepsilon^i) = \sum_{i=1}^k \int_{\mathbb{R}^N} f_\varepsilon(x_i + \varepsilon x, u_i, \nabla u_i). \end{aligned}$$

Using our assumption **(U)** and the dominated convergence theorem, one gets as $\varepsilon \rightarrow 0$:

$$F(u) \leq \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \sum_{i=1}^k \int_{\mathbb{R}^N} f(x_i, u_i, \nabla u_i) \leq \sum_{i=1}^k H(x_i, m_i) + k\eta = \mathbf{M}^H(u) + k\eta.$$

The conclusion follows by arbitrariness of $\eta > 0$. \square

4.3. Proof of the main Γ -convergence result. We now explain how **Theorem 1.2** follows from **Proposition 4.1** and **Proposition 4.2**.

Proof of Theorem 1.2. The lower bound $\Gamma(\mathcal{C}'_0) - \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^{H_f^-}$ follows from **Proposition 4.1**, and the upper bound $\Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon \leq \mathbf{M}^{H_{f,\text{lsc}}}$ from **Proposition 4.2**, where the assumption **(U)** is a consequence of **(H₄)**, **(H₇)** and **(H₈)**. In the case $N = 2$, by **Proposition 2.6** and assumption **(H₆)**, we have $H_f^- = H_f$, and $H_f \geq H_{f,\text{lsc}}$ by definition, so that both $\Gamma - \liminf$ and $\Gamma - \limsup$ (for weak and narrow topologies) coincide. In the case $N = 1$, by **Proposition 2.7** either we have again $H_f^- = H_f$ on $\mathbb{R}^N \times \mathbb{R}_+$ by **(H₆)** and **(H₃)**, or we have $H_f(x_0, \cdot) \equiv +\infty$ on $(0, +\infty)$ for some $x_0 \in \mathbb{R}$. In that case, by **(H₇)** we necessarily have $\mathcal{E}_\varepsilon(u) = +\infty$ for every u such that $\int_{\mathbb{R}^N} u > 0$, and $H_f(x, \cdot) \equiv H_f(x_0, \cdot)$ on \mathbb{R}_+ for every $x \in \mathbb{R}$. It implies that both $\Gamma - \liminf$ and $\Gamma - \limsup$ (for the weak and narrow topologies) coincide with⁸ $\chi_{\{0\}} = \mathbf{M}^{H_f}$ on $\mathcal{M}_+(\mathbb{R}^N)$. Similarly, they coincide on finite negative measures. \square

5. EXAMPLES, COUNTEREXAMPLES AND APPLICATIONS

5.1. Scale-invariant Lagrangians and necessity of assumption **(H₆).** Our assumption **(H₆)** is not very standard, but we need a condition of this type in order to get Γ -convergence of the rescaled energies \mathcal{E}_ε towards \mathbf{M}^{H_f} , as shown by the following class of scale-invariant Lagrangians:

$$(5.1) \quad f_\varepsilon(x, u, \xi) = f(u, \xi) \quad \text{with} \quad f(u, \xi) = \begin{cases} u^{p(\frac{1}{p^*}-1)} |\xi|^p & \text{if } u > 0, \\ 0 & \text{else,} \end{cases}$$

where $p \in (1, N)$, $N \in \mathbb{N}^*$ and $p^* := \frac{pN}{N-p}$. By straightforward computations, $\mathcal{E}_\varepsilon(u) = \mathcal{E}_f(u) := \int_{\mathbb{R}^N} f(u, \nabla u)$ for every $\varepsilon > 0$ and $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^N)$ in that case.

Moreover, the associated minimal cost function H_f is not trivial. Indeed, applying the Gagliardo–Nirenberg–Sobolev inequality,

$$\left(\int_{\mathbb{R}^N} |v|^{p^*} \right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} |\nabla v|^p \right)^{\frac{1}{p}}, \quad \forall v \in L^{p^*} \cap W_{\text{loc}}^{1,1}(\mathbb{R}^N),$$

to the function⁹ $v = u^{\frac{1}{p^*}}$, we obtain that for every $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+) \cap L^1(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} u \right)^{\frac{p}{p^*}} \leq \left(\frac{C}{p^*} \right)^p \int_{\{u>0\}} u^{\frac{p}{p^*}-p} |\nabla u|^p = \left(\frac{C}{p^*} \right)^p \mathcal{E}_f(u).$$

⁸ $\chi_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise.

⁹Actually, we apply it to $v_\varepsilon = \phi_\varepsilon(u)$ where ϕ_ε is a suitable approximation of $(\cdot)^{\frac{1}{p^*}}$ and take $\varepsilon \rightarrow 0$.

Hence, for every $m > 0$, we have $H_f(m) > 0$, and even $H_f(m) < +\infty$ since any function $u = v^{p^*}$, with $v \in W^{1,p}(\mathbb{R}^N, \mathbb{R}_+)$, has finite energy. Replacing u by mu in the infimum defining H_f in (1.2), we actually obtain

$$(5.2) \quad H_f(m) = m^{1-\frac{p}{p^*}} H_f(1), \quad 0 < H_f(1) < +\infty.$$

In that case, it is clear that the Γ -limit of $\mathcal{E}_\varepsilon \equiv \mathcal{E}$ in the weak or narrow topology of $\mathcal{M}_+(\mathbb{R}^N)$, that is the lower semicontinuous relaxation of \mathcal{E}_f , does not coincide with \mathbf{M}^{H_f} ; indeed, the first functional is finite on diffuse measures whose density has finite energy, while the second functional is always infinite for non-trivial diffuse measures since $H'_f(0) = +\infty$.

These scaling invariant Lagrangians are ruled out by our assumption (H₆). All the other assumptions are satisfied except (H₅). Note that the following perturbation of f ,

$$\tilde{f}(u, \xi) = (1 + u^{p(\frac{1}{p^*}-1)})|\xi|^p$$

satisfies all the assumptions except (H₆), and provides a counterexample to the Γ -convergence. Indeed, $\mathbf{M}_{H_{\tilde{f}}} \geq \mathbf{M}_{H_f}$ is still infinite on diffuse measures, while (the relaxation of) $\mathcal{E}_{\tilde{f}}$ is finite for any diffuse measure whose density has finite energy.

We stress that an assumption like (H₆) is actually needed, even for the lower semicontinuity of the function H_f – recall that if \mathbf{M}_{H_f} is a Γ -limit, then it must be lower semicontinuous by [Bra02, Proposition 1.28], which in turn implies that the function H_f is lower semicontinuous by Proposition 2.5. Indeed, consider the Lagrangians

$$f(x, u, \xi) = (1 + u^{p(\frac{1}{p^*}-1)})|\xi|^{p(x)},$$

with $p \in \mathcal{C}^0(\mathbb{R}^N, (1, N))$ such that $p(0) = p \in (1, N)$ and $p(x) > p$ when $x \neq 0$. Then, we have $H_f(0, m) = m^{1-\frac{p}{p^*}} H(1)$, but $H_f(x, \cdot) \equiv 0$ if $x \neq 0$ as can be easily seen via the change of function $\varepsilon^N u(\varepsilon \cdot)$, with $\varepsilon > 0$ small.

5.2. General concave costs in dimension one. It has been proved in [Wir19] that for any continuous concave function $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $H(0) = 0$, there exists a function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $c(0) = 0$, $u \mapsto \frac{c(u)}{u}$ is lower semicontinuous and non-increasing on $(0, +\infty)$, and for every $m \geq 0$,

$$H(m) = \inf \left\{ \int_{\mathbb{R}} |u'|^2 + c(u) : u \in W_{\text{loc}}^{1,1}(\mathbb{R}, \mathbb{R}_+), \int_{\mathbb{R}} u = m \right\}.$$

The Lagrangians of the form $f_\varepsilon(x, u, \xi) = |\xi|^2 + c(u)$, in dimension $N = 1$, satisfy all our assumptions (H₁)–(H₈), hence our Γ -convergence result stated in Theorem 1.2 yields the Γ -convergence of the functionals

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon^3 |u'|^2 + \frac{c(\varepsilon u)}{\varepsilon}, \quad u \in W^{1,2}(\mathbb{R}, \mathbb{R}_+),$$

towards \mathbf{M}^H for both the weak and narrow convergence of measures. Therefore, we may find an elliptic approximation of any concave H -mass. Let us stress that c is determined in [Wir19] from H through several operations including a deconvolution problem, but no closed form solution is given in general; nonetheless, an explicit solution is provided if c is affine by parts.

In higher dimension $N \geq 2$, [Proposition 2.8](#) tells us that the class of functions $H = H_f$ with f satisfying [\(H₁\)–\(H₈\)](#) is smaller, namely, H must satisfy:

$$(5.3) \quad \exists m_* \geq 0, \quad \begin{cases} H \text{ is linear on } [0, m_*], \\ H \text{ is strictly concave } (m_*, +\infty). \end{cases}$$

We have no positive or negative answer to the inverse problem, consisting in finding f satisfying our assumptions such that $H_f = H$, for a given continuous concave function $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying [\(5.3\)](#).

5.3. Homogeneous costs in any dimension. In this section, we provide Lagrangians f to obtain the α -mass $\mathbf{M}^\alpha := \mathbf{M}^{t \rightarrow t^\alpha}$ in any dimension N for a wide range of exponents, including exponents $\alpha \in (1 - \frac{1}{N}, 1]$. We consider for every $p \in [1, +\infty)$, $s \in (-\infty, 1]$ and $N \in \mathbb{N}^*$, the energy defined for every $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+)$ by

$$(5.4) \quad \mathcal{E}_f(u) := \int_{\mathbb{R}^N} f(u, \nabla u) := \int_{\mathbb{R}^N} |\nabla u|^p + u^s.$$

Notice that for $p > 1$, f satisfies all our hypotheses [\(H₁\)–\(H₆\)](#) (without dependence on x), [\(H₆\)](#) holding in dimension $N \geq 2$ with $\rho(t) = t$ for example. Thus by [Theorem 1.2](#) the re-scaled energies Γ -converge to the H_f -mass.

One may compute H_f substituting u by v such that $u = m\lambda^N v(\lambda \cdot)$ in [\(1.2\)](#), where

$$(5.5) \quad \lambda = m^{\frac{s/p-1}{1+N-sN/p}}.$$

Straightforward computations give $\int_{\mathbb{R}^N} v = 1$ if $\int_{\mathbb{R}^N} u = m$, and

$$\mathcal{E}_f(u) = m^\alpha \mathcal{E}_f(v), \quad \text{where } \alpha = \frac{1 - \frac{s}{p} + \frac{s}{N}}{1 - \frac{s}{p} + \frac{1}{N}},$$

thus

$$H_f(m) = cm^\alpha, \quad \text{where } c = H_f(1).$$

We look for cases when the cost is non-trivial, i.e. neither identically zero nor infinite on $(0, +\infty)$. Take an auxiliary exponent $q \in [1, +\infty)$ and $\alpha \in [0, 1]$ such that $1 = \alpha q + (1 - \alpha)s$. By Hölder inequality,

$$\int_{\mathbb{R}^N} u = \int_{\mathbb{R}^N} u^{\alpha q} u^{(1-\alpha)s} \leq \left(\int_{\mathbb{R}^N} u^q \right)^\alpha \left(\int_{\mathbb{R}^N} u^s \right)^{1-\alpha}.$$

Moreover, choosing $q \in (1, p^*)$ if $p < N$ and any $q \in (1, +\infty)$ if $p \geq N$, by the Gagliardo–Nirenberg–Sobolev inequality, for every $u \in W_{\text{loc}}^{1,1} \cap L^1(\mathbb{R}^N, \mathbb{R}_+)$,

$$\left(\int_{\mathbb{R}^N} u^q \right)^{\frac{1}{q}} = \|u\|_{L^q} \leq C \|\nabla u\|_{L^p}^\beta \|u\|_{L^1}^{1-\beta},$$

with $\beta \in (0, 1)$ such that $\frac{1}{q} = \beta \left(\frac{1}{p} - \frac{1}{N} \right) + (1 - \beta)$. Hence,

$$\left(\int_{\mathbb{R}^N} u \right)^{1-q\alpha(1-\beta)} \leq C \left(\int_{\mathbb{R}^N} |\nabla u|^p \right)^{\frac{q\alpha\beta}{p}} \left(\int_{\mathbb{R}^N} u^s \right)^{1-\alpha},$$

and the cost is non-zero for every $m > 0$.

In the case $s \in [0, 1]$, any $u = v^r$ with $v \in \mathcal{C}_c^1(\mathbb{R}^N)$ is a competitor with finite energy, thus \mathcal{E}_f is non-trivial for every $p \in [1, +\infty)$. In the case $s < 0$, consider the

competitor $u : x \mapsto (1 - |x|)_+^\gamma$ for $\gamma > 0$ to be fixed later. Then $\int_{\mathbb{R}^N} |\nabla u|^p < +\infty$ if and only if $t \mapsto (1-t)^{(\gamma-1)p}$ is integrable at 1^- , i.e. $(\gamma-1)p > -1 \iff \gamma > 1-1/p$, and $\int_{\{u>0\}} u^s < +\infty$ if and only if $\gamma s > -1 \iff \gamma < -1/s$. Therefore, one may find $\gamma > 0$ satisfying both conditions, and ensure that H_f is non-trivial, if

$$-p' < s < 0.$$

To summarize, we have shown that H_f is non-trivial if:

$$s \in (-p', 1].$$

Since α is monotone in s , one may easily compute the range of α . If p and N are fixed, α ranges over $\left(\frac{N-1}{N+1-1/p}, 1\right]$ when $s \in (-p', 1]$. Notice that when $N = 1$ we obtain the whole range $\alpha \in (0, 1]$, and at least the range $\left[1 - \frac{1}{N}, 1\right]$ for every p in dimension $N \geq 2$. Finally, we obtain a range $\alpha \in \left(1 - \frac{2}{N+1}, 1\right]$ when p ranges over $(1 + \infty)$ in dimension N .

5.4. Branched transport approximation: H -masses of normal 1-currents.

Branched Transport is a variant of classical optimal transport (see [San15] and Section 4.4.2 therein for a brief presentation of branched transport, and [BCM09] for a vast exposition) where the transport energy concentrates on a network, i.e. a 1-dimensional subset of \mathbb{R}^d , which has a graph structure when optimized with prescribed source and target measures. It can be formulated as a minimal flow problem,

$$\min \left\{ \mathbf{M}_1^H(w) : \operatorname{div}(w) = \mu^- - \mu^+ \right\},$$

where μ^\pm are probability measures on \mathbb{R}^d , $H : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is mass-concave, and the H -mass \mathbf{M}_1^H is this time defined for finite vector measures $w \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$ whose distributional divergence is also a finite measure; in the language of currents, it is called a 1-dimensional normal current. Any such measure may be decomposed into a 1-rectifiable part $\theta\xi \cdot \mathcal{H}^1 \llcorner \Sigma$ where $\theta(x) \geq 0$ and $\xi(x)$ is a unit tangent vector to Σ for \mathcal{H}^1 -a.e. $x \in \Sigma$, and a 1-diffuse part w^\perp satisfying $|w^\perp|(A) = 0$ for every 1-rectifiable set A :

$$w = \theta\xi \cdot \mathcal{H}^1 \llcorner M + w^\perp.$$

The H -mass is then defined by:

$$(5.6) \quad \mathbf{M}_1^H(w) := \int_\Sigma H(x, \theta(x)) \, d\mathcal{H}^1(x) + \int_{\mathbb{R}^d} H'(x, 0) \, d|w^\perp|.$$

In the case $H(x, m) = m^\alpha$ with $0 < \alpha < 1$, a family of approximations of these functional has been introduced in [OS11]:

$$(5.7) \quad \mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} \varepsilon^{\gamma_1} |\nabla w|^2 + \varepsilon^{-\gamma_2} |w|^\beta & \text{if } w \in W_{\text{loc}}^{1,2}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

with $\beta = \frac{2-2d+2\alpha d}{3-d+\alpha(d-1)}$, $\gamma_1 = (d-1)(1-\alpha)$ and $\gamma_2 = 3-d+\alpha(d-1)$. It has been shown in [OS11; Mon17] that the functionals \mathcal{E}_ε Γ -converge as $\varepsilon \rightarrow 0^+$, in the topology of weak convergence of u and its divergence, to a non-trivial multiple of the α -mass $\mathbf{M}_1^\alpha := \mathbf{M}_1^H$ with $H(x, m) = m^\alpha$ in dimension $d = 2$. The result extends to any dimension d , by [Mon15], thanks to a slicing method that relates the energy \mathcal{E}_ε with the energy of the sliced measures $u = (w \cdot \nu)_+$ supported on the

slices $V_a = \{x \in \mathbb{R}^d : x \cdot \nu = a\} \simeq \mathbb{R}^N$, for any given unit vector $\nu \in \mathbb{R}^d$, defined by

$$\bar{\mathcal{E}}_\varepsilon(u) = \int_{\mathbb{R}^N} \varepsilon^{\gamma_1} |\nabla u|^2 + \varepsilon^{-\gamma_2} |u|^\beta.$$

The functionals $\bar{\mathcal{E}}_\varepsilon$ Γ -converge as $\varepsilon \rightarrow 0^+$, in the narrow topology, to $c\mathbf{M}^\alpha$ for some non-trivial c , as shown in [Section 5.3](#), and one may recover every α -mass in this way for $\alpha \in \left(\frac{2d-4}{2d+1}, 1\right]$, and in particular every so-called super-critical exponents for Branched Transport in dimension d , that is $\alpha \in (1 - 1/d, 1]$.

The same slicing method would allow to extend our Γ -convergence result stated in [Theorem 1.2](#) to functionals defined on vector measure

$$(5.8) \quad \mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} f_\varepsilon(x, \varepsilon^{d-1}|w|(x), \varepsilon^d|\nabla w|(x))\varepsilon^{1-d} dx & \text{if } w \in W_{\text{loc}}^{1,1}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

for Lagrangians $f_\varepsilon \rightarrow f$ fitting the framework of [Theorem 1.2](#). The expected Γ -limit, for the weak topology of measures and their divergence measure, would be the functional $\mathbf{M}_1^{H_f}$, with H_f defined in [\(1.2\)](#). Note that this approach would provide approximations of H -masses for more general continuous and concave cost functions $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $H(0) = 0$. By [\[Wir19\]](#), we would obtain all such H -masses when $N = 1$ (corresponding to $d = 2$).

5.5. A Cahn-Hilliard model for droplets. Following the works [\[BDS96\]](#) in the one-dimensional case and [\[Dub98\]](#) in higher dimension, we consider functionals on $\mathcal{M}_+(\mathbb{R}^N)$ of the form:

$$(5.9) \quad \mathcal{W}_\varepsilon(u) = \begin{cases} \int_{\mathbb{R}^N} \varepsilon^{-\rho}(W(u) + \varepsilon|\nabla u|^2) & \text{if } u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+), \\ +\infty & \text{otherwise,} \end{cases}$$

where $W : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a Borel function satisfying $W(t) \sim_{t \rightarrow +\infty} u^s$ for some exponent $s \in (-\infty, 1)$. In [\[BDS96; Dub98\]](#), it is in particular proven, under some assumptions on the slope of W at 0 and its regularity, that the family $(\mathcal{W}_\varepsilon)_{\varepsilon > 0}$ Γ -converges to a non-trivial multiple of the α -mass, $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$, when $s \in (-2, 1)$ and $\rho = \rho(s, N) := \frac{N(1-s)}{(N+2)+N(1-s)}$. In this section, we recover this Γ -convergence result using our general model.

Replacing ε with $\bar{\varepsilon} := \varepsilon^{(N+2)+N(1-s)}$ and noticing that $1 - \rho = \frac{N+2}{(N+2)+N(1-s)}$, one gets for every $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^N, \mathbb{R}_+)$:

$$\begin{aligned} \mathcal{W}_{\bar{\varepsilon}}(u) &= \int_{\mathbb{R}^N} \varepsilon^{-N(1-s)}W(u) + \varepsilon^{N+2}|\nabla u|^2 \\ &= \int_{\mathbb{R}^N} ([\varepsilon^{Ns}W(\varepsilon^{-N}\varepsilon^N u)] + |\varepsilon^{N+1}\nabla u|^2)\varepsilon^{-N} \\ &= \int_{\mathbb{R}^N} f_\varepsilon^W(x, \varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N}, \end{aligned}$$

where f_ε^W is defined for every $x \in \mathbb{R}^N$, $u \in \mathbb{R}_+$, $\xi \in \mathbb{R}^N$ by

$$f_\varepsilon^W(x, u, \xi) := W_\varepsilon(u) + |\xi|^2 \quad \text{and} \quad W_\varepsilon(u) := \varepsilon^{Ns}W(\varepsilon^{-N}u).$$

Therefore if we take $f_\varepsilon = f_\varepsilon^W$ in our general model [\(1.3\)](#) we exactly get $\mathcal{W}_{\bar{\varepsilon}} = \mathcal{E}_{\bar{\varepsilon}}$. The fact that $W(u) \sim u^s$ as $u \rightarrow +\infty$ implies that W_ε converges pointwise to the

map $w_s : u \mapsto u^s$ if $u > 0$, $w_s(0) = 0$, hence f_ε^W converges to $f_s : (x, u, \xi) \mapsto w_s(u) + |\xi|^2$.

Theorem 5.1. *Assume that $W : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies:*

- (HW₁) W is lower semicontinuous,
- (HW₂) $\{W = 0\} = \{0\}$,
- (HW₃) $W(u) \sim_{u \rightarrow +\infty} u^s$ for some $s \in (-\infty, 1)$,
- (HW₄) $\sup_{u>0} \frac{W(u)}{u^s} < +\infty$,
- (HW₅) $0 < \liminf_{u \rightarrow 0^+} \frac{W(u)}{u}$.

Then $(W_\varepsilon)_{\varepsilon>0}$ Γ -converges to $\mathbf{M}^{H_{f_s}}$, for both topologies \mathcal{C}'_0 and \mathcal{C}'_b , and if $s \in (-2, 1]$ then $\mathbf{M}^{H_{f_s}}$ is a non-trivial multiple of \mathbf{M}^α where $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$.

To prove this theorem, we start with a simple lemma.

Lemma 5.2. *Assume that W satisfies (HW₁)–(HW₅). Then for every $\delta \in (0, 1)$, there exists $c_\delta \in (0, +\infty)$ such that for every $\varepsilon > 0$ and every $u \in \mathbb{R}_+$,*

$$(5.10) \quad \delta(u^s \wedge c_\delta \varepsilon^{-N(1-s)} u) \leq W_\varepsilon(u).$$

Proof. Fix $\delta \in (0, 1)$. There exists $M > 0$ such that $\delta u^s \leq W(u)$ for every $u \geq M$. Besides, the map $w : u \mapsto W(u)/u$ is lower semicontinuous and positive on $(0, M]$ by (HW₁) and (HW₂), and since $\liminf_{u \rightarrow 0} w(u) > 0$ by (HW₅), w is necessarily bounded from below on $(0, M]$ by some constant $c > 0$. As a consequence $W_\varepsilon(u) \geq \delta u^s$ if $u \geq \varepsilon^N M$ and $W_\varepsilon(u) \geq c \varepsilon^{N(s-1)} u$ if $u \leq \varepsilon^N M$, hence:

$$\forall u \in \mathbb{R}, \quad W_\varepsilon(u) \geq \delta(u^s \wedge c \varepsilon^{-N(1-s)} u). \quad \square$$

Proof of Theorem 5.1. By (HW₄), there exists a constant C such that $f_\varepsilon^W \leq C f_s$ for every ε , and since f_ε^W does not depend on the x variable and converges pointwise to f_s , (U) is satisfied and our Γ –lim sup result stated in Proposition 4.2 yields

$$\mathbf{M}^{H_{f_s}} \geq \Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon.$$

Fix $\delta \in (0, 1)$. By Lemma 5.2, there exists c_δ such that

$$\forall x, u, \xi, \quad f_\varepsilon^W(x, u, \xi) \geq \delta(|\xi|^2 + (u^s \wedge c_\delta \varepsilon^{-N(1-s)} u)) =: f_\varepsilon^\delta(x, u, \xi).$$

It is easy to check that f_ε^δ satisfies (H₁), (H₂) and (H₅) for every $\varepsilon > 0$. Moreover $f_\varepsilon^\delta \uparrow \delta f_s$ and $(f_\varepsilon^\delta)'_-(\cdot, 0^+, 0) = \delta c_\delta \varepsilon^{-N(1-s)} \uparrow (+\infty) = (\delta f_s)'_-(\cdot, 0^+, 0)$ as $\varepsilon \rightarrow 0$, thus (H₈) holds for the family $(f_\varepsilon^\delta)_{\varepsilon>0}$, and by applying our Γ –lim inf result stated in Proposition 4.1 to the energies $\mathcal{E}_\varepsilon^\delta$ induced by f_ε^δ we get:

$$\Gamma(\mathcal{C}'_0) - \liminf \mathcal{E}_\varepsilon \geq \Gamma(\mathcal{C}'_0) - \liminf \mathcal{E}_\varepsilon^\delta \geq \mathbf{M}^{H_{\delta f_s}}.$$

We get the result by taking the limit $\delta \rightarrow 1$, noticing that $(f_s)'_-(\cdot, 0^+, 0) = +\infty$, so that $H_{\delta f_s}^- = H_{\delta f_s} = \delta H_{f_s}$ and $\mathbf{M}^{H_{\delta f_s}^-} = \mathbf{M}^{\delta H_{f_s}} = \delta \mathbf{M}^{H_{f_s}}$. \square

Remark 5.3. We recover the Γ -convergence results of [BDS96] and [Dub98] when $s \in (-2, 1)$ under slightly more general assumptions: besides (HW₂) and (HW₃), the authors impose the existence of a non-trivial slope $\lim_{u \rightarrow 0} \frac{W(u)}{u} \in (0, +\infty)$ and a regularity condition (either W is of class \mathcal{C}^1 or continuous and non-decreasing close to 0), which are stronger than (HW₁), (HW₄) and (HW₅). Let us stress

however that these works also tackle the cases $s < -2$ in any dimension, where the exponent ρ has to be fixed to $\rho(-2, N)$, and the case $s = -2$ in dimension one, where a logarithmic factor must be introduced, replacing $\varepsilon^{-\rho}$ with $\varepsilon^{-\rho(-2,1)}|\log \varepsilon|^{-1} = \varepsilon^{-1/2}|\log \varepsilon|^{-1}$. This implies that in our model we get a trivial Γ -limit when $s \leq -2$, namely $H_{f_s} \equiv +\infty$ on $(0, +\infty)$.

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