Mass concentration in rescaled first order integral functionals

Antonin Monteil, Paul Pegon

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We consider first order local minimization problems min $\int_{\mathbb{R}^N} f(u, \nabla u)$ under a mass constraint $\int_{\mathbb{R}^N} u = m \in \mathbb{R}$. We prove that the minimal energy function $H(m)$ is always concave on $(-\infty, 0)$ and $(0, +\infty)$, and that relevant rescalings of the energy, depending on a small parameter ε, Γ-converge in the weak topology of measures towards the H-mass, defined for atomic measures $\sum_i m_i \delta_{x_i}$ as $\sum_i H(m_i)$. We also consider space dependent Lagrangians $f(x, u, \nabla u)$, which cover the case of space dependent H-masses $\sum_i H(x_i, m_i)$, and also the case of a family of Lagrangians $(f_\varepsilon)_\varepsilon$ converging as $\varepsilon \to 0$. The Γ-convergence result holds under mild assumptions on f, and covers several situations including homogeneous H-masses in any dimension $N \geq 2$ for exponents above a critical threshold, and all concave H-masses in dimension $N = 1$. Our result yields in particular the concentration of Cahn-Hilliard fluids into droplets, and is related to the approximation of branched transport by elliptic energies.

Contents

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Notation

$B_r(x)$	open ball of radius r centered at x;
B_r	open ball $B_r(0)$;
$\mathcal{M}(\mathbb{R}^N)$	set of finite signed Borel measures on $\mathbb{R}^N$;
$\mathcal{M}_+(\mathbb{R}^N)$	set of finite positive Borel measures on $\mathbb{R}^N$;
$\tau_x\mu$	Borel measure $A \mapsto \mu(A-x)$ if $\mu \in \mathcal{M}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$;
$c_B\mu$	Borel measure $\tau_{-x}(\mu \llcorner B)$ if B is the ball $B_r(x)$;
$\mu_\ell \xrightarrow{\mathcal{C}_0'} \mu$	weak convergence of measures, i.e. weak-$\star$ convergence in duality with the space $\mathcal{C}_0(\mathbb{R}^N)$ of continuous functions vanishing at infinity;
$\mu_\ell \xrightarrow{\mathcal{C}_b'} \mu$	narrow convergence of measures, i.e. weak-$\star$ convergence in duality with thhe space of continuous and bounded function $\mathcal{C}_b(\mathbb{R}^N)$;
Σ	set of increasing maps $\sigma : \mathbb{N} \to \mathbb{N}$;
$\sigma_1 \preceq \sigma_2$	$\sigma_1, \sigma_2 \in \Sigma$ are such that $\sigma_1(\llbracket n, +\infty \llbracket) \subseteq \sigma_2(\mathbb{N})$ for some $n \in \mathbb{N}$.

1 Introduction

1.1 Setting

Let $N \in \mathbb{N}^*$ and let $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ be a Borel function. Consider the following energy functional, defined for any fixed $x \in \mathbb{R}^N$ on the set of finite Borel measures $\mathcal{M}(\mathbb{R}^N)$ on $\mathbb{R}^N$ by

$$\mathcal{E}_f^x(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} f(x, u(y), \nabla u(y))\,\mathrm{d}y & \text{if } u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

The minimization of this energy energy under a mass constraint gives rise to the notion of minimal cost function, defined by

$$H_f(x, m) := \inf \left\{ \mathcal{E}_f^x(u) \ : \ u \in W^{1,1}_{\mathrm{loc}} \cap L^1(\mathbb{R}^N) \text{ such that } \int_{\mathbb{R}^N} u = m \right\} \in [0, +\infty]. \tag{1.2}$$

Note that the minimization can be restricted to non-negative functions when $m \geq 0$ and non-positive functions when $m \leq 0$, as explained in Remark 2.2.

As a preliminary result, which deserves interest on its own, we will establish the following:

Theorem 1.1. *Let $x \in \mathbb{R}^N$. The map $m \mapsto H_f(x,m)$ is concave non-decreasing on $(0,+\infty)$. If we further assume that $f(x,0,0) = 0$ then $H_f(x,0) = 0$ and $H_f(x,\cdot)$ is either identically $+\infty$ on $(0,+\infty)$ or continuous on $[0,+\infty)$; it is in particular lower semicontinuous on $\mathbb{R}_+$. A similar statement holds on $(-\infty,0]$.*

The proof is very simple and works with no further assumptions on f.

Our main purpose is to prove that under some conditions, if $(f_\varepsilon)_{\varepsilon>0}$ is a family of functions $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0,+\infty]$ converging pointwise to f as $\varepsilon \to 0$, then the rescaled energy functionals $\mathcal{E}_\varepsilon$, defined for each $\varepsilon > 0$ on $\mathcal{M}(\mathbb{R}^N)$ by

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} f_\varepsilon(x, \varepsilon^N u(x), \varepsilon^{N+1}\nabla u(x))\varepsilon^{-N}\,\mathrm{d}x & \text{if } u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.3}$$

Γ-converge as $\varepsilon \to 0$, for the narrow or weak convergence of measures, to the H_f-mass, defined on $\mathcal{M}(\mathbb{R}^N)$ by (see Definition 2.3):

$$\mathbf{M}^{H_f}(u) \coloneqq \int_{\mathbb{R}^N} H_f(x, u(\{x\}))\,\mathrm{d}\,\mathcal{H}^0(x) + \int_{\mathbb{R}^N} H'_f(x,0^+)\,\mathrm{d}u^d_+(x) + \int_{\mathbb{R}^N} H'_f(x,0^-)\,\mathrm{d}u^d_-(x).$$

where $u = u^a + u^d$ is the decomposition of u into its atomic part u^a and its diffuse part u^d, and $u^d = u^d_+ - u^d_-$ is the decomposition of u^d into positive and negative parts, and $H'_f(x,0^\pm) = \lim_{m\to 0^\pm} \frac{H_f(x,m)}{|m|} \in [0,+\infty]$.

This kind of singular limit in integral functionals is reminiscent of several variational models with physical relevance which have been the object of intensive mathematical analysis, such as Cahn-Hilliard fluids with concentration on droplets [BDS96] or on singular interfaces [MM77], toy models for micromagnetism and liquid crystals like Aviles-Giga [AG99] and Landau-de Gennes [BPP12], or Ginzburg-Landau theory of supraconductivity [Hél94].

The fact that $\mathbf{M}^{H_f}$ is expected to be the Γ-limit of $\mathcal{E}_\varepsilon$ is due to the following observation: if $B_r(x_0) \subseteq \mathbb{R}^N$ and $u_\varepsilon(x) \coloneqq \varepsilon^{-N} v_\varepsilon(\varepsilon^{-1}(x - x_0))$, then $\int_{B_r(x_0)} u_\varepsilon = \int_{B_{r/\varepsilon}} v_\varepsilon$ and

$$\int_{B_r(x_0)} f_\varepsilon(x, \varepsilon^N u_\varepsilon(x), \varepsilon^{N+1}\nabla u_\varepsilon(x))\varepsilon^{-N}\,\mathrm{d}x = \int_{B_{r/\varepsilon}} f_\varepsilon(x_0 + \varepsilon y, v_\varepsilon(y), \nabla v_\varepsilon(y))\,\mathrm{d}y,$$

so that the energy contribution of a mass $m \geq 0$ contained in a ball $B_r(x_0)$ should be of the order of $H_f(x_0,m)$, where r is arbitrary.

Nevertheless, it is not true in general that $\mathbf{M}^{H_f}$ is the Γ-limit of the functionals $\mathcal{E}_\varepsilon$ (see Section 1.3 below). We will need a couple of assumptions on f and f_ε detailed in the next section.

1.2 Assumptions and main result

Our first two assumptions are rather standard and guarantee the sequential lower semicontinuity of the functionals $\mathcal{E}^x_f$,

(H$_1$) $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ is lower semicontinuous,

(H$_2$) $f(x, u, \cdot)$ is convex for every $x \in \mathbb{R}^N, u \in \mathbb{R}$.

In order for vanishing parts to have no energetic contribution, we will impose

(H$_3$) $f(x, 0, 0) = 0$ for every $x \in \mathbb{R}^N$.

We also need continuity in the spatial variable,

(H$_4$) $f(\cdot, u, \xi)$ is continuous for every $u \in \mathbb{R}_+, \xi \in \mathbb{R}^N$.

Next, we need a compactness assumption which ensures relative compactness in the weak topology of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for sequences of bounded energy $\mathcal{E}^x_f$ and bounded mass; it will also be needed in obtaining lower bounds for the energy (see Proposition 3.8):

(H$_5$) there exist $\alpha, \beta \in (0, +\infty)$, $p \in (1, +\infty)$ such that for all $(x, u, \xi) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N$,

$$f(x, u, \xi) \geq \alpha|\xi|^p - \beta u.$$

We also impose a condition on the slope of $f(x, \cdot, \xi)$ at the origin which will be needed in order to identify the initial slope of $H_f(x, \cdot)$ (see Section 2.3), and rules out some non-trivial scale invariant Lagrangians for which the expected Γ-convergence result fails (see Section 1.3),

(H$_6$) for every $x_0 \in \mathbb{R}^N$,

$$f'_-(x_0, 0^\pm, 0) := \liminf_{(x,u,\xi)\to(x_0,0^\pm,0)} \frac{f(x, u, \xi)}{|u|} \geq \limsup_{u\to 0^\pm} \sup_{|\xi|\leq 1} \frac{f(x_0, u, \rho(|u|)\xi)}{|u|}, \tag{1.4}$$

with $\rho \equiv 0$ if $N = 1$ and for some $\rho \in \mathcal{C}((0, 1], (0, +\infty))$ satisfying

$$\int_0^1 \Big(\int_y^1 \frac{\mathrm{d}t}{\rho(t)} \Big)^N \mathrm{d}y < +\infty \quad \text{if } N \geq 2.$$

Since our aim is not to care much about the dependence on x, we shall impose a spatial quasi-homogeneity condition:

(H$_7$) there exists $C < +\infty$ such that for every $x, y \in \mathbb{R}^N, u \in \mathbb{R}, \xi \in \mathbb{R}^N$,

$$f(y, u, \xi) \leq C(f(x, u, \xi) + u).$$

Last of all, we need the family of functions $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ to converge towards f in a suitable sense, namely, we assume

(H$_8$) $f_\varepsilon \uparrow f$ and $f'_{\varepsilon,-}(\cdot, 0^\pm, 0) \uparrow f'_-(\cdot, 0^\pm, 0)$ as $\varepsilon \to 0$.

Notice that this assumption is empty if f_ε does not depend on ε.

Our main result is the following:

Theorem 1.2. *If $(f_\varepsilon)_{\varepsilon>0}$ satisfies (H$_8$) with each f_ε satisfying (H$_1$)–(H$_5$) and the limit f satisfying (H$_6$)-(H$_7$), then $\mathbf{M}^{H_f}$ is the Γ-limit as $\varepsilon \to 0$ of the functionals $\mathcal{E}_\varepsilon$, defined in (1.3), for both the weak convergence and the narrow convergence of measures.*

In particular, as a Γ-limit, the functional $\mathbf{M}^{H_f}$ must be lower semicontinuous for the weak convergence of measures (and so for the narrow convergence as well). This implies that H_f is lower semicontinuous on $\mathbb{R}^N \times \mathbb{R}$ (see Proposition 2.5).

We point out that for the $\Gamma - \limsup$, we need weaker assumptions on f_ε and f (see Proposition 4.2), which will be useful for some applications (see Section 5.5).

1.3 Examples, counterexamples and applications

We start with two counterexamples, justifying the importance of (H_6) and (H_3), and we then provide several examples satisfying our assumptions. Decomposing measures into positive and negative parts, allows us to restrict our attention to positive measures, which we will do in all our examples. This can be thought as taking Lagrangians such that $f(x,u,\xi) = +\infty$ when $u < 0$.

Scale invariant Lagrangians. In the particular case where $f_\varepsilon \equiv f$ and $f(x,u,\xi) = u^{-p(1-\frac{1}{p^\star})}|\xi|^p$, with $p \in (1,N)$ and $p^\star = \frac{pN}{N-p}$, we find that

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(x, \varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N} = \int_{\mathbb{R}^N} u^{-p(1-\frac{1}{p^\star})}|\nabla u|^p = \mathcal{E}_f(u),$$

i.e. the rescaled energies $\mathcal{E}_\varepsilon$ do not depend on $\varepsilon > 0$. A scaling analysis also shows that the associated cost function satisfies $H_f(m) = m^{1-\frac{p}{N}}H_f(1)$. Moreover, it can be seen that $0 < H_f(1) < +\infty$, which implies that the Γ-limit of $\mathcal{E}_\varepsilon$, which is nothing but the lower semicontinuous relaxation of $\mathcal{E}_f$, does not coincide with $\mathbf{M}_{H_f}$. Considering the perturabation of f given by $\tilde{f}(x,u,\xi) = f(x,u,\xi)+|\xi|^p$, we find a Lagrangian satisfying all our assumptions except (H_6) (note that $|\xi|^p$ is needed in (H_5)), and such that the associated rescaled energies do not Γ-converge to $\mathbf{M}_{H_{\tilde{f}}}$ (see Section 5.1). Hence, an assumption like (H_6) is required in our Γ-convergence result. We will even see that the lower semicontinuity of H_f and $\mathbf{M}_{H_f}$ is not guaranteed without (H_6).

Lagrangians giving energy to vanishing functions. By assumption (H_3) no energy is given to any part where a function u vanishes. It is a necessary condition for $\mathbf{M}^{H_f}$ to be lower semicontinuous (a necessary condition to be a Γ-limit) and not identically $+\infty$. Indeed if $\mathbf{M}^{H_f}$ is lower semicontinuous and finite for some measure $u \in \mathcal{M}(\mathbb{R}^N)$ then thanks to Theorem 2.1 we know that $\mathbf{M}^{H_f}(tu) \leq \mathbf{M}^{H_f}(u)$ for every $t \in (0,1)$ and thus $\mathbf{M}^{H_f}(0) \leq \liminf_{t\to 0^+} \mathbf{M}^{H_f}(tu) \leq \mathbf{M}^{H_f}(u) < +\infty$. Thus $\mathbf{M}^{H_f}$ is not identically $+\infty$ if and only if $\mathbf{M}^{H_f}(0) < +\infty$, i.e. $\int_{\mathbb{R}^N} H_f(x,0)\,\mathrm{d}\,\mathcal{H}^0(x) < +\infty$. But since $H_f(x,0) = (+\infty) \times f(x,0,0)$ this can only happen if $f(\cdot,0,0) \equiv 0$. This justifies imposing (H_3).

Concave H-masses in dimension one. Consider the energy

$$\mathcal{E}_f(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + c(u) \quad \text{with Lagrangian} \quad f(x,u,\xi) = |\xi|^2 + c(u).$$

In dimension $N = 1$, it is shown in [Wir19] that for any concave continuous function H with $H(0) = 0$, there exists a suitable $c \geq 0$ such that $H_f = H$. As explained in Section 5.2, Theorem 1.2 implies that the rescaled energies

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(\varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N} = \int_{\mathbb{R}^N} \varepsilon^{N+2}|\nabla u|^2 + \varepsilon^{-N}c(\varepsilon^N u). \tag{1.5}$$

Γ-converge to $\mathbf{M}^H$, leading to an elliptic approximation of any concave H-mass in dimension one. In dimension $N \geq 2$, we will show that H_f must be concave on $(0,+\infty)$, and strictly concave after the possible initial interval where it is linear (see Proposition 2.8) ; however, we have no solution to the inverse problem, consisting in characterizing the class of attainable cost functions $H = H_f$ for Lagrangians f satisfying our assumptions.

Homogeneous H-masses in any dimension. We consider the functional

$$\mathcal{E}_f(u) = \int_{\mathbb{R}^N} f(u, \nabla u) = \int_{\mathbb{R}^N} |\nabla u|^p + u^s, \quad p > 1, \ s \in (-p', 1]. \tag{1.6}$$

Then, the rescaled energies

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(\varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N} = \int_{\mathbb{R}^N} \varepsilon^{pN+p-N}|\nabla u|^p + \varepsilon^{-(1-s)N}u^s$$

Γ-converge to a non-trivial multiple of some α-mass $\mathbf{M}^\alpha := \mathbf{M}^{t\mapsto t^\alpha}$ where the exponent $\alpha = (1 - \frac{s}{p} + \frac{s}{N})(1 - \frac{s}{p} + \frac{1}{N})^{-1}$ ranges over $(1 - \frac{2}{N+1}, 1]$ when (s, p) varies in its range and $N \geq 2$. More cases, with details, are given in Section 5.3.

Cahn-Hilliard approximations of droplets models. Following the works of [BDS96; Dub98], we consider the functionals

$$\mathcal{W}_\varepsilon(u) = \int_{\mathbb{R}^N} \varepsilon^{-\rho}(W(u) + \varepsilon|\nabla u|^2), \tag{1.7}$$

where $W(t) \sim_{t\to+\infty} t^s$ for some exponent $s \in (-2, 1)$. As shown in Section 5.5, we way rewrite these functionals to fit our general framework, and recover known Γ-convergence results, under slightly more general assumptions, as stated in Theorem 5.1. The Γ-limit is a nontrivial multiple of the α-mass with $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$.

Elliptic approximations of Branched Transport. The energy of Branched Transport (see [BCM09] for an account of the theory), in its Eulerian formulation, is an H-mass defined this time on vector measures w whose divergence is also a measure,

$$\mathbf{M}_1^H(w) := \int_\Sigma H(x, \theta(x))\, \mathrm{d}\mathcal{H}^1(x) + \int_{\mathbb{R}^d} H'(x, 0^+)\, \mathrm{d}|w^\perp|, \tag{1.8}$$

where $w = \theta\xi \cdot \mathcal{H}^1 \llcorner \Sigma + w^\perp$ is the decomposition of w into its 1-rectifiable and 1-diffuse parts (see Section 5.4 for more details). An elliptic approximation of Modica-Mortola type has been introduced in [OS11] for $H(m) = m^\alpha, \alpha \in (0, 1)$, and their Γ-convergence result in dimension $d = 2$ has been extended to any dimension in [Mon15] by a slicing method which relates the energy of w to the energy of its slicings. The same slicing method, together with Theorem 1.2, would allow to prove the Γ-convergence of the functionals

$$\mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} f_\varepsilon(x, \varepsilon^{d-1}|w|(x), \varepsilon^d|\nabla w|(x))\varepsilon^{1-d}\, \mathrm{d}x & \text{if } w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.9}$$

toward $\mathbf{M}_1^{H_f}$ for Lagrangians $f_\varepsilon \to f$ satisfying (H_1)–(H_8), thus covering a wide range of concave H-masses.

1.4 Structure of the paper

In Section 2, we prove the concavity of the cost function H_f with respect to the mass variable m in full generality (Theorem 2.1), we establish useful properties of general H-masses, and we identify the slope at the origin of H_f in terms of f under our assumption (Proposition 2.6 and Proposition 2.7). In Section 3, we apply a concentration-compactness principle to provide a profile decomposition theorem for sequences of positive measures (Theorem 3.2), which is used to obtain our main lower bound for the energy $\mathcal{E}_f$ (Proposition 3.10) and also yields an existence criterion for profiles with minimal energy under a mass constraint (Proposition 3.12). Section 4 is dedicated to proving lower and upper bounds on the rescaled energies $\mathcal{E}_\varepsilon$ (Proposition 4.1 and Proposition 4.2) that imply in particular our main Γ-convergence result (Theorem 1.2). Last of all, in Section 5, we provide several examples of energy functionals that fall into our framework, as summarized in the previous section.

2 Minimal cost function and H-mass

In this section, we study the properties of general H-masses, of costs H_f associated with general Lagrangians f, and we relate the slope of H_f at $m = 0$ to that of f at $(u, \xi) = (0, 0)$ in the variable u, under particular conditions.

2.1 Concavity and lower semicontinuity of the cost function

Our concavity result stated in Theorem 1.1 is a consequence of:

Theorem 2.1. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \to [0, +\infty]$ be Borel measurable and for every $m \in \mathbb{R}_+$,*

$$H_f(m) := \inf \left\{ \mathcal{E}_f(u) := \int_{\mathbb{R}^N} f(u, \nabla u) \ : \ u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+), \int_{\mathbb{R}^N} u = m \right\}. \tag{2.1}$$

Then H_f is either identically $+\infty$ on $(0, +\infty)$, or it is continuous, concave, non-decreasing on $(0, +\infty)$ with $\lim_{m \to 0^+} H_f(m) = 0$.

The fact that Theorem 2.1 implies Theorem 1.1 is due to the following remark.

Remark 2.2. The minimization in (1.2) can be restricted to non-negative (resp. non-positive) functions when $m \geq 0$ (resp. $m \leq 0$). For instance fix $x \in \mathbb{R}^N$, $m \geq 0$, and take $u \in W^{1,1}_{\mathrm{loc}} \cap L^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u = m$. If $u = u_+ - u_-$ is the decomposition of $u \in W^{1,1}_{\mathrm{loc}} \cap L^1(\mathbb{R}^N)$ into positive and negative parts, then $\mathcal{E}^x_f(u) = \mathcal{E}_g(u_+) + \mathcal{E}_{\tilde{g}}(u_-)$, with $g(u, \xi) = f(x, u, \xi)$ and $\tilde{g}(u, \xi) = f(x, -u, -\xi)$ for $(u, \xi) \neq (0, 0)$, $\tilde{g}(0, 0) = 0$. In particular $\mathcal{E}^x_f(u) \geq H_g(m_+) + H_{\tilde{g}}(m_-)$ where $m_\pm = \int_{\mathbb{R}^N} u_\pm$. Since H_g is non-decreasing on $(0, +\infty)$ and $H_{\tilde{g}}$ is non-negative, we have thus $\mathcal{E}^x_f(u) \geq H_g(m)$ since $m_+ \geq m$. A similar reasoning holds for non-positive functions when $m \leq 0$.

Note that, considering Lagrangians f taking infinite values, Theorem 2.1 covers the case where we have a constraint $(u, \nabla u) \in A$, where $A \subseteq \mathbb{R} \times \mathbb{R}^N$ is Borel measurable.

Proof. We first prove that H_f is concave on $(0, +\infty)$. Let $m > 0$ and $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$ such that $\int_{\mathbb{R}^N} u = m$. We pick a non-zero vector $v \in \mathbb{R}^N$ and for every $t \in \mathbb{R}$, we set $u^t(\cdot) = u(\cdot + tv)$ and

$$u \wedge u^t(\cdot) = \inf\{u(\cdot), u^t(\cdot)\}, \quad u \vee u^t(\cdot) = \sup\{u(\cdot), u^t(\cdot)\}.$$

We have $u \wedge u^t + u \vee u^t = u + u^t$. Hence

$$\int_{\mathbb{R}^N} u \wedge u^t + \int_{\mathbb{R}^N} u \vee u^t = 2 \int_{\mathbb{R}^N} u = 2m. \tag{2.2}$$

Moreover, it is standard that $u \wedge u^t = u - (u^t - u)_- \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\nabla(u \wedge u^t) = \nabla u$ a.e. in $\{u \le u^t\}$ and $\nabla(u \wedge u^t) = \nabla u^t$ a.e. in $\{u > u^t\}$. Since $u \vee u^t = u + u^t - u \wedge u^t$, we have similar identities for $u \vee u^t$, and we obtain

$$\mathcal{E}_f(u \wedge u^t) + \mathcal{E}_f(u \vee u^t) = \mathcal{E}_f(u) + \mathcal{E}_f(u^t) = 2\mathcal{E}_f(u). \tag{2.3}$$

Now, let $M : t \mapsto \int_{\mathbb{R}^N} u \wedge u^t$. In view of (2.2), (2.3), and by definition of H, we have proved

$$H_f(M(t)) + H_f(2m - M(t)) \le 2\mathcal{E}_f(u). \tag{2.4}$$

Now, by continuity of translations in L^1 and since the map $(x, y) \mapsto x \wedge y$ is Lipschitz on $\mathbb{R}^2$, we have that M is continuous on $\mathbb{R}$ with $M(0) = m$. Moreover $\lim_{t\to+\infty} M(t) = 0$ by dominated convergence. So, by the intermediate value theorem $M(\mathbb{R}) \supseteq (0, m]$. Hence, we have proved $H_f(\theta) + H_f(2m - \theta) \le 2\mathcal{E}_f(u)$ for every $\theta \in (0, m]$. Taking the infimum over u such that $\int_{\mathbb{R}^N} u = m$, we obtain

$$\frac{H_f(\theta) + H_f(2m - \theta)}{2} \le H_f(m), \quad \forall \theta \in (0, m],$$

that is, H_f is midpoint concave on $(0, +\infty)$. Since H_f is also bounded below (by 0), we can deduce that H_f is concave $(0, +\infty)$ (see [RV73, Section 72]).

We now justify that if $H_f(m) < +\infty$ for some $m > 0$, then $\lim_{m\to0^+} H_f(m) = 0$. By concavity, this will imply that H_f is finite, continuous and non-decreasing on $(0, +\infty)$. Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$ such that $\int_{\mathbb{R}^N} u = m > 0$ and $\mathcal{E}_f(u) < +\infty$, and set

$$t_* \coloneqq \sup\{t \ge 0 \ : \ M(t) > 0\} \in [0, +\infty], \quad \text{where} \quad M(t) = \int_{\mathbb{R}^N} u \wedge u^t.$$

Since M is continuous with $M(0) = \int_{\mathbb{R}^N} u > 0$ and $\lim_{t\to+\infty} M(t) = 0$ as seen above, we have that $t_* \in (0, +\infty]$, $\lim_{t\to t_*} M(t) = 0$ and $M(t)$ does not vanish near t_*. Moreover, if $t_* = +\infty$, since $u^t \to 0$ locally in measure, by dominated convergence,

$$\limsup_{m\to0^+} H_f(m) \le \limsup_{t\to+\infty} \mathcal{E}_f(u \wedge u^t) = \limsup_{t\to+\infty} \int_{\{u<u^t\}} f(u, \nabla u) + \int_{\{u^{-t}\ge u\}} f(u, \nabla u) = 0.$$

If $t_* < +\infty$, we have $u \wedge u^{t_*} = 0$ a.e. and $u^t \to u^{t_*}$ locally in measure as $t \to t_*$ by continuity of translation in L^1. Thus using dominated convergence again,

$$\begin{aligned}\limsup_{m\to0^+} H_f(m) \le \limsup_{t\to(t_*)^-} \mathcal{E}_f(u \wedge u^t) &= \limsup_{t\to(t_*)^-} \int_{\{u<u^t\}} f(u, \nabla u) + \int_{\{u^{-t}\ge u\}} f(u, \nabla u) \\ &= \int_{\{u<u^{t_*}\}} f(u, \nabla u) + \int_{\{u^{-t_*}\ge u\}} f(u, \nabla u).\end{aligned}$$

Besides,

$$+\infty > \int_{\{u<u^{t_*}\}} f(u, \nabla u) + \int_{\{u^{-t_*}\ge u\}} f(u, \nabla u) = \mathcal{E}_f(u \wedge u^{t_*}) = f(0, 0) \times (+\infty),$$

which implies that $f(0, 0) = 0 = H_f(0)$ and $\limsup_{m\to0^+} H_f(m) = 0$. □

2.2 H-transform and H-mass

Definition 2.3. Let $H : \mathbb{R}^N \times \mathbb{R} \to [0, +\infty]$ be a Borel measurable function having left/right slopes at the origin defined for each $x \in \mathbb{R}^N$ by

$$H'(x, 0^\pm) := \lim_{m \to 0^\pm} \frac{H(x, m)}{|m|} \in [0, +\infty]. \tag{2.5}$$

We define the H-transform of a finite signed Borel measure $u \in \mathcal{M}(\mathbb{R}^N)$ as:

$$H(u) := H(\cdot, u(\{\cdot\}))\, \mathcal{H}^0 + H'(\cdot, 0^+)u^d_+ + H'(\cdot, 0^-)u^d_-,$$

where $u = u^a + u^d$ is the decomposition of u into its atomic part u^a and its diffuse (or non-atomic) part u^d, $u^d = u^d_+ - u^d_-$ is the Hahn decomposition of u^d into positive and negative parts.

The H-mass of u is then defined as the total variation of $H(u)$, that is:

$$\mathbf{M}^H(u) := \int_{\mathbb{R}^N} H(x, u(\{x\}))\, \mathrm{d}\,\mathcal{H}^0(x) + \int_{\mathbb{R}^N} H'(x, 0^+)\, \mathrm{d}u^d_+(x) + \int_{\mathbb{R}^N} H'(x, 0^-)\, \mathrm{d}u^d_-(x).$$

$\mathbf{M}^H(u)$ is a natural spatially non-homogeneous extension (depending on the position x) of the H-mass of k-dimensional flat currents[1] from Geometric Measure Theory, introduced by [Fle66] (see also the more recent works [DH03; Col+17]).

From [BB90], we have the following result[2]:

Proposition 2.4 ([BB90])**.** *Assume that $H : \mathbb{R}^N \times \mathbb{R} \to [0, +\infty]$ is lower semicontinuous and that for each $x \in \mathbb{R}^N$, $H(x, 0) = 0$ and the map $m \mapsto H(x, m)$ is concave on $(0, +\infty)$ and $(-\infty, 0)$. Then $\mathbf{M}^H$ is sequentially lower semicontinuous on $\mathcal{M}(\mathbb{R}^N)$ for the weak topology.*

From another work from the same authors [BB93, Theorem 3.2], we know that under some further assumptions on H, $\mathbf{M}^H$ is the relaxation for the weak topology of the functional

$$\mathbf{M}^H_{\text{atom}}(u) = \begin{cases} \sum_{i=1}^k H(x_i, m_i) & \text{if } u = \sum_{i=1}^k m_i \delta_{x_i} \text{ with } k \in \mathbb{N}^*,\ x_i \in \mathbb{R}^N,\ m_i \in \mathbb{R}, \\ +\infty & \text{otherwise.} \end{cases}$$

We need a slightly different result[3], namely that for any function $H : \mathbb{R}^N \times \mathbb{R} \to [0, +\infty]$ satisfying all the assumptions of Proposition 2.4 exept the lower semicontinuity, the relaxation of $\mathbf{M}^H_{\text{atom}}$ for the narrow sequential convergence is $\mathbf{M}^{H_{\text{lsc}}}$, where H_{lsc} is the lower semicontinuous envelope of H, which can be expressed as

$$H_{\text{lsc}}(x, m) = \sup\{G(x, m) \ :\ G \leq H \text{ with } G \text{ lower semicontinuous.}\}. \tag{2.6}$$

It is straightforward to check that if $H(x, 0) = 0$ and $H(x, \cdot)$ is concave on $(-\infty, 0)$ and $(0, +\infty)$, then these properties hold also for H_{lsc}.

[1] In the case $k = 0$, since signed measures are merely 0-currents with finite mass.

[2] In the notations of this paper, we take $\mu = 0$ and $f(x, s) = |s|^2$; we have $\varphi_{f,\mu}(x, 0) = 0$ and $\varphi_{f,\mu}(x, s) = +\infty$ if $s \neq 0$.

[3] In [BB93, Theorem 3.2], H is assumed to be lower semicontinuous and the authors make a further coercivity assumption (assumption (3.5) in the paper) that we want to avoid.

Proposition 2.5. *Let $H : \mathbb{R}^N \times \mathbb{R} \to [0, +\infty]$ be a function such that for each $x \in \mathbb{R}^N$, $H(x, 0) = 0$ and the map $m \mapsto H(x, m)$ is concave on $(0, +\infty)$ and $(-\infty, 0)$. Then, the sequentially lower semicontinuous envelope of $\mathbf{M}^H_{\mathrm{atom}}$ in the narrow topology of $\mathcal{M}(\mathbb{R}^N)$ is given by $\mathbf{M}^{H_{\mathrm{lsc}}}$, namely we have:*

$$\mathbf{M}^{H_{\mathrm{lsc}}} = \sup\Big\{F \ : \ F \leq \mathbf{M}^H_{\mathrm{atom}},\ F \text{ sequentially narrowly l.s.c. on } \mathcal{M}(\mathbb{R}^N)\Big\}. \tag{2.7}$$

We point out that for a general H, for $\mathbf{M}^H$ to be sequentially lower semicontinuous (for the narrow topology) it is necessary that H is lsc on $\mathbb{R}^N \times (0, +\infty)$. However, neither the subadditivity of H in m nor its lower semicontinuity on $\mathbb{R}^N \times \mathbb{R}_+$ are necessary. Indeed, $\mathbf{M}^H$ is sequentially lower semicontinuous if for instance $H(x, m) = +\infty$ when $x \neq 0, m > 0$, $H(x, 0) = 0$ when $x \neq 0$ and $H(0, \cdot)$ is any lower semicontinuous function. Nevertheless the mass-subadditivity and lower semicontinuity would be necessary if H did not depend on x.

Proof of Proposition 2.5. Since H_{lsc} is lower semicontinuous and mass-subadditive, we know from Proposition 2.4 that $\mathbf{M}^{H_{\mathrm{lsc}}}$ is sequentially lower semicontinuous in the weak topology hence also in the narrow topology of $\mathcal{M}(\mathbb{R}^N)$. Since $\mathbf{M}^{H_{\mathrm{lsc}}} \leq \mathbf{M}^H_{\mathrm{atom}}$, we deduce that $\mathbf{M}^{H_{\mathrm{lsc}}}$ is lower or equal than the right hand side in (2.7).

In order to prove the opposite inequality, we take $F : \mathcal{M}(\mathbb{R}^N) \to \mathbb{R}_+$ the sequentially lower semicontinuous envelope of $\mathbf{M}^H_{\mathrm{atom}}$ in the narrow topology. We shall see that $F \leq \mathbf{M}^{H_{\mathrm{lsc}}}$.

We first prove that $F \leq \mathbf{M}^{H_{\mathrm{lsc}}}_{\mathrm{atom}}$. For this, we let $u = \sum_{i=1}^k m_i \delta_{x_i}$ be a finitely atomic positive measure and we let $u_n := \sum_{i=1}^k m_{i,n} \delta_{x_{i,n}}$ where for each $i \in \{1, \dots, k\}$, $(x_{i,n})_{n\in\mathbb{N}}$ is a sequence of points converging to x_i and $m_{i,n}$ is a sequence of non-negative numbers converging to m_i such that $H_{\mathrm{lsc}}(x_i, m_i) = \lim_{n\to\infty} H(x_{i,n}, m_{i,n})$. Then $(u_n)_{n\in\mathbb{N}}$ converges narrowly to u and, by lower semicontinuity,

$$F(u) \leq \liminf_{n\to\infty} F(u_n) \leq \liminf_{n\to\infty} \mathbf{M}^H_{\mathrm{atom}}(u_n) = \lim_{n\to\infty} \sum_{i=1}^k H(x_{i,n}, m_{i,n}) = \sum_{i=1}^k H_{\mathrm{lsc}}(x_i, m_i),$$

so that $F(u) \leq \mathbf{M}^{H_{\mathrm{lsc}}}_{\mathrm{atom}}(u)$ as wanted.

We now prove that $F \leq \mathbf{M}^{H_{\mathrm{lsc}}}$. Let $u \in \mathcal{M}(\mathbb{R}^N)$, let $u = u^a + u^d$ be the decomposition of u into its atomic part $u^a = \sum_{i=1}^k m_i \delta_{x_i}$, with $k \in \mathbb{N} \cup \{+\infty\}$ (here, $k = 0$ if there is no atom), and its diffuse part u^d, and let $u^d = u^d_+ - u^d_-$ be the decomposition of u^d into positive and negative parts. We then discretize $u^d_\pm$ by taking $n \in \mathbb{N}^*$, a partition $(Q^n_i)_{i\in\{1,\dots,(n2^n)^N\}}$ of $[-n, n)^N$ by means of cubes of the form $Q^n_i = c^n_i + 2^{-n}[-1, 1)^N$ with $c^n_i \in \mathbb{R}^N$, and we define

$$u_n := \sum_{i=1}^{n\wedge k} m_i \delta_{x_i} + \sum_{i=1}^{(n2^n)^N} u^d_+(Q^n_i)\delta_{x^n_i} + \sum_{i=1}^{(n2^n)^N} u^d_-(Q^n_i)\delta_{y^n_i},$$

where for each $i \in \{1, \dots, (n2^n)^N\}$, $x^n_i, y^n_i \in \bar{Q}^n_i$ are some points such that

$$H_{\mathrm{lsc}}{}'(x^n_i, 0^+) = \inf_{x\in\bar{Q}^n_i} H_{\mathrm{lsc}}{}'(x, 0^+), \quad H_{\mathrm{lsc}}{}'(y^n_i, 0^-) = \inf_{x\in\bar{Q}^n_i} H_{\mathrm{lsc}}{}'(x, 0^-). \tag{2.8}$$

Such points exist since $\bar{Q}^n_i$ is compact and since by concavity,

$$H_{\mathrm{lsc}}{}'(x, 0^\pm) = \sup_{\pm m>0} \frac{H_{\mathrm{lsc}}(x, m)}{|m|}, \tag{2.9}$$

so that $H'_{\mathrm{lsc}}(\cdot,0^{\pm})$ are lower semicontinuous as suprema of lower semicontinuous functions.

The sequence $(u_n)_{n\in\mathbb{N}}$ converges narrowly to u. We deduce using in turn the lower semicontinuity of the functional F, the inequality $F(u)\le \mathbf{M}^{H_{\mathrm{lsc}}}_{\mathrm{atom}}(u)$, the subadditivity of $H_{\mathrm{lsc}}(x,\cdot)$, (2.9), (2.8), and the monotone convergence, that

$$\begin{aligned}
F(u) &\le \liminf_{n\to\infty} \sum_{i=1}^{n\wedge k} H_{\mathrm{lsc}}(x_i,m_i) + \sum_{i=1}^{(n2^n)^N} H_{\mathrm{lsc}}(x_i^n, u_+^d(Q_i^n)) + \sum_{i=1}^{(n2^n)^N} H_{\mathrm{lsc}}(y_i^n, u_-^d(Q_i^n)),\\
&\le \liminf_{n\to\infty} \sum_{i=1}^{n\wedge k} H_{\mathrm{lsc}}(x_i,m_i) + \sum_{i=1}^{(n2^n)^N} H_{\mathrm{lsc}}{}'(x_i^n,0^+)u_+^d(Q_i^n) + \sum_{i=1}^{(n2^n)^N} H_{\mathrm{lsc}}{}'(y_i^n,0^-)u_-^d(Q_i^n)\\
&\le \liminf_{n\to\infty} \sum_{i=1}^{n\wedge k} H_{\mathrm{lsc}}(x_i,m_i) + \sum_{i=1}^{(n2^n)^N} \int_{Q_i^n} H_{\mathrm{lsc}}{}'(x,0^+)\,\mathrm{d}u_+^d + \int_{Q_i^n} H_{\mathrm{lsc}}{}'(x,0^-)\,\mathrm{d}u_-^d\\
&= \sum_{i=1}^{k} H_{\mathrm{lsc}}(x_i,m_i) + \int_{\mathbb{R}^N} H_{\mathrm{lsc}}{}'(x,0^+)\,\mathrm{d}u_+^d + \int_{\mathbb{R}^N} H_{\mathrm{lsc}}{}'(x,0^-)\,\mathrm{d}u_-^d\\
&= \mathbf{M}^{H_{\mathrm{lsc}}}(u).
\end{aligned}$$

□

2.3 Slope at the origin of the minimal cost function

Proposition 2.6. *Let $f:\mathbb{R}_+\times\mathbb{R}^N\to[0,+\infty]$ be a lower semicontinuous function such that $f(0,0)=0$, with $N\ge 2$. For every function $\rho\in\mathcal{C}((0,1],(0,+\infty))$ such that*

$$\int_0^1 \Big(\int_y^1 \frac{\mathrm{d}t}{\rho(t)}\Big)^N \mathrm{d}y < +\infty, \tag{2.10}$$

the function H_f defined in (2.1) satisfies

$$\lim_{m\to 0^+} \frac{H_f(m)}{m} \le \limsup_{u\to 0^+} \sup_{\xi\in\mathbb{S}^{N-1}} \frac{f(u,\rho(u)\xi)}{u}. \tag{2.11}$$

Proof. For every $y\ge 0$, we let

$$F(y) = \int_y^1 \frac{\mathrm{d}t}{\rho(t)} \in [0,+\infty].$$

The function F is decreasing, and belongs to $\mathcal{C}^1((0,1])$ and $L^N((0,1])$ by assumption. We now consider the solution of the ODE $v_\varepsilon' = -\rho(v_\varepsilon)$, with $v_\varepsilon(0)=\varepsilon$, given by

$$v_\varepsilon(r) = \begin{cases} F^{-1}(F(\varepsilon)+r), & \text{if } 0\le r < F(0)-F(\varepsilon),\\ 0 & \text{if } r\ge F(0)-F(\varepsilon).\end{cases}$$

Notice that $v_\varepsilon\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ because it is nonincreasing and bounded, hence it has finite total variation, and it is of class $\mathcal{C}^1$ except possibly at $r_\varepsilon := F(0)-F(\varepsilon)$, where it has no jump. As a consequence the radial profile defined by $u_\varepsilon(x) := v_\varepsilon(|x|)$ belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and we

compute, using the change of variables $s = v_\varepsilon(r)$ (i.e. $r = F(s) - F(\varepsilon)$) and an integration by parts combined with monotone convergence.

$$\begin{aligned} m_\varepsilon \coloneqq \int_{\mathbb{R}^N} u_\varepsilon &= |\mathbb{S}^{N-1}| \int_0^\infty v_\varepsilon(r) r^{N-1} \,\mathrm{d}r \\ &= -|\mathbb{S}^{N-1}| \int_0^\varepsilon s(F(s) - F(\varepsilon))^{N-1} F'(s) \,\mathrm{d}s \\ &= |\mathbb{S}^{N-1}| \lim_{t \downarrow 0} \left(\int_t^\varepsilon \frac{(F(s) - F(\varepsilon))^N}{N} \,\mathrm{d}s - \left[s \frac{(F(s) - F(\varepsilon))^N}{N} \right]_t^\varepsilon \right) \\ &= |\mathbb{S}^{N-1}| \int_0^\varepsilon \frac{(F(s) - F(\varepsilon))^N}{N} \,\mathrm{d}s \xrightarrow[\varepsilon \to 0]{} 0. \end{aligned}$$

The equality on the last line holds because $\lim_{t \to 0^+} \int_t^\varepsilon (F - F(\varepsilon))^N < +\infty$ (since $F \in L^N((0,1])$), hence $\lim_{t \to 0} t(F(t) - F(\varepsilon))^N$ exists by existence of the limit in the previous line, and it must be zero (again, because $F \in L^N((0,1])$).

Moreover, since $\sup_{[0,+\infty)} v_\varepsilon = \varepsilon$,

$$\mathcal{E}(u_\varepsilon) = \int_0^\infty \int_{\mathbb{S}^{N-1}} f(v_\varepsilon(r), v_\varepsilon'(r)\xi) r^{N-1} \,\mathrm{d}\mathcal{H}^{N-1}(\xi) \,\mathrm{d}r \le m_\varepsilon \sup_{u \le \varepsilon,\, |\xi| = 1} \frac{f(u, \rho(u)\xi)}{u}.$$

By assumption, we deduce that

$$\limsup_{m \to 0^+} \frac{H(m)}{m} \le \limsup_{\varepsilon \to 0^+} \frac{\mathcal{E}(u_\varepsilon)}{m_\varepsilon} \le \limsup_{u \to 0^+} \sup_{\xi \in \mathbb{S}^{N-1}} \frac{f(u, \rho(u)\xi)}{u}. \qquad \square$$

In dimension $N = 1$, we need no other assumption than $H < +\infty$, as stated below.

Proposition 2.7. *Let $f : \mathbb{R}_+ \times \mathbb{R} \to [0, +\infty]$ be Borel measurable. The function H_f defined by (2.1) (with $N = 1$) is either identically infinite on $(0, +\infty)$, or it satisfies (2.11) with $\rho \equiv 0$.*

Proof. One can assume that there exists $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}_+)$ with $0 < \int_{\mathbb{R}} u < +\infty$ and $\mathcal{E}(u) < +\infty$. In particular, up to changing the value of u on a negligible set, u is continuous on $\mathbb{R}$. Let $\varepsilon \in (0, \sup_{\mathbb{R}} u)$, set $A_\varepsilon \coloneqq \{x : u(x) = \varepsilon\}$ which is non-empty by the intermediate value theorem and integrability of u, and define

$$a_\varepsilon = \begin{cases} \inf A_\varepsilon & \text{if } \inf A_\varepsilon > -\infty, \\ \text{any point in } (-\infty, -\varepsilon^{-1}) \cap A_\varepsilon & \text{otherwise,} \end{cases}$$

$$b_\varepsilon = \begin{cases} \sup A_\varepsilon & \text{if } \sup A_\varepsilon < +\infty, \\ \text{any point in } (\varepsilon^{-1}, +\infty) \cap A_\varepsilon & \text{otherwise.} \end{cases}$$

By continuity and integrability of u, $u(a_\varepsilon) = u(b_\varepsilon) = \varepsilon$ and $u < \varepsilon$ on $\mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]$. Moreover $a_\varepsilon, b_\varepsilon$ converge to points $-\infty \le a \le b \le +\infty$, hence $u = 0$ on $\mathbb{R} \setminus (a, b)$ and by dominated convergence, since $\nabla u = 0$ a.e. on $\{u = 0\}$,

$$+\infty > \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]} u + f(u, \nabla u) = f(0,0)\, \mathcal{L}(\mathbb{R} \setminus (a,b)).$$

Notice that this limit is necessary zero. Let $m > 0$. If ε is small enough, then $\int_{\mathbb{R}\setminus[a_\varepsilon,b_\varepsilon]} u < m$ so that we can take $R_\varepsilon > 0$ such that $\varepsilon R_\varepsilon = m - \int_{\mathbb{R}\setminus[a_\varepsilon,b_\varepsilon]} u$. We then define

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if } x \le a_\varepsilon, \\ \varepsilon & \text{if } a_\varepsilon < x < a_\varepsilon + R_\varepsilon, \\ u(b_\varepsilon + x - (a_\varepsilon + R_\varepsilon)) & \text{if } x \ge a_\varepsilon + R_\varepsilon, \end{cases}$$

so that $\int_{\mathbb{R}} v_\varepsilon = m$. Moreover,

$$\mathcal{E}(v_\varepsilon) = \mathcal{E}(u, \mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]) + R_\varepsilon f(\varepsilon, 0).$$

Hence, as $R_\varepsilon = \frac{m+o(1)}{\varepsilon}$ as $\varepsilon \to 0$,

$$H_f(m) \le \limsup_{\varepsilon \to 0^+} \mathcal{E}(v_\varepsilon) = m \limsup_{\varepsilon \to 0^+} \frac{f(\varepsilon, 0)}{\varepsilon}.$$ □

2.4 Strict concavity of the cost function in dimension $N \ge 2$

We show that in dimension $N \ge 2$, the cost function must be strictly concave away from the possible initial interval where it is linear:

Proposition 2.8. *Assume that $N \ge 2$ and that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ satisfies (H1), (H2), (H5) and (H6). Let*

$$m_* = \sup\{m \ge 0 \ : \ H_f(x_0, \cdot) \text{ is linear on } [0, m]\},$$

where $x_0 \in \mathbb{R}^N$ is fixed. Then, $H_f(x_0, \cdot)$ is strictly concave on $(m_*, +\infty)$. A similar statement holds on $\mathbb{R}_-$.

A similar result does not hold in dimension 1 since any continuous concave function $H : \mathbb{R}_+ \to \mathbb{R}_+$ with $H(0) = 0$ can be written as $H = H_f$ with f satisfying all our assumptions (H1)–(H8) (see Section 5.2).

In the rest of this section, we will systematically ommit the dependence on x_0 of $H_f(x_0, m)$ since x_0 is fixed.

We denote by $\mathcal{M}_m^f$ the set of minimizers of mass $m \in \mathbb{R}_+$:

$$\mathcal{M}_m^f := \Big\{ u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+) \ : \ \mathcal{E}_f(u) = H_f(m) \text{ and } \int_{\mathbb{R}^N} u = m \Big\}. \tag{2.12}$$

The proof of Proposition 2.8 is based on the following observation:

Lemma 2.9. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \to [0, +\infty]$ be Borel measurable and let $u_i \in \mathcal{M}_{m_i}^f$ with $m_i \in \mathbb{R}_+$ for $i = 1, 2$. Let also $u_* := \min\{u_1, u_2\}$, $u^* := \max\{u_1, u_2\}$, $m_* := \int_{\mathbb{R}^N} u_*$ and $m^* := \int_{\mathbb{R}^N} u^*$. If H_f is affine on $[m_*, m^*]$ then $u_* \in \mathcal{M}_{m_*}^f$ and $u^* \in \mathcal{M}_{m^*}^f$.*

Proof of Lemma 2.9. We use the same observations as in the proof of Theorem 2.1. In particular, we have $m_* + m^* = m_1 + m_2$; since H_f is affine on $[m_*, m^*]$, it yields

$$H_f(m_*) + H_f(m^*) = H_f(m_1) + H_f(m_2).$$

But we have also

$$H_f(m_*) + H_f(m^*) \le \mathcal{E}_f(u_*) + \mathcal{E}_f(u^*) = \mathcal{E}_f(u_1) + \mathcal{E}_f(u_2) = H_f(m_1) + H_f(m_2),$$

so that the inequalities we used, i.e. $H_f(m_*) \le \mathcal{E}_f(u_*)$ and $H_f(m^*) \le \mathcal{E}_f(u^*)$, are actually equalities. □

We also use an elementary Sobolev type inequality:

Lemma 2.10. *Let $N \ge 2$, $p \in (1, +\infty)$ and $\omega \subset \mathbb{R}^{N-1}$ be a bounded open set. For every $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R} \times \omega)$,*

$$\int_\omega \|u(\cdot, x')\|_{L^\infty(\mathbb{R})}\, \mathrm{d}x' \le \|u\|_{L^1(\mathbb{R}\times\omega)} + |\omega|^{\frac{p-1}{p}} \Big\|\frac{\partial u}{\partial x_1}\Big\|_{L^p(\mathbb{R}\times\omega)}.$$

Proof of Lemma 2.10. We prove the lemma when $u \in \mathcal{C}^1(\mathbb{R} \times \omega)$; the general case follows by approximation. For every $x_1, y_1 \in \mathbb{R}$, $x' \in \omega$, we have

$$u(x_1, x') = u(y_1, x') + \int_{y_1}^{x_1} \frac{\partial u}{\partial x_1}(t, x')\, \mathrm{d}t.$$

By averaging in the variable y_1, we deduce

$$|u(x_1, x')| \le \int_{x_1-\frac12}^{x_1+\frac12} |u(y_1, x')|\, \mathrm{d}y_1 + \int_{x_1-\frac12}^{x_1+\frac12} \Big|\frac{\partial u}{\partial x_1}(t, x')\Big|\, \mathrm{d}t.$$

The result follows from Hölder inequality after integrating over ω. □

Proof of Proposition 2.8. Assume by contradiction that the concave function H_f is not strictly concave on $(m_*, +\infty)$ which means that there exists $m \in (m_*, +\infty)$ and $\eta > 0$ such that H_f is affine on $[m - \eta, m + \eta]$. (Note that $\eta \le m - m_*$ by definition of m_*.) Moreover, we will see in Proposition 3.12 that $\mathcal{M}^f_m$ is not empty. We let $u \in \mathcal{M}^f_m$.

As before, we shall use the notations $\wedge$ and $\vee$ for the minimum and maximum; we also let $(e_1, \dots, e_N)$ be the canonic basis of $\mathbb{R}^N$. Since the translation operator $u \mapsto u(\cdot + \tau)$ is continuous in $L^1(\mathbb{R}^N)$ (for every $\tau \in \mathbb{R}^N$), since the map $(x, y) \mapsto x \wedge y$ is Lipschitz on $\mathbb{R}^N \times \mathbb{R}^N$, and since the set $\mathcal{M}_{m+\frac{\eta}{2}}$ is compact in L^1 up to translations in view of the proof of Proposition 3.12, we deduce that there exists $\tau_0 > 0$ such that

$$\|u \wedge u(\cdot + \tau e_2)\|_{L^1(\mathbb{R}^N)} > m \quad \text{for all } \tau \in (0, \tau_0) \text{ and } u \in \mathcal{M}^f_{m+\frac{\eta}{2}}. \tag{2.13}$$

We now construct by induction a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ and a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{M}^f_m$ such that

$$t_{n+1} \ge t_n + \tau_0 \quad \text{and} \quad u_n(x) \le U(x) \wedge U(x + t_n e_2) \quad \forall x \in \mathbb{R}^N, \tag{2.14}$$

where we have set

$$U(x) := \operatorname*{ess\,sup}_{t\in\mathbb{R}} u(x + te_1).$$

To this aim, we first set $u_0 := u$ and $t_0 = 0$. Then, if we assume that t_n and u_n are constructed as before, we first pick an $\tau^1_n \in \mathbb{R}_+$ such that $v_n := u_n \vee u_n(\cdot + \tau^1_n e_1)$ satisfies $\int_{\mathbb{R}^N} v_n = m + \frac{\eta}{2}$, which is possible since $\eta \le m$, as we argued in the proof of Theorem 2.1. Similarly,

we pick a $\tau_n^2 \in \mathbb{R}_+$ such that $u_{n+1} := v_n \wedge v_n(\cdot + \tau_n^2 e_2)$ satisfies $\int_{\mathbb{R}^N} u_{n+1} = m$, and we set $t_{n+1} = t_n + \tau_n^2$. By Lemma 2.9, $v_n \in \mathcal{M}_{m+\frac{\eta}{2}}^f$ and $u_n \in \mathcal{M}_m^f$. By (2.13), we have $\tau_n^2 \geq \tau_0$, thus insuring the first condition in (2.14). For the second condition, we observe that for all $x = (x_1, x') \in \mathbb{R}^N$,

$$u_{n+1}(x) \leq \sup_{t\in\mathbb{R}} u_n(x + te_1) \wedge u_n(x + te_1 + \tau_n^2 e_2) \leq U(x) \wedge U(x + t_{n+1}e_2),$$

where in the last inequality we have used the induction hypothesis (2.14).

We now show that the sequence $(u_n \mathcal{L}^N)_{n\in\mathbb{N}}$ is vanishing which will contradict the compactness of $\mathcal{M}_m^f$ in L^1 up to translations.

For this, we let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N$ and $(u_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(u_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{x+[0,1)^N} u_n = \lim_{k\to\infty} \int_{x_k+[0,1)^N} u_{n_k}.$$

By (H_5), we have $\frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^N)$. Using this fact, the estimate (2.14), and Lemma 2.10 with ω a unit cube in $\mathbb{R}^{N-1}$, we obtain

$$\begin{aligned}
\lim_{k\to\infty} \int_{x_k+[0,1)^N} u_{n_k} &\leq \liminf_{k\to\infty} \int_{x_k+[0,1)^N} U \wedge \int_{x_k+t_{n_k}e_2+[0,1)^N} U \\
&\leq \liminf_{k\to\infty} \left(\|u\|_{L^1(\{0\leq(x-x_k)\cdot e_2\leq 1\})} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\{0\leq(x-x_k)\cdot e_2\leq 1\})} \right) \\
&\wedge \liminf_{k\to\infty} \left(\|u\|_{L^1(\{t_{n_k}\leq(x-x_k)\cdot e_2\leq t_{n_k}+1\})} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\{t_{n_k}\leq(x-x_k)\cdot e_2\leq t_{n_k}+1\})} \right),
\end{aligned}$$

and the conclusion follows since the sequences $(x_k \cdot e_2)_{k\in\mathbb{N}}$ and $(t_{n_k} + x_k \cdot e_2))_{k\in\mathbb{N}}$ cannot be both bounded as $\lim_{k\to\infty} t_{n_k} = \infty$. □

3 Lower bound for the energy and existence of optimal profiles

Our main tool to localize the energy and obtain a lower bound relies on a profile decomposition for bounded sequences of positive measures, which is reminiscent of the concentration compactness principle of P.-L. Lions. This differs from classical strategies to localize the energy which are based on suitable cut-offs. Naturally, this concentration compactness result also provides a criterion for the existence of optimal profiles in (1.2).

3.1 Profile decomposition by concentration compactness

We prove a profile decomposition theorem for bounded sequences of positive measures over $\mathbb{R}^N$, which is essentially equivalent to [Mar14, Theorem 1.5] in the Euclidean case. We have added an extra information on mass conservation that will be useful, and provide a self-contained simple proof. We start with a definition.

Definition 3.1. A sequence of positive measures $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{M}_+(\mathbb{R}^N)$ is vanishing if

$$\sup_{x\in\mathbb{R}^N} \mu_n(B_1(x)) \xrightarrow[n\to\infty]{} 0.$$

Any bounded sequence of positive measures over $\mathbb{R}^N$ may be decomposed (up to subsequence) into a countable collection of narrowly converging "bubbles" and a vanishing part, accounting for the total mass of the sequence, as stated in the following theorem.

Theorem 3.2. *For every bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive Borel measures on $\mathbb{R}^N$, there exists a subsequence $(\mu_n)_{n\in\sigma(\mathbb{N})}$, $\sigma\in\Sigma$, a non-decreasing sequence of integers $(k_n)_{n\in\sigma(\mathbb{N})}$ converging to some $k\in\mathbb{N}\cup\{+\infty\}$, a sequence of non-trivial positive Borel measures $(\mu^i)_{0\le i<k}$, and for every $n\in\sigma(\mathbb{N})$, a collection of balls $(B_n^i)_{0\le i<k_n}$ centered at points of $\operatorname{supp}\mu_n$ such that, writing for all $n\in\sigma(\mathbb{N})$,*

$$\mu_n=\mu_n^b+\mu_n^v,\quad \text{where } \mu_n^b=\sum_{0\le i<k_n}\mu_n\llcorner B_n^i, \tag{3.1}$$

(A) bubbles emerge: $(c_{B_n^i}\mu_n)_{n\in\sigma(\mathbb{N})}\xrightarrow[n\to\infty]{\mathcal{C}_b'}\mu^i$ *for every* $i<k$,[4]

(B) bubbles split: $\min_{0\le i<j<k_n}\operatorname{dist}(B_n^i,B_n^j)\xrightarrow[n\to\infty]{}+\infty$,

(C) bubbles diverge: $\min_{0\le i<k_n}\operatorname{diam}(B_n^i)\xrightarrow[n\to\infty]{}+\infty$,

(D) the bubbling mass is conserved: $\|\mu_n^b\|\xrightarrow[\ell\to\infty]{}\sum_{0\le i<k}\|\mu^i\|$,

(E) the remaining part is vanishing: $\sup_{x\in\mathbb{R}^N}\mu_n^v(B_1(x))\xrightarrow[n\to\infty]{}0$.

Before proving Theorem 3.2, we introduce the "bubbling" function of a sequence of finite signed measures $(\mu_n)_{n\in\mathbb{N}}$:

$$m((\mu_n)_{n\in\mathbb{N}}):=\sup\Big\{\|\mu\|\ :\ (\tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)})_{\ell\in\mathbb{N}}\xrightarrow{\mathcal{C}_0'}\mu,\ \sigma\in\Sigma,\ x_{\sigma(\ell)}\in\mathbb{R}^N\ (\forall\ell)\Big\}. \tag{3.2}$$

Although we will use this function on signed measures, we will start from a sequence of positive measures and use the following characterization of vanishing sequences, which holds only in the case of positive measures:

Lemma 3.3. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of finite positive measures is vanishing if and only if $m((\mu_n)_{n\in\mathbb{N}})=0$.*

Proof. Assume that $(\mu_n)_{n\in\mathbb{N}}$ is vanishing and that $(\tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)})_{\ell\in\mathbb{N}}\xrightarrow{\mathcal{C}_0'}\mu$ for some $\sigma\in\Sigma$ and some sequence of points $(x_{\sigma(\ell)})_{\ell\in\mathbb{N}}$. Then, for every $x\in\mathbb{R}^N$,

$$\mu(B_1(x))\le\liminf_{\ell\to\infty}\tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)}(B_1(x))=\liminf_{\ell\to\infty}\mu_{\sigma(\ell)}(B_1(x+x_{\sigma(\ell)}))=0,$$

i.e. $\mu=0$ and thus $m((\mu_\ell)_{\ell\in\mathbb{N}})=0$.

Conversely, if $(\mu_n)_{n\in\mathbb{N}}$ is not vanishing, then there exists $\varepsilon>0$, $\sigma\in\Sigma$ a sequence of points $(x_n)_{n\in\sigma(\mathbb{N})}$ in $\mathbb{R}^N$ such that $\mu_n(B_1(x_n))\ge\varepsilon$ for every $n\in\sigma(\mathbb{N})$. Up to further extraction, one can assume that $(\tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)})_{\ell\in\mathbb{N}}\xrightarrow{\mathcal{C}_0'}\mu\in\mathcal{M}(\mathbb{R}^N)$. We have

$$\mu(\bar B_1(0))\ge\limsup_{\ell\to\infty}\tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)}(\bar B_1(0))=\limsup_{\ell\to\infty}\mu_{\sigma(\ell)}(\bar B_1(x_{\sigma(\ell)}))\ge\varepsilon>0,$$

which entails $m((\mu_\ell)_{\ell\in\mathbb{N}})\ge\varepsilon>0$. □

[4] Recall that $c_B\mu=(x\mapsto x-y)_\sharp(\mu\llcorner B)$ if $B=B_r(y)$ and $\mu\in\mathcal{M}(\mathbb{R}^N)$.

Proof of Theorem 3.2. If $(\mu_n)_{n\in\mathbb{N}}$ is vanishing, then we take $\sigma = \mathrm{Id}$ and $k = 0$, so that $\mu_{\sigma(\ell)} = \mu_\ell = \mu_\ell^v$, (A) to (D) are empty statements and (E) is satisfied since $(\mu_n)_{n\in\mathbb{N}}$ is vanishing. Assume on the contrary that $(\mu_n)_{n\in\mathbb{N}}$ is not vanishing. We shall construct the bubbles by induction and prove their properties in several steps.

Step 1: construction of bubbles centers. At first step (step 0), since $m((\mu_n)_{n\in\mathbb{N}}) > 0$, there exists $\sigma_0 \in \Sigma$ and a sequence of points $(x_n^0)_{n\in\sigma_0(\mathbb{N})}$, such that

$$(\tau_{-x_n^0}\mu_n)_{n\in\sigma_0(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^0 \in \mathcal{M}(\mathbb{R}^N) \quad \text{with} \quad \|\mu^0\| \geq \frac{1}{2} m((\mu_n)_{n\in\mathbb{N}}). \tag{3.3}$$

We then set $\mu_n^0 := \mu_n - \tau_{x_n^0}\mu^0$ and we continue by induction, starting from the sequence $(\mu_n^0)_{n\in\sigma_0(\mathbb{N})}$. More precisely, assume that for a fixed step $k-1 \in \mathbb{N}$, for every $i \in \mathbb{N}$ with $0 \leq i \leq k-1$, we have built $\mu^i \in \mathcal{M}(\mathbb{R}^N)$, $\sigma_i \in \Sigma$, points $(x_n^i)_{n\in\sigma_i(\mathbb{N})}$ and sequences $(\mu_n^i)_{n\in\sigma_i(\mathbb{N})} \in \mathcal{M}(\mathbb{R}^N)$ such that for every i,

$$\sigma_i \preceq \sigma_{i-1}, \tag{3.4}$$

$$\mu_n^i = \mu_n - \sum_{0\leq j\leq i} \tau_{x_n^j}\mu^j, \quad (\forall n \in \sigma_i(\mathbb{N})), \tag{3.5}$$

$$(\tau_{-x_n^i}\mu_n^{i-1})_{n\in\sigma_i(\mathbb{N})} \xrightarrow{\mathcal{C}'_0} \mu^i, \tag{3.6}$$

$$\|\mu^i\| \geq \frac{1}{2} m((\mu_n^i)_{n\in\sigma_i(\mathbb{N})}) > 0, \tag{3.7}$$

where $\sigma_{-1} := \mathrm{Id}, (\mu_n^{-1}) := (\mu_n)$. If $m((\mu_n^{k-1})_{n\in\sigma_{k-1}(\mathbb{N})}) = 0$, we stop; otherwise, we proceed to the next step k to build $\sigma_k, \mu^k, (x_n^k)_{n\in\sigma_k(\mathbb{N})}, (\mu_n^k)$ as we did at step $k = 0$, starting with $(\mu_n^{k-1})_{n\in\sigma_{k-1}(\mathbb{N})}$. Either the induction stops at some step $k-1 \in \mathbb{N}$ for which $m((\mu_n^{k-1})_{n\in\sigma_{k-1}(\mathbb{N})}) = 0$ or the previous objects are defined for every $i \in \mathbb{N}$, in which case we let $k := +\infty$.

Step 2: splitting of bubbles centers. We prove that

$$\lim_{\sigma_i(\mathbb{N})\ni n\to\infty} \mathrm{dist}(x_n^i, x_n^j) = +\infty \quad \text{for every } i, j \in \mathbb{N} \text{ with } 0 \leq j < i < k. \tag{3.8}$$

Indeed, assume by contradiction that there is a first index $i < k$ such that for some $j_0 < i$, $(\mathrm{dist}(x_n^i, x_n^{j_0}))_{n\in\sigma_i(\mathbb{N})}$ is not divergent. In particular, there exists $\sigma \preceq \sigma_i$ such that $(x_n^i - x_n^{j_0})_{n\in\sigma(\mathbb{N})} \to x \in \mathbb{R}^N$. Moreover, $(\mathrm{dist}(x_n^i, x_n^j))_{n\in\sigma_i(\mathbb{N})} \to \infty$, for every $j < i$, $j \neq j_0$ by minimality of i and the triangle inequality $\mathrm{dist}(x_n^j, x_n^{j_0}) \leq \mathrm{dist}(x_n^j, x_n^i) + \mathrm{dist}(x_n^i, x_n^{j_0})$. Notice by (3.5) that for every $n \in \sigma(\mathbb{N})$,

$$\mu_n^{i-1} = \mu_n^{j_0-1} - \tau_{x_n^{j_0}}\mu^{j_0} - \sum_{j_0<j<i} \tau_{x_n^j}\mu^j,$$

hence taking the translation $\tau_{-x_n^i}$,

$$\tau_{-x_n^i}\mu_n^{i-1} = \tau_{x_n^{j_0}-x_n^i}(\tau_{-x_n^{j_0}}\mu_n^{j_0-1} - \mu^{j_0}) - \sum_{j_0<j<i} \tau_{x_n^j-x_n^i}\mu^j,$$

and passing to the weak limit, knowing that $x_n^{j_0} - x_n^i \to -x$ and $\mathrm{dist}(x_n^j, x_n^i) \to +\infty$ for $j_0 < j < i$,

$$\mu^i = \tau_{-x}(\mu^{j_0} - \mu^{j_0}) - \sum_{j_0<j<i} 0 = 0.$$

This contradicts the fact that $(\tau_{-x_n^i}\mu_n^{i-1})_{n\in\sigma(\mathbb{N})} \overset{\mathcal{C}_0'}{\rightharpoonup} \mu^i \neq 0$ and proves (3.8).

Step 3: weak convergence of bubbles. From (3.6) we get

$$\tau_{-x_n^i}\mu_n^{i-1} = \tau_{-x_n^i}\mu_n - \sum_{0\le j<i} \tau_{-x_n^i+x_n^j}\mu^j, \tag{3.9}$$

and by (3.8), the sum converges weakly to 0, and so

$$(\tau_{-x_n^i}\mu_n)_{n\in\sigma_i(\mathbb{N})} \overset{\mathcal{C}_0'}{\rightharpoonup} \mu^i \quad \text{for every } i\in\mathbb{N} \text{ with } i<k. \tag{3.10}$$

Step 4: construction of the bubbles with mass conservation. We now construct the extraction $\sigma\in\Sigma$ that we need by induction: we set $\sigma(0)=0$ and, assuming that $\sigma(0)<\cdots<\sigma(\ell-1)$, with $\ell\in\mathbb{N}^*$, have been constructed, we set $\sigma(\ell) := n$ with $n\in\sigma_{\ell\wedge k-1}(\mathbb{N})$ large enough so that $n>\sigma(\ell-1)$ and for every $i<\ell\wedge k$,

$$\mu_n(B_\ell(x_n^i)) \le \|\mu^i\| + 2^{-\ell}, \tag{3.11}$$

and

$$\min_{0\le j<i} \operatorname{dist}(x_n^i, x_n^j) \ge 4\ell. \tag{3.12}$$

Such an n exists by (3.8) and (3.10), noticing that $\mu_n(B_\ell(x_n^i)) = (\tau_{-x_n^i}\mu_n)(B_\ell)$. Then for each $n=\sigma(\ell)$, $\ell\in\mathbb{N}$, we set $k_n=\ell\wedge k$, and for each $i\in\{0,\dots,k_n-1\}$,

$$B_n^i := B_\ell(x_n^i).$$

Finally, for every $n\in\sigma(\mathbb{N})$, we decompose μ_n as expected:

$$\mu_n = \mu_n^b + \mu_n^v, \quad \text{where } \mu_n^b = \sum_{0\le i<k_n} \mu_n \llcorner B_n^i.$$

Let us check the four first items (A)–(D). Notice that (C) is fulfilled because $\operatorname{diam}(B^i_{\sigma(\ell)}) = \ell\to+\infty$ as $\ell\to\infty$, and (B) because of (3.12). Since for every $i<k$, $\lim_{\sigma(\mathbb{N})\ni n\to\infty}\operatorname{diam}(B_n^i) = +\infty$ and $c_{B_n^i}\mu_n = (\tau_{-x_n^i}(\mu_n\llcorner B_n^i))$ for every $n\in\sigma_i(\mathbb{N})$, $(c_{B_n^i}\mu_n)_{n\in\sigma(\mathbb{N})}$ converges weakly to μ^i by (3.10), and together with (3.11) it implies that

$$(c_{B_n^i}\mu_n)_{n\in\sigma(\mathbb{N})} \overset{\mathcal{C}_b'}{\rightharpoonup} \mu^i,$$

i.e. (A) is satisfied. Moreover, by (3.11) again,

$$\limsup_{\ell\to\infty} \sum_{0\le i<k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B^i_{\sigma(\ell)}) \le \sum_{0\le i<k} \|\mu^i\| + \limsup_{\ell\to\infty}(\ell\wedge k)2^{-\ell} = \sum_{0\le i<k}\|\mu^i\|,$$

and since $k_n\to k$, by Fatou's lemma we have,

$$\sum_{0\le i<k}\|\mu^i\| \le \liminf_{\ell\to\infty} \sum_{0\le i<k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B^i_{\sigma(\ell)}),$$

which proves (D) because $\sum_{0\le i<k_{\sigma(\ell)}}\mu_{\sigma(\ell)}(B^i_{\sigma(\ell)}) = \|\mu^b_{\sigma(\ell)}\|$.

Step 5: vanishing of the remaining part, proof of (E). By Lemma 3.3, it suffices to prove that $m((\mu_n^v)_{n\in\sigma(\mathbb{N})}) = 0$. We claim that:

$$m((\mu_n^v)_{n\in\sigma(\mathbb{N})}) \le m((\mu_n^i)_{n\in\sigma_i(\mathbb{N})}), \quad \text{for every } i\in\mathbb{N} \text{ with } i<k, \tag{3.13}$$

which concludes since $m((\mu_n^k)_{n\in\sigma_{k-1}(\mathbb{N})}) = 0$ if $k<\infty$, and $m((\mu_n^i))_{n\in\sigma_i(\mathbb{N})}) \to 0$ as $i\to\infty$ if $k=\infty$. Indeed, if $k=\infty$, we have by (3.7) and (D),

$$\frac{1}{2}\sum_{i\in\mathbb{N}} m((\mu_n^i)_{n\in\sigma_i(\mathbb{N})}) \le \sum_{i\in\mathbb{N}} \|\mu^i\| = \lim_{\ell\to\infty} \|\mu^b_{\sigma(\ell)}\| \le \liminf_{\ell\to\infty} \|\mu_{\sigma(\ell)}\| < \infty.$$

Let us show (3.13). Let $\bar\sigma \preceq \sigma$ and $(x_n)_{n\in\bar\sigma(\mathbb{N})}$ be a sequence of points such that

$$(\tau_{-x_n}\mu_n^v)_{n\in\bar\sigma(\mathbb{N})} \xrightarrow{\mathcal{C}_0'} \mu\in\mathcal{M}(\mathbb{R}^N).$$

We need to prove that $\|\mu\| \le m((\mu_n^i)_{n\in\sigma_i(\mathbb{N})})$ for every $i<k$. Assume without loss of generality that $\|\mu\|>0$. Then for every $i<k$,

$$(\mathrm{dist}(x_n, x_n^i))_{n\in\bar\sigma(\mathbb{N})} \to \infty. \tag{3.14}$$

Otherwise, up to subsequence, $(\mathrm{dist}(x_n,x_n^i))_n$ would be bounded by some constant M, and for every $r>0$,

$$(\tau_{-x_n}\mu_n^v)(B_r) \le \mu_n^v(B_{r+M}(x_n^i)) \xrightarrow[n\to\infty]{} 0,$$

because μ_n^v is supported on $\mathbb{R}^N \setminus \cup_{0\le i<k_n} B_n^i$ and $B_{r+M}(x_n^i)\subseteq B_n^i$ for n large enough by (E). Hence μ would be 0, a contradiction. Up to further extraction, one can assume that $(\tau_{-x_n}\mu_n)_{n\in\bar\sigma(\mathbb{N})}$ converges weakly to a measure $\bar\mu\in\mathcal{M}(\mathbb{R}^N)$. Since $\mu_n^v\le\mu_n$, we have $\mu\le\bar\mu$. Moreover by (3.5), for every $i<k$ and $n\in\bar\sigma(\mathbb{N})$ large enough,

$$\tau_{-x_n}\mu_n^i = \tau_{-x_n}\mu_n - \sum_{0\le j\le i} \tau_{x_n^j - x_n}\mu^j,$$

and because of (3.14) the sum converges weakly to 0, so that $\tau_{-x_n}\mu_n^i \xrightarrow{\mathcal{C}_0'} \bar\mu$, and consequently,

$$\|\mu\| \le \|\bar\mu\| \le m((\mu_n^i)_{n\in\sigma_i(\mathbb{N})}),$$

which is what had to be proved.

Step 6: re-centering of the bubbles at points of $\mathrm{supp}\,\mu_n$. By (3.10), $(\tau_{-x_n^i}\mu_n)_{n\in\sigma(\mathbb{N})}$ converges weakly to the non-trivial measure μ_i for every $i<k$, thus

$$R_i/2 := \limsup_{\sigma(\mathbb{N})\ni n\to+\infty} \mathrm{dist}(\mathrm{supp}\,\mu_n, x_n^i) < +\infty. \tag{3.15}$$

Therefore, for every n large enough, there is a point $\tilde x_n^i$ such that $|x_n^i - \tilde x_n^i| < R_i$ and $\tilde x_n^i \in \mathrm{supp}\,\mu_n$. After a further extraction, one may assume that for every i, $|x_n^i-\tilde x_n^i| < R_i < r_i^n$ with $\mathrm{diam}\,B_n^i = 2r_n^i$ for every n, and $(x_n^i - \tilde x_n^i)_{n\in\sigma(\mathbb{N})}$ converges to some $p_i\in\mathbb{R}^N$. Finally, we set $\tilde r_i^n := r_i^n - R_i$ and $\tilde B_n^i := B(\tilde x_n^i, \tilde r_i^n)\subseteq B_n^i$. After replacing the balls B_n^i by $\tilde B_n^i$, (B) and (C) are satisfied by definition. Notice that $(\tau_{-\tilde x_n^i}\mu_n)_{n\in\sigma(\mathbb{N})}$ converges weakly to $\tilde\mu^i := \tau_{p_i}\mu^i$

with $\|\tilde{\mu}^i\| = \|\mu^i\|$, and $\limsup_n \|c_{B_n^i}\mu_n\| = \limsup_n \mu_n(\tilde{B}_n^i) \le \limsup_n \mu_n(B_n^i) = \|\mu^i\|$ hence (A) holds. Besides, using Fatou's lemma,

$$\limsup_n \sum_{i<k_n} \mu_n(\tilde{B}_n^i) \le \limsup_n \sum_{i<k_n} \mu_n(B_n^i) = \sum_{i<k} \|\mu^i\| \\ \le \sum_{i<k} \liminf_n \mu_n(\tilde{B}_n^i) \le \liminf_n \sum_{i<k_n} \mu_n(\tilde{B}_n^i)$$

so that $\lim_n \sum_{i<k_n} \mu_n(\tilde{B}_n^i) = \sum_i \|\mu_i\|$ and (D) is satisfied. In particular, $\lim_n \sum_{i<k_n} \mu_n(B_n^i \setminus \tilde{B}_n^i) = \lim_n \sum_{i<k_n} \mu_n(B_n^i) - \lim_n \sum_{i<k_n} \mu_n(\tilde{B}_n^i) = 0$ and (E) holds as well. □

Remark 3.4. If the sequence of families of balls $(B_n^i)_{0\le i<k_n}$ satisfies the conclusion of the theorem, i.e. (A)–(E), then it is also the case for any family of balls $(\tilde{B}_n^i)_{0\le i<k_n}$ with the same centers as those of B_n^i and with smaller but still divergent radii (i.e. satisfying (C)). It can be easily seen following the arguments at Step 6 of the proof.

3.2 Lower bound by concentration compactness

We will first establish a lower bound for the minimal energy along vanishing sequences defined on varying subsets of $\mathbb{R}^N$. We say that a sequence of Borel functions $(u_n)_{n\in\mathbb{N}}$, each defined on some open set $\Omega_n \subseteq \mathbb{R}^N$, is vanishing if the sequence of measures $(|u_n|\,\mathcal{L}^N \llcorner \Omega_n)_{n\in\mathbb{N}}$ is vanishing in the sense of Definition 3.1, namely if $\|u_n\|_{L^1_{\mathrm{uloc}}(\Omega_n)} \to 0$ as $n \to \infty$, where $L^1_{\mathrm{uloc}}(\Omega)$ is the set of uniformly locally integrable functions on the open set Ω, i.e. Borel functions u on Ω such that

$$\|u\|_{L^1_{\mathrm{uloc}}(\Omega)} := \sup_{x\in\mathbb{R}^N} \int_{\Omega\cap(x+[0,1)^N)} |u| < +\infty. \tag{3.16}$$

It will be convenient to first extend our Sobolev functions to a neighbourhood Ω_δ of Ω where for every $\delta > 0$ and every set $X \subseteq \mathbb{R}^N$, we have set

$$X_\delta := \{x \in \mathbb{R}^N \ : \ \mathrm{dist}(x, X) < \delta\}.$$

We will need to consider sufficiently regular domains for which we have an extension operator $W^{1,p}\cap L^1_{\mathrm{uloc}}(\Omega) \to W^{1,p}\cap L^1_{\mathrm{uloc}}(\Omega_\delta)$. We will only apply it to domains with smooth boundary, in which case we can use a reflexion technique. Since we want quantitative estimates, we will use the notion of reach of a set $X \subseteq \mathbb{R}^N$ (see [Fed59]). We say that X has positive reach if there exists $\delta > 0$ such that every $x \in X_\delta$ has a unique nearest point $\pi(x)$ on X. The greatest δ for which this holds is denoted by $\mathrm{reach}(X)$ and the map $x \in X_{\mathrm{reach}(X)} \mapsto \pi(x) \in X$ is called the nearest point retraction.

Example 3.5. Assume that Ω is a perforated domain $B^0 \setminus \bigcup_{i=1}^k B^i$ where the B^i are disjoint closed balls included in some open ball B^0 (possibly $B^0 = \mathbb{R}^N$). Then,

$$\mathrm{reach}(\partial\Omega) = \inf\{\mathrm{radius}(B^i) \ : \ i = 0, \dots, k\} \cup \{\mathrm{dist}(\partial B^i, \partial B^j) \ : \ i \ne j\}.$$

By [Fed59, Theorem 4.8], we have

i) if $x, y \in X_\delta$ with $0 < \delta < \delta_0 := \mathrm{reach}(X)$, then $|\pi(x) - \pi(y)| \le \frac{\delta_0}{\delta_0 - \delta}|x - y|$,

ii) if $x \in X$ and D_x is the intersection of $X_{\mathrm{reach}(X)}$ with the straight line crossing $\partial\Omega$ orthogonally at x, then $\pi(y) = x$ for every $y \in D_x$.

Lemma 3.6 (Extension)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that its boundary $\partial\Omega$ is $\mathcal{C}^1$ with positive reach. Then, for every $\delta \in (0, \mathrm{reach}(\partial\Omega))$, every $p \in [1, +\infty)$ and every $u \in L^1 \cap W^{1,p}(\Omega)$, there exists $\bar{u} \in L^1 \cap W^{1,p}(\Omega_\delta)$ such that $\bar{u} = u$ a.e. on Ω, and*

$$\|\bar{u}\|_{L^1(\Omega_\delta)} \leq A\|u\|_{L^1(\Omega)}, \quad \|\bar{u}\|_{L^1_{\mathrm{uloc}}(\Omega_\delta)} \leq A\|u\|_{L^1_{\mathrm{uloc}}(\Omega)}, \quad \|\nabla\bar{u}\|_{L^p(\Omega_\delta)} \leq A\|\nabla u\|_{L^p(\Omega)},$$

with a constant $A < +\infty$ depending only on N, δ and $\mathrm{reach}(\partial\Omega)$.

Proof. Let $\sigma : (\partial\Omega)_\delta \to (\partial\Omega)_\delta$ be the reflexion through $\partial\Omega$, defined by $\sigma(x) = 2\pi(x) - x$. By the properties i) and ii) of the nearest point retraction, we have that $\sigma = \sigma^{-1}$ (simply because $\pi(\sigma(x)) = \pi(x)$) and σ is L-Lipschitz with a constant $L < +\infty$ depending on δ and $\mathrm{reach}(\partial\Omega)$ only.

We define $\bar{u}$ by $\bar{u} = u$ on Ω and $\bar{u} = u \circ \sigma$ on $\Omega_\delta \setminus \Omega$[5]. This map is well defined since $\sigma(\Omega_\delta \setminus \Omega) \subseteq \Omega$. Indeed, if we had $x, \sigma(x) \in \Omega_\delta \setminus \Omega$, then the line segment $[x, \sigma(x)]$ would meet $\partial\Omega$ orthogonally at its center $\pi(x)$, and would remain out of Ω elsewhere, because otherwise there would exist a point y belonging either to $\partial\Omega \cap (x, \pi(x))$ or $\partial\Omega \cap (\pi(x), \sigma(x))$ thus contradicting the definition of $\pi(x)$. Such a situation is not possible for a $\mathcal{C}^1$ boundary.

Moreover, by the change of variable formula and the chain rule, $\bar{u}$ satisfies the desired estimates since σ is bi-Lipschitz with its Lipschitz constants controlled in terms of δ and $\mathrm{reach}(\partial\Omega)$. □

We will need a localized version of the Gagliardo–Nirenberg–Sobolev inequality in a particular case:

Lemma 3.7. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\partial\Omega$ is $\mathcal{C}^1$ with positive reach, let $p \in [1, +\infty)$, let $r \geq p(1 + \frac{1}{N})$, and assume that $r \leq \frac{pN}{N-p}$ when $p < N$. Then for every $u \in L^1 \cap W^{1,p}(\Omega)$,*

$$\|u\|_{L^r(\Omega)} \leq C\big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}\big)^\alpha \|u\|^{1-\alpha}_{L^1_{\mathrm{uloc}}(\Omega)},$$

where $\alpha \in (0, 1]$ is the unique parameter such that $\frac{1}{r} = \alpha(\frac{1}{p} - \frac{1}{N}) + (1 - \alpha)$, and the constant $C < +\infty$ depends on N, r, p and $\mathrm{reach}(\partial\Omega)$.

Proof of Lemma 3.7. We let $u \in L^1 \cap W^{1,p}(\Omega)$ and we extend u to $\bar{u} \in L^1 \cap W^{1,p}(\Omega_\delta)$ as in Lemma 3.6, with $\delta := \mathrm{reach}(\Omega)/2$. By the Gagliardo–Nirenberg–Sobolev inequality (see [Nir59]) on the hypercube $Q_\delta = [-\frac{\delta}{\sqrt{N}}, \frac{\delta}{\sqrt{N}})^N$, we have

$$\|\bar{u}\|_{L^r(Q_\delta)} \leq C\|\nabla\bar{u}\|^\alpha_{L^p(Q_\delta)}\|\bar{u}\|^{1-\alpha}_{L^1(Q_\delta)} + C\|\bar{u}\|_{L^1(Q_\delta)}.$$

We then cover Ω with the hypercubes $Q_\delta(c) = c + Q_\delta \subseteq \Omega_\delta$ centered at points c on the grid $\mathcal{C} := \Omega \cap \delta\mathbb{Z}^N$. Since $\alpha \geq \frac{N}{N+1}$, we can check that

$$r\alpha = \frac{r-1}{1 + \frac{1}{N} - \frac{1}{p}} \geq p. \tag{3.17}$$

[5]Note that $\bar{u}$ is not defined on $\partial\Omega$, but this set is negligible.

By superadditivity of $s \mapsto s^{\frac{r\alpha}{p}}$ and of $s \mapsto s^{r\alpha}$, we obtain

$$\begin{aligned}
\|u\|_{L^r(\Omega)}^r &\leq \sum_{c\in\mathcal{C}} \|\bar{u}\|_{L^r(Q_\delta(c))}^r \\
&\leq C' \sum_{c\in\mathcal{C}} \|\nabla \bar{u}\|_{L^p(Q_\delta(c))}^{p\frac{r\alpha}{p}} \|\bar{u}\|_{L^1(Q_\delta(c))}^{r(1-\alpha)} + C' \|\bar{u}\|_{L^1(Q_\delta(c))}^r \\
&\leq C'' \|\nabla \bar{u}\|_{L^p(\Omega_\delta)}^{r\alpha} \|\bar{u}\|_{L^1_{\mathrm{uloc}}(\Omega_\delta)}^{r(1-\alpha)} + C' \|\bar{u}\|_{L^1(\Omega_\delta)}^{r\alpha} \|\bar{u}\|_{L^1_{\mathrm{uloc}}(\Omega_\delta)}^{r(1-\alpha)} \\
&\leq C''' \big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}\big)^{r\alpha} \|u\|_{L^1_{\mathrm{uloc}}(\Omega)}^{r(1-\alpha)}.
\end{aligned}$$

□

Proposition 3.8. *Assume that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ satisfies (H1) and (H5) for some $p \in (1, +\infty)$. Consider a vanishing sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,1}_{\mathrm{loc}}(\Omega_n, \mathbb{R}_\pm)$, where the $\Omega_n \subseteq \mathbb{R}^N$ are open sets with $\mathcal{C}^1$ boundary and such that $\inf_{n\in\mathbb{N}} \mathrm{reach}(\partial\Omega_n) > 0$, and a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of Borel maps $\Phi_n : \Omega_n \to \mathbb{R}^N$ such that $\sup_{y\in\Omega_n} |\Phi_n(y) - x_0| \to 0$ as $n \to +\infty$ for some $x_0 \in \mathbb{R}^N$. If $\theta_n := \int_{\Omega_n} u_n > 0$ for every n and $(\theta_n)_{n\in\mathbb{N}}$ is bounded, then:*

$$\liminf_{n\to+\infty} \frac{1}{\theta_n} \int_{\Omega_n} f(\Phi_n(y), u_n(y), \nabla u_n(y))\, \mathrm{d}y \geq f'_-(x_0, 0^\pm, 0),$$

where $f'_-(x_0, 0^\pm, 0)$ was defined in (1.4).

Proof of Proposition 3.8. Suppose for example that $u_n \geq 0$ a.e. for every n. Without loss of generality, we may assume after extracting a subsequence that:

$$K := \sup_n \frac{1}{\theta_n} \int_{\Omega_n} f(\Phi_n(y), u_n(y), \nabla u_n(y))\, \mathrm{d}y + \theta_n < +\infty. \tag{3.18}$$

We consider the sequence of measures $(\nu_n)_{n\in\mathbb{N}} \in \mathcal{M}_+(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ defined by

$$\nu_n := \frac{1}{\theta_n} (\Phi_n, u_n, \nabla u_n)_\sharp (u_n\, \mathcal{L}^N \llcorner \Omega_n), \quad n \in \mathbb{N}.$$

We are going to show in several steps that $\nu_n \xrightarrow{\mathcal{C}'_b} \delta_{(x_0,0,0)}$ and deduce the result. It suffices to show that the three projections $\nu_n^i := (\pi^i)_\sharp \nu_n$, $i \in \{1, 2, 3\}$ converge narrowly to δ_{x_0}, δ_0 and δ_0 respectively. Indeed, this would imply that (ν_n) converges narrowly to a measure concentrated on $(x_0, 0, 0)$, hence to $\delta_{(x_0,0,0)}$ since the ν_n are probability measures. First of all, since (ν_n) has bounded mass and (θ_n) is bounded, we may take a subsequence (not relabeled) such that $\nu_n \xrightarrow{\mathcal{C}'_0} \nu$ and $\theta_n \to \theta$ as $n \to \infty$ for some $\nu \in \mathcal{M}_+(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ and $\theta \geq 0$.

Step 1: $\nu_n^1 \xrightarrow{\mathcal{C}'_b} \delta_{x_0}$. This is a direct consequence of the fact that ν_n^1 is concentrated on $\Phi_n(\mathbb{R}^N)$ for every n and $\mathrm{dist}(\Phi_n(\mathbb{R}^N), x_0)$ as $n \to \infty$.

Step 2: $\nu_n^2 \xrightarrow{\mathcal{C}'_b} \delta_0$. By (3.18) and our assumption (H5), there is a constant $K_1 > 0$ with

$$\int_{\Omega_n} |\nabla u_n|^p \leq K_1 \int_{\Omega_n} u_n, \quad \forall n \in \mathbb{N}. \tag{3.19}$$

We deduce from Markov's inequality, and Lemma 3.7 applied with $r = p(1+\frac{1}{N})$, corresponding to $\alpha = \frac{N}{N+1}$, that

$$\begin{aligned}\nu_n^2([\eta,+\infty)) = \frac{1}{\theta_n}\int_{\{u_n\geq\eta\}} u_n &= \frac{1}{\theta_n}\int_{\{u_n\geq\eta\}} u_n^{1-r}u_n^r \\ &\leq \frac{1}{\theta_n\eta^{r-1}}\int_{\Omega_n} u_n^r \\ &\leq \frac{C}{\theta_n\eta^{r-1}}\big(\|\nabla u_n\|_{L^p(\Omega_n)} + \|u_n\|_{L^1(\Omega_n)}\big)^{r\alpha}\|u_n\|^{r(1-\alpha)}_{L^1_{\mathrm{uloc}}(\Omega_n)} \\ &\leq \frac{C'}{\eta^{r-1}}(1+\theta_n^{p-1})\|u_n\|^{r(1-\alpha)}_{L^1_{\mathrm{uloc}}(\Omega_n)},\end{aligned}$$

where in the last inequality, we have used the identity $\alpha r = p$ and (3.19).

Since $(u_n)_{n\in\mathbb{N}}$ is vanishing and $(\theta_n)_{n\in\mathbb{N}}$ is bounded, the last term in the previous inequality goes to zero as $n\to\infty$ and it follows that $\nu_n^2 \xrightarrow{\mathcal{C}_b'} \delta_0$.

Step 3: $\nu_n^3 \xrightarrow{\mathcal{C}_b'} \delta_0$. Fix $M>0$ and $\eta>0$. One has by (3.19),

$$\begin{aligned}\nu_n^3([M,+\infty)) = \frac{1}{\theta_n}\int_{\{|\nabla u_n|\geq M\}} u_n &\leq \frac{1}{\theta_n}\int_{\{u_n<\eta\}\cap\{|\nabla u_n|\geq M\}} u_n + \frac{1}{\theta_n}\int_{\{u_n>\eta\}} u_n \\ &\leq \frac{\eta}{\theta_n}\,\mathcal{L}^N(\{|\nabla u_n|\geq M\}) + \nu_n^2([\eta,+\infty)) \\ &\leq \frac{\eta}{\theta_n}\frac{1}{M^p}\int_{\Omega_n}|\nabla u_n|^p + \nu_n^2([\eta,+\infty)) \\ &\leq \frac{\eta K_1}{M^p} + \nu_n^2([\eta,+\infty)).\end{aligned}$$

By the previous step, we know that $\lim_{n\to+\infty}\nu_n^2([\eta,+\infty)) = 0$, hence taking the superior limit as $n\to+\infty$ then $\eta\to 0$ we get $\lim_{n\to+\infty}\nu_n^3([M,+\infty)) = 0$. Since this is true for every $M>0$ we obtain $\nu_n^3 \xrightarrow{\mathcal{C}_b'} \delta_0$.

Step 4: conclusion. By the previous steps, we deduce that $\nu_n \xrightarrow{\mathcal{C}_b'} \delta_{(x_0,0,0)}$ as $n\to+\infty$. We define $g:\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N\to[0,+\infty]$ as the lower semicontinuous envelope of $\mathbb{R}^N\times\mathbb{R}_+^*\times\mathbb{R}^N\ni(x,u,\xi)\mapsto\frac{1}{u}f(x,u,\xi)$. By ($H_1$), we have $g(x,u,\xi)=\frac{1}{u}f(x,u,\xi)$ if $u>0$, and by (1.4), we have $g(x,0,0) = f'_-(x,0^+,0)$ for every $x\in\mathbb{R}^N$. Hence, by lower semicontinuty of g and weak convergence of (ν_n), we get

$$\begin{aligned}\liminf_{n\to\infty}\int_{\Omega_n} f(\Phi_n,u_n,\nabla u_n) &\geq \liminf_{n\to\infty}\int_{\{u_n>0\}}\frac{f(\Phi_n,u_n,\nabla u_n)}{u_n}u_n \\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N} g(x,u,\xi)\,\mathrm{d}\nu_n(x,u,\xi) \\ &\geq \int_{\mathbb{R}^N} g(x,u,\xi)\,\mathrm{d}\delta_{(x_0,0,0)} = f'_-(x_0,0^+,0),\end{aligned}$$

which ends the proof of the lemma. □

As a corollary, we may now relate the slope at 0 of H_f to that of f.

Corollary 3.9. *Assume that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ satisfies (H1), (H5) for some $p \in (1, +\infty)$ and (H6). Fix $x \in \mathbb{R}^N$. If either $N \geq 2$ or ($N = 1$ and $H_f(x, \cdot) \not\equiv +\infty$ on $\mathbb{R}^*_\pm$), then $H'_f(x, 0^\pm) = f'_-(x, 0^\pm, 0)$.*

Proof. The inequality $H'_f(x, 0^\pm) \leq f'_-(x, 0^\pm, 0)$ is a consequence of (H6), Proposition 2.6 and Proposition 2.7, and the converse inequality $H'_f(x, 0^\pm) \geq f'_-(x, 0^\pm, 0)$ comes from Proposition 3.8. □

We now establish our main energy lower bound along sequences with bounded mass (not necessarily vanishing):

Proposition 3.10. *Assume that $(f_\varepsilon)_{\varepsilon>0}$ is a family of functions $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ satisfying (H1), (H2), (H5) and (H8) for some limit f. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers going to zero, $(R_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ be two sequences in $(0, +\infty]$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} R_n - r_n = +\infty$, $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions $u_n \in W^{1,1}_{\mathrm{loc}}(B_{R_n}, \mathbb{R}_\pm)$ with finite limit mass $m := \lim_{n\to\infty} \int_{B_{r_n}} u_n$, and $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of Borel maps $\Phi_n : B_{R_n} \to \mathbb{R}^N$ such that*

$$\sup_{y\in B_{R_n}} |\Phi_n(y) - x_0| \xrightarrow[n\to\infty]{} 0 \quad \text{for some } x_0 \in \mathbb{R}^N. \tag{3.20}$$

Then there exists a family $(u^i)_{0\leq i<k}$ of functions in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_\pm)$ with $k \in \mathbb{N} \cup \{+\infty\}$, such that $m_i := \int_{\mathbb{R}^N} u^i \in \mathbb{R}^*_\pm$ for every i, and

$$m = m_v + \sum_{0\leq i<k} m_i \quad \text{with } m_v \geq 0, \tag{3.21}$$

$$\liminf_{n\to\infty} \int_{B_{R_n}} f_{\varepsilon_n}(\Phi_n, u_n, \nabla u_n) \geq m_v f'_-(x_0, 0^\pm, 0) + \sum_{0\leq i<k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i). \tag{3.22}$$

Proof. Suppose for example that $u_n \geq 0$ a.e. for every n. We first assume, up to subsequence, that the left hand side of (3.22) is a limit. We apply the profile decomposition Theorem 3.2 to the sequence of positive measures $\mu_n = u_n \mathcal{L}^N_{|B_{r_n}}$ where, without loss of generality, we assume the extraction σ to be the identity for convenience, and we use the same notation as in Theorem 3.2. In particular, for each bubble $B^i_n = B_{r^i_n}(x^i_n)$, with $0 \leq i < k_n$, we have $x^i_n \in \operatorname{supp} \mu_n \subseteq B_{r_n}$. By assumption, we have $\lim_{n\to\infty}(R_n - r_n) = +\infty$; hence, up to reducing the radii of the balls B^i_n if necessary, in such a way that their radii still diverge (see Remark 3.4), we can assume that

$$B^i_n \subseteq B_{R_n - 1}, \quad 0 \leq i < k_n. \tag{3.23}$$

For each $0 \leq i < k_n$, we let $u^i_n := u_n(\cdot + x^i_n)$. Assuming without loss of generality that the left hand side of (3.22) is finite, we get that the sequence $(u^i_n)_n$ is bounded in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ by (H5). Hence, after a further extraction if needed, we get that $(u^i_n)_{n\in\mathbb{N}} \rightharpoonup u^i$ weakly in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for some limit u^i, for every $0 \leq i < k = \lim k_n$. Setting $m_i = \int_{\mathbb{R}^N} u^i$ for every i, by (D) in Theorem 3.2, we have

$$m_v := m - \sum_{0\leq i<k} m_i = \lim_{n\to\infty} \int_{B_{r_n} \setminus \cup_{0\leq i<k_n} B^i_n} u_n.$$

Fix $\varepsilon > 0$. We decompose the energy as

$$\int_{B_{R_n}} f_\varepsilon(\Phi_n, u_n, \nabla u_n) = \int_{B_{R_n}\setminus\cup_{0\le i<k_n} B_n^i} f_\varepsilon(\Phi_n, u_n, \nabla u_n) + \sum_{0\le i<k_n} \int_{B_{r_n^i}} f_\varepsilon(\Phi_n(\cdot + x_n^i), u_n^i, \nabla u_n^i). \quad (3.24)$$

Note that the domains $\Omega_n := B_{R_n} \setminus \cup_{0\le i<k} B_n^i$ satisfy $\inf_{n\in\mathbb{N}} \operatorname{reach}(\partial\Omega_n) > 0$ as noticed in Example 3.5, thanks to (3.23) and (B), (C) in Theorem 3.2. Hence, applying Proposition 3.8 to the Lagrangian f_ε, we obtain

$$\liminf_{n\to\infty} \int_{B_{R_n}\setminus\cup_{0\le i<k_n} B_n^i} f_\varepsilon(\Phi_n, u_n, \nabla u_n) \ge m_v (f_\varepsilon)'_-(x_0, 0^+, 0). \quad (3.25)$$

Moreover, by lower semicontinuity of integral functionals (see [But89, Theorem 4.1.1]), in view of (3.20), we have for each i with $0 \le i < k$,

$$\liminf_{n\to\infty} \int_{B_{r_n^i}} f_\varepsilon(\Phi_n(\cdot + x_n^i), u_n^i, \nabla u_n^i) \ge \int_{\mathbb{R}^N} f_\varepsilon(x_0, u^i, \nabla u^i). \quad (3.26)$$

Finally, by (3.24), (3.25), (3.26), (H_8) and by monotone convergence, we deduce that

$$\begin{aligned}\liminf_{n\to\infty} \int_{B_{R_n}} f_{\varepsilon_n}(\Phi_n, u_n, \nabla u_n) &\ge \lim_{\varepsilon\to 0^+} \Big(m_v (f_\varepsilon)'_-(x_0, 0^+, 0) + \sum_{0\le i<k} \int_{\mathbb{R}^N} f_\varepsilon(x_0, u^i, \nabla u^i) \Big) \\ &= m_v f'_-(x_0, 0^+, 0) + \sum_{0\le i<k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i). \qquad \square\end{aligned}$$

3.3 Existence of optimal profiles

For the existence of an optimal profile in (1.2), we need a criterion that rules out splitting and vanishing of minimizing sequences:

Lemma 3.11. *Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function. Then H is subadditive, and if for some $0 < \theta < m$ one has $H(m) = H(m-\theta) + H(\theta)$, then H is linear on $(0, m)$.*

Proof. By concavity, $t \mapsto \frac{H(t)}{t}$ is non-increasing. Hence,

$$H(m) = \theta\frac{H(m)}{m} + (m-\theta)\frac{H(m)}{m} \le \theta\frac{H(\theta)}{\theta} + (m-\theta)\frac{H(m-\theta)}{m-\theta}.$$

But, by assumption, the last inequality is an equality which means that $\frac{H(m)}{m} = \frac{H(\theta)}{\theta} = \frac{H(m-\theta)}{m-\theta}$. In particular, the monotone function $t \mapsto \frac{H(t)}{t}$ must be constant on $[\theta, m]$, i.e. H must be linear on $[\theta, m]$. By concavity this is only possible if H is linear on $[0, m]$. $\square$

We can now state and prove our existence result:

Proposition 3.12. *Assume that $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ satisfies (H_1), (H_2), (H_5) and (H_6). Let $x_0 \in \mathbb{R}^N$ and $m \in \mathbb{R}_+$ (resp. $m \in \mathbb{R}_-$). If $H_f(x_0, \cdot)$ is not linear on $[0, m]$ (resp. $[m, 0]$), then (1.2) admits a solution $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$, i.e. $\int_{\mathbb{R}^N} u = m$ and $\int_{\mathbb{R}^N} f(x_0, u, \nabla u) = H_f(x_0, m)$, such that $u \ge 0$ (resp. $u \le 0$) in $\mathbb{R}^N$.*

Proof. We consider the case $m \geq 0$, the case $m < 0$ can then be deduced by considering $\tilde{f}(x,u,\xi) = f(x,-u,\xi)$. By Remark 2.2, the admissible class in (1.2) can be reduced to non negative functions. In particular, if $m = 0$, then $u = 0$ is the only solution. If $m > 0$, we apply Proposition 3.10 in the following situation: $f_\varepsilon \equiv f$, $R_n \equiv +\infty$, $\Phi_n \equiv x_0$, $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence for the minimization problem in (1.2), and $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive radii going to $+\infty$ such that $\lim_{n\to\infty} \int_{B_{r_n}} u_n = m$. We obtain

$$H_f(x_0,m) \geq m_v f'_-(x_0,0^+,0) + \sum_{0\leq i<k} \int_{\mathbb{R}^N} f(x_0,u^i,\nabla u^i),$$

with $k \in \mathbb{N}\cup\{+\infty\}$, $u^i \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}_+)$ and $m = \sum_{0\leq i<k} m_i + m_v$, where $m_i := \int_{\mathbb{R}^N} u^i$. By Proposition 2.6 and Proposition 2.7, in view of our assumption (H_6), we have $f'_-(x_0,0^+,0) \geq H'_f(x_0,0)$. Moreover, by Theorem 2.1, we have $m_v H'_f(x_0,0) \geq H_f(x_0,m_v)$. Hence, by definition of H_f,

$$H_f(x_0,m) \geq H_f(x_0,m_v) + \sum_{0\leq i<k} H_f(x_0,m_i).$$

Since the concave function $H_f(x_0,\cdot)$ is not linear on $[0,m]$, by Lemma 3.11, we have either $k = 1$ and $m_v = 0$, and we are done, or $k = 0$ and $m = m_v$. But in the latter case, we would have $H_f(x_0,m) = mH'_f(x_0,0)$ which implies that the monotone function $t \mapsto \frac{H_f(x_0,t)}{t}$ is constant on $[0,m]$, i.e. that $H_f(x_0,\cdot)$ is linear on $[0,m]$. This contradicts our assumption. □

4 Γ-convergence of the rescaled energies towards the H-mass

We establish lower and upper bounds for the $\Gamma-\liminf$ and $\Gamma-\limsup$ respectively, from which we deduce the proof of our main Γ-convergence result. The upper bound on the $\Gamma-\limsup$ holds under more general assumptions and will be needed in Section 5.5.

4.1 Lower bound for the $\Gamma-\liminf$

Given a Borel function $f : \mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to [0,+\infty]$, we define for every $(x,m)\in\mathbb{R}^N\times\mathbb{R}$,

$$H^-_f(x,m) := H_f(x,m) \wedge (f'_-(x,0^\pm,0)|m|), \quad \text{if } \pm m \geq 0, \tag{4.1}$$

recalling that H_f is defined in (1.2) and $f'_-(x,0^\pm,0)$ in (1.4), with the usual convention $(\pm\infty)\times 0 = 0$. Notice that under (H_6), in view of Proposition 2.6 and Proposition 2.7 we have $H^-_f(x,m) = H_f(x,m)$.

Proposition 4.1. *Assume that $(f_\varepsilon)_{\varepsilon>0}$ is a family of functions $f_\varepsilon : \mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to [0,+\infty]$ satisfying (H_1), (H_2), (H_5) and (H_8) where $f = \lim_{\varepsilon\to 0} f_\varepsilon$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers going to zero, $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$, and let*

$$e_n(x) := f_{\varepsilon_n}(x,\varepsilon_n^N u_n(x),\varepsilon_n^{N+1}\nabla u_n(x))\varepsilon_n^{-N}, \quad x\in\mathbb{R}^N,$$

be the energy density of u_n. If $u_n\,\mathcal{L}^N \xrightarrow{\mathcal{C}'_0} u \in \mathcal{M}(\mathbb{R}^N)$ and $e_n\,\mathcal{L}^N \xrightarrow{\mathcal{C}'_0} e \in \mathcal{M}(\mathbb{R}^N)$, then

$$e \geq H^-_f(u). \tag{4.2}$$

In particular, $\Gamma(\mathcal{C}'_0) - \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^{H^-_f}$.

Proof of Proposition 4.1. Set $H := H_f^-$. Let us assume first that $u_n \geq 0$ a.e. for every n. To obtain (4.2), it is enough to prove that for every $x_0 \in \mathbb{R}^N$,

$$e(\{x_0\}) \geq H(x_0, u(\{x_0\})). \tag{4.3}$$

and that if $x_0 \in \operatorname{supp} u$ is not an atom of u, then

$$\limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} \geq H'(x_0, 0^+), \tag{4.4}$$

Indeed (4.3) implies that $e \geq (H(u))^a$ (the atomic part of the measure $H(u)$) while (4.4) implies that $e \geq H'(\cdot, 0^+)u^d = (H(u))^d$, by Radon-Nikodỳm theorem (see [AFP00, Theorem 2.22]); these two relations yield $e \geq (H(u))^a + (H(u))^d = H(u)$ as required.

We fix $x_0 \in \operatorname{supp} u$ and proceed in several steps.

Step 1: blow-up near x_0. We first take two sequences of positive radii $(R_\ell)_{\ell\in\mathbb{N}} \to 0$ and $(r_\ell)_{\ell\in\mathbb{N}}$ such that for every $\ell \in \mathbb{N}$, $r_\ell \in (0, R_\ell)$,

$$e(\partial B_{R_\ell}(x_0)) = u(\partial B_{r_\ell}(x_0)) = 0, \tag{4.5}$$

and

$$\lim_{\ell\to\infty} \frac{e(B_{R_\ell}(x_0))}{u(B_{r_\ell}(x_0))} = \limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))}. \tag{4.6}$$

This last property is obtained by taking first a sequence $(\rho_\ell)_\ell$ such that

$$\limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} = \lim_{\ell\to\infty} \frac{e(B_{\rho_\ell}(x_0))}{u(B_{\rho_\ell}(x_0))},$$

then using monotone convergence the measures to get first r_ℓ then R_ℓ such that $0 < r_\ell < R_\ell < \rho_\ell$, $u(B_{r_\ell}(x_0)) \geq (1 - 2^{-\ell})u(B_{\rho_\ell}(x_0))$ and $e(B_{R_\ell}(x_0)) \geq (1 - 2^{-\ell})e(B_{\rho_\ell}(x_0))$.

By weak convergence and (4.5), according to [AFP00, Proposition 1.62 b)], we have for every $\ell \in \mathbb{N}$,

$$\lim_{n\to\infty} e_n(B_{R_\ell}(x_0)) = e(B_{R_\ell}(x_0)) \quad \text{and} \quad \lim_{n\to\infty} \int_{B_{r_\ell}(x_0)} u_n = u(B_{r_\ell}(x_0)).$$

Hence, there exists an extraction $(n_\ell)_{\ell\in\mathbb{N}} \in \Sigma$ such that

$$\lim_{\ell\to\infty} \frac{r_\ell}{\varepsilon_{n_\ell}} = +\infty \quad \text{and} \quad \lim_{\ell\to\infty} \frac{R_\ell - r_\ell}{\varepsilon_{n_\ell}} = +\infty, \tag{4.7}$$

satisfying the following conditions:

$$u(\{x_0\}) = \lim_{\ell\to\infty} \int_{B_{r_\ell}(x_0)} u_{n_\ell}, \quad e(\{x_0\}) = \lim_{\ell\to\infty} e_{n_\ell}(B_{R_\ell}(x_0)), \tag{4.8}$$

and

$$\limsup_{\ell\to\infty} \frac{e(B_{R_\ell}(x_0))}{u(B_{r_\ell}(x_0))} = \lim_{\ell\to\infty} \frac{e_{n_\ell}(B_{R_\ell}(x_0))}{\int_{B_{r_\ell}(x_0)} u_{n_\ell}}. \tag{4.9}$$

We may rewrite the mass and energy in terms of the re-scaled map v_ℓ defined by

$$v_\ell(y) \coloneqq \varepsilon_{n_\ell}^N u_{n_\ell}(x_0 + \varepsilon_{n_\ell} y), \quad y \in \mathbb{R}^N, \ell \in \mathbb{N} \tag{4.10}$$

as follows:

$$\int_{B_{r_\ell}(x_0)} u_{n_\ell} = \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell, \tag{4.11}$$

and

$$e_{n_\ell}(B_{R_\ell}(x_0)) = \int_{B_{\varepsilon_{n_\ell}^{-1} R_\ell}} f_{\varepsilon_{n_\ell}}(x_0 + \varepsilon_{n_\ell} y, v_\ell(y), \nabla v_\ell(y))\,\mathrm{d}y. \tag{4.12}$$

Step 2: proof of (4.3). By Proposition 3.10, we have

$$e(\{x_0\}) = \lim_{\ell\to\infty} e_{n_\ell}(B_{R_\ell}(x_0)) \geq m_v f'_-(x_0, 0^+, 0) + \sum_{0\leq i<k} H_f(x_0, m_i). \tag{4.13}$$

Here $k \in \mathbb{N} \cup \{+\infty\}$ and $m = m_v + \sum_{0\leq i<k} m_i$, with $m_i > 0$, $m_v \geq 0$ and

$$m = \lim_{\ell\to\infty} \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell = u(\{x_0\}).$$

Since the function $H = H_f^-$, defined in (4.1), is the infimum of two functions which are concave in the mass m, it is itself concave in m hence subadditive. From (4.13) we thus arrive at

$$e(\{x_0\}) \geq H(x_0, m_v) + \sum_{0\leq i<k} H(x_0, m_i) \geq H\Big(x_0, m_v + \sum_{0\leq i<k} m_i\Big) = H(x_0, u(\{x_0\})).$$

Step 3: proof of (4.4). Fix $\varepsilon > 0$ and assume that $m = u(\{x_0\}) = 0$. In that case, we apply Proposition 3.8 to the sequence of functions $(v_\ell)_{\ell\in\mathbb{N}}$ defined on the sets $\Omega_\ell = B_{\varepsilon_{n_\ell}^{-1} r_\ell}$ and the function f_ε to get, thanks to (H_8):

$$\begin{aligned}
\limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} &= \lim_{\ell\to\infty} \frac{e_{n_\ell}(B_{R_\ell}(x_0))}{\int_{B_{r_\ell}(x_0)} u_{n_\ell}} \\
&\geq \liminf_{\ell\to\infty} \frac{1}{\int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell} \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} f_\varepsilon(x_0 + \varepsilon_{n_\ell} y, v_\ell(y), \nabla v_\ell(y)) \\
&\geq (f_\varepsilon)'_-(x_0, 0^+, 0).
\end{aligned}$$

Taking the limit $\varepsilon \to 0^+$, we deduce by (H_8) and (4.1):

$$\limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} \geq f'_-(x_0, 0^+, 0) \geq H'(x_0, 0^+). \tag{4.14}$$

In view of the discussion at the beginning of the proof, we have now proved (4.2).

Step 4: proof of (4.2) for signed $(u_n)_n$. Notice that the preceding reasoning for non negative u_n applies also to the case of non-positive u_n. Let us handle the case where the (u_n)'s may change sign. We simply apply the above cases to the positive and negative parts $((u_n)_\pm)_n$ which converge weakly (up to subsequence) to some measures $u^\pm \in \mathcal{M}_+(\mathbb{R}^N)$ which satisfy $u = u^+ - u^-$, so that $e \geq H(\pm u^\pm)$. We know that the positive and negative parts $u_\pm$ of u are optimal decompositions in the sense that $u = u_+ - u_-$ with $u_\pm \leq u^\pm$, and we have $u_+ \perp u_-$. By monotonicity of the H-transform, due to Theorem 2.1, $e \geq H(\pm u^\pm) \geq H(\pm u_\pm)$, and since $H(u_+) \perp H(-u_-)$, we get

$$e \geq H(u_+) + H(-u_-) = H(u).$$

Step 5: lower bound for the $\Gamma - \liminf$. We justify that (4.2) implies the lower bound $\Gamma(\mathcal{C}_0') - \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^H$. Indeed, fix $u \in \mathcal{M}(\mathbb{R}^N)$ and consider a family $(u_\varepsilon)_{\varepsilon>0}$ weakly converging to u as $\varepsilon \to 0$. We need to show that $\mathbf{M}^H(u) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon)$. Assume without loss of generality that the inferior limit is finite and take a sequence of positive numbers $(\varepsilon_n)_{n\in\mathbb{N}} \to 0$ such that this inferior limit is equal to $\lim_{n\to\infty} \mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n})$. Now the energy density e_n associated to $u_n = u_{\varepsilon_n}$ has bounded mass and up to extracting a subsequence one may assume that it converges weakly to some measure $e \in \mathcal{M}_+(\mathbb{R}^N)$. By the previous steps, $e \geq H(u)$, and by lower semicontinuity and monotonicity of the mass:

$$\liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(u_\varepsilon) = \liminf_{n\to\infty} \|e_n\| \geq \|e\| \geq \|H(u)\| = \mathbf{M}^H(u). \qquad \square$$

4.2 Upper bound for the $\Gamma - \limsup$

In this section, we introduce the following substitute for (H_8), where $f, (f_\varepsilon)_{\varepsilon>0}$ are Borel maps from $\mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ to $[0, +\infty]$:

(U) there exists $C < +\infty$ such that for every $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$\limsup_{\varepsilon\to 0^+} f_\varepsilon(x + \varepsilon y, u, \xi) \leq f(x, u, \xi) \quad \text{and} \quad f_\varepsilon(y, u, \xi) \leq C(f(x, u, \xi) + u) \quad \forall \varepsilon > 0.$$

Proposition 4.2. *Assume that $f, (f_\varepsilon)_{\varepsilon>0}$ satisfy (U) and (H_3). If $u \in \mathcal{M}(\mathbb{R}^N)$, then there exists $(u_\varepsilon)_{\varepsilon>0} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $u_\varepsilon \mathcal{L}^N \xrightarrow{\mathcal{C}_b'} u$ when $\varepsilon \to 0$ and which satisfies*

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathbf{M}^{H_{f,\mathrm{lsc}}}(u),$$

where $H_{f,\mathrm{lsc}} \leq H_f$ stands for the lower semicontinuous and mass-subadditive envelope of H_f, defined in (2.6). In other words, we have $\Gamma(\mathcal{C}_b') - \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon \leq \mathbf{M}^{H_{f,\mathrm{lsc}}}$.

Proof of Proposition 4.2. Let $F = \Gamma(\mathcal{C}_b') - \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon$. As an upper Γ-limit, F is sequentially lower semicontinuous in the narrow topology. Hence, by Proposition 2.5, it is enough to prove that $F(u) \leq \mathbf{M}^{H_f}(u)$ whenever u is finitely atomic. Let $u = \sum_{i=1}^k m_i \delta_{x_i}$ with $k \in \mathbb{N}$, $m_i \in \mathbb{R}$, $x_i \in \mathbb{R}^N$, and assume without loss of generality that $x_i \neq x_j$ when $i \neq j$ and $\mathbf{M}^{H_f}(u) < +\infty$. Fix $\eta > 0$. For each $i = 1, \dots, k$, there exists $u_i \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u_i = m_i$ and $\int_{\mathbb{R}^N} f(x_i, u_i, \nabla u_i) \leq H(x_i, m_i) + \eta$. We define for every $i = 1, \dots, k$,

$$u_\varepsilon^i(x) = \varepsilon^{-N} u_i(\varepsilon^{-1}(x - x_i)), \quad x \in \mathbb{R}^N, \tag{4.15}$$

and

$$u_\varepsilon = \sup\{u_\varepsilon^i \ : \ i = 1, \dots, k\}, \tag{4.16}$$

which converge narrowly as measures to u as $\varepsilon \to 0$. We have by change of variables:

$$\mathcal{E}_\varepsilon(u_\varepsilon) \le \sum_{i=1}^k \mathcal{E}_\varepsilon(u_\varepsilon^i) = \sum_{i=1}^k \int_{\mathbb{R}^N} f_\varepsilon(x_i + \varepsilon x, u_i, \nabla u_i).$$

Using our assumption (U) and the dominated convergence theorem, one gets as $\varepsilon \to 0$:

$$F(u) \le \limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \le \sum_{i=1}^k \int_{\mathbb{R}^N} f(x_i, u_i, \nabla u_i) \le \sum_{i=1}^k H(x_i, m_i) + k\eta = \mathbf{M}^H(u) + k\eta.$$

The conclusion follows by arbitrariness of $\eta > 0$. □

4.3 Proof of the main Γ-convergence result

We now explain how Theorem 1.2 follows from Proposition 4.1 and Proposition 4.2.

Proof of Theorem 1.2. The lower bound $\Gamma(\mathcal{C}_0') - \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon \ge \mathbf{M}^{H_f^-}$ follows from Proposition 4.1, and the upper bound $\Gamma(\mathcal{C}_b') - \limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon \le \mathbf{M}^{H_{f,\mathrm{lsc}}}$ from Proposition 4.2, where the assumption (U) is a consequence of (H_4), (H_7) and (H_8). In the case $N = 2$, by Proposition 2.6 and assumption (H_6), we have $H_f^- = H_f$, and $H_f \ge H_{f,\mathrm{lsc}}$ by definition, so that both $\Gamma - \liminf$ and $\Gamma - \limsup$ (for weak and narrow topologies) coincide. In the case $N = 1$, by Proposition 2.7 either we have again $H_f^- = H_f$ on $\mathbb{R}^N \times \mathbb{R}_+$ by (H_6) and (H_3), or we have $H_f(x_0, \cdot) \equiv +\infty$ on $(0, +\infty)$ for some $x_0 \in \mathbb{R}$. In that case, by (H_7) and Remark 2.2 we necessarily have $\mathcal{E}_\varepsilon(u) = +\infty$ for every u such that $\int_{\mathbb{R}^N} u > 0$, and $H_f(x, \cdot) \equiv H_f(x_0, \cdot)$ on $\mathbb{R}_+$ for every $x \in \mathbb{R}$. It implies that both $\Gamma - \liminf$ and $\Gamma - \limsup$ (for the weak and narrow topologies) coincide with[6] $\chi_{\{0\}} = \mathbf{M}^{H_f}$ on $\mathcal{M}_+(\mathbb{R}^N)$. Similarly, they coincide on finite negative measures. □

5 Examples, counterexamples and applications

5.1 Scale-invariant Lagrangians and necessity of assumption (H_6)

Our assumption (H_6) is not very standard, but we need a condition of this type in order to get Γ-convergence of the rescaled energies $\mathcal{E}_\varepsilon$ towards $\mathbf{M}^{H_f}$, as shown by the following class of scale-invariant Lagrangians:

$$f_\varepsilon(x, u, \xi) = f(u, \xi) \quad \text{with} \quad f(u, \xi) = \begin{cases} u^{p(\frac{1}{p^\star} - 1)} |\xi|^p & \text{if } u > 0, \\ 0 & \text{else,} \end{cases} \tag{5.1}$$

where $p \in (1, N)$, $N \in \mathbb{N}^*$ and $p^\star := \frac{pN}{N-p}$. By straightforward computations, $\mathcal{E}_\varepsilon(u) = \mathcal{E}_f(u) := \int_{\mathbb{R}^N} f(u, \nabla u)$ for every $\varepsilon > 0$ and $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ in that case.

[6] $\chi_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise.

Moreover, the associated cost function H_f is not trivial. Indeed, applying the Gagliardo–Nirenberg–Sobolev inequality,

$$\Big(\int_{\mathbb{R}^N} |v|^{p^\star}\Big)^{\frac{1}{p^\star}} \le C\Big(\int_{\mathbb{R}^N} |\nabla v|^p\Big)^{\frac{1}{p}}, \quad \forall v \in L^{p^\star} \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N),$$

to the function[7] $v = u^{\frac{1}{p^\star}}$, we obtain that for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+) \cap L^1(\mathbb{R}^N)$,

$$\Big(\int_{\mathbb{R}^N} u\Big)^{\frac{p}{p^\star}} \le \Big(\frac{C}{p^\star}\Big)^p \int_{\{u>0\}} u^{\frac{p}{p^\star}-p} |\nabla u|^p = \Big(\frac{C}{p^\star}\Big)^p \mathcal{E}_f(u).$$

Hence, for every $m > 0$, we have $H_f(m) > 0$, and even $H_f(m) < +\infty$ since any function $u = v^{p^\star}$, with $v \in W^{1,p}(\mathbb{R}^N, \mathbb{R}_+)$, has finite energy. Replacing u by mu in the infimum defining H_f in (1.2), we actually obtain

$$H_f(m) = m^{1-\frac{p}{N}} H_f(1), \quad 0 < H_f(1) < +\infty. \tag{5.2}$$

In that case, it is clear that the Γ-limit of $\mathcal{E}_\varepsilon \equiv \mathcal{E}$ in the weak or narrow topology of $\mathcal{M}_+(\mathbb{R}^N)$, that is the lower semicontinuous relaxation of $\mathcal{E}_f$, does not coincide with $\mathbf{M}^{H_f}$; indeed, the first functional is finite on diffuse measures whose density has finite energy, while the second functional is always infinite for non-trivial diffuse measures since $H'_f(0) = +\infty$.

These scaling invariant Lagrangians are ruled out by our assumption (H_6). All the other assumptions are satisfied except (H_5). Note that the following perturbation of f,

$$\tilde{f}(u, \xi) = (1 + u^{p(\frac{1}{p^\star}-1)})|\xi|^p$$

satisfies all the assumptions except (H_6), and provides a counterexample to the Γ-convergence. Indeed, $\mathbf{M}_{H_{\tilde{f}}} \ge \mathbf{M}_{H_f}$ is still infinite on diffuse measures, while (the relaxation of) $\mathcal{E}_{\tilde{f}}$ is finite for any diffuse measure whose density has finite energy.

We stress that an assumption like (H_6) is actually needed, even for the lower semicontinuity of the function H_f – recall that if $\mathbf{M}_{H_f}$ is a Γ-limit, then it must be lower semicontinuous by [Bra02, Proposition 1.28], which in turn implies that the function H_f is lower semicontinuous by Proposition 2.5. Indeed, consider the Lagrangians

$$f(x, u, \xi) = (1 + u^{p(\frac{1}{p^\star}-1)})|\xi|^{p(x)},$$

with $p \in \mathcal{C}^0(\mathbb{R}^N, (1, N))$ such that $p(0) = p \in (1, N)$ and $p(x) > p$ when $x \neq 0$. Then, we have $H_f(0, m) = m^{1-\frac{p}{N}} H(1)$, but $H_f(x, \cdot) \equiv 0$ if $x \neq 0$ as can be easily seen via the change of function $\varepsilon^N u(\varepsilon\,\cdot)$, with $\varepsilon > 0$ small.

5.2 General concave costs in dimension one

It has been proved in [Wir19] that for any continuous concave function $H : \mathbb{R}_+ \to \mathbb{R}_+$ with $H(0) = 0$, there exists a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $c(0) = 0$, $u \mapsto \frac{c(u)}{u}$ is lower semicontinuous and non-increasing on $(0, +\infty)$, and for every $m \ge 0$,

$$H(m) = \inf\left\{\int_{\mathbb{R}} |u'|^2 + c(u) \ : \ u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}_+), \ \int_{\mathbb{R}} u = m\right\}.$$

[7] Actually, we apply it to $v_\varepsilon = \phi_\varepsilon(u)$ where ϕ_ε is a suitable approximation of $(\cdot)^{\frac{1}{p^\star}}$ and take $\varepsilon \to 0$.

The Lagrangians of the form $f_\varepsilon(x,u,\xi) = |\xi|^2 + c(u)$, in dimension $N=1$, satisfy all our assumptions (H_1)–(H_8), hence our Γ-convergence result stated in Theorem 1.2 yields the Γ-convergence of the functionals

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon^3 |u'|^2 + \frac{c(\varepsilon u)}{\varepsilon}, \quad u \in W^{1,2}(\mathbb{R}, \mathbb{R}_+),$$

towards $\mathbf{M}^H$ for both the weak and narrow convergence of measures. Therefore, we may find an elliptic approximation of any concave H-mass. Let us stress that c is determined in [Wir19] from H through several operations including a deconvolution problem, but no closed form solution is given in general; nonetheless, an explicit solution is provided if c is affine by parts.

In higher dimension $N \geq 2$, Proposition 2.8 tells us that the class of functions $H = H_f$ with f satisfying (H_1)–(H_8) is smaller, namely, H must satisfy:

$$\exists m_* \geq 0, \quad \begin{cases} H \text{ is linear on } [0, m_*], \\ H \text{ is strictly concave } (m_*, +\infty). \end{cases} \tag{5.3}$$

We have no positive or negative answer to the inverse problem, consisting in finding f satisfying our assumptions such that $H_f = H$, for a given continuous concave function $H : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (5.3).

5.3 Homogeneous costs in any dimension

In this section, we provide Lagrangians f to obtain the α-mass $\mathbf{M}^\alpha := \mathbf{M}^{t \mapsto t^\alpha}$ in any dimension N for a wide range of exponents, including super-critical exponents $\alpha \in \left(1 - \frac{1}{N}, 1\right]$. We consider for every $p \in [1, +\infty)$, $s \in (-\infty, 1]$ and $N \in \mathbb{N}^*$, the energy defined for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$ by

$$\mathcal{E}_{N,ps}(u) := \int_{\mathbb{R}^N} f_{N,p,s}(u, \nabla u) := \int_{\mathbb{R}^N} |\nabla u|^p + u^s. \tag{5.4}$$

Notice that for $p > 1$, $f_{N,p,s}$ satisfies all our hypotheses (H_1)–(H_6) (without dependence on x), (H_6) holding in dimension $N \geq 2$ with $\rho(t) = t$ for example. Thus by Theorem 1.2 the re-scaled energies Γ-converge to the $H_{f_{N,p,s}}$-mass.

One may compute $H_{f_{N,p,s}}$ substituting u by v such that $u = m\lambda^N v(\lambda \cdot)$ in (1.2), where

$$\lambda = m^{\frac{s/p-1}{1+N-sN/p}}. \tag{5.5}$$

Straightforward computations give $\int_{\mathbb{R}^N} v = 1$ if $\int_{\mathbb{R}^N} u = m$, and

$$\mathcal{E}_{N,p,s}(u) = m^{\alpha(N,p,s)} \mathcal{E}_{N,p,s}(v), \qquad \text{where} \quad \alpha(N,p,s) = \frac{1 - \frac{s}{p} + \frac{s}{N}}{1 - \frac{s}{p} + \frac{1}{N}},$$

thus

$$H_{N,p,s}(m) = c_{N,p,s} m^{\alpha(N,p,s)}, \qquad \text{where} \quad c_{N,p,s} = H_{N,p,s}(1).$$

We look for cases when the cost is non-trivial, i.e. neither identically zero nor infinite on $(0,+\infty)$. Take an auxiliary exponent $q \in [1,+\infty)$ and $\alpha \in [0,1]$ such that $1 = \alpha q + (1-\alpha)s$. By Hölder inequality,

$$\int_{\mathbb{R}^N} u = \int_{\mathbb{R}^N} u^{\alpha q} u^{(1-\alpha)s} \leq \left(\int_{\mathbb{R}^N} u^q\right)^{\alpha} \left(\int_{\mathbb{R}^N} u^s\right)^{1-\alpha}.$$

Moreover, choosing $q \in (1,p^\star)$ if $p < N$ and any $q \in (1,+\infty)$ if $p \geq N$, by the Gagliardo–Nirenberg–Sobolev inequality, for every $u \in W^{1,1}_{\mathrm{loc}} \cap L^1(\mathbb{R}^N,\mathbb{R}_+)$,

$$\left(\int_{\mathbb{R}^N} u^q\right)^{\frac{1}{q}} = \|u\|_{L^q} \leq C\|\nabla u\|_{L^p}^{\beta}\|u\|_{L^1}^{1-\beta},$$

with $\beta \in (0,1)$ such that $\frac{1}{q} = \beta\left(\frac{1}{p} - \frac{1}{N}\right) + (1-\beta)$. Hence,

$$\left(\int_{\mathbb{R}^N} u\right)^{1-q\alpha(1-\beta)} \leq C \left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{q\alpha\beta}{p}} \left(\int_{\mathbb{R}^N} u^s\right)^{1-\alpha},$$

and the cost is non-zero for every $m > 0$.

In the case $s \in [0,1]$, any $u = v^r$ with $v \in \mathcal{C}^1_c(\mathbb{R}^N)$ is a competitor with finite energy, thus $\mathcal{E}_{N,p,s}$ is non-trivial for every $p \in [1,+\infty)$. In the case $s < 0$, consider the competitor $u : x \mapsto (1-|x|)_+^\gamma$ for $\gamma > 0$ to be fixed later. Then $\int_{\mathbb{R}^N} |\nabla u|^p < +\infty$ if and only if $t \mapsto (1-t)^{(\gamma-1)p}$ is integrable at 1^-, i.e. $(\gamma-1)p > -1 \iff \gamma > 1-1/p$, and $\int_{\{u>0\}} u^s < +\infty$ if and only if $\gamma s > -1 \iff \gamma < -1/s$. Therefore, one may find $\gamma > 0$ satisfying both conditions, and ensure that $H_{f_{N,p,r,s}}$ is non-trivial, if

$$-p' < s < 0.$$

To summarize, we have shown that $H_{f_{N,p,s}}$ is non-trivial if:

$$s \in (-p', 1].$$

Since $\alpha = \alpha(N,p,s)$ is monotone in s, one may easily compute the range of α. If p and N are fixed, α ranges over $\left(\frac{N-1}{N+1-1/p}, 1\right]$ when $s \in (-p', 1]$. Notice that when $N = 1$ we obtain the whole range $\alpha \in (0,1]$, and at least the range $\left[1 - \frac{1}{N}, 1\right]$ for every p in dimension $N \geq 2$. Finally, we obtain a range $\alpha \in \left(1 - \frac{2}{N+1}, 1\right]$ when p ranges over $(1+\infty)$ in dimension N.

5.4 Branched transport approximation: H-masses of normal 1-currents

Branched Transport is a variant of classical optimal transport (see [San15] and Section 4.4.2 therein for a brief presentation of branched transport, and [BCM09] for a vast exposition) where the transport energy concentrates on a network, i.e. a 1-dimensional subset of $\mathbb{R}^d$, which has a graph structure when optimized with prescribed source and target measures. It can be formulated as a minimal flow problem,

$$\min\left\{\mathbf{M}_1^H(w) \ : \ \mathrm{div}(w) = \mu^- - \mu^+\right\},$$

where $\mu^\pm$ are probability measures on $\mathbb{R}^d$, $H : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is mass-subadditive, and the H-mass $\mathbf{M}_1^H$ is this time defined for finite vector measures $w \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$ whose distributional divergence is also a finite measure; in the language of currents, it is called a 1-dimensional normal current. Any such measure may be decomposed into a 1-rectifiable part $\theta\xi \cdot \mathcal{H}^1 \llcorner \Sigma$ where $\theta(x) \geq 0$ and $\xi(x)$ is a unit tangent vector to Σ for $\mathcal{H}^1$-a.e. $x \in \Sigma$, and a 1-diffuse part $w^\perp$ satisfying $|w^\perp|(A) = 0$ for every 1-rectifiable set A:

$$w = \theta\xi \cdot \mathcal{H}^1_{|M} + w^\perp.$$

The H-mass is then defined by:

$$\mathbf{M}_1^H(w) \coloneqq \int_\Sigma H(x, \theta(x)) \, \mathrm{d}\mathcal{H}^1(x) + \int_{\mathbb{R}^d} H'(x, 0) \, \mathrm{d}|w^\perp|. \tag{5.6}$$

In the case $H(x, m) = m^\alpha$ with $0 < \alpha < 1$, a family of approximations of these functional has been introduced in [OS11]:

$$\mathcal{E}_{\beta,\varepsilon}(w) = \begin{cases} \int_{\mathbb{R}^d} \varepsilon^{\gamma_1} |\nabla w|^2 + \varepsilon^{-\gamma_2} |w|^\beta & \text{if } w \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.7}$$

with $\beta = \frac{2-2d+2\alpha d}{3-d+\alpha(d-1)}$, $\gamma_1 = (d-1)(1-\alpha)$ and $\gamma_2 = 3 - d + \alpha(d-1)$. It has been shown in [OS11; Mon17] that the functionals $\mathcal{F}_{\beta,\varepsilon}$ Γ-converge as $\varepsilon \to 0^+$, in the topology of weak convergence of u and its divergence, to a non-trivial multiple of the α-mass $\mathbf{M}_1^\alpha \coloneqq \mathbf{M}_1^H$ with $H(x, m) = m^\alpha$ in dimension $d = 2$. The result extends to any dimension d, by [Mon15], thanks to a slicing method that relates the energy $\mathcal{E}_{\beta,\varepsilon}$ with the energy of the sliced measures $u = (w \cdot \nu)_+$ supported on the slices $V_a = \{x \in \mathbb{R}^d \ : \ x \cdot \nu = a\} \simeq \mathbb{R}^N$, for any given unit vector $\nu \in \mathbb{R}^d$, defined by

$$\bar{\mathcal{E}}_{\beta,\varepsilon}(u) = \int_{\mathbb{R}^N} \varepsilon^{\gamma_1} |\nabla u|^2 + \varepsilon^{-\gamma_2} |u|^\beta.$$

The functionals $\bar{\mathcal{E}}_{\beta,\varepsilon}$ Γ-converge as $\varepsilon \to 0^+$, in the weak-$\star$ topology of $\mathcal{C}_b'$, to $c\mathbf{M}^\alpha$ for some non-trivial c, as shown in Section 5.3, and one may recover every α-mass in this way for $\alpha \in \left(\frac{2d-4}{2d+1}, 1\right]$, and in particular every so-called super-critical exponents for Branched Transport in dimension d, that is $\alpha \in (1 - 1/d, 1]$.

The same slicing method would allow to extend our Γ-convergence result stated in Theorem 1.2 to functionals defined on vector measure

$$\mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} f_\varepsilon(x, \varepsilon^{d-1}|w|(x), \varepsilon^d |\nabla w|(x)) \varepsilon^{1-d} \, \mathrm{d}x & \text{if } w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.8}$$

for Lagrangians $f_\varepsilon \to f$ fitting the framework of Theorem 1.2. The expected Γ-limit, for the weak topology of measures and their divergence measure, would be the functional $\mathbf{M}_1^{H_f}$, with H_f defined in (1.2). Note that this approach would provide approximations of H-masses for more general continuous and concave cost functions $H : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $H(0) = 0$. By [Wir19], we would obtain all such H-masses when $N = 1$ (corresponding to $d = 2$).

5.5 A Cahn-Hilliard model for droplets

Following the works [BDS96] in the one-dimensional case and [Dub98] in higher dimension, we consider functionals on $\mathcal{M}_+(\mathbb{R}^N)$ of the form:

$$\mathcal{W}_\varepsilon(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \varepsilon^{-\rho}(W(u) + \varepsilon|\nabla u|^2) & \text{if } u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.9}$$

where $W : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel function satisfying $W(t) \sim_{u\to+\infty} u^s$ for some exponent $s \in (-\infty, 1)$. In [BDS96; Dub98], it is in particular proven, under some assumptions on the slope of W at 0 and its regularity, that the family $(\mathcal{W}_\varepsilon)_{\varepsilon>0}$ Γ-converges to a non-trivial multiple of the α-mass, $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$, when $s \in (-2, 1)$ and $\rho = \rho(s, N) := \frac{N(1-s)}{(N+2)+N(1-s)}$. In this section, we recover this Γ-convergence result using our general model.

Replacing ε with $\bar\varepsilon := \varepsilon^{(N+2)+N(1-s)}$ and noticing that $1 - \rho = \frac{N+2}{(N+2)+N(1-s)}$, one gets for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$:

$$\begin{aligned} \mathcal{W}_{\bar\varepsilon}(u) = \int_{\mathbb{R}^N} \varepsilon^{-N(1-s)}W(u) + \varepsilon^{N+2}|\nabla u|^2 &= \int_{\mathbb{R}^N} \left([\varepsilon^{Ns}W(\varepsilon^{-N}\varepsilon^N u)] + |\varepsilon^{N+1}\nabla u|^2\right)\varepsilon^{-N} \\ &= \int_{\mathbb{R}^N} f^W_\varepsilon(x, \varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N}, \end{aligned}$$

where f^W_ε is defined for every $x \in \mathbb{R}^N, u \in \mathbb{R}_+, \xi \in \mathbb{R}^N$ by

$$f^W_\varepsilon(x, u, \xi) := W_\varepsilon(u) + |\xi|^2 \quad \text{and} \quad W_\varepsilon(u) := \varepsilon^{Ns}W(\varepsilon^{-N}u).$$

Therefore if we take $f_\varepsilon = f^W_\varepsilon$ in our general model (1.3) we exactly get $\mathcal{W}_{\bar\varepsilon} = \mathcal{E}_\varepsilon$. The fact that $W(u) \sim u^s$ as $u \to +\infty$ implies that W_ε converges pointwise to the map $k_s : u \mapsto u^s$ if $u > 0$, $k_s(0) = 0$, hence f^W_ε converges to $f_s : (x, u, \xi) \mapsto k_s(u) + |\xi|^2$.

Theorem 5.1. *Assume that $W : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:*

(HW$_1$) *W is lower semicontinuous,*

(HW$_2$) $\{W = 0\} = \{0\}$,

(HW$_3$) $W(u) \sim_{u\to+\infty} u^s$ *for some* $s \in (-\infty, 1)$,

(HW$_4$) $\displaystyle\sup_{u>0} \frac{W(u)}{u^s} < +\infty$,

(HW$_5$) $\displaystyle 0 < \liminf_{u\to0^+} \frac{W(u)}{u}$.

Then $(\mathcal{W}_\varepsilon)_{\varepsilon>0}$ Γ-converges to $\mathbf{M}^{H_{f_s}}$, for both topologies $\mathcal{C}'_0$ and $\mathcal{C}'_b$, and if $s \in (-2, 1]$ then $\mathbf{M}^{H_{f_s}}$ is a nontrivial multiple of $\mathbf{M}^\alpha$ where $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$.

To prove this theorem, we start with a simple lemma.

Lemma 5.2. *Assume that W satisfies (HW$_1$)–(HW$_5$). Then for every $\delta \in (0, 1)$, there exists $c_\delta \in (0, +\infty)$ such that for every $\varepsilon > 0$ and every $u \in \mathbb{R}_+$,*

$$\delta(u^p \wedge c_\delta\varepsilon^{-N(1-s)}u) \le W_\varepsilon(u). \tag{5.10}$$

Proof. Fix $\delta \in (0,1)$. There exists $M > 0$ such that $\delta u^s \le W(u)$ for every $u \ge M$. Besides, the map $w : u \mapsto W(u)/u$ is lower semicontinuous and positive on $(0, M]$ by (HW$_1$) and (HW$_2$), and since $\liminf_{u\to 0} w(u) > 0$ by (HW$_5$), w is necessarily bounded from below on $(0, M]$ by some contant $c > 0$. As a consequence $W_\varepsilon(u) \ge \delta u^s$ if $u \ge \varepsilon^N M$ and $W_\varepsilon(u) \ge c\varepsilon^{N(s-1)}u$ if $u \le \varepsilon^N M$, hence:

$$\forall u \in \mathbb{R}, \quad W_\varepsilon(u) \ge \delta(u^s \wedge c\varepsilon^{-N(1-s)}u). \qquad \square$$

Proof of Theorem 5.1. By (HW$_4$), there exists a constant C such that $f_\varepsilon^W \le C f_s$ for every ε, and since f_ε^W does not depend on the x variable and converges pointwise to f_s, (U) is satisfied and our $\Gamma - \limsup$ result stated in Proposition 4.2 yields

$$\mathbf{M}^{H_{f_s}} \ge \Gamma(\mathcal{C}_b') - \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon.$$

Fix $\delta \in (0,1)$. By Lemma 5.2, there exists c_δ such that

$$\forall x, u, \xi, \quad f_\varepsilon^W(x,u,\xi) \ge \delta(|\xi|^2 + (u^s \wedge c_\delta \varepsilon^{-N(1-s)}u) =: f_\varepsilon^\delta(x,u,\xi).$$

It is easy to check that f_ε^δ satisfies (H$_1$), (H$_2$) and (H$_5$) for every $\varepsilon > 0$. Moreover $f_\varepsilon^\delta \uparrow \delta f_s$ and $(f_\varepsilon^\delta)'_-(\cdot, 0^+, 0) = \delta c_\delta \varepsilon^{-N(1-s)} \uparrow +\infty = (\delta f_s)'_-(\cdot, 0^+, 0)$ as $\varepsilon \to 0$, thus (H$_8$) holds for the family $(f_\varepsilon^\delta)_{\varepsilon>0}$, and by applying our $\Gamma - \liminf$ result stated in Proposition 4.1 to the energies $\mathcal{E}_\varepsilon^\delta$ induced by f_ε^δ we get:

$$\Gamma(\mathcal{C}_0') - \liminf \mathcal{E}_\varepsilon \ge \Gamma(\mathcal{C}_0') - \liminf \mathcal{E}_\varepsilon^\delta \ge \mathbf{M}^{H^-_{\delta f_s}}.$$

We get the result by taking the limit $\delta \to 1$, noticing that $(f_s)'_-(\cdot, 0^+, 0) = +\infty$, so that $H^-_{\delta f_s} = H_{\delta f_s} = \delta H_{f_s}$ and $\mathbf{M}^{H^-_{\delta f_s}} = \mathbf{M}^{\delta H_{f_s}} = \delta \mathbf{M}^{H_{f_s}}$. $\square$

Remark 5.3. We recover the Γ-convergence results of [BDS96] and [Dub98] when $s \in (-2, 1)$ under slightly more general assumptions: besides (HW$_2$) and (HW$_3$), the authors impose the existence of a nontrivial slope $\lim_{u\to 0} \frac{W(u)}{u} \in (0, +\infty)$ and a regularity condition (either W is of class $\mathcal{C}^1$ or continuous and nondecreasing close to 0), which are stronger than (HW$_1$), (HW$_4$) and (HW$_5$). Let us stress however that these works also tackle the cases $s < -2$ in any dimension, where the exponent ρ has to be fixed to $\rho(-2, N)$, and the case $s = -2$ in dimension one, where a logarithmic factor must be introduced, replacing $\varepsilon^{-\rho}$ with $\varepsilon^{-\rho(-2,1)}|\log\varepsilon|^{-1} = \varepsilon^{-1/2}|\log\varepsilon|^{-1}$. This implies that in our model we get a trivial Γ-limit when $s \le -2$, namely $H_{f_s} \equiv +\infty$ on $(0, +\infty)$.

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