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A STABILIZER INTERPRETATION OF DOUBLE SHUFFLE LIE ALGEBRAS

BENJAMIN ENRIQUEZ AND HIDEKAZU FURUSHO

ABSTRACT. According to Racinet’s work, the scheme of double shuffle and regularization relations between cyclotomic analogues of multiple zeta values has the structure of a torsor over a pro-unipotent \mathbb{Q} -algebraic group DMR_0 , which is an algebraic subgroup of a pro-unipotent \mathbb{Q} -algebraic group of outer automorphisms of a free Lie algebra. We show that the harmonic (stuffle) coproduct of double shuffle theory may be viewed as an element of a module over the above group, and that DMR_0 identifies with the stabilizer of this element. We identify the tangent space at origin of DMR_0 with the stabilizer Lie algebra of the harmonic coproduct, thereby obtaining an alternative proof of Racinet’s result stating that this space is a Lie algebra (the double shuffle Lie algebra).

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1. INTRODUCTION

Multiple L -values (MLV in short) $L(k_1, \dots, k_m; \zeta_1, \dots, \zeta_m)$ are the complex numbers defined by the following series

$$L(k_1, \dots, k_m; \zeta_1, \dots, \zeta_m) := \sum_{n_1 > \dots > n_m > 0} \frac{\zeta_1^{n_1} \dots \zeta_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}}$$

for $m, k_1, \dots, k_m \in \mathbb{Z}_{>0}$ and ζ_1, \dots, ζ_m in the group μ_N of N -th roots of unity in \mathbb{C} , where N is an integer ≥ 1 . They converge if and only if $(k_1, \zeta_1) \neq (1, 1)$. Multiple zeta values are regarded as a special case for $N = 1$. These values have recently garnered much interest due to their appearance in various fields of physics and mathematics ([BK, Dr, LM]). In connection with motive theory ([De, DeG]), linear and algebraic relations among MLV's are particularly important. The extended (regularized) double shuffle relations ([IKZ, R]) might be one of the most fascinating ones.

Racinet's formalism [R] is quite useful to state these relations. The basic ingredients of this formalism are: a noncommutative formal power series algebra $\mathbb{C}\langle\langle X \rangle\rangle$ over variables x_σ ($\sigma \in \mu_N$), another noncommutative formal power series algebra $\mathbb{C}\langle\langle Y \rangle\rangle$ over another set of variables, a map $\mathbb{C}\langle\langle X \rangle\rangle^\times \rightarrow \mathbb{C}\langle\langle Y \rangle\rangle^\times$ denoted $G \mapsto G_\star$ (see Def. 5.3), a coproduct map $\Delta : \mathbb{C}\langle\langle X \rangle\rangle \rightarrow \mathbb{C}\langle\langle X \rangle\rangle^{\hat{\otimes} 2}$ for which the elements of X are primitive and another coproduct map $\Delta_\star : \mathbb{C}\langle\langle Y \rangle\rangle \rightarrow \mathbb{C}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2}$ (see §2.2). The central object of Racinet's approach is a certain invertible non-commutative formal power series \mathcal{S} in $\mathbb{C}\langle\langle X \rangle\rangle^\times$, constructed through iterated integrals and whose coefficients are expressed in terms of MLV's; in particular, the coefficient of $x_0^{k_1-1} x_{\zeta_1} x_0^{k_2-1} x_{\zeta_1 \zeta_2} \dots x_0^{k_m-1} x_{\zeta_1 \dots \zeta_m}$ is

$$L(k_1, \dots, k_m; \zeta_1, \dots, \zeta_m).$$

The regularized double shuffle relations for MLV's can then be formulated as

$$(1.1) \quad \Delta(\mathcal{S}) = \mathcal{S} \otimes \mathcal{S} \text{ and } \Delta_\star(\mathcal{S}_\star) = \mathcal{S}_\star \otimes \mathcal{S}_\star$$

(equalities in $\mathbb{C}\langle\langle X \rangle\rangle^{\hat{\otimes} 2}$, resp., $\mathbb{C}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2}$). The series \mathcal{S} has constant term equal to 1, coefficients of x_0 and x_1 equal to 0, and coefficient of $x_0 x_1$ equal to $-(2\pi\sqrt{-1})^2/24$. For \mathbb{k} a commutative \mathbb{Q} -algebra and $\lambda \in \mathbb{k}$, the set of series in $\mathbb{k}\langle\langle X \rangle\rangle$ satisfying (1.1) and the above coefficient conditions, with $2\pi\sqrt{-1}$ replaced by λ , is denoted by $\text{DMR}_\lambda(\mathbb{k})$. So \mathcal{S} belongs to $\text{DMR}_{2\pi\sqrt{-1}}(\mathbb{C})$. The structure of the regularized double shuffle relations is clarified by the following result:

Theorem 1.1. ([R] *Th. I, §3.2.3*) *The scheme $\mathbb{k} \mapsto \text{DMR}_0(\mathbb{k})$ forms a pro-unipotent subgroup scheme of MT. For \mathbb{k} a \mathbb{Q} -commutative algebra and $\lambda \in \mathbb{k}$, the set $\text{DMR}_\lambda(\mathbb{k})$ is a torsor (principal homogeneous space) over the group $\text{DMR}_0(\mathbb{k})$.*

Here MT is¹ the pro-unipotent \mathbb{Q} -algebraic group scheme

$$\mathbb{k} \mapsto \text{MT}(\mathbb{k}) := (\{\text{series in } \mathbb{k}\langle\langle X \rangle\rangle \text{ with constant term equal to } 1\}, \otimes),$$

¹DMR and MT stand for the French “double mélange et régularisation” and “groupe de Magnus tordu”.

where the group structure \otimes is such that there is a group morphism from $\text{MT}(\mathbb{k})$ to the group of automorphisms of the topological \mathbb{k} -algebra $\mathbb{k}\langle\langle X \rangle\rangle$ taking x_0 to itself and each x_σ , $\sigma \in \mu_N$, to a conjugate of itself, and which are invariant with respect to the natural action of μ_N (see §4.1).

Thm. 1.1 shows that the scheme $\text{DMR}_{2\pi\sqrt{-1}}(\mathbb{C})$, which corresponds to the double shuffle relations of MLVs, is isomorphic to the group $\text{DMR}_0(\mathbb{C})$. This group is pronipotent, and therefore isomorphic to its Lie algebra, which is the degree completion of a positively graded Lie algebra $\mathfrak{d}\mathfrak{mr}_0$. It is therefore an important question, which is still open today, to determine the size of this Lie algebra (more precisely, its Poincaré polynomial, as $\mathfrak{d}\mathfrak{mr}_0$ is graded).

In order to better understand $\mathfrak{d}\mathfrak{mr}_0$, it seems useful to improve our understanding of its Lie algebra structure. Here it seems helpful to recall the situation with the Grothendieck-Teichmüller Lie algebra \mathfrak{grt}_1 . This Lie algebra was introduced by Drinfeld ([Dr]) and later shown to be contained, up to the automorphism $(x_0, x_1) \mapsto (x_0, -x_1)$, in $\mathfrak{d}\mathfrak{mr}_0$ when $N = 1$ ([F]). A transparent interpretation of its Lie algebra structure was given by Ihara ([I]): \mathfrak{grt}_1 is isomorphic to any of the Lie algebras of symmetric special outer derivations of \mathfrak{p}_n for $n \geq 5$, where \mathfrak{p}_n is the graded Lie algebra associated to the pronipotent completion of the pure sphere braid group with n strands.

It is also instructive to recall the situation with the Kashiwara-Vergne Lie algebra \mathfrak{krv}_2 arising in the work of Alekseev and Torossian ([AT]). The inclusion $\mathfrak{grt}_1 \subset \mathfrak{krv}_2$ was proved in [AT], while the inclusion $\mathfrak{d}\mathfrak{mr}_0 \subset \mathfrak{krv}_2$ was proved in [Sch]. In contrast to the previous situation, the Lie algebra status of \mathfrak{krv}_2 is rather transparent from its definition, as it is related to a Lie algebra cocycle.

Let us review the known proofs of the Lie algebra nature of $\mathfrak{d}\mathfrak{mr}_0$. The space $\mathfrak{d}\mathfrak{mr}_0$ is introduced in [R] as the tangent space of DMR_0 , viewed as a subspace of MT , at the unit element. The proof of [R] that $\mathfrak{d}\mathfrak{mr}_0$ is a Lie algebra appeals to coderivations of the coalgebra $(\mathbb{C}\langle\langle Y \rangle\rangle, \Delta_*)$; in the case when $N = 1$, this proof was streamlined in the appendix of [F]. The statement that $\mathfrak{d}\mathfrak{mr}_0$ is a Lie algebra was also announced without proof in [E]; in the case when $N = 1$, it was later proved using the techniques of Ecalle's theory of moulds in [SaS].

In this paper, we introduce a graded Lie subalgebra $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ of a graded Lie algebra \mathfrak{mt} , whose completion is the Lie algebra of MT , and show that $\mathfrak{d}\mathfrak{mr}_0$ *coincides, up to addition of an abelian Lie algebra, with the stabilizer of an element in a $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ -module (see Thm. 3.10)*; this gives an alternative proof of the Lie algebra nature of $\mathfrak{d}\mathfrak{mr}_0$. We then prove a group version of this result: for any commutative \mathbb{Q} -algebra \mathbb{k} , we construct a subgroup $(\exp(\hat{\mathfrak{L}\mathfrak{ib}}_{\mathbb{k}}(X)), \otimes)$ of $\text{MT}(\mathbb{k})$ and prove that $\text{DMR}_0(\mathbb{k})$ *coincides (up to a central additive group) with the stabilizer of an element of an $(\exp(\hat{\mathfrak{L}\mathfrak{ib}}_{\mathbb{k}}(X)), \otimes)$ -module*. More precisely:

1) there is a group representation of $\text{MT}(\mathbb{k})$ on the topological \mathbb{k} -module $\mathbb{k}\langle\langle Y \rangle\rangle$; we denote by $g \mapsto S_g^Y$ the corresponding group morphism $\text{MT}(\mathbb{k}) \rightarrow \text{Aut}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle)$ (see (5.16));

2) there is a group morphism $\Theta_{\mathbb{k}} : (\exp(\hat{\mathfrak{L}\mathfrak{ib}}_{\mathbb{k}}(X)), \otimes) \rightarrow \text{MT}(\mathbb{k})$ (see §4.3.2); this morphism should be viewed as a correction of the canonical inclusion related with regularization;

3) the set $\widetilde{\text{DMR}}_0(\mathbb{k}) := \{e^{\beta x_1} \cdot g \cdot e^{\alpha x_0} \mid \alpha, \beta \in \mathbb{k}, g \in \text{DMR}_0(\mathbb{k})\}$ (where \cdot is the usual product in $\mathbb{k}\langle\langle X \rangle\rangle^\times$) is a subgroup of $(\exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X)), \otimes)$, isomorphic to $\text{DMR}_0(\mathbb{k}) \times \mathbb{k}^2$ (see §5.2);

we then prove (see §5.4):

Theorem 1.2. *The subgroup $\widetilde{\text{DMR}}_0(\mathbb{k})$ coincides with the stabilizer of the element Δ_* of the space $\text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2})$ of all continuous \mathbb{k} -linear maps $\mathbb{k}\langle\langle Y \rangle\rangle \rightarrow \mathbb{k}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2}$, equipped with the action of $(\exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}(X)), \otimes)$, pulled back by $\Theta_{\mathbb{k}}$ of the natural action of $\text{MT}(\mathbb{k})$, namely*

$$\widetilde{\text{DMR}}_0(\mathbb{k}) = \{g \in \exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X)) \mid (S_{\Theta_{\mathbb{k}}(g)}^Y)^{\otimes 2} \circ \Delta_* = \Delta_* \circ S_{\Theta_{\mathbb{k}}(g)}^Y \text{ (in } \text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2})\text{)}\}.$$

Therefore $\widetilde{\text{DMR}}_0(\mathbb{k})$ may be interpreted as the set of $g \in \exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X))$ such that the diagram

$$\begin{array}{ccc} \mathbb{k}\langle\langle Y \rangle\rangle & \xrightarrow{\Delta_*} & \mathbb{k}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2} \\ \downarrow S_{\Theta_{\mathbb{k}}(g)}^Y & & \downarrow (S_{\Theta_{\mathbb{k}}(g)}^Y)^{\otimes 2} \\ \mathbb{k}\langle\langle Y \rangle\rangle & \xrightarrow{\Delta_*} & \mathbb{k}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2} \end{array}$$

is commutative, i.e. such that $S_{\Theta_{\mathbb{k}}(g)}^Y$ is an automorphism of the topological coalgebra $(\mathbb{k}\langle\langle Y \rangle\rangle, \Delta_*)$.

The realization of this program is done in the following steps. §2 presents background material necessary for the proof of the Lie algebra version of Thm. 1.2, which is obtained in §3 (Thm. 3.10). In §4, we construct the group homomorphism $\Theta_{\mathbb{k}}$, a necessary step for the proof of Thm. 1.2 in §5.

2. LIE ALGEBRAS OF DERIVATIONS OF FREE LIE ALGEBRAS

In this section, we present the material needed for the comparison of dmt_0 and $\text{stab}(\Delta_*)$. In §2.1 and §2.2, we recall the formalism of double shuffle theory: the Ihara bracket on the augmentation ideal $\mathbb{Q}\langle X \rangle_0$ of the associative algebra $\mathbb{Q}\langle X \rangle$ and on the free Lie algebra $\mathfrak{L}\mathfrak{i}\mathfrak{b}(X)$, the maps $\pi_Y, \text{corr} : \mathbb{Q}\langle X \rangle \rightarrow \mathbb{Q}\langle Y \rangle$, the endomorphism \mathbf{q} of $\mathbb{Q}\langle Y \rangle$, and the harmonic coproduct Δ_* of $\mathbb{Q}\langle Y \rangle$. In §2.3, we construct modules $\mathbb{Q}\langle X \rangle, \mathbb{Q}\langle Y \rangle$ over the space $\mathbb{Q}\langle X \rangle_0$, viewed as a Lie algebra via its Ihara bracket, and a morphism $\mathbb{Q}\langle X \rangle \xrightarrow{\mathbf{q} \circ \pi_Y} \mathbb{Q}\langle Y \rangle$ between these modules. In §2.4 we construct a Lie algebra morphism $\theta : \mathfrak{L}\mathfrak{i}\mathfrak{b}(X) \rightarrow \mathbb{Q}\langle X \rangle_0$, where both sides are equipped with the Lie algebra structures arising from the Ihara bracket (Prop. 2.5). One then pulls back the module $\mathbb{Q}\langle Y \rangle$ over $\mathbb{Q}\langle X \rangle_0$ to get a module over $\mathfrak{L}\mathfrak{i}\mathfrak{b}(X)$, and constructs from there a new module $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$ over this Lie algebra containing a particular element Δ_* . In §2.5, we construct the stabilizer subalgebra $\text{stab}(\Delta_*) \subset \mathfrak{L}\mathfrak{i}\mathfrak{b}(X)$ of this element. Finally, in §2.6, we show a relation between θ , a morphism $\text{sec} : \mathbb{Q}\langle Y \rangle \rightarrow \mathbb{Q}\langle X \rangle$ and an endomorphism $\tilde{\mathbf{p}}$ of $\mathbb{Q}\langle X \rangle$.

2.1. Ihara brackets and outer derivation Lie algebras.

2.1.1. Ihara brackets. Let Γ be a finite commutative group, whose product will be denoted multiplicatively. Let X be the alphabet $\{x_0\} \sqcup \{x_\gamma \mid \gamma \in \Gamma\}$, indexed by $\{0\} \sqcup \Gamma$. Let $\mathbb{Q}\langle X \rangle$ be the free noncommutative associative algebra with unit over the alphabet X and $\mathfrak{L}\mathfrak{i}\mathfrak{b}(X)$ be the Lie subalgebra of $\mathbb{Q}\langle X \rangle$ generated by X (see [R], §2.2.3).

The group Γ acts on X by $\sigma \cdot x_0 = x_0$, $\sigma \cdot x_\gamma = x_{\sigma\gamma}$ for $\gamma \in \Gamma$. This action extends to an action on $\mathbb{Q}\langle X \rangle$ and $\mathfrak{L}\mathfrak{ib}(X)$, which will be denoted $\sigma \mapsto t_\sigma$ (see [R], §3.1.1).

For $\psi \in \mathbb{Q}\langle X \rangle$, let d_ψ be the derivation of $\mathbb{Q}\langle X \rangle$ given by

$$\forall \sigma \in \Gamma, \quad d_\psi(x_0) = 0, \quad d_\psi(x_\sigma) = [x_\sigma, t_\sigma(\psi)]$$

(see [R], §3.1.12.2) and let s_ψ be the linear endomorphism of $\mathbb{Q}\langle X \rangle$ defined by

$$s_\psi(v) := \psi v + d_\psi(v)$$

for $v \in \mathbb{Q}\langle X \rangle$ (see [R], §3.1.12.1).

Let $\mathbb{Q}\langle X \rangle_0$ be the subspace of $\mathbb{Q}\langle X \rangle$ of all series with zero constant term (this is denoted $\mathfrak{mt}(k)$ in [R], §3.1.8). On $\mathbb{Q}\langle X \rangle_0$, one defines the (Ihara) bracket

$$(2.1) \quad \langle \psi_1, \psi_2 \rangle := s_{\psi_1}(\psi_2) - s_{\psi_2}(\psi_1)$$

(see [R], (3.1.10.2)). This restricts to a bracket on the subspace $\mathfrak{L}\mathfrak{ib}(X)$ of $\mathbb{Q}\langle X \rangle_0$. We have therefore a Lie algebra inclusion

$$(2.2) \quad (\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle) \subset (\mathbb{Q}\langle X \rangle_0, \langle, \rangle) =: \mathfrak{mt}.$$

2.1.2. Outer derivation Lie algebras. Call an associative (resp., Lie) algebra derivation of $\mathbb{Q}\langle X \rangle$ (resp., $\mathfrak{L}\mathfrak{ib}(X)$) *tangential* if it takes any of the generators x_γ , $\gamma \in \Gamma \cup \{0\}$ to an element of $[x_\gamma, \mathbb{Q}\langle X \rangle]$ (resp., $[x_\gamma, \mathfrak{L}\mathfrak{ib}(X)]$). The tangential derivations of either kind form a Lie algebra, of which the subspace of inner derivations is an ideal. In each case, the quotient algebra has a natural action of Γ . The subalgebra of invariants of this action is denoted $\mathfrak{outder}^*(\mathbb{Q}\langle X \rangle)^\Gamma$ (resp., $\mathfrak{outder}^*(\mathfrak{L}\mathfrak{ib}(X))^\Gamma$). As $\mathbb{Q}\langle X \rangle$ is the universal enveloping algebra of $\mathfrak{L}\mathfrak{ib}(X)$, any derivation of the latter algebra extends uniquely to a derivation of the former algebra, which induces an injective Lie algebra morphism $\mathfrak{outder}^*(\mathfrak{L}\mathfrak{ib}(X))^\Gamma \hookrightarrow \mathfrak{outder}^*(\mathbb{Q}\langle X \rangle)^\Gamma$.

Introduce a \mathbb{Z}^2 -grading in the algebras $\mathfrak{L}\mathfrak{ib}(X)$ and $\mathbb{Q}\langle X \rangle$ by $\deg(x_0) := (1, 0)$, $\deg(x_\gamma) := (0, 1)$ for $\gamma \in \Gamma$. This grading is compatible with the Lie bracket \langle, \rangle in both cases. Moreover, it induces a \mathbb{Z}^2 -grading on $\mathfrak{outder}^*(\mathfrak{L}\mathfrak{ib}(X))^\Gamma$ and $\mathfrak{outder}^*(\mathbb{Q}\langle X \rangle)^\Gamma$.

Lemma 2.1. *There are isomorphisms of \mathbb{Z}^2 -graded Lie algebras*

$$(2.3) \quad (\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle) \simeq \mathfrak{outder}^*(\mathfrak{L}\mathfrak{ib}(X))^\Gamma \oplus (\mathbb{Q}x_0 \oplus \mathbb{Q}x_1),$$

and

$$(2.4) \quad \mathfrak{mt} \simeq \mathfrak{outder}^*(\mathbb{Q}\langle X \rangle)^\Gamma \oplus (\mathbb{Q}[x_0]_0 \oplus \mathbb{Q}[x_1]_0),$$

in which $\mathbb{Q}[x_0]_0, \mathbb{Q}[x_1]_0$ are the spaces of polynomials in x_0, x_1 with no constant terms, and where the spaces $\mathbb{Q}x_0 \oplus \mathbb{Q}x_1$ and $\mathbb{Q}[x_0]_0 \oplus \mathbb{Q}[x_1]_0$ are equipped with abelian Lie algebra structures. The Lie algebra inclusion (2.2) is compatible with the isomorphisms (2.3) and (2.4) and with the natural inclusions of the respective summands of the right-hand side of (2.3) in the summands of the right-hand side of (2.4).

Proof. The map $\psi \mapsto$ (class of d_ψ) induces a surjective Lie algebra morphism $(\mathbb{Q}\langle X \rangle_0, \langle, \rangle) \rightarrow \mathfrak{outder}^*(\mathbb{Q}\langle X \rangle)^\Gamma$. Its kernel is $\mathbb{Q}[x_0]_0 \oplus \mathbb{Q}[x_1]_0$, which is contained in the center of $(\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$.

For $n \geq 0$, the components of $\mathbb{Q}[x_0]_0$ and of $\mathbb{Q}\langle X \rangle_0$ of bidegree $(n, 0)$ are the same, so $\mathfrak{outder}^*(\mathbb{Q}\langle X \rangle)^\Gamma[n, 0] = 0$ for $n \geq 0$ (where $X[n, m]$ is the subspace of a bigraded space X with bidegree (n, m)).

The direct sum of homogenous components of $\mathbb{Q}\langle X \rangle$ of bidegrees $(0, n)$, $n \geq 0$ identifies with the free associative algebra $\mathbb{Q}\langle x_\gamma, \gamma \in \Gamma \rangle$. Let I be the two-sided ideal of this algebra generated by the family $\{x_\gamma \mid \gamma \in \Gamma \setminus \{1\}\}$. This is a \mathbb{Q} -linear subspace of $\mathbb{Q}\langle X \rangle$, complementary to $\mathbb{Q}[x_1]$, so $I \oplus \mathbb{Q}[x_1]_0 = \bigoplus_{n \geq 0} \mathbb{Q}\langle X \rangle_0[0, n]$. It follows that $\mathfrak{g} := I \oplus (\bigoplus_{n, m > 0} \mathbb{Q}\langle X \rangle_0[n, m])$ is such that $\mathfrak{g} \oplus (\mathbb{Q}[x_0]_0 \oplus \mathbb{Q}[x_1]_0) = \mathbb{Q}\langle X \rangle_0$. One also has $\langle I, I \rangle \subset I$, so that \mathfrak{g} is a Lie subalgebra of $\mathbb{Q}\langle X \rangle_0$. This proves both that the latter equality is a direct sum decomposition of the Lie algebra $(\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$, and that the natural projection $\mathfrak{g} \rightarrow \text{outder}^*(\mathbb{Q}\langle X \rangle)^\Gamma$ is an isomorphism, which proves (2.4).

The proof of (2.3) is similar, the inclusion $I \subset \mathbb{Q}\langle X \rangle_0$ being replaced by the inclusion in the Lie algebra $\mathfrak{Lie}(x_\gamma, \gamma \in \Gamma)$ of its ideal generated by $\{x_\gamma \mid \gamma \in \Gamma \setminus \{1\}\}$. \square

2.2. Harmonic coproduct, correction and projection operators. Let Y be the alphabet $\{y_{n, \sigma} \mid (n, \sigma) \in \mathbb{N}_{>0} \times \Gamma\}$, indexed by $\mathbb{N}_{>0} \times \Gamma$. We define $\mathbb{Q}\langle Y \rangle$ to be the free algebra over this alphabet. We now construct linear maps fitting in the following diagram:

$$\begin{array}{ccc} & \mathfrak{q} & \\ & \curvearrowright & \\ \mathbb{Q}\langle X \rangle & \xrightarrow[\text{corr}]{\pi_Y} & \mathbb{Q}\langle Y \rangle \xrightarrow{\Delta_*} \mathbb{Q}\langle Y \rangle^{\otimes 2} \\ & \curvearrowleft & \\ & (-)_* & \end{array}$$

- the algebra $\mathbb{Q}\langle Y \rangle$ is viewed as a subalgebra of $\mathbb{Q}\langle X \rangle$ by the assignment $y_{n, \sigma} \mapsto x_0^{n-1} x_\sigma$ for $n \geq 0, \sigma \in \Gamma$. The map π_Y is then determined by the conditions $\pi_Y(\mathbb{Q}\langle X \rangle x_0) = 0$ and that $\mathbb{Q}\langle Y \rangle \hookrightarrow \mathbb{Q}\langle X \rangle \xrightarrow{\pi_X} \mathbb{Q}\langle Y \rangle$ is the identity (see [R], §§2.2.5 and 4.2.1);
- the map $\text{corr} : \mathbb{Q}\langle X \rangle \rightarrow \mathbb{Q}\langle Y \rangle$ is defined to be the linear map taking the word $x_0^{n-1} x_1$ to $\frac{(-1)^{n-1}}{n} (y_{1,1})^n$ if $n \geq 1$, and any other word to 0; in particular, it takes 1 to 0 (see [R], §3.1.1);
- the map $\mathfrak{q} : \mathbb{Q}\langle Y \rangle \rightarrow \mathbb{Q}\langle Y \rangle$ is defined to be the linear map taking 1 to 1 and by

$$\mathfrak{q}(y_{s_1, \sigma_1} \cdots y_{s_r, \sigma_r}) = y_{s_1, \sigma_1} y_{s_2, \sigma_2} \sigma_1^{-1} \cdots y_{s_r, \sigma_r} \sigma_{r-1}^{-1}$$

for any $r \geq 1$, s_1, \dots, s_r in $\mathbb{N}_{>0}$, and $\sigma_1, \dots, \sigma_r$ in Γ (see [R], §2.2.7).

- the map $(-)_* : \mathbb{Q}\langle X \rangle \rightarrow \mathbb{Q}\langle Y \rangle$ is given by $\psi \mapsto \psi_*$, where

$$(2.5) \quad \psi_* := \mathfrak{q}(\pi_Y(\psi)) + \text{corr}(\psi);$$

one has therefore $(-)_* = \mathfrak{q} \circ \pi_Y + \text{corr}$ (see [R], (3.3.1.2));

- the map $\Delta_* : \mathbb{Q}\langle Y \rangle \rightarrow \mathbb{Q}\langle Y \rangle^{\otimes 2}$ is the unique algebra morphism such that for any $n \geq 1$ and $\sigma \in \Gamma$

$$\Delta_*(y_{n, \sigma}) = y_{n, \sigma} \otimes 1 + 1 \otimes y_{n, \sigma} + \sum_{\substack{n' + n'' = n \mid n', n'' > 0, \\ \sigma', \sigma'' \in \Gamma \mid \sigma' \sigma'' = \sigma}} y_{n', \sigma'} \otimes y_{n'', \sigma''}.$$

The map Δ_* is called the *harmonic coproduct* (see [R], §2.3.1).

2.3. Modules over the Lie algebra $(\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$. The space $\mathbb{Q}\langle X \rangle$ is a left module over the Lie algebra $(\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$ for the action

$$\mathbb{Q}\langle X \rangle_0 \rightarrow \text{End}(\mathbb{Q}\langle X \rangle), \quad \psi \mapsto s_\psi$$

(see [R], (3.1.9.2)).

For $\varphi \in \mathbb{Q}\langle X \rangle_0$, there exists a unique linear endomorphism s_φ^Y of $\mathbb{Q}\langle Y \rangle$ such that the following diagram commutes

$$(2.6) \quad \begin{array}{ccc} \mathbb{Q}\langle X \rangle & \xrightarrow{s_\varphi} & \mathbb{Q}\langle X \rangle \\ \mathfrak{q} \circ \pi_Y \downarrow & & \downarrow \mathfrak{q} \circ \pi_Y \\ \mathbb{Q}\langle Y \rangle & \xrightarrow{s_\varphi^Y} & \mathbb{Q}\langle Y \rangle \end{array}$$

(see [R], §4.1.1). One shows:

Lemma 2.2. *The map $\mathbb{Q}\langle X \rangle_0 \rightarrow \text{End}(\mathbb{Q}\langle Y \rangle)$, $\psi \mapsto s_\psi^Y$ defines a module structure on $\mathbb{Q}\langle Y \rangle$ over the Lie algebra $(\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$. The map $\mathbb{Q}\langle X \rangle \xrightarrow{\mathfrak{q} \circ \pi_Y} \mathbb{Q}\langle Y \rangle$ is then a module morphism over this Lie algebra.*

2.4. A Lie algebra homomorphism $\theta : (\mathfrak{L}\mathfrak{i}\mathfrak{b}(X), \langle, \rangle) \rightarrow (\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$. The following statement is a consequence of the fact that the elements x_0^n, x_1^n are central in $(\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$:

Lemma 2.3. *Let $(f_n)_{n>0}$ and $(g_n)_{n>0}$ be two collections of characters $f_n, g_n : (\mathfrak{L}\mathfrak{i}\mathfrak{b}(X), \langle, \rangle) \rightarrow \mathbb{Q}$. The map*

$$\theta_{(f_n)_n, (g_n)_n} : (\mathfrak{L}\mathfrak{i}\mathfrak{b}(X), \langle, \rangle) \rightarrow (\mathbb{Q}\langle X \rangle_0, \langle, \rangle), \quad \psi \mapsto \psi + \sum_{n>0} f_n(\psi)x_1^n + \sum_{n>0} g_n(\psi)x_0^n$$

is a Lie algebra homomorphism.

Lemma 2.4. *Let $n \geq 1$. Define² maps $f_n^0, g_n^0 : \mathfrak{L}\mathfrak{i}\mathfrak{b}(X) \rightarrow \mathbb{Q}$ by $f_n^0 : \psi \mapsto (\psi|x_0^{n-1}x_1)$ for $n > 0$, $g_1^0 : \psi \mapsto (\psi|x_0)$, and $g_n^0 : \psi \mapsto 0$ if $n > 1$. These maps are Lie algebra characters $f_n^0, g_n^0 : (\mathfrak{L}\mathfrak{i}\mathfrak{b}(X), \langle, \rangle) \rightarrow \mathbb{Q}$.*

Proof. According to Lemma 2.1, there is a decomposition

$$(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X), \langle, \rangle) \simeq \mathbf{outder}^*(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X))^\Gamma \oplus (\mathbb{Q}x_0 \oplus \mathbb{Q}x_1),$$

where the first summand is supported in bidegrees (n, m) with $n, m > 0$, and the second summand is supported in bidegrees $(1, 0)$ and $(0, 1)$. We will denote by $\text{Supp}(X)$ the bidegree support of a bigraded vector space X .

Let $n > 1$. As f_n^0 is supported in bidegrees $(n-1, 1)$ with $n-1 > 0$, this linear form factors through the summand $\mathbf{outder}^*(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X))^\Gamma$, and since $\text{Supp}(\mathbf{outder}^*(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X))^\Gamma) \subset \mathbb{Z}_{>0}^2$, one has $(n-1, 1) \notin \text{Supp}(\mathbf{outder}^*(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X))^\Gamma) + \text{Supp}(\mathbf{outder}^*(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X))^\Gamma)$, therefore f_n^0 is a character of the Lie algebra $\mathbf{outder}^*(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X))^\Gamma$, and therefore is a character of $(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X), \langle, \rangle)$. By bidegree reasons, the linear forms f_1^0 and g_1^0 factor through the abelian Lie algebra $\mathbb{Q}x_0 \oplus \mathbb{Q}x_1$, on which they are characters. These linear forms are therefore also characters of $(\mathfrak{L}\mathfrak{i}\mathfrak{b}(X), \langle, \rangle)$. \square

²For f an element of $\mathbb{Q}\langle X \rangle$ and w a word in X , we denote by $(f|w)$ the coefficient of w in f , so $f = \sum_w (f|w)w$. Viewing $\mathfrak{L}\mathfrak{i}\mathfrak{b}(X)$ as a subspace of $\mathbb{Q}\langle X \rangle$, we use the same notation for $f \in \mathfrak{L}\mathfrak{i}\mathfrak{b}(X)$.

Combining Lemma 2.3 and Lemma 2.4, and putting $\theta := \theta_{(f_n^0)_n, (g_n^0)_n}$, we get:

Proposition 2.5. *The map $\theta : (\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle) \rightarrow (\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$ given by*

$$\theta(\psi) := \psi + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (\psi | x_0^{n-1} x_1) x_1^n - (\psi | x_0) x_0$$

is a Lie algebra homomorphism.

2.5. The stabilizer Lie algebra of Δ_* . Via pull-back through the morphism θ , the morphism

$$\mathbf{q} \circ \pi_Y : \mathbb{Q}\langle X \rangle \rightarrow \mathbb{Q}\langle Y \rangle$$

of modules over $(\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$ may be viewed as a morphism of modules over $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ (see Lemma 2.2). The action of $\mathfrak{L}\mathfrak{ib}(X)$ on $\mathbb{Q}\langle X \rangle$ is given by $\psi \mapsto s_{\theta(\psi)}$, while its action on $\mathbb{Q}\langle Y \rangle$ is given by $\psi \mapsto s_{\theta(\psi)}^Y$.

The space $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$ of linear maps $\mathbb{Q}\langle Y \rangle \rightarrow \mathbb{Q}\langle Y \rangle^{\otimes 2}$ is then equipped with a module structure over $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$. Namely, the action of $\psi \in \mathfrak{L}\mathfrak{ib}(X)$ on $f \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$ is given by

$$\psi \cdot f := (s_{\theta(\psi)}^Y \otimes \text{id} + \text{id} \otimes s_{\theta(\psi)}^Y) \circ f - f \circ s_{\theta(\psi)}^Y.$$

The stabilizer Lie algebra $\mathfrak{stab}(\Delta_*)$ of $\Delta_* \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$ is then a Lie subalgebra of $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$, given by

$$\mathfrak{stab}(\Delta_*) = \{\psi \in \mathfrak{L}\mathfrak{ib}(X) \mid (s_{\theta(\psi)}^Y \otimes \text{id} + \text{id} \otimes s_{\theta(\psi)}^Y) \circ \Delta_* = \Delta_* \circ s_{\theta(\psi)}^Y\}.$$

2.6. Relation between $\tilde{\mathbf{p}}$, sec and θ . Let ∂_0 be the derivation of $\mathbb{Q}\langle X \rangle$ defined by $\partial_0(x_0) = 1$, $\partial_0(x_\sigma) = 0$ for $\sigma \in \Gamma$ (see [R], §4.2.1). Let $\tilde{\text{sec}}$ be the linear endomorphism of $\mathbb{Q}\langle X \rangle$ given by

$$\tilde{\text{sec}}(f) := \sum_{i \geq 0} \frac{(-1)^i}{i!} \partial_0^i(f) x_0^i$$

for $f \in \mathbb{Q}\langle X \rangle$. Recall that $\mathbb{Q}\langle X \rangle$ contains subspaces $\mathbb{Q}\langle Y \rangle$ and $\text{Ker}(\partial_0)$. It is proved in [R], §4.2.1, that $\tilde{\text{sec}}(\mathbb{Q}\langle Y \rangle) \subset \text{Ker}(\partial_0)$. We denote by

$$\text{sec} : \mathbb{Q}\langle Y \rangle \rightarrow \text{Ker}(\partial_0)$$

the resulting linear map. The following statement was proved in [R], Prop. 4.2.2:

Lemma 2.6. *The map sec and the restriction $(\pi_Y)_{|\text{Ker}(\partial_0)} : \text{Ker}(\partial_0) \rightarrow \mathbb{Q}\langle Y \rangle$ are inverse linear isomorphisms.*

Let \mathbf{p} be the linear endomorphism of $\mathbb{Q}\langle Y \rangle$ given by

$$(2.7) \quad \mathbf{p}(y_{n_1, \sigma_1} \cdots y_{n_r, \sigma_r}) = y_{n_1, \sigma_1} y_{n_2, \sigma_1 \sigma_2} \cdots y_{n_r, \sigma_1 \cdots \sigma_r},$$

for any $n_i \in \mathbb{N}_{>0}$ and $\sigma_i \in \Gamma$. Then \mathbf{p} and \mathbf{q} are linear automorphisms of $\mathbb{Q}\langle Y \rangle$, which are inverse to each other (see [R], §2.2.7).

For $\psi \in \text{Ker}(\partial_0)$, one has $\text{sec} \circ \mathbf{p}(\psi_*) = \text{sec} \circ \mathbf{p}(\mathbf{q} \circ \pi_Y(\psi) + \text{corr}(\psi))$ (identity in $\text{Ker}(\partial_0)$). Since $\mathbf{p} \circ \mathbf{q} = \text{id}_{\mathbb{Q}\langle Y \rangle}$ and by Lemma 2.6, one has $\text{sec} \circ \mathbf{p}(\mathbf{q} \circ \pi_Y(\psi)) = \psi$. The restriction of \mathbf{p} and of sec to $\mathbb{Q}[y_{1,1}]$ is the identity. As $\text{corr}(\psi)$ belongs to $\mathbb{Q}[y_{1,1}]$, it follows that $\text{sec} \circ \mathbf{p}(\text{corr}(\psi)) = \text{corr}(\psi)$. All this implies that

$$(2.8) \quad \forall \psi \in \text{Ker}(\partial_0), \quad \text{sec} \circ \mathbf{p}(\psi_*) = \psi + \text{corr}(\psi)$$

(identity in $\text{Ker}(\partial_0)$). Define $\mathfrak{L}\mathfrak{ib}(X) \ominus \mathbb{Q}x_0$ as the sum of all the homogeneous components of $\mathfrak{L}\mathfrak{ib}(X)$ for the degree in $\mathbb{N}^{\{0\} \cup \Gamma}$, of degree $\neq \epsilon_0$. Then $\mathfrak{L}\mathfrak{ib}(X) \ominus \mathbb{Q}x_0 \subset \text{Ker}(\partial_0)$. One then has

$$\forall \psi \in \mathfrak{L}\mathfrak{ib}(X) \ominus \mathbb{Q}x_0, \quad \text{sec} \circ \mathbf{p}(\psi_*) = \psi + \text{corr}(\psi)$$

(identity in $\text{Ker}(\partial_0)$). For any $\psi \in \mathfrak{L}\mathfrak{ib}(X) \ominus \mathbb{Q}x_0$, one has $\psi + \text{corr}(\psi) = \theta(\psi)$ (identity in $\mathbb{Q}\langle X \rangle$), therefore

$$\forall \psi \in \mathfrak{L}\mathfrak{ib}(X) \ominus \mathbb{Q}x_0, \quad \text{sec} \circ \mathbf{p}(\psi_*) = \theta(\psi)$$

(identity in $\mathbb{Q}\langle X \rangle$). For $\psi = x_0$, one has $\text{sec} \circ \mathbf{p}(\psi_*) = \text{sec}(0) = 0 = \theta(\psi)$. It follows:

$$(2.9) \quad \forall \psi \in \mathfrak{L}\mathfrak{ib}(X), \quad \text{sec} \circ \mathbf{p}(\psi_*) = \theta(\psi)$$

(identity in $\mathbb{Q}\langle X \rangle$).

Define $\tilde{\mathbf{p}}$ to be the endomorphism of $\mathbb{Q}\langle X \rangle$ given by

$$\tilde{\mathbf{p}}(x_0^{n_1} x_{\sigma_1} x_0^{n_2} \cdots x_0^{n_r} x_{\sigma_r} x_0^{n_{r+1}}) = x_0^{n_1} x_{\sigma_1} x_0^{n_2} \cdots x_0^{n_r} x_{\sigma_1 \cdots \sigma_r} x_0^{n_{r+1}},$$

for any n_i in \mathbb{N} and σ_i in Γ . This endomorphism of $\mathbb{Q}\langle X \rangle$ commutes both with ∂_0 and with the endomorphism of $\mathbb{Q}\langle X \rangle$ given by right multiplication by x_0 . It follows that $\tilde{\mathbf{p}}$ commutes with the endomorphism $\widetilde{\text{sec}}$ of $\mathbb{Q}\langle X \rangle$. As the restriction of $\tilde{\mathbf{p}}$ to $\mathbb{Q}\langle Y \rangle \subset \mathbb{Q}\langle X \rangle$ is \mathbf{p} , it follows that $\widetilde{\text{sec}} \circ \mathbf{p} = \tilde{\mathbf{p}} \circ \text{sec}$. Combining this with (2.9), one obtains:

Lemma 2.7. *For any $\psi \in \mathfrak{L}\mathfrak{ib}(X)$, one has $\tilde{\mathbf{p}} \circ \text{sec}(\psi_*) = \theta(\psi)$ (identity in $\mathbb{Q}\langle X \rangle$).*

3. COMPARISON OF LIE ALGEBRAS

This section is devoted to the comparison of the Lie algebras $\mathfrak{d}\mathfrak{mr}_0$ and $\mathfrak{stab}(\Delta_*)$. In §3.1, we recall the definition of Racinet's spaces $\mathfrak{d}\mathfrak{mr}_0$ and $\mathfrak{d}\mathfrak{mr}$; we then compare the Lie algebra $\mathfrak{stab}(\Delta_*)$ introduced in §2.5 with these spaces. In §3.2, we use the fact that $\mathfrak{d}\mathfrak{mr}_0$ and $\mathfrak{d}\mathfrak{mr}$ coincide in certain degrees to compute $\mathfrak{stab}(\Delta_*)$ in these degrees. In §§3.3 and 3.4, we compute $\mathfrak{stab}(\Delta_*)$ in the remaining degrees. This leads us in §3.5 to Thm. 3.10 computing $\mathfrak{stab}(\Delta_*)$ in terms of $\mathfrak{d}\mathfrak{mr}_0$. We then derive Cor. 3.11 stating that $\mathfrak{d}\mathfrak{mr}_0$ is a Lie subalgebra of $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$, thus recovering Racinet's result (Prop. 4.1 i) in [R]).

3.1. The double inclusion $\mathfrak{d}\mathfrak{mr}_0 \subset \mathfrak{stab}(\Delta_*) \subset \mathfrak{d}\mathfrak{mr} \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1$.

Definition 3.1. (see [R], §3.3.1)

$$\mathfrak{d}\mathfrak{mr} := \{ \psi \in \mathfrak{L}\mathfrak{ib}(X) \mid (\psi|x_0) = (\psi|x_1) = 0, \quad \Delta_*(\psi_*) = \psi_* \otimes 1 + 1 \otimes \psi_* \}$$

Definition 3.2. (see [R], §3.3.8) *One defines $\mathfrak{d}\mathfrak{mr}_0$ to be the subspace of $\mathfrak{d}\mathfrak{mr}$ of all elements ψ satisfying the relation*

$$(\psi_*|y_{n,\sigma}) + (-1)^n (\psi_*|y_{n,\sigma^{-1}}) = 0$$

for $n = 1$ and any $\sigma \in \Gamma$ if $|\Gamma| \geq 3$, and for $(n, \sigma) = (2, 1)$ if $|\Gamma| \leq 2$.

In [R], Prop. 4.3.1 (see also [F], Prop. A.5 when $\Gamma = 1$), it is proved:

Proposition 3.3. *If $\psi \in \mathfrak{d}\mathfrak{mr}_0$, then*

$$(3.1) \quad (s_{\mathbf{p}\text{osec}(\psi_*)}^Y \otimes \text{id} + \text{id} \otimes s_{\mathbf{p}\text{osec}(\psi_*)}^Y) \circ \Delta_* = \Delta_* \circ s_{\mathbf{p}\text{osec}(\psi_*)}^Y$$

(equality of maps $\mathbb{Q}\langle Y \rangle \rightarrow \mathbb{Q}\langle Y \rangle^{\otimes 2}$).

By Lemma 2.7, condition (3.1) is equivalent to condition

$$(s_{\theta(\psi)}^Y \otimes \text{id} + \text{id} \otimes s_{\theta(\psi)}^Y) \circ \Delta_* = \Delta_* \circ s_{\theta(\psi)}^Y$$

defining $\mathbf{stab}(\Delta_*)$. Combining this with Prop. 3.3, we get the inclusion

$$(3.2) \quad \mathfrak{d}\mathbf{mr}_0 \subset \mathbf{stab}(\Delta_*).$$

Assume now that ψ belongs to $\mathbf{stab}(\Delta_*)$. It satisfies the identity

$$(s_{\theta(\psi)}^Y \otimes \text{id} + \text{id} \otimes s_{\theta(\psi)}^Y) \circ \Delta_* = \Delta_* \circ s_{\theta(\psi)}^Y.$$

Applying this identity to $1 \in \mathbb{Q}\langle Y \rangle$, one obtains

$$(3.3) \quad s_{\theta(\psi)}^Y(1) \otimes 1 + 1 \otimes s_{\theta(\psi)}^Y(1) = \Delta_*(s_{\theta(\psi)}^Y(1))$$

(identity in $\mathbb{Q}\langle Y \rangle^{\otimes 2}$). Let φ be an element of $\mathbb{Q}\langle X \rangle$. Applying diagram (2.6) to $1 \in \mathbb{Q}\langle X \rangle$, one obtains $s_\varphi^Y(\mathbf{q}(\pi_Y(1))) = \mathbf{q} \circ \pi_Y(s_\varphi(1))$, which using $\pi_Y(1) = \mathbf{q}(1) = 1$ and $s_\varphi(1) = \varphi$, implies:

$$\forall \varphi \in \mathbb{Q}\langle X \rangle, \quad s_\varphi^Y(1) = \mathbf{q} \circ \pi_Y(\varphi).$$

Applying this identity to $\varphi := \theta(\psi)$, one obtains $s_{\theta(\psi)}^Y(1) = \mathbf{q} \circ \pi_Y(\theta(\psi))$. By Lemma 2.7, this is equal to $\mathbf{q} \circ \pi_Y(\tilde{\mathbf{p}} \circ \text{sec}(\psi_*))$. We have $\pi_Y \circ \tilde{\mathbf{p}} = \mathbf{p} \circ \pi_Y$; moreover $\mathbf{q} \circ \mathbf{p} = \text{id}_{\mathbb{Q}\langle Y \rangle}$, and Lemma 2.6 implies that $\pi_Y \circ \text{sec} = \text{id}_{\mathbb{Q}\langle Y \rangle}$, so that $s_{\theta(\psi)}^Y(1) = \psi_*$.

Identity (3.3) then implies that

$$\psi_* \otimes 1 + 1 \otimes \psi_* = \Delta_*(\psi_*)$$

(identity in $\mathbb{Q}\langle Y \rangle^{\otimes 2}$). Therefore:

$$(3.4) \quad \mathbf{stab}(\Delta_*) \subset \{\psi \in \mathfrak{L}\mathbf{ib}(X) \mid \Delta_*(\psi_*) = \psi_* \otimes 1 + 1 \otimes \psi_*\}.$$

(3.2) and (3.4) then imply

$$(3.5) \quad \mathfrak{d}\mathbf{mr}_0 \subset \mathbf{stab}(\Delta_*) \subset \mathfrak{d}\mathbf{mr} \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1.$$

3.2. Computation of $\mathbf{stab}(\Delta_*)$ for large degrees. The double inclusion (3.5) deals with vector subspaces of the space $\mathfrak{L}\mathbf{ib}(X)$. This space is equipped with a grading for which the generators x_0 and x_σ for $\sigma \in \Gamma$ have degree 1; this is the total degree of the bidegree from §2.1.2. The spaces from (3.5) are then graded subspaces of $\mathfrak{L}\mathbf{ib}(X)$. The space $\mathfrak{L}\mathbf{ib}(X)$ is the direct sum of its components of degrees ≥ 1 and the extremal spaces $\mathfrak{d}\mathbf{mr}_0$ and $\mathfrak{d}\mathbf{mr} \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1$ of the double inclusion (3.5) coincide in each degree ≥ 2 if $|\Gamma| \geq 3$, and in each degree ≥ 3 if $|\Gamma| = 1, 2$ (here $|\Gamma|$ is the cardinality of Γ).

It follows:

- Lemma 3.4.**
- If $|\Gamma| \geq 3$, then $\mathbf{stab}(\Delta_*)[n] = \mathfrak{d}\mathbf{mr}_0[n]$ any $n \geq 2$.
 - If $|\Gamma| = 1, 2$, then $\mathbf{stab}(\Delta_*)[n] = \mathfrak{d}\mathbf{mr}_0[n]$ any $n \geq 3$.

Here the notation $[n]$ means the component of degree n .

3.3. Computation of $\text{stab}(\Delta_*)$ in degree 1. The degree 1 component of $\mathfrak{d}\mathfrak{mr} \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1$ coincides with that of $\mathfrak{Lib}(X)$, which is

$$\bigoplus_{g \in \Gamma \cup \{0\}} \mathbb{Q}x_g.$$

On the other hand, according to §3.1, the degree 1 component of $\mathfrak{d}\mathfrak{mr}_0$ is

$$\text{Span}\{x_g + x_{g^{-1}} \mid g \in \Gamma \setminus \{1\}\}.$$

Recall that $\mathbb{Q}\langle Y \rangle$ may be viewed as the subspace $\mathbb{Q} \oplus \bigoplus_{\alpha \in \Gamma} \mathbb{Q}\langle X \rangle x_\alpha$ of $\mathbb{Q}\langle X \rangle$ (see §2.2). If now ψ belongs to $\mathbb{Q}\langle Y \rangle$ and α belongs to Γ , the element $d_\psi(x_\alpha) = [x_\alpha, t_\alpha(\psi)]$ also belongs to $\mathbb{Q}\langle Y \rangle$. As d_ψ is a derivation, it follows that d_ψ preserves $\mathbb{Q}\langle Y \rangle$. We denote by

$$\tilde{d}_\psi : \mathbb{Q}\langle Y \rangle \rightarrow \mathbb{Q}\langle Y \rangle$$

the resulting linear map. On the other hand, the endomorphism of $\mathbb{Q}\langle X \rangle$ given by $v \mapsto \psi v$ also preserves the subspace $\mathbb{Q}\langle Y \rangle$. As the endomorphism s_ψ of $\mathbb{Q}\langle X \rangle$ is the sum of d_ψ and of $(v \mapsto \psi v)$, it follows that s_ψ preserves $\mathbb{Q}\langle Y \rangle$. We denote by

$$\tilde{s}_\psi : \mathbb{Q}\langle Y \rangle \rightarrow \mathbb{Q}\langle Y \rangle$$

the resulting linear map. The purpose of this section is to compute the degree 1 component of the space $\mathfrak{stab}(\Delta_*)$, which lies between the extremal spaces of (3.5).

3.3.1. s_ψ^Y when ψ lies in $\mathbb{Q}\langle Y \rangle$. Combining the diagram describing the compatibility of s_ψ and \tilde{s}_ψ with the diagram (2.6), we obtain the diagram

$$\begin{array}{ccccc} \mathbb{Q}\langle Y \rangle & \hookrightarrow & \mathbb{Q}\langle X \rangle & \xrightarrow{\mathbf{q} \circ \pi_Y} & \mathbb{Q}\langle Y \rangle \\ \tilde{s}_\psi \downarrow & & s_\psi \downarrow & & \downarrow s_\psi^Y \\ \mathbb{Q}\langle Y \rangle & \hookrightarrow & \mathbb{Q}\langle Y \rangle & \xrightarrow{\mathbf{q} \circ \pi_Y} & \mathbb{Q}\langle Y \rangle \end{array}$$

Since the composed map $\mathbb{Q}\langle Y \rangle \hookrightarrow \mathbb{Q}\langle X \rangle \xrightarrow{\pi_Y} \mathbb{Q}\langle Y \rangle$ is the identity of $\mathbb{Q}\langle Y \rangle$, the composition of the horizontal maps of this diagram equals \mathbf{q} , so that

$$(3.6) \quad \forall \psi \in \mathbb{Q}\langle Y \rangle, \quad s_\psi^Y = \mathbf{q} \circ \tilde{s}_\psi \circ \mathbf{q}^{-1}.$$

3.3.2. Condition for s_ψ^Y to be a coderivation. For $g \in \Gamma$, one has

$$s_{x_g}(1) = x_g \quad \text{and} \quad \forall \alpha \in \Gamma, \quad s_{x_g}(x_\alpha) = x_g x_\alpha + [x_\alpha, x_{\alpha g}].$$

As x_g lies in $\mathbb{Q}\langle Y \rangle$ (it identifies with the element $y_{1,g}$), the endomorphism s_{x_g} of $\mathbb{Q}\langle X \rangle$ restricts to an endomorphism \tilde{s}_{x_g} of $\mathbb{Q}\langle Y \rangle$, which satisfies

$$\tilde{s}_{x_g}(1) = y_{1,g} \quad \text{and} \quad \forall \alpha \in \Gamma, \quad \tilde{s}_{x_g}(y_{1,\alpha}) = y_{1,g} y_{1,\alpha} + [y_{1,\alpha}, y_{1,\alpha g}].$$

Equation (3.6) then implies:

$$s_{x_g}^Y(1) = y_{1,g} \quad \text{and} \quad \forall \alpha \in \Gamma, \quad s_{x_g}^Y(y_{1,\alpha}) = y_{1,g} y_{1,\alpha g^{-1}} + y_{1,\alpha} y_{1,g} - y_{1,\alpha g} y_{1,g^{-1}}.$$

Let $\alpha \in \Gamma$. As $y_{1,\alpha}$ is primitive for Δ_* , one has

$$(s_{x_g}^Y \otimes \text{id} + \text{id} \otimes s_{x_g}^Y) \circ \Delta_*(y_{1,\alpha}) = s_{x_g}^Y(y_{1,\alpha}) \otimes 1 + 1 \otimes s_{x_g}^Y(y_{1,\alpha}) + \text{sym}(y_{1,\alpha} \otimes y_{1,g}),$$

where $\text{sym}(a \otimes b) := a \otimes b + b \otimes a$. On the other hand,

$$\begin{aligned} \Delta_*(s_{x_g}^Y(y_{1,\alpha})) &= s_{x_g}^Y(y_{1,\alpha}) \otimes 1 + 1 \otimes s_{x_g}^Y(y_{1,\alpha}) \\ &\quad + \text{sym}(y_{1,\alpha} \otimes y_{1,g}) + \text{sym}(y_{1,g} \otimes y_{1,\alpha g^{-1}}) - \text{sym}(y_{1,g^{-1}} \otimes y_{1,\alpha g}). \end{aligned}$$

It follows that

$$(3.7) \quad (\Delta_* \circ s_{x_g}^Y - (s_{x_g}^Y \otimes \text{id} + \text{id} \otimes s_{x_g}^Y) \circ \Delta_*)(y_{1,\alpha}) = \text{sym}(y_{1,g} \otimes y_{1,\alpha g^{-1}}) - \text{sym}(y_{1,g^{-1}} \otimes y_{1,\alpha g}).$$

Let now ψ be an element of $\text{Span}\{x_g | g \in \Gamma\}$ and assume that s_ψ^Y is a coderivation for Δ_* , namely that $\Delta_* \circ s_\psi^Y = (s_\psi^Y \otimes \text{id} + \text{id} \otimes s_\psi^Y) \circ \Delta_*$. Set $\psi = \sum_{g \in \Gamma} c_g x_g$. Using (3.7), the equation $(\Delta_* \circ s_\psi^Y - (s_\psi^Y \otimes \text{id} + \text{id} \otimes s_\psi^Y) \circ \Delta_*)(y_{1,\alpha}) = 0$ expresses as follows

$$\forall \alpha \in \Gamma, \quad \sum_{g \in \Gamma} c_g (x_g \otimes \alpha x_{g^{-1}} - x_{g^{-1}} \otimes x_{\alpha g} + x_{\alpha g^{-1}} \otimes x_g - x_{\alpha g} \otimes x_{g^{-1}}) = 0$$

(equality in $\mathbb{Q}\Gamma^{\otimes 2}$). Set $A := \sum_{g \in \Gamma} c_g (x_g \otimes x_{g^{-1}} - x_{g^{-1}} \otimes x_g)$. Multiplying the above identity by $1 \otimes x_{\alpha^{-1}}$, it reexpresses as

$$(3.8) \quad \forall \alpha \in \Gamma, \quad A = (x_\alpha \otimes x_{\alpha^{-1}})A.$$

For any $g \in \Gamma$, set $a_g := c_g - c_{g^{-1}}$. Then $A = \sum_{g \in \Gamma} a_g (x_g \otimes x_{g^{-1}})$. As Γ is abelian, (3.8) translates into the condition $a_g = a_{\alpha^{-1}g}$ for any $(g, \alpha) \in \Gamma^2$. This means that the map $g \mapsto a_g$ is constant. On the other hand, $a_1 = 0$, therefore $a_g = 0$ for any $g \in \Gamma$. This implies that $c_g = c_{g^{-1}}$ for any $g \in \Gamma$. We have proved:

Lemma 3.5. *If ψ is an element of $\text{Span}\{x_g | g \in \Gamma\}$ such that s_ψ^Y is a coderivation for Δ_* , then it belongs to $\text{Span}\{x_g + x_{g^{-1}} | g \in \Gamma\}$.*

The restriction to X of the map $\theta : \mathfrak{L}\mathfrak{ib}(X) \rightarrow \mathbb{Q}\langle X \rangle_0$ is given by

$$(3.9) \quad x_0 \mapsto 0, \quad x_1 \mapsto 2x_1, \quad x_g \mapsto x_g \quad \text{for } g \in \Gamma \setminus \{1\}.$$

The degree 1 part of $\mathfrak{stab}(\Delta_*)$ is the set of all $\underline{\psi} \in \text{Span}\{x_g | g \in \Gamma\}$, such that $s_{\theta(\underline{\psi})}^Y$ is a coderivation. Combining Lemma 3.5 with (3.9), we obtain:

Lemma 3.6. *The degree 1 part of $\mathfrak{stab}(\Delta_*)$ is contained in $\mathbb{Q}x_0 \oplus \text{Span}\{x_g + x_{g^{-1}} | g \in \Gamma\}$.*

On the other hand, it follows from the definition of $\mathfrak{d}\mathfrak{m}\mathfrak{r}_0$ that its degree 1 part is equal to

$$\text{Span}\{x_g + x_{g^{-1}} | g \in \Gamma \setminus \{1\}\}$$

(see §3.1). Combining this equality, the first inclusion of (3.5) and Lemma 3.6, we obtain:

Lemma 3.7. *Let Γ be arbitrary. The components of degree 1 of $\mathfrak{stab}(\Delta_*)$ and $\mathfrak{d}\mathfrak{m}\mathfrak{r}_0$ are related by*

$$\mathfrak{stab}(\Delta_*)[1] = \mathfrak{d}\mathfrak{m}\mathfrak{r}_0[1] \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1.$$

3.4. Computation of $\mathfrak{stab}(\Delta_*)$ in degree 2.

3.4.1. *The case $|\Gamma| = 1$.* Specializing the inclusion $\mathbf{stab}(\Delta_*) \subset \mathfrak{d}\mathbf{mr} \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1 \subset \mathfrak{L}\mathbf{ib}(X)$ in degree 2, we obtain the inclusion

$$\mathbf{stab}(\Delta_*)[2] \subset \mathfrak{d}\mathbf{mr}[2] \subset \mathfrak{L}\mathbf{ib}(X)[2]$$

as x_0, x_1 are of degree 1.

One has $\mathfrak{L}\mathbf{ib}(X)[2] = \mathbb{Q} \cdot [x_0, x_1]$. If $\psi := [x_0, x_1]$, then $\psi_* = y_2 - \frac{1}{2}y_1^2$ (we drop the elements of Γ from the notation of generators of $\mathbb{Q}\langle Y \rangle$ as $\Gamma = 1$), which is Δ_* -primitive. Therefore $\mathfrak{d}\mathbf{mr}[2] = \mathbb{Q} \cdot [x_0, x_1]$.

One computes $\sigma([x_0, x_1]) = [x_0, x_1] - \frac{1}{2}x_1^2$. The image of x_1 by $s_{\sigma([x_0, x_1])}$ is

$$s_{\sigma([x_0, x_1])}(x_1) = \sigma([x_0, x_1])x_1 + [x_1, \sigma([x_0, x_1])] = x_1\sigma([x_0, x_1]) = x_1[x_0, x_1] - \frac{1}{2}x_1^3.$$

The image of this element under $\mathbf{q} \circ \pi_Y$ is $y_1y_2 - \frac{1}{2}y_1^3$. So

$$\mathbf{q} \circ \pi_Y \circ s_{\sigma([x_0, x_1])}(x_1) = y_1y_2 - \frac{1}{2}y_1^3.$$

On the other hand, the image of x_1 by $\mathbf{q} \circ \pi_Y$ is y_1 . It then follows from diagram (2.6) that

$$s_{\sigma([x_0, x_1])}^Y(y_1) = y_1y_2 - \frac{1}{2}y_1^3.$$

Whereas y_1 is primitive for Δ_* , the element $y_1y_2 - \frac{1}{2}y_1^3$ is not, as its image by Δ_* contains for example the term $y_1 \otimes y_2$. It follows that $s_{\sigma([x_0, x_1])}^Y$ is not a coderivation for Δ_* , so that $[x_0, x_1] \notin \mathbf{stab}(\Delta_*)[2]$. Therefore

Lemma 3.8. *When $|\Gamma| = 1$, one has*

$$\mathbf{stab}(\Delta_*)[2] = 0 = \mathfrak{d}\mathbf{mr}_0[2].$$

3.4.2. *The case $|\Gamma| = 2$.* Specializing the inclusion $\mathbf{stab}(\Delta_*) \subset \mathfrak{d}\mathbf{mr} \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1 \subset \mathfrak{L}\mathbf{ib}(X)$ in degree 2, we again obtain the inclusion

$$\mathbf{stab}(\Delta_*)[2] \subset \mathfrak{d}\mathbf{mr}[2] \subset \mathfrak{L}\mathbf{ib}(X)[2].$$

We set $\Gamma = \{+, -\}$ so that x_1 can be denoted x_+ . One has

$$\mathfrak{L}\mathbf{ib}(X)[2] = \mathbb{Q}[x_0, x_+] \oplus \mathbb{Q}[x_0, x_-] \oplus \mathbb{Q}[x_+, x_-].$$

Set

$$\psi_1 := [x_0, x_+], \quad \psi_2 := [x_0, x_-], \quad \psi_3 := [x_+, x_-].$$

Then $(\psi_1)_* = y_{2,+} - \frac{1}{2}(y_{1,+})^2$, $(\psi_2)_* = y_{2,-}$, $(\psi_3)_* = y_{1,+}y_{1,-} - (y_{1,-})^2$. Then

$$\Delta_*((\psi_1)_*) = (\psi_1)_* \otimes 1 + 1 \otimes (\psi_1)_* + y_{1,-} \otimes y_{1,-},$$

$$\Delta_*((\psi_2)_*) = (\psi_2)_* \otimes 1 + 1 \otimes (\psi_2)_* + y_{1,+} \otimes y_{1,-} + y_{1,-} \otimes y_{1,+},$$

$$\Delta_*((\psi_3)_*) = (\psi_3)_* \otimes 1 + 1 \otimes (\psi_3)_* + y_{1,+} \otimes y_{1,-} + y_{1,-} \otimes y_{1,+} - 2y_{1,-} \otimes y_{1,-}.$$

A linear combination of $(\psi_1)_*$, $(\psi_2)_*$, $(\psi_3)_*$ is therefore primitive for Δ_* iff it is a linear combination of $2(\psi_1)_* - (\psi_2)_* + (\psi_3)_*$. It follows that

$$\mathfrak{d}\mathbf{mr}[2] = \mathbb{Q} \cdot \psi, \quad \text{where } \psi := 2\psi_1 - \psi_2 + \psi_3.$$

One computes $\sigma(\psi) = \psi + \frac{(-1)^1}{2}(\psi|x_0x_+)x_+^2$, therefore

$$\sigma(\psi) = \psi - x_+^2.$$

The image of x_+ by $s_{\sigma(\psi)}$ is $\sigma(\psi)x_+ + d_{\sigma(\psi)}(x_+) = x_+\sigma(\psi)$ so

$$s_{\sigma(\psi)}(x_+) = 2(x_+x_0x_+ - x_+x_+x_0) - (x_+x_0x_- - x_+x_-x_0) + (x_+x_+x_- - x_+x_-x_+).$$

The image of this element by π_Y is

$$2y_{1,+}y_{2,+} - y_{1,+}y_{2,-} + (y_{1,+})^2y_{1,-} - y_{1,+}y_{1,-}y_{1,+}.$$

The image of this element by \mathbf{q} is

$$2y_{1,+}y_{2,+} - y_{1,+}y_{2,-} + (y_{1,+})^2y_{1,-} - y_{1,+}(y_{1,-})^2.$$

Since the image of x_+ by $\mathbf{q} \circ \pi_Y$ is $y_{1,+}$, (2.6) implies

$$s_{\sigma(\psi)}^Y(y_{1,+}) = 2y_{1,+}y_{2,+} - y_{1,+}y_{2,-} + (y_{1,+})^2y_{1,-} - y_{1,+}(y_{1,-})^2.$$

Whereas $y_{1,+}$ is primitive for Δ_* , its image by $s_{\sigma(\psi)}^Y$ is not: the image by Δ_* of $s_{\sigma(\psi)}^Y(y_{1,+})$ contains for example the word $y_{1,+} \otimes y_{2,+}$. It follows that $s_{\sigma(\psi)}^Y$ is not a coderivation for Δ_* , so that $\psi \notin \mathbf{stab}(\Delta_*)[2]$. Therefore:

Lemma 3.9. *When $|\Gamma| = 2$, one has*

$$\mathbf{stab}(\Delta_*)[2] = 0 = \mathfrak{d}\mathbf{mr}_0[2].$$

3.5. The equality $\mathbf{stab}(\Delta_*) = \mathfrak{d}\mathbf{mr}_0 \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1$. Combining Lemma 3.4, Lemma 3.7, Lemma 3.8, and Lemma 3.9, we obtain:

Theorem 3.10. *Let Γ be arbitrary. The Lie algebras $\mathbf{stab}(\Delta_*)$ and $\mathfrak{d}\mathbf{mr}_0 \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1$ are equal:*

$$\mathbf{stab}(\Delta_*) = \mathfrak{d}\mathbf{mr}_0 \oplus \mathbb{Q}x_0 \oplus \mathbb{Q}x_1.$$

Note that $\mathbb{Q}x_0 \oplus \mathbb{Q}x_1$ is contained in the center of $(\mathfrak{L}\mathbf{ib}(X), \langle, \rangle)$. The subspace $\mathbf{stab}(\Delta_*)$ of $(\mathfrak{L}\mathbf{ib}(X), \langle, \rangle)$ is obviously a Lie subalgebra. It is graded and lives in degrees ≥ 1 . Therefore projection to any quotient of its degree 1 component is a Lie algebra morphism to an abelian Lie algebra, and the kernel of such a projection is an ideal in $\mathbf{stab}(\Delta_*)$, and therefore a Lie subalgebra of $(\mathfrak{L}\mathbf{ib}(X), \langle, \rangle)$. Applying this to the quotient of $\mathbf{stab}(\Delta_*)[1]$ by its subspace $\mathbb{Q}x_0 \oplus \mathbb{Q}x_1$, we obtain:

Corollary 3.11. *$\mathfrak{d}\mathbf{mr}_0$ is a Lie subalgebra of $(\mathfrak{L}\mathbf{ib}(X), \langle, \rangle)$.*

As mentioned in the introduction, this result was first proved in [R].

Remark 3.12. Let Γ, Γ' be finite commutative groups. Let us specify the dependence of X on the group Γ by denoting it X_Γ . To a group morphism $\varphi : \Gamma \rightarrow \Gamma'$, one associates the algebra morphisms

$$\varphi_* : \mathbb{Q}\langle X_\Gamma \rangle \rightarrow \mathbb{Q}\langle X_{\Gamma'} \rangle \quad \text{and} \quad \varphi^* : \mathbb{Q}\langle X_{\Gamma'} \rangle \rightarrow \mathbb{Q}\langle X_\Gamma \rangle,$$

defined by

$$\varphi_* : x_0 \mapsto d \cdot x_0, \quad x_\gamma \mapsto x_{\varphi(\gamma)} \quad \text{for } \gamma \in \Gamma,$$

where $d := |\mathbf{Ker}\varphi|$, and

$$\varphi^* : x_0 \mapsto x_0, \quad x_{\gamma'} \mapsto \sum_{\gamma \in \varphi^{-1}(\gamma')} x_\gamma \quad \text{for } \gamma' \in \Gamma'.$$

The maps $\varphi_* : \mathbb{Q}\langle X_\Gamma \rangle \rightarrow \mathbb{Q}\langle X_{\Gamma'} \rangle$ and $\varphi^* : \mathbb{Q}\langle X_{\Gamma'} \rangle \rightarrow \mathbb{Q}\langle X_\Gamma \rangle$ are Lie algebra morphisms, the spaces $\mathbb{Q}\langle X_\Gamma \rangle$ and $\mathbb{Q}\langle X_{\Gamma'} \rangle$ being equipped with the Ihara brackets. As

the maps φ_* and φ^* restrict to linear maps $\mathfrak{L}\mathfrak{ib}(X_\Gamma) \rightleftarrows \mathfrak{L}\mathfrak{ib}(X_{\Gamma'})$, the maps φ_* and φ^* define Lie algebra morphisms between the Ihara Lie algebras $(\mathfrak{L}\mathfrak{ib}(X_\Gamma), \langle, \rangle)$ and $(\mathfrak{L}\mathfrak{ib}(X_{\Gamma'}), \langle, \rangle)$.

If Γ is a finite commutative group, then for any integer $d \geq 1$, the group Γ^d is defined as the image of the endomorphism of Γ given by multiplication by d . There is an injective morphism $i_d : \Gamma^d \rightarrow \Gamma$ given by inclusion, and a surjective morphism $p^d : \Gamma \rightarrow \Gamma^d$ given by corestriction of the multiplication by d . To them are associated morphisms $i_d^*, p_*^d : \mathfrak{L}\mathfrak{ib}(X_\Gamma) \rightarrow \mathfrak{L}\mathfrak{ib}(X_{\Gamma^d})$ of Lie algebras, both sides being equipped with the Ihara bracket. Any linear form $\ell_d : \mathbb{Q}X_\Gamma \rightarrow \mathbb{Q}$ extends to a linear form $\tilde{\ell}_d : \mathfrak{L}\mathfrak{ib}(X_\Gamma) \rightarrow \mathbb{Q}$ by the condition that $\tilde{\ell}_d$ vanishes on the components of degree > 1 . As the image of the Ihara Lie bracket is concentrated in degrees > 1 , the form $\tilde{\ell}_d$ is a character of the Ihara bracket on $\mathfrak{L}\mathfrak{ib}(X_\Gamma)$. Since on the other hand $x_0 \in \mathfrak{L}\mathfrak{ib}(X_\Gamma)$ is a central element, the space

$$\mathfrak{L}\mathfrak{ib}(X_\Gamma)_d := \{x \in \mathfrak{L}\mathfrak{ib}(X_\Gamma) \mid p_*^d(x) = i_d^*(x) + \tilde{\ell}_d(x)x_0\}$$

is a Lie subalgebra of $(\mathfrak{L}\mathfrak{ib}(X_\Gamma), \langle, \rangle)$. Set $\ell_d(x) := \sum_{\sigma \in \Gamma \mid \sigma^{n/d}=1} (x \mid x_\sigma)$, and

$$\mathfrak{L}\mathfrak{ib}(X_\Gamma)_{\text{dist}} := \bigcap_{d \mid |\Gamma|} \mathfrak{L}\mathfrak{ib}(X_\Gamma)_d.$$

Then $\mathfrak{L}\mathfrak{ib}(X_\Gamma)_{\text{dist}}$ is a Lie subalgebra of $\mathfrak{L}\mathfrak{ib}(X_\Gamma)$. The Lie algebra $\mathfrak{d}\mathfrak{m}\mathfrak{t}\mathfrak{d}$ from [R], §3.3.1, is the intersection $\mathfrak{L}\mathfrak{ib}(X_\Gamma)_{\text{dist}} \cap \mathfrak{d}\mathfrak{m}\mathfrak{t}\mathfrak{d}$. \square

4. EXPONENTIATION OF THE LIE ALGEBRA HOMOMORPHISM θ

Thm. 3.10 says that $\mathfrak{d}\mathfrak{m}\mathfrak{t}\mathfrak{d}$ identifies with the stabilizer of an element in a module over $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$, obtained from a $\mathfrak{m}\mathfrak{t}$ -module by the pull-back under $\theta : (\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle) \rightarrow \mathfrak{m}\mathfrak{t}$. In order to prove the group version of this result, we construct in this section the group analogue of θ . In §4.1, we recall useful notions from the theory of \mathbb{Q} -group schemes. In §4.2, we show how a collection of characters of the group scheme $(\mathbb{k} \mapsto (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes))$ gives rise to a morphism of this group scheme to $\mathfrak{M}\mathfrak{T}$. In §4.3, we construct an explicit collection of such characters, giving rise to a morphism Θ to $\mathfrak{M}\mathfrak{T}$, whose associated Lie algebra morphism is θ (Prop. 4.13).

4.1. \mathbb{Q} -group schemes and their Lie algebras.

4.1.1. An affine \mathbb{Q} -group scheme is a functor $\{\text{commutative } \mathbb{Q}\text{-algebras}\} \rightarrow \{\text{groups}\}$ which is representable by a commutative Hopf algebra. Such a functor $\mathbf{G} : \{\text{commutative } \mathbb{Q}\text{-algebras}\} \rightarrow \{\text{groups}\}$ being fixed, the kernel $\text{Ker}(\mathbf{G}(\mathbb{Q}[\epsilon]/(\epsilon^2)) \rightarrow \mathbf{G}(\mathbb{Q}))$ is naturally equipped with a \mathbb{Q} -Lie algebra structure; this is the Lie algebra $(\text{Lie}(\mathbf{G}), [,])$ of \mathbf{G} . If \mathbf{G}, \mathbf{H} are two \mathbb{Q} -group schemes, then a morphism $\mathbf{G} \rightarrow \mathbf{H}$ is a morphism from the Hopf algebra of \mathbf{H} to that of \mathbf{G} . It gives rise to a natural transformation between the functors $\{\text{commutative } \mathbb{Q}\text{-algebras}\} \rightarrow \{\text{groups}\}$ attached to \mathbf{G}, \mathbf{H} .

4.1.2. Recall that a \mathbb{Q} -Lie algebra is called nilpotent if its lower central series stabilizes to 0. A \mathbb{Q} -Lie algebra is called pronilpotent if it is complete and separated for the topology defined by the members of the lower central series. Such a Lie algebra is then a projective limit of nilpotent Lie algebras, which are its quotients by the members of the lower central series.

If \mathfrak{g} is a pronilpotent \mathbb{Q} -Lie algebra and \mathbf{k} is a commutative \mathbb{Q} -algebra, define $\mathfrak{g} \hat{\otimes} \mathbf{k}$ to be the inverse limit $\lim_{\leftarrow} (\mathfrak{g}/\mathfrak{g}^{(n)}) \otimes \mathbf{k}$, where $\mathfrak{g}^{(n)} := [\mathfrak{g}, [\mathfrak{g}, \dots, \mathfrak{g}]]$ (n arguments). This is a complete separated topological Lie algebra over \mathbf{k} .

For $x, y \in \mathfrak{g} \hat{\otimes} \mathbf{k}$, the Campbell-Baker-Hausdorff (CBH) series $\text{cbh}(x, y) = \log(e^x e^y)$ is a well-defined element of $\mathfrak{g} \hat{\otimes} \mathbf{k}$. When equipped with the product map $\text{cbh} : (\mathfrak{g} \hat{\otimes} \mathbf{k})^2 \rightarrow \mathfrak{g} \hat{\otimes} \mathbf{k}$, $(x, y) \mapsto \text{cbh}(x, y)$, the set $\mathfrak{g} \hat{\otimes} \mathbf{k}$ is a group. The assignment $\{\text{commutative } \mathbb{Q}\text{-algebras}\} \rightarrow \{\text{groups}\}$, $\mathbf{k} \mapsto (\mathfrak{g} \hat{\otimes} \mathbf{k}, \text{cbh}) =: \mathbf{G}_n(\mathbf{k})$ is a \mathbb{Q} -group scheme, denoted \mathbf{G}_n and represented by the Hopf algebra given by the direct limit $\lim_{\rightarrow} U(\mathfrak{g}/\mathfrak{g}^{(n)})^\circ$, where U is the universal enveloping algebra functor and $^\circ$ denotes the restricted dual (so for \mathfrak{a} a finite dimensional Lie algebra, $U(\mathfrak{a})^\circ$ is the set of linear maps $U(\mathfrak{a}) \rightarrow \mathbb{Q}$ which vanish on a finite codimensional ideal). The assignment $\mathfrak{n} \mapsto \mathbf{G}_n$ is a functor $\{\text{pronilpotent } \mathbb{Q}\text{-Lie algebras}\} \rightarrow \{\text{affine } \mathbb{Q}\text{-group schemes}\}$. The Lie algebra of \mathbf{G}_n is then \mathfrak{n} . A \mathbb{Q} -group scheme in the image of $\mathfrak{n} \mapsto \mathbf{G}_n$ will be called prounipotent (unipotent if \mathfrak{g} is nilpotent).

A projective limit of unipotent \mathbb{Q} -group schemes is prounipotent, and any \mathbb{Q} -group subscheme of a prounipotent \mathbb{Q} -group scheme is prounipotent.

4.1.3. Let \mathbb{k} be a commutative \mathbb{Q} -algebra. One defines $\mathbb{k}\langle X \rangle$ to be the tensor product algebra $\mathbb{k} \otimes \mathbb{Q}\langle X \rangle$ and $\mathbb{k}\langle\langle X \rangle\rangle$ to be the degree completion of $\mathbb{k}\langle X \rangle$, where $x_0, x_\sigma, \sigma \in \Gamma$ are all of degree 1. The free algebra structure on $\mathbb{k}\langle X \rangle$ induces on $\mathbb{k}\langle\langle X \rangle\rangle$ the structure of a complete topological algebra. The product in this algebra is denoted by \cdot . The group $(\mathbb{k}\langle\langle X \rangle\rangle^\times, \cdot)$ of invertible elements of $\mathbb{k}\langle\langle X \rangle\rangle$ is its subset of series whose constant term belongs to the group \mathbb{k}^\times of invertible elements of \mathbb{k} .

For G in $\mathbb{k}\langle\langle X \rangle\rangle^\times$, define aut_G to be the automorphism of the topological algebra $\mathbb{k}\langle\langle X \rangle\rangle$ given by $x_0 \mapsto x_0, x_\sigma \mapsto t_\sigma(G)^{-1} x_\sigma t_\sigma(G)$.

Define a map

$$(4.1) \quad \mathbb{k}\langle\langle X \rangle\rangle^\times \times \mathbb{k}\langle\langle X \rangle\rangle \rightarrow \mathbb{k}\langle\langle X \rangle\rangle, \quad (G, H) \mapsto G \circledast H := G \cdot \text{aut}_G(H).$$

Then one checks that for any G, H in $\mathbb{k}\langle\langle X \rangle\rangle^\times$ and K in $\mathbb{k}\langle\langle X \rangle\rangle$, one has

$$(4.2) \quad (G \circledast H) \circledast K = G \circledast (H \circledast K)$$

and that (4.1) restricts and corestricts to a map $(\mathbb{k}\langle\langle X \rangle\rangle^\times)^2 \rightarrow \mathbb{k}\langle\langle X \rangle\rangle^\times$. This implies:

Lemma 4.1. (see [R], §3.1.2) $(\mathbb{k}\langle\langle X \rangle\rangle^\times, \circledast)$ is a group.

Remark 4.2. One checks the identity $\text{aut}_{G \circledast H} = \text{aut}_G \circ \text{aut}_H$ for any G, H in $\mathbb{k}\langle\langle X \rangle\rangle^\times$. It follows that the map $G \mapsto \text{aut}_G$ defines a morphism $(\mathbb{k}\langle\langle X \rangle\rangle^\times, \circledast) \rightarrow \text{Aut}(\mathbb{k}\langle\langle X \rangle\rangle)^\Gamma$, where the target is the group of automorphisms of $\mathbb{k}\langle\langle X \rangle\rangle$ commuting with the action of Γ by $\sigma \mapsto t_\sigma$ (see §2.1).

One checks that the functor $\{\text{commutative } \mathbb{Q}\text{-algebras}\} \rightarrow \{\text{groups}\}$, $\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle^\times, \circledast)$ is representable by a Hopf algebra, therefore:

Lemma 4.3. The functor $\{\text{commutative } \mathbb{Q}\text{-algebras}\} \rightarrow \{\text{groups}\}$, $\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle^\times, \circledast)$ is a \mathbb{Q} -group scheme.

For \mathbb{k} a commutative \mathbb{Q} -algebra, let $\mathbb{k}\langle\langle X \rangle\rangle_1^\times$ be the subset of $\mathbb{k}\langle\langle X \rangle\rangle^\times$ of all series with constant term equal to 1. Then $\mathbb{k}\langle\langle X \rangle\rangle_1^\times$ is a subgroup of $(\mathbb{k}\langle\langle X \rangle\rangle^\times, \cdot)$. One checks that $\mathbb{k}\langle\langle X \rangle\rangle_1^\times$ is a subgroup of $(\mathbb{k}\langle\langle X \rangle\rangle^\times, \otimes)$.

Moreover, the functor $\mathbb{k} \mapsto \mathbb{k}\langle\langle X \rangle\rangle_1^\times$ is also represented by a Hopf algebra, therefore

Lemma 4.4. *The functor $\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$ is a \mathbb{Q} -group subscheme of the group scheme $\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle^\times, \otimes)$.*

This group scheme is denoted \mathbf{MT} in [R], so $\mathbf{MT}(\mathbb{k}) = (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$.

For $n \geq 1$, let $\mathbb{k}\langle\langle X \rangle\rangle_{1, \geq n}^\times$ be the subset of $\mathbb{k}\langle\langle X \rangle\rangle_1^\times$ consisting of all the series which are $\equiv 1$ modulo degree $\geq n$. Then $\mathbb{k}\langle\langle X \rangle\rangle_{1, \geq n}^\times$ is a normal subgroup of $(\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$, and the functor $\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_{1, \geq n}^\times / \mathbb{k}\langle\langle X \rangle\rangle_{1, \geq n}^\times, \otimes)$ is a unipotent \mathbb{Q} -group scheme. It follows:

Lemma 4.5. *The group \mathbb{Q} -scheme $\mathbf{MT} := (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes))$ is prounipotent.*

Recall that $\mathbb{k}\langle\langle X \rangle\rangle$ is equipped with a Hopf algebra structure, with product \cdot and co-product Δ such that the elements $x_0, x_\sigma, \sigma \in \Gamma$, are primitive. Then $(\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \cdot)$ contains as a subgroup the set of group-like elements for this coproduct, namely $\mathcal{G}(\mathbb{k}\langle\langle X \rangle\rangle, \Delta) = \{G \in \mathbb{k}\langle\langle X \rangle\rangle^\times \mid \Delta(G) = G \otimes G\}$.

The exponential map relative to \cdot sets up a bijection

$$(\exp)_{\mathbb{k}} : \mathbb{k}\langle\langle X \rangle\rangle_0 \xrightarrow{\sim} \mathbb{k}\langle\langle X \rangle\rangle_1^\times,$$

where the index 0 means formal series with vanishing constant term; this bijection restricts to a bijection

$$(\exp)_{\mathbb{k}} : \hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X) \xrightarrow{\sim} \mathcal{G}(\mathbb{k}\langle\langle X \rangle\rangle, \Delta),$$

where $\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X) \subset \mathbb{k}\langle\langle X \rangle\rangle_0$ is the degree completion of $\mathfrak{L}\mathfrak{ib}_{\mathbb{k}}(X) := \mathfrak{L}\mathfrak{ib}(X) \otimes \mathbb{k}$. One checks that $\mathcal{G}(\mathbb{k}\langle\langle X \rangle\rangle, \Delta) = \exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X))$ is a subgroup of $(\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$. Therefore:

Lemma 4.6. *The functor $\mathbb{k} \mapsto (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)$ is a \mathbb{Q} -group subscheme of $\mathbf{MT} = (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes))$.*

As the \mathbb{Q} -group scheme \mathbf{MT} is prounipotent, we get:

Lemma 4.7. *The \mathbb{Q} -group scheme $\mathbb{k} \mapsto (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)$ is prounipotent.*

4.1.4. The Lie algebras of the \mathbb{Q} -group schemes $\mathbf{MT} = (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes))$ and $\mathbb{k} \mapsto (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)$ may be computed using the ring of dual numbers. One gets

$$(4.3) \quad \text{Lie}(\mathbf{MT}) = (\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle) = \widehat{\mathfrak{m}\mathfrak{t}}$$

and

$$\text{Lie}(\mathbb{k} \mapsto (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)) = (\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle).$$

These are the degree completions of the Lie algebras from (2.2).

Since the \mathbb{Q} -group schemes $\mathbf{MT} = (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes))$ and $\mathbb{k} \mapsto (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)$ are both prounipotent, they are isomorphic to the \mathbb{Q} -group schemes $\widehat{\mathbf{G}}_{\widehat{\mathfrak{m}\mathfrak{t}}} = (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_0, \text{cbh}_{\otimes}))$ and $\mathbf{G}_{(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)} = (\mathbb{k} \mapsto (\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X), \text{cbh}_{\otimes}))$ respectively, where in both cases, cbh_{\otimes} denotes the CBH series relative to the bracket \langle, \rangle .

4.1.5. The \mathbb{Q} -group scheme isomorphism between \widehat{G}_{mt} and MT is an assignment $\mathbb{k} \mapsto (\exp_{\otimes})_{\mathbb{k}}$, where $(\exp_{\otimes})_{\mathbb{k}}$ is a bijection $\mathbb{k}\langle\langle X \rangle\rangle_0 \rightarrow \mathbb{k}\langle\langle X \rangle\rangle_1^{\times}$, which intertwines cbh_{\otimes} and \otimes . One checks that the permutation $(\exp_{\mathbb{k}})^{-1} \circ (\exp_{\otimes})_{\mathbb{k}}$ of $\mathbb{k}\langle\langle X \rangle\rangle_0$ is given by $x \mapsto x + \sum_{n \geq 2} e_n(x)$, where $e_n(x)$ is a homogeneous polynomial self-map of $\mathbb{k}\langle\langle X \rangle\rangle_0$ of degree n with rational coefficients.

In particular, the differentials at 0 of the two maps $\exp_{\mathbb{k}}, (\exp_{\otimes})_{\mathbb{k}} : \mathbb{k}\langle\langle X \rangle\rangle_0 \rightarrow \mathbb{k}\langle\langle X \rangle\rangle_1^{\times}$ coincide.

4.2. A \mathbb{Q} -group scheme morphism to MT.

4.2.1. *From characters $(\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}, +)$ to group homomorphisms $(\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}\langle\langle X \rangle\rangle_1^{\times}, \otimes)$.* Let $G, H \in \mathbb{k}\langle\langle X \rangle\rangle^{\times}$ and $f_G, f_H \in \mathbb{k}[[x_1]]^{\times}$, $g_G, g_H \in \mathbb{k}[[x_0]]^{\times}$. One has

$$(4.4) \quad \begin{aligned} (f_G \cdot G \cdot g_G) \otimes (f_H \cdot H \cdot g_H) &= f_G \cdot G \cdot g_G \cdot \text{aut}_{f_G \cdot G \cdot g_G}(f_H \cdot H \cdot g_H) \\ &= f_G \cdot G \cdot g_G \cdot \text{aut}_{f_G \cdot G \cdot g_G}(f_H) \text{aut}_{f_G \cdot G \cdot g_G}(H) \text{aut}_{f_G \cdot G \cdot g_G}(g_H). \end{aligned}$$

Since

$$\text{aut}_{f_G \cdot G \cdot g_G}(f_H) = (f_G \cdot G \cdot g_G)^{-1} \cdot f_H \cdot f_G \cdot G \cdot g_G,$$

$$\text{aut}_{f_G \cdot G \cdot g_G}(H) = \text{aut}_{G \cdot g_G}(H) = (g_G)^{-1} \cdot \text{aut}_G(H) \cdot g_G, \quad \text{aut}_{f_G \cdot G \cdot g_G}(g_H) = g_H,$$

the right hand side of (4.4) is equal to $f_H \cdot f_G \cdot G \cdot \text{aut}_G(H) \cdot g_G \cdot g_H$.

Therefore:

Lemma 4.8. *Let $G, H \in \mathbb{k}\langle\langle X \rangle\rangle^{\times}$ and $f_G, f_H \in \mathbb{k}[[x_1]]^{\times}$, $g_G, g_H \in \mathbb{k}[[x_0]]^{\times}$. Then:*

$$(f_G \cdot G \cdot g_G) \otimes (f_H \cdot H \cdot g_H) = f_H \cdot f_G \cdot (G \otimes H) \cdot g_G \cdot g_H.$$

Let $\underline{f}_n, \underline{g}_n : (\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}, +)$ ($n \geq 1$) be two collections of group homomorphisms. We construct from them group homomorphisms $\underline{f}, \underline{g} : (\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes) \rightarrow \mathbb{k}[[t]]^{\times}$, via

$$(4.5) \quad \underline{f}(G) := \exp\left(\sum_{n \geq 1} \underline{f}_n(G)t^n\right), \quad \underline{g}(G) := \exp\left(\sum_{n \geq 1} \underline{g}_n(G)t^n\right).$$

Lemma 4.8 implies:

Lemma 4.9. *If $\underline{f}_n, \underline{g}_n : (\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}, +)$ ($n \geq 1$) are two collections of group homomorphisms and the maps $\underline{f}, \underline{g}$ are defined by (4.5), then the map*

$$(4.6) \quad G \mapsto \underline{f}(G)|_{t \rightarrow x_1} \cdot G \cdot \underline{g}(G)|_{t \rightarrow x_0}$$

defines a group homomorphism

$$\Theta_{(\underline{f}_n)_n, (\underline{g}_n)_n} : (\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}\langle\langle X \rangle\rangle_1^{\times}, \otimes).$$

Here the indices $|t \rightarrow x_i$, $i = 0, 1$ stand for the homomorphisms $\mathbb{k}[[t]] \rightarrow \mathbb{k}\langle\langle X \rangle\rangle$, $t \mapsto x_i$ ($i = 0, 1$).

4.2.2. *Construction of a \mathbb{Q} -group scheme morphism* $(\mathbb{k} \mapsto (\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes)) \rightarrow (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes))$. Recall that the additive group is the \mathbb{Q} -group scheme given by $\mathbb{k} \mapsto (\mathbb{k}, +)$. Assume that

$$f_n, g_n : (\mathbb{k} \mapsto (\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes)) \rightarrow \mathbb{G}_a$$

are \mathbb{Q} -group scheme morphisms; they give rise to compatible families of group homomorphisms $f_n(\mathbb{k}), g_n(\mathbb{k}) : (\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}, +)$ and therefore to a group homomorphism

$$\Theta_{(f_n(\mathbb{k}))_n, (g_n(\mathbb{k}))_n} : (\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes).$$

For $g \in \exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X))$, one checks that $\Theta_{(f_n(\mathbb{k}))_n, (g_n(\mathbb{k}))_n}(g)$ can be expressed algebraically in terms of the coefficients of g . It follows:

Lemma 4.10. *Let $f_n, g_n : (\mathbb{k} \mapsto (\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes)) \rightarrow \mathbb{G}_a$ be \mathbb{Q} -group scheme morphisms. There exists a \mathbb{Q} -group scheme morphism*

$$\Theta_{(f_n)_n, (g_n)_n} : (\mathbb{k} \mapsto (\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes)) \rightarrow (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)),$$

such that for any commutative \mathbb{Q} -algebra \mathbb{k} , the specialization to \mathbb{k} of $\Theta_{(f_n)_n, (g_n)_n}$ coincides with $\Theta_{(f_n(\mathbb{k}))_n, (g_n(\mathbb{k}))_n}$.

Remark 4.11. When the f_n, g_n are trivial, then $\Theta_{(f_n(\mathbb{k}))_n, (g_n(\mathbb{k}))_n}$ is nothing but the inclusion $(\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes) \subset (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$.

The Lie algebra homomorphisms associated with f_n, g_n are characters

$$\text{Lie}(f_n), \text{Lie}(g_n) : (\mathcal{L}\hat{\mathbf{ib}}(X), \langle, \rangle) \rightarrow \mathbb{Q}.$$

Then (4.6) implies that the Lie algebra homomorphism associated to $\Theta_{(f_n)_n, (g_n)_n}$ is

$$\text{Lie}(\Theta_{(f_n)_n, (g_n)_n}) : (\mathcal{L}\hat{\mathbf{ib}}(X), \langle, \rangle) \rightarrow (\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle),$$

$$(4.7) \quad \psi \mapsto \psi + \sum_{n \geq 1} \text{Lie}(f_n)(\psi)x_1^n + \sum_{n \geq 1} \text{Lie}(g_n)(\psi)x_0^n.$$

4.3. Construction of a \mathbb{Q} -group scheme morphism Θ exponentiating θ .

4.3.1. *Exponentiation of some characters* $(\mathcal{L}\hat{\mathbf{ib}}(X), \langle, \rangle) \rightarrow \mathbb{Q}$.

Lemma 4.12. *Let $n \geq 1$. There are \mathbb{Q} -group scheme morphisms*

$$f_n^0, g_n^0 : (\mathbb{k} \mapsto (\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes)) \rightarrow \mathbb{G}_a,$$

such that for any \mathbb{k} , the group homomorphisms $f_n^0(\mathbb{k}), g_n^0(\mathbb{k}) : (\exp(\mathcal{L}\hat{\mathbf{ib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}, +)$ are given by

$$f_n^0(\mathbb{k}) : G \mapsto (G|x_0^{n-1}x_1), \quad g_1^0(\mathbb{k}) : G \mapsto (G|x_0), \quad g_n^0(\mathbb{k}) : G \mapsto 0 \quad \text{for } n \geq 2.$$

The associated Lie algebra homomorphisms coincide with the characters $f_n^0, g_n^0 : (\mathcal{L}\hat{\mathbf{ib}}(X), \langle, \rangle) \rightarrow \mathbb{Q}$ defined in Lemma 2.4.

Proof. If \mathfrak{G} is a pronilpotent \mathbb{Q} -Lie algebra, and if $\mathfrak{G} \rightarrow \mathbb{Q}$ is a Lie algebra homomorphism (character), then there is a unique morphism of \mathbb{Q} -group schemes $(\mathbb{k} \rightarrow (\mathfrak{G} \hat{\otimes} \mathbb{k}, \text{cbh})) \rightarrow \mathbb{G}_a$ with differential the initial map $\mathfrak{G} \rightarrow \mathbb{Q}$. The homomorphisms f_n^0, g_n^0 from Lemma 2.4 therefore give rise to \mathbb{Q} -group scheme morphisms $f_n^0, g_n^0 : (\mathbb{k} \mapsto (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)) \rightarrow \mathbb{G}_a$.

Let $n \geq 1$ and for \mathbb{k} a commutative \mathbb{Q} -algebra, let $(\varphi_n)_{\mathbb{k}} : (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes) \rightarrow \mathbb{k}$ be given by $(\varphi_n)_{\mathbb{k}}(G) := (G|x_0^{n-1}x_1)$ for $G \in \exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X))$.

Let $G_0, H \in \exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X))$, where $(\log(G_0)|x_0) = 0$. Then

$$(G_0 \otimes H|x_0^{n-1}x_1) = (G_0|x_0^{n-1}x_1) + \sum_{k=0}^{n-1} (G_0|x_0^k)(\text{aut}_{G_0}(H)|x_0^{n-1-k}x_1).$$

All the $(G_0|x_0^k)$ are zero for $k > 0$, so

$$(G_0 \otimes H|x_0^{n-1}x_1) = (G_0|x_0^{n-1}x_1) + (\text{aut}_{G_0}(H)|x_0^{n-1}x_1).$$

One also computes $(\text{aut}_{G_0}(H)|x_0^{n-1}x_1) = (H|x_0^{n-1}x_1)$, so

$$(4.8) \quad (G_0 \otimes H|x_0^{n-1}x_1) = (G_0|x_0^{n-1}x_1) + (H|x_0^{n-1}x_1).$$

Let now $G \in \exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X))$ be arbitrary. Let $\alpha := (\log(G_0)|x_0)$. Set $G_0 := G \cdot \exp(-\alpha x_0)$. We have $G = G_0 \cdot \exp(\alpha x_0) = G_0 \otimes \exp(\alpha x_0)$. Then

$$(G \otimes H|x_0^{n-1}x_1) = (G_0 \otimes \exp(\alpha x_0) \otimes H|x_0^{n-1}x_1) = (G_0|x_0^{n-1}x_1) + (\exp(\alpha x_0) \otimes H|x_0^{n-1}x_1)$$

after (4.8). One has $\exp(\alpha x_0) \otimes H = H \cdot \exp(\alpha x_0)$ (recall that x_0 is central), so this is

$$(G_0|x_0^{n-1}x_1) + (H \cdot \exp(\alpha x_0)|x_0^{n-1}x_1).$$

The last term is obviously equal to $(H|x_0^{n-1}x_1)$, so

$$(G \otimes H|x_0^{n-1}x_1) = (G_0|x_0^{n-1}x_1) + (H|x_0^{n-1}x_1).$$

Also, $(G_0|x_0^{n-1}x_1) = (G_0 \cdot \exp(\alpha x_0)|x_0^{n-1}x_1) = (G|x_0^{n-1}x_1)$. So finally

$$(\varphi_n)_{\mathbb{k}}(G \otimes H) = (G \otimes H|x_0^{n-1}x_1) = (G|x_0^{n-1}x_1) + (H|x_0^{n-1}x_1) = (\varphi_n)_{\mathbb{k}}(G) + (\varphi_n)_{\mathbb{k}}(H)$$

so that $(\varphi_n)_{\mathbb{k}}$ is a character.

Let also $(\gamma_n)_{\mathbb{k}} : (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes) \rightarrow \mathbb{k}$ be given by $(\gamma_1)_{\mathbb{k}}(G) := (G|x_0)$ for $G \in \exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X))$, and by $(\gamma_n)_{\mathbb{k}}(G) := 0$ for $n > 1$ and $G \in \exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X))$.

Let $G, H \in \exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X))$. One has $G \otimes H = G \cdot \text{aut}_G(H)$, and the series $G, \text{aut}_G(H)$ have constant term 1, so $(G \otimes H|x_0) = (G|x_0) + (\text{aut}_G(H)|x_0)$. One checks that $(\text{aut}_G(H)|x_0) = (H|x_0)$, so $(G \otimes H|x_0) = (G|x_0) + (H|x_0)$. This implies that $(\gamma_1)_{\mathbb{k}}$ is a character. It is also clear that the maps $(\gamma_n)_{\mathbb{k}}$ are characters for $n > 1$.

The assignments $\mathbb{k} \mapsto (\varphi_n)_{\mathbb{k}}, \mathbb{k} \mapsto (\gamma_n)_{\mathbb{k}}$ are functorial. The Lie algebra of the assignment $\mathbb{k} \mapsto (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)$ coincides with $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$.

According to §4.1.5, the maps $(\exp)_{\mathbb{k}}$ and $(\exp_{\otimes})_{\mathbb{k}}$ have the same differential at origin. This enables one to compute the Lie algebra homomorphisms induced by $\mathbb{k} \mapsto (\varphi_n)_{\mathbb{k}}, (\gamma_n)_{\mathbb{k}}$, which are found to be equal to f_n^0, g_n^0 .

Since the assignments $\mathbb{k} \mapsto \varphi_n(\mathbb{k})$ and $\mathbb{k} \mapsto f_n^0(\mathbb{k})$ on the one hand, $\mathbb{k} \mapsto \gamma_n(\mathbb{k})$ and $\mathbb{k} \mapsto g_n^0(\mathbb{k})$ on the other hand, are both functorial and induce the same Lie algebra homomorphisms, they coincide. \square

4.3.2. *Construction of Θ .* In Lemma 4.12, \mathbb{Q} -group scheme morphisms f_n^0, g_n^0 are constructed. Plugging them in the construction of Lemma 4.10, one obtains a \mathbb{Q} -group scheme morphism $\Theta_{(f_n^0)_n, (g_n^0)_n} : (\mathbb{k} \mapsto (\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes)) \rightarrow (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes))$, which will be denoted Θ . According to §4.2.2, the infinitesimal of Θ is the Lie algebra homomorphism θ from Prop. 2.5. Summarizing:

Proposition 4.13. *There exists a \mathbb{Q} -group scheme morphism*

$$\Theta : (\mathbb{k} \mapsto (\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes)) \rightarrow (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)),$$

such that for any commutative \mathbb{Q} -algebra \mathbb{k} , the group homomorphism

$$\Theta_{\mathbb{k}} : (\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$$

is given by

$$(4.9) \quad G \mapsto \exp\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (G|x_0^{n-1}x_1)x_1^n\right) \cdot G \cdot \exp(- (G|x_0)x_0).$$

The differential of Θ coincides with the Lie algebra homomorphism θ in Prop. 2.5.

5. COMPARISON OF \mathbb{Q} -GROUP SCHEMES

The group analogue of Thm. 3.10 is Thm. 1.2, which states that for \mathbb{k} any commutative \mathbb{Q} -algebra, the subgroup $\widehat{\text{DMR}}_0(\mathbb{k})$ of $(\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes)$ coincides with the stabilizer of Δ_* , viewed as an element of the pull-back under $\Theta_{\mathbb{k}}$ of the $\text{MT}(\mathbb{k})$ -module $\text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\otimes 2})$. This result is proved in several steps. In §5.1, we show that in the context of certain modules over a pronilpotent Lie algebra, stabilizer group functors coincide with the group functors arising from stabilizer Lie algebras. In §5.2, we introduce the modification $\widehat{\text{DMR}}_0$ of DMR_0 corresponding to the product of the Lie algebra \mathfrak{dmr}_0 with the abelian Lie algebra $\mathbb{Q}x_0 \oplus \mathbb{Q}x_1$. In §5.3, we construct a module over $(\exp(\widehat{\mathfrak{Lib}}_{\mathbb{k}}(X)), \otimes)$, together with a submodule, to which the results of §5.1 can be applied. In §5.4, we put together the various constructions to prove Thm. 1.2.

5.1. Stabilizer group functors and schemes.

5.1.1. *Module functors associated to a module over a pronilpotent \mathbb{Q} -Lie algebra.* Let \mathfrak{n} be a pronilpotent \mathbb{Q} -Lie algebra. Let $\mathcal{C}_{\mathfrak{n}}$ be the category of topological \mathfrak{n} -modules M , which are complete and separated, such that the sequence of subspaces $(\mathfrak{n}^k \cdot M)_{k \geq 0}$ is a neighborhood of origin, and such that for each $k \geq 0$, the quotient $\mathfrak{n}^k \cdot M / \mathfrak{n}^{k+1} \cdot M$ is finite dimensional. The category $\mathcal{C}_{\mathfrak{n}}$ is symmetric and monoidal.

For \mathbb{k} a commutative \mathbb{Q} -algebra, we define a \mathbb{k} -module $M \hat{\otimes} \mathbb{k} := \lim_{\leftarrow} (M / \mathfrak{n}^k \cdot M) \otimes \mathbb{k}$. The \mathbb{k} -module $M \hat{\otimes} \mathbb{k}$ is then a module over the group $\mathbf{G}_{\mathfrak{n}}(\mathbb{k}) = (\mathfrak{n} \hat{\otimes} \mathbb{k}, \text{cbh})$, the action of $x \in \mathfrak{n} \hat{\otimes} \mathbb{k}$ on $M \hat{\otimes} \mathbb{k}$ being given by the element $\sum_{n \geq 0} \rho_{\mathbb{k}}(x)^n / n! \in \text{Aut}_{\mathbb{k}}(M \hat{\otimes} \mathbb{k})$, where $\rho_{\mathbb{k}} : \mathfrak{n} \hat{\otimes} \mathbb{k} \rightarrow \text{End}_{\mathbb{k}}(M \hat{\otimes} \mathbb{k})$ is the action map derived from the action map $\rho : \mathfrak{n} \rightarrow \text{End}_{\mathbb{Q}}(M)$ by extension of scalars. The assignment $\mathbb{k} \mapsto M \hat{\otimes} \mathbb{k}$ is a module functor over the group functor $\mathbb{k} \mapsto \mathbf{G}_{\mathfrak{n}}(\mathbb{k})$.

5.1.2. *Stabilizer group functors.* Let \mathfrak{n} be a nilpotent Lie algebra, let $M \in \text{Ob}(\mathcal{C}_{\mathfrak{n}})$ and let $v \in M$ be a vector. For \mathbb{k} a commutative \mathbb{Q} -algebra, set

$$(5.1) \quad \text{Stab}(v)(\mathbb{k}) := \{g \in \mathbf{G}_{\mathfrak{n}}(\mathbb{k}) \mid g \cdot v = v \text{ (equality in } M \hat{\otimes} \mathbb{k})\}.$$

The assignment $\mathbb{k} \mapsto \text{Stab}(v)(\mathbb{k})$ is then a subgroup functor of the group functor $\mathbb{k} \mapsto \mathbf{G}_{\mathfrak{n}}(\mathbb{k})$.

On the other hand, set $\widehat{\mathfrak{stab}}(v) := \{x \in \mathfrak{n} \mid x \cdot v = 0\}$. Then $\widehat{\mathfrak{stab}}(v)$ is a closed Lie subalgebra of \mathfrak{n} , which is therefore pronilpotent. With this Lie algebra is associated a \mathbb{Q} -group scheme $\mathbf{G}_{\widehat{\mathfrak{stab}}(v)}$, which is a group subscheme of $\mathbf{G}_{\mathfrak{n}}$.

Lemma 5.1. *The subgroup functors $\mathbb{k} \mapsto \text{Stab}(v)(\mathbb{k})$ and $\mathbb{k} \mapsto \mathbf{G}_{\widehat{\mathfrak{stab}}(v)}(\mathbb{k})$ of the group functor $\mathbb{k} \mapsto \mathbf{G}_{\mathfrak{n}}(\mathbb{k})$ are equal, so for any commutative \mathbb{Q} -algebra \mathbb{k} , one has*

$$\text{Stab}(v)(\mathbb{k}) = \mathbf{G}_{\widehat{\mathfrak{stab}}(v)}(\mathbb{k})$$

(equality of subgroups of $\mathbf{G}_{\mathfrak{n}}(\mathbb{k})$).

Proof. For $k \geq 0$, set $M^{(k)} := M/\mathfrak{n}^k \cdot M$ and let $v^{(k)}$ be the image of $v \in M$. Then $M^{(k)} \in \text{Ob}(\mathcal{C}_{\mathfrak{n}})$. One associates with the pair $(M^{(k)}, v^{(k)})$ the following data:

- the Lie subalgebra $\widehat{\mathfrak{stab}}(v) \subset \mathfrak{n}$, and therefore the \mathbb{Q} -group scheme inclusion $\mathbf{G}_{\widehat{\mathfrak{stab}}(v)} \subset \mathbf{G}_{\mathfrak{n}}$;
- the subgroup functor $\mathbb{k} \mapsto \text{Stab}(v^{(k)})(\mathbb{k})$ of the group functor $\mathbb{k} \mapsto \mathbf{G}_{\mathfrak{n}}(\mathbb{k})$.

One has $\bigcap_{k \geq 0} \widehat{\mathfrak{stab}}(v^{(k)}) = \widehat{\mathfrak{stab}}(v)$, which implies that for any commutative \mathbb{Q} -algebra \mathbb{k} , one has

$$(5.2) \quad \bigcap_{k \geq 0} \mathbf{G}_{\widehat{\mathfrak{stab}}(v^{(k)})}(\mathbb{k}) = \mathbf{G}_{\widehat{\mathfrak{stab}}(v)}(\mathbb{k})$$

(equality of subsets of $\mathbf{G}_{\mathfrak{n}}(\mathbb{k})$).

On the other hand, for any commutative \mathbb{Q} -algebra \mathbb{k} , one has

$$(5.3) \quad \bigcap_{k \geq 0} \text{Stab}(v^{(k)})(\mathbb{k}) = \text{Stab}(v)(\mathbb{k})$$

(equality of subsets of $\mathbf{G}_{\mathfrak{n}}(\mathbb{k})$). Recall that $\mathbf{G}_{\mathfrak{n}}(\mathbb{k}) = (\mathfrak{n} \hat{\otimes} \mathbb{k}, \text{cbh})$. Let $x \in \mathfrak{n} \hat{\otimes} \mathbb{k}$. Then:

- Assume that $x \in \mathbf{G}_{\widehat{\mathfrak{stab}}(v^{(k)})}(\mathbb{k})$. This means that $x \in \widehat{\mathfrak{stab}}(v^{(k)}) \hat{\otimes} \mathbb{k}$. Then $x \cdot v^{(k)} = \sum_{n \geq 0} (1/n!) \rho_{\mathbb{k}}^{(k)}(x)^n \cdot v^{(k)}$ (equality in $M^{(k)} \hat{\otimes} \mathbb{k}$), where $\rho_{\mathbb{k}}^{(k)} : \mathfrak{n} \hat{\otimes} \mathbb{k} \rightarrow \text{End}_{\mathbb{k}}(M^{(k)} \hat{\otimes} \mathbb{k})$ is the action map of $\mathfrak{n} \hat{\otimes} \mathbb{k}$ on $M^{(k)} \hat{\otimes} \mathbb{k}$. Since $\rho_{\mathbb{k}}^{(k)}(v) = 0$, one has $x \cdot v^{(k)} = v^{(k)}$, so $x \in \text{Stab}(v^{(k)})(\mathbb{k})$. This shows $\mathbf{G}_{\widehat{\mathfrak{stab}}(v^{(k)})}(\mathbb{k}) \subset \text{Stab}(v^{(k)})(\mathbb{k})$.
- Assume now that $x \in \text{Stab}(v^{(k)})(\mathbb{k})$. Then $x \cdot v^{(k)} = v^{(k)}$; moreover, for any $\ell \geq 1$, $x \cdots x \cdot v^{(k)} = v^{(k)}$ (ℓ factors x). The ℓ -th power of x in $\mathbf{G}_{\mathfrak{n}}(\mathbb{k})$ corresponds to $\ell \cdot x \in \mathfrak{n} \hat{\otimes} \mathbb{k}$. So $(\ell \cdot x) \cdot v^{(k)} = v^{(k)}$, therefore $\sum_{n \geq 0} \ell^n (\rho_{\mathbb{k}}^{(k)}(x)^n / n!) (v^{(k)}) = v^{(k)}$. One has $\rho_{\mathbb{k}}^{(k)}(x)^n (v^{(k)}) = 0$ for $n \geq k$, therefore

$$\forall \ell \geq 1, \quad \sum_{n=1}^{k-1} \ell^n (\rho_{\mathbb{k}}^{(k)}(x)^n / n!) (v^{(k)}) = 0.$$

As the matrix with elements $(\ell^n)_{\ell \in [1, k-1], n \in [1, k-1]}$ is invertible, one gets $\rho_{\mathbb{k}}^{(k)}(x)(v^{(k)}) = 0$, therefore $x \in \widehat{\mathbf{stab}}(v^{(k)}) \hat{\otimes} \mathbb{k} \simeq \mathbf{G}_{\widehat{\mathbf{stab}}(v^{(k)})}(\mathbb{k})$. This shows $\mathbf{Stab}(v^{(k)})(\mathbb{k}) \subset \mathbf{G}_{\widehat{\mathbf{stab}}(v^{(k)})}(\mathbb{k})$.

All this implies the equality $\mathbf{G}_{\widehat{\mathbf{stab}}(v^{(k)})}(\mathbb{k}) = \mathbf{Stab}(v^{(k)})(\mathbb{k})$, which combined with (5.2) and (5.3), yields the conclusion. \square

5.2. The group scheme \mathbf{DMR}_0 and its modification $\widetilde{\mathbf{DMR}}_0$. The Lie algebra of the affine \mathbb{Q} -group scheme \mathbf{MT} is $(\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle)$, which is pronilpotent. By (4.3), we have then

$$\mathbf{G}_{(\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle)} \simeq \mathbf{MT} = (\mathbb{k} \mapsto (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)).$$

Let \mathfrak{G} be a closed Lie subalgebra of $(\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle)$, let $\mathbf{G}_{\mathfrak{G}}$ be the corresponding \mathbb{Q} -group subscheme of $\mathbf{G}_{(\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle)}$ and let \mathbf{G} be the group subscheme of \mathbf{MT} , image of $\mathbf{G}_{\mathfrak{G}}$ under the isomorphism $\mathbf{G}_{(\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle)} \simeq \mathbf{MT}$, so that the diagram

$$\begin{array}{ccc} \mathbf{G}_{(\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle)} & \xrightarrow{\sim} & \mathbf{MT} \\ \uparrow & & \uparrow \\ \mathbf{G}_{\mathfrak{G}} & \xrightarrow{\sim} & \mathbf{G} \end{array}$$

commutes.

Then for any commutative \mathbb{Q} -algebra \mathbb{k} , we have a group inclusion

$$(\mathbf{G}(\mathbb{k}), \otimes) \subset (\mathbf{MT}(\mathbb{k}), \otimes) = (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes).$$

Let now $\tilde{\mathfrak{G}} := \mathfrak{G} + (\mathbb{Q}x_0 \oplus \mathbb{Q}x_1)$. As x_0 and x_1 are central in $(\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle)$, $\tilde{\mathfrak{G}}$ is a closed Lie subalgebra of $(\mathbb{Q}\langle\langle X \rangle\rangle_0, \langle, \rangle)$. Let $\tilde{\mathbf{G}}$ be the corresponding group subscheme of \mathbf{MT} .

Lemma 5.2. *For any commutative \mathbb{Q} -algebra \mathbb{k} , we have*

$$(5.4) \quad \tilde{\mathbf{G}}(\mathbb{k}) = \{e^{\beta x_1} \cdot g \cdot e^{\alpha x_0} \mid \alpha, \beta \in \mathbb{k}, g \in \mathbf{G}(\mathbb{k})\}$$

(equality of subsets of $\mathbf{MT}(\mathbb{k})$). In the right-hand side of (5.4), the product \cdot is that of $\mathbb{k}\langle\langle X \rangle\rangle^\times$ (see §4.1.3).

Proof. The algebraic group scheme \mathbf{MT} is pro-unipotent, therefore if \mathbb{k} is any commutative \mathbb{Q} -algebra, the exponential map \exp_{\otimes} sets up a bijection $\exp_{\otimes} : \widehat{\mathbf{mt}}(\mathbb{k}) \xrightarrow{\sim} \mathbf{MT}(\mathbb{k})$ between the group of \mathbb{k} -points of this group scheme and the Lie algebra $\widehat{\mathbf{mt}}(\mathbb{k})$ of \mathbb{k} -points of its Lie algebra, which is equal to $\mathbb{k}\langle\langle X \rangle\rangle_0$. One has for any $\alpha, \beta \in \mathbb{k}$,

$$(5.5) \quad \exp_{\otimes}(\alpha x_0) = \exp(\alpha x_0), \quad \exp_{\otimes}(\beta x_1) = \exp(\beta x_1)$$

and for any $g \in \mathbf{MT}(\mathbb{k})$,

$$\exp_{\otimes}(\alpha x_0) \otimes g = g \cdot \exp(\alpha x_0), \quad g \otimes \exp_{\otimes}(\beta x_1) = \exp(\beta x_1) \cdot g$$

(identities in $\mathbf{MT}(\mathbb{k}) = 1 + \mathbb{k}\langle\langle X \rangle\rangle_0$); in the right-hand sides, the products are that of $\mathbb{k}\langle\langle X \rangle\rangle$. If now $\alpha, \beta, \alpha', \beta' \in \mathbb{k}$ and $g, g' \in \mathbf{MT}(\mathbb{k})$,

$$\begin{aligned} & (e^{\beta x_1} \cdot g \cdot e^{\alpha x_0}) \otimes (e^{\beta' x_1} \cdot g' \cdot e^{\alpha' x_0}) \\ &= \exp_{\otimes}(\beta x_1) \otimes g \otimes \exp_{\otimes}(\alpha x_0) \otimes \exp_{\otimes}(\beta' x_1) \otimes g' \otimes \exp_{\otimes}(\alpha' x_0) \\ &= \exp_{\otimes}((\beta + \beta')x_1) \otimes g \otimes g' \otimes \exp_{\otimes}((\alpha + \alpha')x_0) = e^{(\beta + \beta')x_1} \cdot (g \otimes g') \cdot e^{(\alpha + \alpha')x_0}, \end{aligned}$$

where the second equality follows from the centrality of x_0, x_1 in $\widehat{\mathbf{mt}}(\mathbb{k})$.

This implies that the assignment $\mathbb{k} \mapsto$ (right-hand side of (5.4)) is a subgroup functor of $\mathbb{k} \mapsto \mathbf{MT}(\mathbb{k})$. The associated Lie algebra is $\tilde{\mathfrak{G}}$, so this functor coincides with $\mathbb{k} \mapsto \tilde{\mathbf{G}}(\mathbb{k})$. \square

Assume that there is an embedding $\iota : \Gamma \rightarrow \mathbb{C}^\times$. With this datum and $\lambda \in \mathbb{k}$ is associated in [R], Déf. 3.2.1, the set $\mathbf{DMR}_\lambda^\iota(\mathbb{k})$. When $\lambda = 0$, this set does not depend on ι and will be denoted simply $\mathbf{DMR}_0(\mathbb{k})$. One has:

Definition 5.3. (see [R], Déf. 3.2.1) *The set $\mathbf{DMR}_0(\mathbb{k})$ is the set of elements G of $\mathbb{k}\langle\langle X \rangle\rangle^\times$, such that:*

$$(G|1) = 1, \quad (G|x_0) = (G|x_1) = 0, \quad (G|x_0 x_1) = 0, \quad (G|x_\sigma - x_{\sigma-1}) = 0$$

for any $\sigma \in \Gamma$, and

$$\Delta(G) = G \otimes G, \quad \Delta_*(G_\star) = G_\star \otimes G_\star$$

(identities in $\mathbb{k}\langle\langle X \rangle\rangle^{\hat{\otimes} 2}$ and $\mathbb{k}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2}$), where

$$(5.6) \quad G_\star = \exp\left(\sum_{n \geq 2} \frac{(-1)^{n-1}}{n} (\pi_Y(G)|y_{n,1}) y_{1,1}^n\right) \cdot \mathbf{q}(\pi_Y(G))$$

(an element of $\mathbb{k}\langle\langle Y \rangle\rangle^\times$).³

Then:

Lemma 5.4. (see [R], §§3.2.3 and 3.2.8) *This assignment $\mathbb{k} \mapsto \mathbf{DMR}_0(\mathbb{k})$ corresponds to a subgroup scheme \mathbf{DMR}_0 of \mathbf{MT} , with Lie algebra \mathfrak{dmt}_0 .*

We derive from there:

Lemma 5.5. *The subgroup functor of $\mathbb{k} \mapsto \mathbf{MT}(\mathbb{k})$ corresponding to the degree completion of the Lie algebra $\widetilde{\mathfrak{dmt}}_0 := \mathfrak{dmt}_0 \oplus (\mathbb{Q}x_0 \oplus \mathbb{Q}x_1)$ is $\mathbb{k} \mapsto \widetilde{\mathbf{DMR}}_0(\mathbb{k})$, where*

$$\widetilde{\mathbf{DMR}}_0(\mathbb{k}) = \{e^{\beta x_1} \cdot g \cdot e^{\alpha x_0} \mid \alpha, \beta \in \mathbb{k}, g \in \mathbf{DMR}_0(\mathbb{k})\}$$

for \mathbb{k} any commutative \mathbb{Q} -algebra.

5.3. Topological modules over the Lie algebra $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$.

³In [R], the notation $\psi \mapsto \psi_\star$ is used to denote both the ‘additive’ map $\psi \mapsto \psi_\star$, where ψ_\star is as in (2.5) and the ‘multiplicative’ map $G \mapsto G_\star$, where G_\star is as in (5.6). We made the choice of giving two different notations to these maps in order to avoid confusions.

5.3.1. *Graded modules over $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$.* Recall from §2.1 the Lie algebras $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ and $\mathfrak{m}\mathfrak{t} = (\mathbb{Q}\langle X \rangle_0, \langle, \rangle)$, from §2.4 the morphism $\theta : (\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle) \rightarrow \mathfrak{m}\mathfrak{t}$; from Lemma 2.2 the $\mathfrak{m}\mathfrak{t}$ -module structure on $\mathbb{Q}\langle Y \rangle$ given by the map $\mathbb{Q}\langle X \rangle_0 \ni \psi \mapsto s_\psi^Y \in \text{End}(\mathbb{Q}\langle Y \rangle)$, from §2.5 the induced $\mathfrak{m}\mathfrak{t}$ -module structure on $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$, as well as the pull-back $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ -module structure on the same space induced by θ . All the objects in this construction are \mathbb{Z} -graded: the gradings on $\mathfrak{L}\mathfrak{ib}(X)$ and $\mathbb{Q}\langle X \rangle_0$ are defined by $\deg(x_\gamma) = 1$ for $\gamma \in \Gamma \cup \{0\}$, and the grading on $\mathbb{Q}\langle Y \rangle$ is defined by $\deg(y_{n,\gamma}) = n$ for $n \geq 1, \gamma \in \Gamma \cup \{0\}$. The nonzero homogeneous components of $\mathfrak{L}\mathfrak{ib}(X)$ (resp., of $\mathfrak{m}\mathfrak{t}$, of $\mathbb{Q}\langle Y \rangle$) correspond to degrees in $\mathbb{Z}_{>0}$ (resp., $\mathbb{Z}_{>0}, \mathbb{Z}_{\geq 0}$), while $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$ has nonzero components of all degrees in \mathbb{Z} .

The element Δ_* of $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$ is of degree 0. Let $U(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ be the universal enveloping algebra of $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ and let

$$(5.7) \quad M := U(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle) \cdot \Delta_*.$$

Then M is a graded sub- $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ -module of $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$; it has nonzero components only in degrees ≥ 0 .

5.3.2. *Topological modules over $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$.* In §4.1.3, we introduced the degree completion $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$ of $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$, and in §4.1.4, the degree completion $\widehat{\mathfrak{m}\mathfrak{t}}$ of $\mathfrak{m}\mathfrak{t}$. Being completions of positively graded Lie algebras, both Lie algebras are pronilpotent. The morphism θ from §2.4 extends to a continuous morphism

$$\hat{\theta} : (\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle) \rightarrow \widehat{\mathfrak{m}\mathfrak{t}}.$$

Let $\mathbb{Q}\langle\langle Y \rangle\rangle$ be the degree completion of $\mathbb{Q}\langle Y \rangle$. Then $\mathbb{Q}\langle\langle Y \rangle\rangle$ is a topological module over $\widehat{\mathfrak{m}\mathfrak{t}}$. Let $\mathbb{Q}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2}$ be the degree completion of $\mathbb{Q}\langle\langle Y \rangle\rangle^{\otimes 2}$ and let $\text{Hom}_{\mathbb{Q}}^{\text{cont}}(\mathbb{Q}\langle\langle Y \rangle\rangle, \mathbb{Q}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2})$ be the set of continuous \mathbb{Q} -linear maps $\mathbb{Q}\langle\langle Y \rangle\rangle \rightarrow \mathbb{Q}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2}$. Using the notation $V[i]$ for the homogeneous component of degree i of a \mathbb{Z} -graded \mathbb{Q} -vector space V , one identifies this set with the subspace of $\prod_{i,j \geq 0} \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle[i], \mathbb{Q}\langle Y \rangle^{\otimes 2}[j])$ of all collections $\underline{\varphi} = (\varphi_{ij})_{i,j \geq 0}$, satisfying:

$$(5.8)$$

there exists a map $F_{\underline{\varphi}} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ with $\lim_{n \rightarrow +\infty} F_{\underline{\varphi}}(n) = +\infty$, such that $\varphi_{ij} = 0$ if $j < F_{\underline{\varphi}}(i)$.

The inclusion of $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$ in this space may then be described as follows: this is the subspace of all collections $(\varphi_{ij})_{i,j \geq 0}$, such that there exists an integer $N \geq 0$, such that $\varphi_{ij} = 0$ for $|i - j| > N$.

The space $\text{Hom}_{\mathbb{Q}}^{\text{cont}}(\mathbb{Q}\langle\langle Y \rangle\rangle, \mathbb{Q}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2})$ is then equipped with the structure of a topological $\widehat{\mathfrak{m}\mathfrak{t}}$ -module, which can be pulled back to a topological $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$ -module structure through $\hat{\theta}$.

Set $\hat{M} := \prod_{i \geq 0} M[i]$, where M is the $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ -module defined in (5.7) and the $M[i]$'s are its homogeneous components. Then \hat{M} is a topological $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$ -module.

The canonical inclusion $M \subset \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$ extends to an inclusion

$$\hat{M} \subset \text{Hom}_{\mathbb{Q}}^{\text{cont}}(\mathbb{Q}\langle\langle Y \rangle\rangle, \mathbb{Q}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2})$$

which is compatible with the topological module structures of both sides over $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$.

Note that \hat{M} is an object of the category $\mathcal{C}_{(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)}$ associated to the pronilpotent Lie algebra $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ in §5.1.1.

5.4. Identification of $\mathbb{k} \mapsto \widetilde{\text{DMR}}_0(\mathbb{k})$ with a stabilizer group functor. Recall from §1.1 the inclusion of graded Lie algebras $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle) \subset \mathfrak{mt}$, and from Lemma 5.5 and §1.4 the definitions of the graded Lie subalgebras $\widetilde{\text{dmr}}_0$ and $\mathfrak{stab}(\Delta_*)$ in $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$.

In this section, we introduce the following notation $\mathfrak{stab}(v, V) := \{x \in \mathfrak{g} \mid x \cdot v = 0\}$ for any collection of a graded Lie algebra \mathfrak{g} , a graded \mathfrak{g} -module V , and a homogeneous element v in V . Then $\mathfrak{stab}(\Delta_*) = \mathfrak{stab}(\Delta_*, \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2}))$, the underlying Lie algebra being $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$.

According to Thm. 3.10, we have $\mathfrak{stab}(\Delta_*, \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})) = \widetilde{\text{dmr}}_0$.

Recall from §5.3.1 the $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$ -submodule $M \subset \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})$. As $\Delta_* \in M$, one has $\mathfrak{stab}(\Delta_*, \text{Hom}_{\mathbb{Q}}(\mathbb{Q}\langle Y \rangle, \mathbb{Q}\langle Y \rangle^{\otimes 2})) = \mathfrak{stab}(\Delta_*, M)$, therefore

$$(5.9) \quad \mathfrak{stab}(\Delta_*, M) = \widetilde{\text{dmr}}_0$$

In §4.1.3, we introduced the degree completion $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$ of $(\mathfrak{L}\mathfrak{ib}(X), \langle, \rangle)$, and in §5.3.2, the degree completion \hat{M} of M , which is a topological module over $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$. According to §5.1.2, one attaches to these data and to the element $\Delta_* \in \hat{M}$ the pronilpotent Lie algebra $\widehat{\mathfrak{stab}}(\Delta_*)$, which is a closed Lie subalgebra of $(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)$. It follows from the fact that M is graded and Δ_* is homogeneous that this Lie algebra coincides with the degree completion of $\mathfrak{stab}(\Delta_*, M)$.

Taking degree completions in (5.9), we obtain

$$(5.10) \quad \widehat{\mathfrak{stab}}(\Delta_*) = \widetilde{\text{dmr}}_0^{\wedge} \quad (\text{equality of closed Lie subalgebras of } (\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)),$$

where $\widetilde{\text{dmr}}_0^{\wedge}$ is the degree completion of $\widetilde{\text{dmr}}_0$. We have therefore an equality of pronilpotent Lie algebras $\widetilde{\text{dmr}}_0^{\wedge} = \widehat{\mathfrak{stab}}(\Delta_*) \hookrightarrow (\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle) \hookrightarrow \widehat{\mathfrak{mt}}$. Applying the functor $\mathfrak{n} \mapsto \mathbf{G}_{\mathfrak{n}}$ of §4.1.2 and specializing to a commutative \mathbb{Q} -algebra \mathbb{k} , one obtains the equality of groups

$$(5.11) \quad \mathbf{G}_{\widetilde{\text{dmr}}_0^{\wedge}}(\mathbb{k}) = \mathbf{G}_{\widehat{\mathfrak{stab}}(\Delta_*)}(\mathbb{k}) \hookrightarrow \mathbf{G}_{(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)}(\mathbb{k}) \hookrightarrow \mathbf{G}_{\widehat{\mathfrak{mt}}}(\mathbb{k}).$$

According to Lemma 5.1 and to (5.1), one has

$$(5.12) \quad \mathbf{G}_{\widehat{\mathfrak{stab}}(\Delta_*)}(\mathbb{k}) = \{g \in \mathbf{G}_{(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)}(\mathbb{k}) \mid g \cdot \Delta_* = \Delta_* \quad (\text{equality in } \hat{M} \hat{\otimes} \mathbb{k})\}.$$

According to §4, there is a group isomorphism $\mathbf{G}_{\widehat{\mathfrak{mt}}}(\mathbb{k}) \simeq \text{MT}(\mathbb{k}) = (\mathbb{k}\langle\langle X \rangle\rangle_1^{\times}, \otimes)$. This group isomorphism takes $\mathbf{G}_{(\hat{\mathfrak{L}}\mathfrak{ib}(X), \langle, \rangle)}(\mathbb{k})$ to $(\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)$ and $\mathbf{G}_{\widetilde{\text{dmr}}_0^{\wedge}}(\mathbb{k})$ to $(\widetilde{\text{DMR}}_0(\mathbb{k}), \otimes)$ (see Def. 5.3 and Lemma 5.5).

On the other hand, (5.12) implies that the image of $\mathbf{G}_{\widehat{\mathfrak{stab}}(\Delta_*)}(\mathbb{k})$ is

$$(5.13) \quad \{g \in (\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes) \mid g \cdot \Delta_* = \Delta_* \quad (\text{equality in } \hat{M} \hat{\otimes} \mathbb{k})\}.$$

The space $\mathbb{k}\langle\langle Y \rangle\rangle$ is a topological module over $(\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)$. This leads to a $(\exp(\hat{\mathfrak{L}}\mathfrak{ib}_{\mathbb{k}}(X)), \otimes)$ -module structure over $\text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2})$. This space can be identified with the subspace of $\prod_{i, j \geq 0} \text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle Y \rangle[i], \mathbb{k}\langle Y \rangle^{\otimes 2}[j])$ defined by conditions (5.8). Then there is a module inclusion

$$\hat{M} \hat{\otimes} \mathbb{k} \subset \text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\hat{\otimes} 2}).$$

It follows that the module $\widehat{M} \widehat{\otimes} \mathbb{k}$ may be replaced by $\text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\widehat{\otimes} 2})$ in the description of the image of $\widehat{\mathbf{G}}_{\text{stab}(\Delta_*)}(\mathbb{k})$ given in (5.13).

The image of the equality in diagram (5.11) under the isomorphism $\widehat{\mathbf{G}}_{\widehat{\mathfrak{m}\mathfrak{t}}}(\mathbb{k}) \simeq (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$ is then the equality

$$(5.14) \quad \widehat{\text{DMR}}_0(\mathbb{k}) = \{g \in (\exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X)), \otimes) \mid g \cdot \Delta_* = \Delta_* \quad (\text{equality in } \text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\widehat{\otimes} 2}))\}.$$

According to (4.2), the map

$$(5.15) \quad \mathbb{k}\langle\langle X \rangle\rangle_1^\times \times \mathbb{k}\langle\langle X \rangle\rangle \rightarrow \mathbb{k}\langle\langle X \rangle\rangle, \quad (g, H) \mapsto g \otimes H =: S_g(H)$$

defines on $\mathbb{k}\langle\langle X \rangle\rangle$ a structure of $(\mathbb{k}\langle\langle X \rangle\rangle_1^\times \times, \otimes)$ -module. The induced $\widehat{\mathfrak{m}\mathfrak{t}}$ -module structure over $\mathbb{k}\langle\langle X \rangle\rangle$ is obtained from the $\mathfrak{m}\mathfrak{t}$ -module structure on $\mathbb{Q}\langle X \rangle$ denoted $\psi \mapsto s_\psi$ in §2.3 by tensor product with \mathbb{k} and completion. This implies that (5.15) is the module functor associated to $\psi \mapsto s_\psi$ according to the procedure of §5.1.1. As in §2.3, one shows that for $g \in \mathbb{k}\langle\langle X \rangle\rangle_1^\times$, there exists a unique automorphism S_g^Y of $\mathbb{k}\langle\langle Y \rangle\rangle$, such that the diagram

$$\begin{array}{ccc} \mathbb{k}\langle\langle X \rangle\rangle & \xrightarrow{S_g} & \mathbb{k}\langle\langle X \rangle\rangle \\ \mathfrak{q} \circ \pi_Y \downarrow & & \downarrow \mathfrak{q} \circ \pi_Y \\ \mathbb{k}\langle\langle Y \rangle\rangle & \xrightarrow{S_g^Y} & \mathbb{k}\langle\langle Y \rangle\rangle \end{array}$$

commutes. The map

$$(5.16) \quad \mathbb{k}\langle\langle X \rangle\rangle_1^\times \times \mathbb{k}\langle\langle Y \rangle\rangle \rightarrow \mathbb{k}\langle\langle Y \rangle\rangle, \quad (g, K) \mapsto S_g^Y(K)$$

defines on $\mathbb{k}\langle\langle Y \rangle\rangle$ a structure of $(\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$ -module, and gives rise to the module functor associated to $\psi \mapsto s_\psi^Y$. The map S_g^Y may be described explicitly as follows

$$S_g^Y = \mathfrak{q} \circ \tilde{S}_g^Y \circ \mathfrak{q}^{-1},$$

where $\tilde{S}_g^Y \in \text{End}_{\mathbb{k}}(\mathbb{k}\langle\langle Y \rangle\rangle)$ is defined by

$$\tilde{S}_g^Y(K) := \pi_Y(g \otimes \tilde{K})$$

for $K = K((y_{n,\sigma})_{n \geq 1, \sigma \in \Gamma}) \in \mathbb{k}\langle\langle Y \rangle\rangle$ and where $\tilde{K} := K((x_0^{n-1} x_\sigma)_{n \geq 1, \sigma \in \Gamma}) \in \mathbb{k}\langle\langle X \rangle\rangle$.

According to Prop. 4.13, the map $\Theta_{\mathbb{k}}$ given by (4.9) defines a group morphism $(\exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X)), \otimes) \rightarrow (\mathbb{k}\langle\langle X \rangle\rangle_1^\times, \otimes)$. It follows that the map

$$(\exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X)), \otimes) \times \mathbb{k}\langle\langle Y \rangle\rangle \rightarrow \mathbb{k}\langle\langle Y \rangle\rangle, \quad (h, K) \mapsto S_{\Theta_{\mathbb{k}}(h)}^Y(K)$$

defines on $\mathbb{k}\langle\langle Y \rangle\rangle$ the structure of a $(\exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X)), \otimes)$ -module, corresponding to the $(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}(X), \langle, \rangle)$ -module structure on $\mathbb{Q}\langle Y \rangle$ given by $\psi \mapsto s_{\theta(\psi)}^Y$.

The action of $(\exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X)), \otimes)$ on $\text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\widehat{\otimes} 2})$ is then given by $g \cdot u := (S_{\Theta_{\mathbb{k}}(g)}^Y)^{\otimes 2} \circ u \circ (S_{\Theta_{\mathbb{k}}(g)}^Y)^{-1}$. Therefore the right hand side of (5.14) is equal to

$$\{g \in (\exp(\widehat{\mathfrak{L}\mathfrak{i}\mathfrak{b}}_{\mathbb{k}}(X)), \otimes) \mid (S_{\Theta_{\mathbb{k}}(g)}^Y)^{\otimes 2} \circ \Delta_* = \Delta_* \circ S_{\Theta_{\mathbb{k}}(g)}^Y \quad (\text{equality in } \text{Hom}_{\mathbb{k}}^{\text{cont}}(\mathbb{k}\langle\langle Y \rangle\rangle, \mathbb{k}\langle\langle Y \rangle\rangle^{\widehat{\otimes} 2}))\}.$$

Combining this with (5.14), we get the statement of Thm. 1.2.

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