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New smoothing methods for solving the linear complementarity problem with $\mathcal{P}_0$-matrix

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Based on smoothing techniques, we propose two new methods to solve linear complementarity problems (LCP) called TLCP and Soft-LCP. The idea of these two new methods takes inspiration from interior-point methods in optimization. The technique that we propose avoids any parameter management while ensuring good theoretical convergence results. In our approach, we do not need any complicated strategy to update the smoothing parameter r since we will consider it as a new variable. Our methods are validated by extensive numerical tests, in which we compare our methods to several other classical methods.

Keywords: Linear complementarity problem; Newton's method; smoothing functions; $\mathcal{P}_0$-matrix; Interior-Point Method, Soft-Max.

AMS Subject Classification: 47H05; 90C33.

1. Introduction

The linear complementarity problem consists in finding a vector in a finite-dimensional real vector space that satisfies a certain system of inequalities. Specifically, given a vector $q \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$, the linear complementarity problem, abbreviated LCP, is to find a vector $x \in \mathbb{R}^n$ such that

$$0 \leq x \perp (Mx + q) \geq 0. \tag{1}$$

The linear complementarity problem has many important applications in engineering and equilibrium modeling [11, 30], and many numerical methods are developed to solve LCPs [2, 4]. Although the effectiveness of complementarity algorithms has improved substantially in recent years, the fact remains that increasingly more difficult problems are being proposed that are exceeding the capabilities of these algorithms. As a result, there is a real need to propose new methods and algorithms to address complicated and difficult situations. To solve LCP, there are essentially three different classes of methods: equation-based methods (smoothing), merit functions, and projection-type methods. Our goal in this paper is to present new and very simple smoothing and approximation schemes to solve LCP and to produce efficient numerical methods.

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Many algorithms have been proposed to solve problem LCP [7, 29]. They may be based on pivoting techniques [8, 20], which often suffer from the combinatorial aspect of the problem, on interior point methods, which originate from an algorithm introduced by Karmarkar in linear optimization [17], see also [19] for one of the first accounts on the use of interior-point methods to solve LCP. Some researchers try to solve LCPs by reformulating them as an unconstrained optimization [14], and on nonsmooth Newton approaches [10], and rewrite the complementarity conditions as a system of smooth equations [21], such as the one considered here. See [7, 32] for other iterative methods.
In this work, we propose two new algorithms called TLCP and Soft-LCP for solving the LCP. The principle of these algorithms are as follows: first, we proposed two smoothing techniques to regularize the complementary condition, we replace

$$0 \leq x \perp z \geq 0$$

by

$$\theta_r(x) + \theta_r(z) = 1, \quad r \searrow 0,$$

and

$$\forall \rho > 0, \qquad x = \max(0, x - \rho z) \simeq r \log\left(1 + e^{\frac{x-\rho z}{r}}\right), \quad r \searrow 0,$$

where θ_r, log, and $e^{\cdot}$ operate componentwise on x and z; then we give a strategy that decreases r during iterations and ensures the nonnegatives of variables. The main difference in our approach is that we do not need any complicated strategy to update the parameter r since we will consider it as a new variable. Finally, the two new algorithms are solved using the standard Newton method. To enforce a global convergence behavior, we also recommend using Armijo's line search.
This article is structured as follows. In section 2 of this paper we gives some definitions and properties of the smoothing functions. In section 3, we present our two approximation for the problem LCP and give the new formulation of the problem LCP. In section 4, we propose two new methods to solve the LCP. In section 5, we propose two generic algorithms to solve LCP and prove some convergence results. In section 6, we provide some numerical results where we present a comparison on some randomly generated problems of our two methods with other approaches that have been suggested recently in [5, 9] and we study two concrete examples, the first one is a second-order ordinary differential equation and the second is an obstacle problem also, we tested our algorithms on several absolute value equations problems. Finally, we conclude our paper.

2. Preliminaries and Problem Setting

In this section, we present some necessary definitions and lemmas.
A matrix $M \in \mathbb{R}^{n\times n}$ is said to be positive definite if $\langle x, Mx\rangle > 0$ for all nonzero $x \in \mathbb{R}^n$. $M \in \mathbb{R}^{n\times n}$ is called a $\mathcal{P}$-matrix if all its minors are positive. As a consequence, if M is positive definite, then M is a $\mathcal{P}$-matrix.
A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_0-$matrix if every of its principal minors is nonnegative.

First, we state a result for the unique solution of an LCP, the following result was proved by Cottle, Pang, and Stone [7, 31]. Next, we give the definition of θ-smoothing function and Soft-Max function that will use to approximate the complementarity condition.

THEOREM 2.1 *(Theorem 3.3.7, [7]). A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}$-matrix if and only if the LCP* (1) *has a unique solution for every $q \in \mathbb{R}^n$.*

2.1 *Definition of θ-smoothing function*

We introduce the function θ with the following properties (these functions were used in [15, 16]).
Let $\theta : \mathbb{R} \to]-\infty, 1[$, be a non-decreasing continuous smooth concave function such that

$$\theta(t) < 0 \ \text{ if } \ t < 0, \ \theta(0) = 0 \ \text{ and } \ \lim_{t\to+\infty} \theta(t) = 1.$$

One possible way to build such function is to consider non-increasing probability density functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ and then take the corresponding cumulative distribution function

$$\theta(t) = \int_0^t f(x)dx.$$

By definition of f we can verify that

$$\lim_{t\to+\infty} \theta(t) = \int_0^{+\infty} f(x)dx = 1,$$

and

$$\theta(0) = \int_0^0 f(x)dx = 0.$$

The non-decreasing hypothesis gives the concavity of θ. We then extend this functions for negative values in a smooth way.
Example of this family are $\theta^1(t) = t/(t+1)$ if $t \geq 0$ and $\theta^1(t) = t$ if $t < 0$.
We introduce $\theta_r(t) := \theta(\frac{t}{r})$ for $r > 0$. This definition is similar to the perspective functions in convex analysis. This functions satisfy

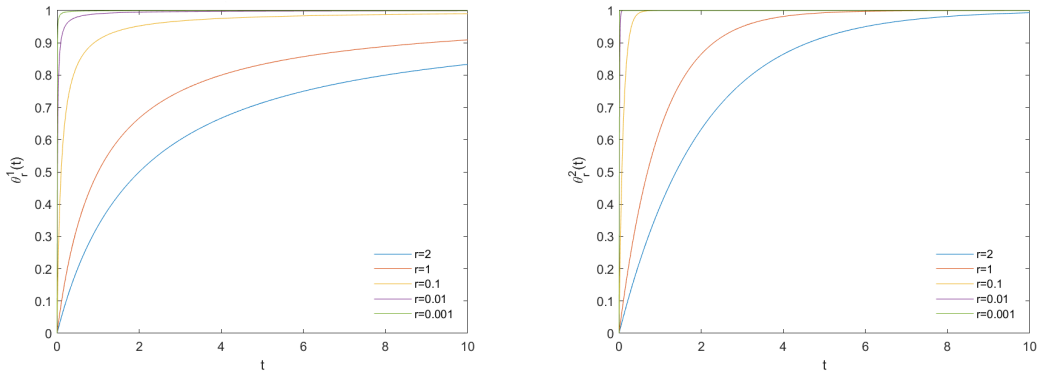
$$\theta_r(0) = 0 \ \ \forall r > 0 \text{ and } \lim_{r\searrow 0} \theta(t) = 1 \ \ \forall t > 0.$$

There are some examples of such functions

$$\theta_r^1(t) = \frac{t}{t+r} \text{ if } t \geq 0 \quad \text{ and } \quad \theta_r^1(t) = \frac{t}{r} \text{ if } t < 0,$$

$$\theta_r^2(t) = 1 - e^{-t/r}, \ t \in \mathbb{R}.$$

The function θ_r^1 will be extensively used in this paper and is illustrated in Figure. 1 for several values of r.



(a) Function θ_r^1. (b) Function θ_r^2.

Figure 1. Function θ_r for several values of r.

It can be seen on Figure 1 that the function θ_r behave as a step function when r becomes small.

2.1.1 θ-smoothing of a complementarity condition

A θ-smoothing function paves the way for a smooth approximation of a complementarity condition. Let $(x, z) \in \mathbb{R}^2$ be two scalars such that

$$0 \leq x \perp z \geq 0, \tag{2}$$

that is,

$$x \geq 0, \quad z \geq 0, \quad xz = 0.$$

In the (x, z)-plane, the set of points obeying (2) is the union of the two semi-axes $\{x \geq 0,\ z = 0\}$ and
$\{x = 0,\ z \geq 0\}$. Visually, the nonsmoothness of (2) is manifested by the "kink" at the corner $(x, z) = (0, 0)$.
We consider two possible smooth approximations of (2), depending how it is rewritten in terms of θ-function.

LEMMA 2.2 *[16] Given $x, z \in \mathbb{R}_+$ and the parameter $r > 0$, we have the equivalence*

$$xz = 0 \iff \lim_{r \searrow 0} (\theta_r(x) + \theta_r(z)) \leq 1.$$

LEMMA 2.3 *[16] θ_r is sub-additive for non-negative values, i.e. given $x,\ z \geq 0$ it holds that*

$$\theta_r(x) + \theta_r(z) \ \geq \ \theta_r(x + z),$$

and with equality if and only if $x = 0$ or $z = 0$,

$$xz = 0 \iff \theta_r(x) + \theta_r(z) = \theta_r(x + z).$$

Now, let's consder the following equation on the one-dimensional case. Let $x, z \in \mathbb{R}_+$ be such that

$$\theta_r(x) + \theta_r(z) = 1. \tag{3}$$

For instance, let's take θ_r^1. The equality (3) is then equivalent to

$$xz = r^2.$$

So, when r goes to 0, we simply get $xz = 0$.
Our objective is to approximate the complementarity constraints by using these theta functions then we will present the max function which will be the basic idea of our second approximation.

2.2 *Soft-Max Function*

Let f be a function defined as:

$$f(x_1, ..., x_n) = \max(x_1, ..., x_n),$$

obviously, the max function is non-differentiable. We approximate the max function by a smooth function, noted Soft-Max function as introduced in [6]:

$$\forall r > 0, \qquad f_r(x_1, ..., x_n) = r \log \left(\sum_{i=1}^{n} e^{\, x_i/r} \right).$$

Indeed: $\forall r > 0$ and $\forall x \in \mathbb{R}^n$,

$$r \log \left(\sum_{i=1}^{n} e^{\, x_i/r} \right) \leq r \log \left(n \max_i e^{\, x_i/r} \right) = \max_i x_i + r \log n$$

and

$$\max_i x_i \leq r \log \left(\sum_{i=1}^{n} e^{\, x_i/r} \right) \leq r \log \left(\sum_{i=1}^{n} e^{\, x_i/r} \right) + r \log n.$$

Then

$$|\max_i x_i - f_r(x)| \leq r \log n, \quad \forall\ i = 1, ..., n.$$

Thus f_r is a uniformly smoothing approximation function of f. Notice that the accuracy of the Soft-Max approximation depends on scale r.

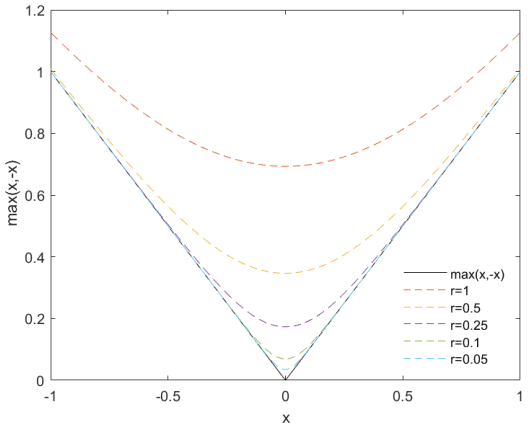


Figure 2. Smoothing by Soft-Max function.

3. An approximate formulation

In this section, we present our two formulations for LCP (1), the first with the θ-function and the second with the Soft-Max function.
Consider the linear complementarity problem LCP, which is to find a solution of the system $F(\mathbf{X}) = 0$, with

$$F(\mathbf{X}) = \begin{bmatrix} Mx + q - z \\ x.z \end{bmatrix}, \tag{4}$$

where $\mathbf{X} = (x,\ z) \in \mathbb{R}^{2n}_{+}$. Recall that the Hadamard product $x.z$ of two vectors x and z is the vector having its ith component equal to $x_i z_i$.

3.1 *Approximation of LCP using θ-function*

We reformulate the problem LCP using θ_r function, we regularize each complementarity constraint by considering

$$x_i z_i = 0, \quad \text{by} \quad \theta_r(x_i) + \theta_r(z_i) = 1, \qquad \forall\ i = 1, ...n, \quad r \searrow 0.$$

in fact $x_i z_i = 0$ should be approximated by

$$\theta_r(x_i) + \theta_r(z_i) \leq 1, \qquad \text{(both can be zeros)}$$

but we use an implicit assumption of strict complementarity. Using this approximation we obtain the following formulation:

$$(P^r_\theta) \qquad \begin{cases} Mx + q = z, \\ x \geq 0, \quad z \geq 0, \quad r \searrow 0 \\ \theta_r(x) + \theta_r(z) - \mathbf{1} = 0. \end{cases} \tag{5}$$

Here, it is understood that θ_r operates componentwise on x and z, while $\mathbf{1} \in \mathbb{R}^n$ is the vector whose entries are all equal to 1. We consider the family $\{F_\theta^r(.),\ r > 0\}$, where

$$F_\theta^r(\mathbf{X}) = \begin{bmatrix} Mx + q - z \\ r(\theta_r(x) + \theta_r(z) - \mathbf{1}) \end{bmatrix}, \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} x \\ z \end{bmatrix}. \tag{6}$$

is a regularized function of F defined in (4). It is highly recommended that the smoothed complementarity equations in (6) be premultiplied by r, so as to control the magnitude of their partial derivatives.
Indeed, for all $t \geq 0$,

$$\theta_r^{'}(t) = \frac{1}{r}\theta^{'}\left(\frac{t}{r}\right),$$

can be seen to blow up when $r \searrow 0$, while $r\theta_r^{'}(t)$ tends to the finite limit $\theta^{'}(0)$.

3.2 *Approximation of LCP using Soft-Max function*

It is obvious that the vectors x and z satisfy complementarity condition if and only if

$$\forall \rho > 0, \qquad\qquad x = \max(0, x - \rho z).$$

Using the Soft-Max function defined below, we approximate the max function by $r \log\left(1 + e^{\frac{x-\rho z}{r}}\right)$ we obtain

$$(P_s^r) \qquad \begin{cases} Mx + q = z, \quad r \searrow 0, \quad \rho > 0 \\ x = r \log\left(1 + e^{\frac{x-\rho z}{r}}\right), \end{cases} \tag{7}$$

by the same way as for (6), log and $e^{\cdot}$ operate componentwise on x and z. We consider the family $\{F_s^r(.),\ r > 0\}$, where

$$F_s^r(\mathbf{X}) = \begin{bmatrix} Mx + q - z \\ x - r \log\left(1 + e^{\frac{x-\rho z}{r}}\right) \end{bmatrix}, \text{ and } \quad \mathbf{X} = \begin{bmatrix} x \\ z \end{bmatrix}, \tag{8}$$

is a regularized function of F defined in (4).

LEMMA 3.1 *Let $F_s^r(\mathbf{X})$ define by (8). Then, the Jacobian matrix of $F_s^r(\mathbf{X})$ is*

$$\nabla F_s^r(\mathbf{X}) = \begin{pmatrix} M & -I \\ D_a(\mathbf{X}) & D_b(\mathbf{X}) \end{pmatrix}$$

where $D_a(\mathbf{X}) = diag\{a_1(\mathbf{X}), ..., a_n(\mathbf{X})\}$ and $D_b(\mathbf{X}) = diag\{b_1(\mathbf{X}), ..., b_n(\mathbf{X})\}$ are two

diagonal matrices, and

$$a_i(\mathbf{X}) = \frac{1}{1+e^{\frac{x_i-\rho z_i}{r}}}, \qquad b_i(\mathbf{X}) = \frac{\rho e^{\frac{x_i-\rho z_i}{r}}}{1+e^{\frac{x_i-\rho z_i}{r}}}, \qquad i = 1, ...n.$$

Let $F_\theta^r(\mathbf{X})$ define by (6)*. Then, the Jacobian matrix of $F_\theta^r(\mathbf{X})$ is*

$$\nabla F_s^r(\mathbf{X}) = \begin{pmatrix} M & -I \\ Q_k(\mathbf{X}) & Q_l(\mathbf{X}) \end{pmatrix}$$

where $Q_k(\mathbf{X}) = diag\{k_1(\mathbf{X}), ..., k_n(\mathbf{X})\}$ and $Q_l(\mathbf{X}) = diag\{l_1(\mathbf{X}), ..., l_n(\mathbf{X})\}$ are two diagonal matrices, and

$$k_i(\mathbf{X}) = \frac{r^2}{(x_i+r)^2}, \qquad l_i(\mathbf{X}) = \frac{r^2}{(z_i+r)^2}, \qquad i = 1, ...n.$$

LEMMA 3.2 *Let $M \in \mathbb{R}^{n\times n}$ be a $\mathcal{P}_0-$matrix. Then any matrix in the following form is nonsingular:*

$$N_s + N_t M,$$

where $N_s \in \mathbb{R}^{n\times n}$ is a positive (negative) diagonal matrix, and $N_t \in \mathbb{R}^{n\times n}$ is a nonnegative (non-positive) diagonal matrix.

Proof. Let $N_s = \text{diag}(s_1, s_2, ..., s_n)$ and $N_t = \text{diag}(t_1, t_2, ..., t_n)$. If N_s is positive, and N_t is nonnegative, then $s_i > 0$ and $t_i \geq 0$ for all $i = 1, 2, ..., n$.
Let $v \in \mathbb{R}^n$ be a vector such that $(N_s + N_t M)v = 0$. Then, we have $v_i = -\frac{t_i}{s_i}(Mv)_i$.
It yields $v_i^2 = -\frac{t_i}{s_i} v_i (Mv)_i$. If $t_i = 0$, then $v_i = 0, \quad \forall i = 1, ..., n$.
If $v_i \neq 0$, we have $\frac{t_i}{s_i} > 0$. Owing to $v_i^2 \geq 0$, we have $v_i(Mv)_i \leq 0$. If $v_i(Mv)_i = 0$, then $v_i = 0$. Otherwise, $v_i(Mv)_i < 0$ contradicts the property of M. Based on the above discussion, it is concluded that $v = 0$, then $N_s + N_t M$ is a nonsingular matrix. ■

By Lemma 3.2, we can obtain a property of F_s^r and F_θ^r if M is a $\mathcal{P}_0$-matrix.

THEOREM 3.3 *Let M be a $\mathcal{P}_0$-matrix. Then, for $r > 0$, the Jacobian matrix $\nabla F_s^r(\mathbf{X})$ (resp. $\nabla F_\theta^r(\mathbf{X})$) is nonsingular.*

Proof. For $r > 0$ and from Lemma 3.1, it follows that the diagonal matrix $D_a(\mathbf{X})$ (resp. $Q_k(\mathbf{X})$) is non-negative, and $D_b(\mathbf{X})$ (resp. $Q_l(\mathbf{X})$) is non-negative diagonal matrix.
We have $\det(\nabla F_s^r(\mathbf{X})) = \det(D_a(\mathbf{X}) + MD_b(\mathbf{X}))$ (resp. $\det(\nabla F_\theta^r(\mathbf{X})) = \det(Q_k(\mathbf{X}) + MQ_l(\mathbf{X}))$), since M is a $\mathcal{P}_0$-matrix and from Lemma 3.2, it follows that $D_a(\mathbf{X}) + MD_b(\mathbf{X})$ (resp. $Q_k(\mathbf{X}) + MQ_l(\mathbf{X})$) is nonsingular. Hence $\nabla F_s^r(\mathbf{X})$ (resp. $\nabla F_\theta^r(\mathbf{X})$ is nonsingular. ■

4. Solving LCP via New Algorithm

In this section, we present the idea of our algorithms for optimization problems to solve the LCP, but here we don't have any objective function to minimize. Our methods take inspiration from Interior Point Methods.
We recall that the Interior Point Methods have replaced the original nonsmooth problem LCP by a sequence of regularized problems

$$F_r(\mathbf{X}) = 0, \tag{9}$$

where

$$\mathbf{X} = \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{2n}_+, \quad F_r(\mathbf{X}) = \begin{bmatrix} Mx + q - z \\ x.z - r\mathbf{1} \end{bmatrix}, \tag{10}$$

where $r \geq 0$ is the smoothing parameter. The Jacobian matrix of F_r with respect to $\mathbf{X}$, does not depend on r and can be denoted by

$$\nabla_{\mathbf{X}} F_r(\mathbf{X}) = \begin{pmatrix} M & -I \\ Z & X \end{pmatrix}, \tag{11}$$

where $Z = \mathrm{diag}(z)$ and $X = \mathrm{diag}(x)$, i.e. the diagonal matrix of z (resp. x).

4.1 *When the parameter becomes a variable*

In the system (9), the status of the parameter r is very distinct from that of the variable $\mathbf{X}$. While $\mathbf{X}$ is computed "automatically" by a Newton iteration, r has to be updated "manually" in an ad-hoc manner.
Our goal is to find a strategy that decreases r during iterations and ensures the nonnegative of variables. However, we must adjust the strategy when the model or its parameters are changed. To avoid this trouble, we consider r as an unknown of the system instead of a parameter.
We feel that it would be judicious to incorporate the parameter r into the variables. Let us therefore consider the enlarged vector of unknowns

$$\mathbb{X} = \begin{bmatrix} \mathbf{X} \\ r \end{bmatrix} \in \mathbb{R}^{2n} \times \mathbb{R}_+, \tag{12}$$

and then consider a system of $2n + 1$ equations

$$\mathbb{F}_\theta(\mathbb{X}) = 0, \quad (\text{resp. } \mathbb{F}_s(\mathbb{X}) = 0), \tag{13}$$

to be on $\mathbb{X}$. To this end, let us remind ourselves that our ultimate goal is to solve $F^0_\theta(\mathbf{X}) = 0$ (resp. $F^0_s(\mathbf{X}) = 0$), together with the inequalities $x \geq 0,\ z \geq 0$. Thus, it is really natural to first consider

$$\mathbb{F}_\theta(\mathbb{X}) = \begin{bmatrix} Mx + q - z \\ r(\theta^1_r(x) + \theta^1_r(z) - \mathbf{1}) \\ r \end{bmatrix}, \tag{14}$$

and

$$\mathbb{F}_s(\mathbb{X}) = \begin{bmatrix} Mx + q - z \\ x - r\log\left(1 + e^{\frac{x-\rho z}{r}}\right) \\ r \end{bmatrix}. \tag{15}$$

This construction turns out to be to naive. Indeed, if we start from some r^0 and solve the smooth system (14) and (15) by the smooth Newton method, since the last equation is linear, we end up with $r^1 = 0$ at the first iteration. Once the boundary of the interior region is reached, we are "stuck" there.
To prevent r from rushing to zero in just one iteration, we could set

$$\mathbb{F}_\theta(\mathbb{X}) = \begin{bmatrix} Mx + q - z \\ r(\theta_r^1(x) + \theta_r^1(z) - \mathbf{1}) \\ r^2 \end{bmatrix}, \tag{16}$$

we restrict our choice of θ-function to $\theta_r^1(x) = \frac{x}{x+r}$, and

$$\mathbb{F}_s(\mathbb{X}) = \begin{bmatrix} Mx + q - z \\ x - r\log\left(1 + e^{\frac{x-\rho z}{r}}\right) \\ r^2 \end{bmatrix}, \tag{17}$$

At this stage, system (16) (resp. (17)) is not yet fully adequate. Indeed, the last equation is totally decoupled from the others. Everything happens as if r follows a prefixed sequence, generated by the Newton iterates of the scalar equation $r^2 = 0$, regardless of $\mathbf{X}$. It is desirable to couple r and $\mathbf{X}$ in a tighter way. In this respect, we advocate

$$\mathbb{F}_\theta(\mathbb{X}) = \begin{bmatrix} Mx + q - z \\ r(\theta_r^1(x) + \theta_r^1(z) - \mathbf{1}) \\ \frac{1}{2}\|x^-\|^2 + \frac{1}{2}\|z^-\|^2 + r^2 \end{bmatrix}, \tag{18}$$

and

$$\mathbb{F}_s(\mathbb{X}) = \begin{bmatrix} Mx + q - z \\ x - r\log\left(1 + e^{\frac{x-\rho z}{r}}\right) \\ \frac{1}{2}\|x^-\|^2 + \frac{1}{2}\|z^-\|^2 + r^2 \end{bmatrix}, \tag{19}$$

where

$$\|x^-\|^2 = \sum_{i=1}^{n} \min^2(x_i, 0), \quad \|z^-\|^2 = \sum_{i=1}^{n} \min^2(z_i, 0).$$

This choice has the benefit of taking into account the nonnegativity condition $x \geq 0$ and $z \geq 0$.

Indeed, the last equation of (18) and (19) implies that, as long as $r \geq 0$, we are ascertained that $x^- = z^- = 0$. This amounts to saying that $x \geq 0$ and $z \geq 0$. Should a component of x or z become negative during the iteration, this equation would contribute to "penalize" it. Since r is now considered as a variable and the scalar function $t \mapsto \frac{1}{2}|\min(t,0)|^2$ is differentiable and its derivative is equal to $\min(t,0)$. From this observation, the two Jacobian matrices of $\mathbb{F}_\theta$ and $\mathbb{F}_s$ are:

$$\nabla_{\mathbb{X}}\mathbb{F}_\theta(\mathbb{X}) = \begin{pmatrix} M_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ \operatorname{diag}\left(\dfrac{r^2}{(x+r)^2}\right) & \operatorname{diag}\left(\dfrac{r^2}{(z+r)^2}\right) & W\mathbf{e} \\ (x^-)^{\mathrm{T}} & (z^-)^{\mathrm{T}} & 2r \end{pmatrix}, \tag{20}$$

and

$$\nabla_{\mathbb{X}}\mathbb{F}_s(\mathbb{X}) = \begin{pmatrix} M_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ \operatorname{diag}\left(\dfrac{1}{1+e^{\frac{x-\rho z}{r}}}\right) & \operatorname{diag}\left(\dfrac{\rho e^{\frac{x-\rho z}{r}}}{1+e^{\frac{x-\rho z}{r}}}\right) & V\mathbf{e} \\ (x^-)^{\mathrm{T}} & (z^-)^{\mathrm{T}} & 2r \end{pmatrix}, \tag{21}$$

where x^- is the vector of components $x_i^- = \min(x_i, 0)$ and similarly for z^-,

$$V = \operatorname{diag}\left(-\log(1+e^{\frac{x-\rho z}{r}}) + \frac{\frac{x-\rho z}{r} e^{\frac{x-\rho z}{r}}}{1+e^{\frac{x-\rho z}{r}}}\right), \qquad W = \operatorname{diag}\left(\frac{x^2}{(x+r)^2} + \frac{z^2}{(z+r)^2} - 1\right),$$

and $\mathbf{e}$ is a n-dimensional vector whose entries are equal to 1.
If $\mathbb{F}_\theta(\mathbb{X}) = 0$ (resp. $\mathbb{F}_s(\mathbb{X}) = 0$) where $\mathbb{X} \in \mathbb{R}^{2n}_+ \times \mathbb{R}_+$ we obtain $r = 0$ and $x^- = z^- = 0$. Hence in this case, $\nabla_{\mathbb{X}}\mathbb{F}_\theta(\mathbb{X})$ becomes singular (resp. $\nabla_{\mathbb{X}}\mathbb{F}_s(\mathbb{X})$ becomes singular) since $\det\nabla_{\mathbb{X}}\mathbb{F}_\theta(\mathbb{X}) = 0$ (resp. $\det\nabla_{\mathbb{X}}\mathbb{F}_s(\mathbb{X}) = 0$). To solve this issue, we add a small enough positive parameter ε in the last equation. We get

$$\frac{1}{2}\|x^-\|^2 + \frac{1}{2}\|z^-\|^2 + r^2 + \varepsilon r = 0. \tag{22}$$

Hence, we define the following systems

$$\mathbb{F}_\theta(\mathbb{X}) = \begin{bmatrix} Mx + q - z \\ r(\theta_r^1(x) + \theta_r^1(z) - \mathbf{1}) \\ \frac{1}{2}\|x^-\|^2 + \frac{1}{2}\|z^-\|^2 + r^2 + \varepsilon r \end{bmatrix}, \tag{23}$$

and

$$\mathbb{F}_s(\mathbb{X}) = \begin{bmatrix} Mx + q - z \\ x - r\log\left(1+e^{\frac{x-\rho z}{r}}\right) \\ \frac{1}{2}\|x^-\|^2 + \frac{1}{2}\|z^-\|^2 + r^2 + \varepsilon r \end{bmatrix}. \tag{24}$$

5. Convergence

In this section, we propose two generic algorithms to solve LCP and prove some convergence results.
From now on, the enlarged equation (23) and (24) are selected as the reference system in the design of our new algorithms. The idea is simply to apply the standard Newton method to the smooth system (23) and (24). To enforce a global convergence behavior, we also recommend using Armijo's line search.
Now, we present our algorithms for our methods described above:

Algorithm 1 Nonparametric TLCP with Armijo line search

---

1. Chose $\mathbb{X}^0 = (\mathbf{X}^0, r^0),\ \mathbf{X}^0 > 0,\ r^0 =< x^0, z^0 > /n,\ \tau \in (1, 1/2), \varrho \in (0, 1).$ Set $k = 0.$
2. If $\mathbb{F}_\theta(\mathbb{X}^k) = 0,$ stop.
3. Find a direction $d^k \in \mathbb{R}^{2n+1}$ such that

$$\mathbb{F}_\theta(\mathbb{X}^k) + \nabla_{\mathbb{X}}\mathbb{F}_\theta(\mathbb{X}^k)d^k = 0.$$

4. Choose $\zeta^k = \varrho^{j_k} \in (0, 1),$ where $j_k \in \mathbb{N}$ is the smallest integer such that

$$\Theta_\theta(\mathbb{X}^k + \varrho^{j_k} d^k) \leq (1 - 2\tau\varrho^{j_k})\ \Theta_\theta(\mathbb{X}^k).$$

5. Set $\mathbb{X}^{k+1} = \mathbb{X}^k + \zeta^k d^k$ and $k \leftarrow k + 1.$ Go to step 2.

Algorithm 2 Nonparametric Soft-LCP method with Armijo line search

---

1. Chose $\mathbb{X}^0 = (\mathbf{X}^0, r^0),\ \mathbf{X}^0 > 0,\ r^0 =< x^0, z^0 > /n,\ \tau \in (1, 1/2),\ \varrho \in (0, 1).$ Set $k = 0.$
2. If $\mathbb{F}_s(\mathbb{X}^k) = 0,$ stop.
3. Find a direction $d^k \in \mathbb{R}^{2n+1}$ such that

$$\mathbb{F}_s(\mathbb{X}^k) + \nabla_{\mathbb{X}}\mathbb{F}_s(\mathbb{X}^k)d^k = 0.$$

4. Choose $\zeta^k = \varrho^{j_k} \in (0, 1),$ where $j_k \in \mathbb{N}$ is the smallest integer such that

$$\Theta_s(\mathbb{X}^k + \varrho^{j_k} d^k) \leq (1 - 2\tau\varrho^{j_k})\ \Theta_s(\mathbb{X}^k).$$

5. Set $\mathbb{X}^{k+1} = \mathbb{X}^k + \zeta^k d^k$ and $k \leftarrow k + 1.$ Go to step 2.

Where the merit functions used in the line search are:

$$\Theta_\theta(\mathbb{X}) = \frac{1}{2}\|\mathbb{F}_\theta(\mathbb{X})\|^2.$$

and

$$\Theta_s(\mathbb{X}) = \frac{1}{2}\|\mathbb{F}_s(\mathbb{X})\|^2.$$

A detailed description of Nonparametric Soft-LCP is given in Algorithm 2. A few comments are in order:

- The initial point $\mathbb{X}^0 = (\mathbf{X}^0, r^0)$ must be an interior point, namely, $\mathbf{X}^0 > 0$ and the initial parameter $r^0 =< x^0, z^0 > /n$ has the correct order of magnitude.
- If $\mathbf{X}^k > 0$, then $(x^k)^- = (z^k)^- = 0$ and

$$d^k = \begin{bmatrix} d\mathbf{X}^k \\ dr^k \end{bmatrix} = - \begin{bmatrix} \nabla F_s^r(\mathbf{X}^k) & \partial_r F_s^r(\mathbf{X}^k) \\ 0 & \varepsilon + 2r^k \end{bmatrix}^{-1} \begin{bmatrix} F_s^r(\mathbf{X}^k) \\ \varepsilon r^k + (r^k)^2 \end{bmatrix},$$

  provided that the Jacobian matrix is invertible. The increment for the parameter is then

$$dr^k = -\frac{\varepsilon r^k + (r^k)^2}{\varepsilon + 2r^k}.$$

- There is no need to truncate the Newton direction d^k to preserve positivity for x^{k+1} and z^{k+1}, since nonnegativity is "guaranteed" at convergence. However, if we wish all the iterates are nonnegative, then we are free to carry out an additional damping after Step 4 (Armijo's line search).

PROPOSITION 5.1 *Let $M \in \mathbb{R}^{n\times n}$ be a $\mathcal{P}_0-$matrix. Then, step 3 in Algorithm 1 (resp. Algorithm 2) is well-defined.*

Proof. From the update rule of Algorithm 1 (resp. Algorithm 2) we know that for all $k \geq 0,\ r^k > 0, \mathbf{X}^k > 0$, and $\varepsilon > 0$,

$$\det \nabla \mathbb{F}_s(\mathbb{X}^k) = (\varepsilon + 2r^k) \det \nabla F_s^{r^k}(\mathbf{X}^k),$$

and

$$\det \nabla \mathbb{F}_\theta(\mathbb{X}^k) = (\varepsilon + 2r^k) \det \nabla F_\theta^{r^k}(\mathbf{X}^k).$$

In view of Theorem 3.3, we know that $\nabla F_s^{r^k}(\mathbf{X}^k)$ (resp. $\nabla F_\theta^{r^k}(\mathbf{X}^k)$) is nonsingular. Thus Step 3 of Algorithm 1 (resp. Algorithm 2) is well-defined. ■

Now we would like to study the asymptotic behavior of the Jacobian matrix $\mathbb{F}_s$ (resp. $\mathbb{F}_\theta$) when r goes to 0 and we need a lemma that is used to prove our main result.

LEMMA 5.2 *We consider the following system*

$$\begin{aligned} &Z.X = 0 \\ &Z \geq 0, \quad X \geq 0, \end{aligned} \tag{25}$$

where $Z = diag(z)$ and $X = diag(x)$.
Assume that Z, X are strictly complementary (i.e. $\exists\ \alpha > 0$ such that $Z + X > \alpha$). Then J is singular if and only if T is singular, where

$$J = \begin{pmatrix} M & -I \\ Z & X \end{pmatrix}, \quad \textit{and} \quad T = \begin{pmatrix} M & -I \\ \phi(Z) & \phi(X) \end{pmatrix},$$

such that

$$\phi(t) = \begin{cases} 1 & \text{if} \quad t \neq 0 \\ 0 & \text{if} \quad t = 0, \end{cases}$$

here ϕ operates componentwise on t, and it verifies the following system

$$\begin{array}{l} \phi(Z).\phi(X) = 0 \\ \phi(Z) \geq 0, \quad \phi(X) \geq 0. \end{array}$$

Proof. By the strict complementarity hypothesis, we range the rows and the columns of J and T as follows

$$J_\sigma = \begin{pmatrix} M_\sigma & -I_\sigma \\ \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & X_2 \end{pmatrix} \end{pmatrix},$$

where $X_2 > 0$ and $Z_1 > 0$, and

$$(T)_\sigma = \begin{pmatrix} M_\sigma & -I_\sigma \\ \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & 0 & & 0 \end{pmatrix} & \begin{pmatrix} 0 & & 0 & \\ & \ddots & & \\ & & 1 & 0 \\ 0 & & & \ddots \\ & & 0 & 1 \end{pmatrix} \end{pmatrix}.$$

The determinant of the two matrices J_σ and $(T)_\sigma$ are equal to

$$\det(J_\sigma) = \begin{vmatrix} M_\sigma & -I_\sigma \\ \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & X_2 \end{pmatrix} \end{vmatrix} = \pm \prod_{i \in \mathbb{I}_1} x_i \prod_{i \in \mathbb{I}_2} z_i \ \det(C),$$

$$\det(T_\sigma) = \begin{vmatrix} M_\sigma & -I_\sigma \\ \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & 0 & & 0 \end{pmatrix} & \begin{pmatrix} 0 & & 0 & \\ & \ddots & & \\ & & 1 & 0 \\ 0 & & & \ddots \\ & & 0 & 1 \end{pmatrix} \end{vmatrix} = \pm \prod_{i \in \mathbb{I}_1} \phi(x_i) \prod_{i \in \mathbb{I}_2} \phi(z_i) \ \det(C),$$

where C is a certain matrix, $\mathbb{I}_1 = \{i \ \mid \ x_i > 0\}$ and $\mathbb{I}_2 = \{i \ \mid \ z_i > 0\}$. Since

$$\pm \prod_{i \in \mathbb{I}_1} x_i \prod_{i \in \mathbb{I}_2} z_i \qquad \text{and} \qquad \prod_{i \in \mathbb{I}_1} \phi(x_i) \prod_{i \in \mathbb{I}_2} \phi(z_i),$$

are nonzeros, then we can conclude that J and T are invertibles and singulars at the same time. ■

Below is a result about the Jacobian matrix of $\mathbb{F}_s(\mathbb{X})$, when r goes to 0 and which will be useful for later purposes.

THEOREM 5.3 *Suppose that $\mathbf{X}^* = (x^*, z^*)$ is a solution of LCP which satisfies the strict complementarity (i.e. $\exists\ \alpha > 0$ such that $x_i^* + z_i^* > \alpha$, $\forall i \in \{1, ..., n\}$), and $\nabla_{\mathbf{X}} F_0(\mathbf{X}^*)$ define by (11) (the Jacobian matrix of the Interior-Point Method) is invertible.*
Then $\lim_{r\to 0} \nabla_{\mathbb{X}} \mathbb{F}_s(\mathbf{X}^*, r)$ is invertible, i.e. the two Jacobian matrices are singular or non-sigular at the same time.

Proof. Let

$$\mathbb{F}_s(\mathbb{X}) = \begin{bmatrix} \mathbb{F}_{s,\ 1}(\mathbb{X}) \\ \mathbb{F}_{s,\ 2}(\mathbb{X}) \\ \mathbb{F}_{s,\ 3}(\mathbb{X}) \end{bmatrix} = \begin{bmatrix} Mx + q - z \\ x - r \log\left(1 + e^{\frac{x - \rho z}{r}}\right) \\ \frac{1}{2}\|x^-\|^2 + \frac{1}{2}\|z^-\|^2 + r^2 + \varepsilon r \end{bmatrix}.$$

The Jacobian matrix of $\mathbb{F}_s$ is:

$$\nabla_{\mathbb{X}} \mathbb{F}_s(\mathbb{X}) = \begin{pmatrix} M_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ \nabla_x \mathbb{F}_{s,2}(\mathbb{X}) & \nabla_z \mathbb{F}_{s,2}(\mathbb{X}) & \nabla_r \mathbb{F}_{s,2}(\mathbb{X}) \\ (x^-)^{\mathrm{T}} & (z^-)^{\mathrm{T}} & 2r + \varepsilon \end{pmatrix}.$$

Let us to calculate $\lim_{r\to 0} \nabla_{\mathbb{X}} \mathbb{F}_s(\mathbf{X}^*, r)$:

(1) The derivative of $\mathbb{F}_{s,\ 2}(\mathbf{X}, r)$ with respect to x is:

$$\nabla_x \mathbb{F}_{s,2}(x^*, z^*, r) = \operatorname{diag}\left(\frac{1}{1 + e^{\frac{x^* - \rho z^*}{r}}}\right)_{n\times n},$$

when r goes to 0 and in view of the strict complementary of $\mathbf{X}^* = (x^*, z^*)$, the only two cases to consider are:

- $x_i^* \to 0$, and $z_i^* > 0\ \forall i \in \{1, ..., n\}$ then
$$\lim_{\substack{r\to 0 \\ x_i^*\to 0 \\ z_i^*\neq 0}} (\nabla_x \mathbb{F}_{s,2}(x^*, z^*, r))_{ii} = \lim_{r\to 0} \frac{1}{1 + e^{-\frac{\rho z_i^*}{r}}} = 1.$$
- $x_i^* > 0$, and $z_i^* \to 0\ \forall i \in \{1, ..., n\}$ then
$$\lim_{\substack{r\to 0 \\ z_i^*\to 0 \\ x_i^*\neq 0}} (\nabla_x \mathbb{F}_{s,2}(x^*, z^*, r))_{ii} = \lim_{r\to 0} \frac{1}{1 + e^{\frac{x_i^*}{r}}} = 0.$$

(2) The derivative of $\mathbb{F}_{s,\ 2}(\mathbf{X}, r)$ with respect to z is:

$$\nabla_z \mathbb{F}_{s,2}(x^*, z^*, r) = \operatorname{diag}\left(\frac{\rho e^{\frac{x^* - \rho z^*}{r}}}{1 + e^{\frac{x^* - \rho z^*}{r}}}\right)_{n\times n},$$

as below, the only two cases to consider are:

- $x_i^* \to 0$, and $z_i^* > 0\ \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ x_i^*\to 0\\ z_i^*\neq 0}} (\nabla_z \mathbb{F}_{s,2}(x^*, z^*, r))_{ii} = \lim_{r\to 0} \rho \frac{e^{-\frac{\rho z_i^*}{r}}}{1 + e^{-\frac{\rho z_i^*}{r}}} = 0.$$

- $x_i^* > 0$, and $z_i^* \to 0\ \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ z_i^*\to 0\\ x_i^*\neq 0}} (\nabla_z \mathbb{F}_{s,2}(x^*, z^*, r))_{ii} = \lim_{r\to 0} \rho \frac{e^{\frac{x_i^*}{r}}}{1 + e^{\frac{x_i^*}{r}}} = \rho.$$

We take $\rho = 2$ to ensure the convergence (see Figure 5).

(3) The derivative of $\mathbb{F}_{s,\ 2}(\mathbf{X}, r)$ with respect to r is:

$$\nabla_r \mathbb{F}_{s,2}(x^*, z^*, r) = \begin{pmatrix} -\log(1 + e^{\frac{x^* - \rho z^*}{r}}) + \dfrac{\frac{x^* - \rho z^*}{r} e^{\frac{x^* - \rho z^*}{r}}}{1 + e^{\frac{x^* - \rho z^*}{r}}} \end{pmatrix}_{n\times 1},$$

when r goes to 0 and in view of the strict complementary of $\mathbf{X}^* = (x^*, z^*)$, the only two cases to consider are:

- $x_i^* \to 0$, and $z_i^* > 0\ \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ x_i^*\to 0\\ z_i^*\neq 0}} (\nabla_r \mathbb{F}_{s,2}(x^*, z^*, r))_{ii} = \lim_{r\to 0} \left[-\log(1 + e^{-\frac{\rho z_i^*}{r}}) - \frac{\rho z_i^*}{r} \frac{e^{-\frac{\rho z_i^*}{r}}}{1 + e^{-\frac{\rho z_i^*}{r}}} \right] = 0.$$

- $x_i^* > 0$, and $z_i^* \to 0\ \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ z_i^*\to 0\\ x_i^*\neq 0}} (\nabla_r \mathbb{F}_{s,2}(x^*, z^*, r))_{ii} = \lim_{r\to 0} \left[-\log(1 + e^{\frac{x_i^*}{r}}) + \frac{x_i^*}{r} \frac{e^{\frac{x_i^*}{r}}}{1 + e^{\frac{x_i^*}{r}}} \right] = 0.$$

Finally, thanks to the assumption $\mathbf{X}^* = (x^*, z^*)$ is a solution of LCP, we have $x^* \geq 0$ and $z^* \geq 0$, so that $x^- = z^- = 0$. Hence

$$\lim_{r\to 0} \det\ (\nabla_{\mathbb{X}} \mathbb{F}_s(\mathbf{X}^*, r)) = \left| \begin{bmatrix} \begin{pmatrix} M_{n\times n} & -I_{n\times n} \\ \phi(Z^*) & \phi(X^*) \end{pmatrix} & 0 \\ 0 \quad 0 & \varepsilon \end{bmatrix} \right| = \varepsilon \left| \begin{pmatrix} M & -I \\ \phi(Z^*) & \phi(X^*) \end{pmatrix} \right|,$$

where $\phi(.)$ is defined in Lemma (5.2), $Z^* = \mathrm{diag}(z^*)$ and $X^* = \mathrm{diag}(x^*)$.
From Lemma (5.2), we conclude that if $\nabla_{\mathbf{X}} F_0(\mathbf{X}^*)$ is invertible then $\lim_{r\to 0} \nabla_{\mathbb{X}} \mathbb{F}_s(\mathbf{X}^*, r)$ is invertible. This means, that if the Interior Point Method converges our method converges. ■

Here we present the same result but for the system $\mathbb{F}_\theta(\mathbb{X})$.

THEOREM 5.4 *Suppose that $\mathbf{X}^* = (x^*, z^*)$ is a solution of LCP which satisfies the strict complementarity (i.e. $\exists\ \alpha > 0$ such that $x_i^* + z_i^* > \alpha$, $\forall i \in \{1, ..., n\}$), and $\nabla_{\mathbf{X}} F_0(\mathbf{X}^*)$ define by (11) (the Jacobian matrix of the interior-point methods) is invertible.*
Then $\lim_{r\to 0} \nabla_{\mathbb{X}} \mathbb{F}_\theta(\mathbf{X}^*, r)$ is invertible, i.e. the two Jacobian matrices are singular or non-sigular at the same time.

Proof. Let S defined as

$$S = \{(x_i, z_i, r)/\ \ \theta_r^1(x_i) + \theta_r^1(z_i) = 1, \forall i \in \{1, ..., n\}\},$$

by Lemma (2.3), we have

$$\theta_r^1(x_i) + \theta_r^1(z_i) = 1 \iff x_i z_i = r^2, \quad \forall i \in \{1, ..., n\}.$$

We can therefore define the set S in the form:

$$S = \{(x_i, z_i, r)/\ \ x_i z_i - r^2 = 0, \forall i \in \{1, ..., n\}\}.$$

Since $\mathbf{X}^* = (x^*, z^*)$ is a solution of LCP, we deduce that (x^*, z^*, r) is near to S, then

$$x^* z^* - r^2 = o(r)\,,$$

i.e. $x^* z^* - r^2$ is negligent by r. In view of the assumption of the strict complementary of $\mathbf{X}^* = (x^*, z^*)$, we only have two cases to consider. If $z_i^* > 0$ then $x_i^* = o(r)$ and if $x_i^* > 0$ then $z_i^* = o(r)$. Let

$$\mathbb{F}_\theta(\mathbb{X}) = \begin{bmatrix} \mathbb{F}_{\theta,\ 1}(\mathbb{X}) \\ \mathbb{F}_{\theta,\ 2}(\mathbb{X}) \\ \mathbb{F}_{\theta,\ 3}(\mathbb{X}) \end{bmatrix} = \begin{bmatrix} Mx + q - z \\ \dfrac{rx}{x+r} + \dfrac{rz}{z+r} - r\mathbf{1} \\ \frac{1}{2}\|x^-\|^2 + \frac{1}{2}\|z^-\|^2 + r^2 + \varepsilon r \end{bmatrix}.$$

The jacobian matrix of $\mathbb{F}_\theta$ is:

$$\nabla_{\mathbb{X}} \mathbb{F}_\theta(\mathbb{X}) = \begin{pmatrix} M_{n\times n} & -I_{n\times n} & 0_{n\times 1} \\ \nabla_x \mathbb{F}_{\theta,2}(\mathbb{X}) & \nabla_z \mathbb{F}_{\theta,2}(\mathbb{X}) & \nabla_r \mathbb{F}_{\theta,2}(\mathbb{X}) \\ (x^-)^{\mathrm{T}} & (z^-)^{\mathrm{T}} & 2r + \varepsilon \end{pmatrix},$$

Let us to calculate $\lim_{r\to 0} \nabla_{\mathbb{X}} \mathbb{F}_\theta(\mathbf{X}^*, r)$:

(1) The derivative of $\mathbb{F}_{\theta,\ 2}(\mathbf{X}, r)$ with respect to x is:

$$\nabla_x \mathbb{F}_{\theta,2}(x^*, z^*, r) = \operatorname{diag}\left(\left(\frac{r}{x^* + r}\right)^2\right)_{n\times n},$$

when r goes to 0 and in view of the strict complementary of $\mathbf{X}^* = (x^*, z^*)$, the only two cases to consider are:

- $x_i^* \to 0$, and $z_i^* > 0 \quad \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ x_i^*\to 0\\ z_i^*\neq 0}} (\nabla_x \mathbb{F}_{\theta,2}(x^*, z^*, r))_{ii} = \lim_{\substack{r\to 0\\ x_i^*\to 0\\ z_i^*\neq 0}} \left(\frac{r}{o(r)+r}\right)^2 = \lim_{r\to 0}\left(\frac{r}{r}\right)^2 = 1.$$

- $x_i^* > 0$, and $z_i^* \to 0 \quad \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ z_i^*\to 0\\ x_i^*\neq 0}} (\nabla_x \mathbb{F}_{\theta,2}(x^*, z^*, r))_{ii} = \lim_{\substack{r\to 0\\ z_i^*\to 0\\ x_i^*\neq 0}} \left(\frac{r}{x_i^*+r}\right)^2 = 0.$$

(2) The derivative of $\mathbb{F}_{\theta,\ 2}(\mathbf{X}, r)$ with respect to z is:

$$\nabla_z \mathbb{F}_{\theta,2}(x^*, z^*, r) = \operatorname{diag}\left(\left(\frac{r}{z^*+r}\right)^2\right)_{n\times n},$$

as below, the only two cases to consider are:

- $x_i^* \to 0$, and $z_i^* > 0 \quad \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ x_i^*\to 0\\ z_i^*\neq 0}} (\nabla_z \mathbb{F}_{\theta,2}(x^*, z^*, r))_{ii} = \lim_{\substack{r\to 0\\ x_i^*\to 0\\ z_i^*\neq 0}} \left(\frac{r}{z_i^*+r}\right)^2 = 0.$$

- $x_i^* > 0$, and $z_i^* \to 0 \quad \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ z_i^*\to 0\\ x_i^*\neq 0}} (\nabla_z \mathbb{F}_{\theta,2}(x^*, z^*, r)) = \lim_{\substack{r\to 0\\ z_i^*\to 0\\ x_i^*\neq 0}} \left(\frac{r}{o(r)+r}\right)^2 = 1.$$

(3) The derivative of $\mathbb{F}_{\theta,\ 2}(\mathbf{X}, r)$ with respect to r is:

$$\nabla_r \mathbb{F}_{\theta,2}(x^*, z^*, r) = \left(\left(\frac{x^*}{x^*+r}\right)^2 + \left(\frac{z^*}{z^*+r}\right)^2 - \mathbf{1}\right)_{n\times 1}.$$

when r goes to 0 and in view of the strict complementary of $\mathbf{X}^* = (x^*, z^*)$, the only two cases to consider are:

- $x_i^* \to 0$, and $z_i^* > 0 \quad \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ x_i^*\to 0\\ z_i^*\neq 0}} (\nabla_r \mathbb{F}_{\theta,2}(x^*, z^*, r))_{ii} = \lim_{\substack{r\to 0\\ x_i^*\to 0\\ z_i^*\neq 0}} \left(\left(\frac{o(r)}{o(r)+r}\right)^2 + \left(\frac{z_i^*}{z_i^*+r}\right)^2 - 1\right) = 0.$$

- $x_i^* > 0$, and $z_i^* \to 0 \quad \forall i \in \{1, ..., n\}$ then

$$\lim_{\substack{r\to 0\\ z_i^*\to 0\\ x_i^*\neq 0}} (\nabla_r \mathbb{F}_{\theta,2}(x^*, z^*, r))_{ii} = \lim_{\substack{r\to 0\\ z_i^*\to 0\\ x_i^*\neq 0}} \left(\left(\frac{x_i^*}{x_i^* + r} \right)^2 + \left(\frac{o(r)}{o(r) + r} \right)^2 - 1 \right) = 0.$$

Finally, since $\mathbf{X}^* = (x^*, z^*)$ is a solution of LCP, we have $x^* \geq 0$ and $z^* \geq 0$, so that $x^- = z^- = 0$. Hence

$$\lim_{r\to 0} \det\ (\nabla_{\mathbb{X}} \mathbb{F}_\theta(\mathbf{X}^*, r)) = \left| \begin{bmatrix} \begin{pmatrix} M_{n\times n} & -I_{n\times n} \\ \phi(Z^*) & \phi(X^*) \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \varepsilon \end{bmatrix} \right| = \varepsilon \left| \begin{pmatrix} M & -I \\ \phi(Z^*) & \phi(X^*) \end{pmatrix} \right|.$$

From Lemma (5.2), we conclude that if $\nabla_{\mathbf{X}} F_0(\mathbf{X}^*)$ is invertible then $\lim_{r\to 0} \nabla_{\mathbb{X}} \mathbb{F}_\theta(\mathbf{X}^*, r)$ is invertible. Hence the two Jacobian matrices are singular or nonsingular at the same time. This means, that if the Interior Point Method converges our method converges. ■

5.1 *Global convergence analysis*

DEFINITION 1 (Regular zero). Let $\bar{\mathbf{X}} \in \mathbb{R}^{2n}$ be a zero of F, that is, $F(\bar{\mathbf{X}}) = 0$. If the Jacobian matrix $\nabla F(\bar{\mathbf{X}})$ is nonsingular, $\bar{\mathbf{X}}$ is said to be a regular zero of F.

The main interest of Algorithm 1 and Algorithm 2 lies in the prospect of global convergence, as envisioned by the theory that we are developing now. This global convergence theory, due to bonnans [3], is primarily based on the regularity of zeros [Definition 1]. We reproduce most of Bonnans' theory here, in view of its importance to our algorithm.

THEOREM 5.5 *(Theorem 6.9, [3]). Let $\mathbb{F}_s : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ (resp. $\mathbb{F}_\theta : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$) be a continuously-differentiable function.*

(i) (Local analysis) Let $\bar{\mathbb{X}}$ be a regular zero of $\mathbb{F}_s$ (resp. $\mathbb{F}_\theta$). If $\mathbb{X}^0$ is close enough to $\bar{\mathbb{X}}$, then $\zeta^k = 1$ for all k, and $\mathbb{X}^k$ converge to $\bar{\mathbb{X}}$ super-linearly (and we recover the standard Newton method).

(ii) (Limit point) Let $\tilde{\mathbb{X}}$ be a limit point of sequence $\{\mathbb{X}^k\}$. If $\nabla \mathbb{F}_s(\tilde{\mathbb{X}})$ (resp. $\nabla \mathbb{F}_\theta(\tilde{\mathbb{X}})$) is invertible, then $\tilde{\mathbb{X}}$ is a regular zero of $\mathbb{F}$. If $\tilde{\mathbb{X}}$ is a regular zero of $\mathbb{F}_s$ (resp. $\mathbb{F}_\theta$) , then $\zeta^k = 1$ for k big enough and $\mathbb{X}^k$ converge to $\tilde{\mathbb{X}}$ super-linearly.

(iii) (General behavior) At least one of three possibilities below holds:

a) $\mathbb{F}_s(\mathbb{X}^k) \to 0$ (resp. $\mathbb{F}_\theta(\mathbb{X}^k) \to 0$).
b) $\| d(\mathbb{X}^k) \|$ is unbounded.
c) The sequence $\{\mathbb{X}^k\}$ converge to $\tilde{\mathbb{X}}$ where $\nabla \mathbb{F}_s(\tilde{\mathbb{X}})$ (resp. $\nabla \mathbb{F}_\theta(\tilde{\mathbb{X}})$) is not invertible.

The three items of the Theorem illustrate the conditions and the qualities of convergence of the algorithm. Item (i) corresponds to the behavior of the algorithm near a regular zero. Item (ii) states the rate of convergence in some particular situations.

Item (iii) summarizes all of the possible scenarios when running the algorithm. In particular, if $\nabla \mathbb{F}_s(\mathbb{X})$ (resp. $\nabla \mathbb{F}_\theta(\mathbb{X})$) is invertible everywhere (or at least during the iterations of the algorithm) and $\| \mathbb{F}_s(\mathbb{X}) \| \to \infty$ as $\| \mathbb{X} \| \to \infty$, (resp. $\| \mathbb{F}_\theta(\mathbb{X}) \| \to \infty$ as $\| \mathbb{X} \| \to \infty$), then only the possibility (a) of (iii) can occur; conditons of (ii) are satisfied so that if the algorithm converges, it will converge super-linearly to regular zero.

6. Numerical Results and Applications

Through this article, we studied two methods Soft-LCP and TLCP to solve the LCP, we present in this section some numerical experiments. First, we present a comparison on some randomly generated problems of our two methods with other approaches that have been suggested recently in [5, 9].
Then, we study two concrete examples, the first one is a second order ordinary differential equation and the second is an obstacle problem that can be formulated as LCP (1).
Finally We tested our algorithms on several absolute value equations problems. Our results are very promising and outperform standard methods.
For all the numerical tests and all the considered methods, the used codes are simple Matlab codes. We restrict our choice of θ-function to $\theta_r^1(x) = \frac{x}{x+r}$.
Our aim is to validate our approach and run some preliminary comparison with other methods, and not to optimize the performance of the algorithm.

6.1 *Comparisons on standard LCP*

We generate for several problem sizes, $n = 32,\ 64,\ 128,\ 256$, the data (M, q) in order to have a solution for LCP as follows:

```
R=rand(n, n);
M=R'*R+n*eye(n);
h=rand(n) ;
z=round(h).*rand(n, 1);
t=(1-round(h)).*rand(n, 1);
q=-M*t+z;
```

We compare our two methods denoted Soft-LCP and TLCP with other methods:

- TLCP2 method which is the same algorithm with a different formulation for the complementarity

  $$\theta_r(x_i) + \theta_r(z_i) - \theta_r(x_i + z_i) = 0.$$

  It is a reformulation of the complementarity and not a relaxation (we can use a fixed r).
- The classical interior-point method IPM [9].
- The classical Fischer-Burmeister method [5] when the complementariy $x_i z_i$ is regularized by

  $$\sqrt{x_i^2 + z_i^2 + r^2} - (x_i + z_i) = 0.$$

The main idea of all these methods is to regularize the complementarity condition $x^T z = 0$ and solve a system of equations using Newton's method.

We use an Infeasible IPM and the classical Fischer-Burmeister method referred to as FB-Algor to compare with our methods. For TLCP2, we have fixed r to 1. We take for all this methods the initial point $(x_0,\ z_0) = 1,\quad$ where $1 \in \mathbb{R}^n$ is the vector whose components are all equal to 1 and $r_0 = \langle x_0,\ z_0\rangle/n$ and the precision is set as 10^{-6}.
The comparative results are given in the Table 1 to 5. We are interested in the following aspects: the comp.err, computed as $|x^T z|$, feas.err computed as $\|Mx+q-z\|$ the number of iterations computed as nb-iter and the time.

Table 1. Results from Soft-LCP with n=32, 64, 128, 256.

n	errcomp	erfeas	r	nb-iter	time(s)
32	4.8594e-07	3.1676e-14	0.0021	9	0.0081
64	1.6217e-05	9.4826e-13	0.0072	10	0.0292
128	1.1151e-07	5.5129e-12	0.0014	14	0.0364
256	5.1985e-06	9.3893e-12	0.0020	23	0.1900

Table 2. Results from FB-Algor with n=32, 64, 128, 256.

n	comp.err	feas.err	r	nb-iter	time(s)
32	4.5973e-08	1.2373e-07	1.324e-06	12	0.0218
64	9.3296e-08	2.8274e-07	9.715e-06	14	0.0305
128	5.1455e-08	1.8937e-07	4.799e-06	15	0.0372
256	2.2314e-07	5.5151e-07	4.050e-06	17	0.1294

Table 3. Results from TLCP with n=32, 64, 128, 256.

n	comp.err	feas.err	r	nb-iter	time(s)
32	3.0137e-07	1.0516e-13	4.8821e-04	11	0.0083
64	1.1888e-07	7.1035e-13	4.8813e-04	11	0.0121
128	4.5973e-07	4.8634e-12	4.8793e-04	11	0.0244
256	4.4479e-07	9.9347e-12	2.4369e-04	12	0.0986

Table 4. Results from TLCP2 with n=32, 64, 128, 256.

n	comp.err	feas.err	r	nb-iter	time(s)
32	7.0412e-09	8.7429e-09	1	11	0.0154
64	5.7344e-09	2.1192e-09	1	12	0.0429
128	2.3476e-07	1.02336e-08	1	11	0.1993
256	8.0873e-08	8.92116e-07	1	38	1.6799

Table 5. Results from IPM with n=32, 64, 128, 256.

n	comp.err	feas.err	r	nb-iter	time(s)
32	9.4531e-07	0	0	198	0.5606
64	9.8840e-07	1.3455e-12	0	212	0.9313
128	9.1254e-07	4.1933e-10	0	248	3.6056
256	9.4437e-07	0	0	238	8.7353

In the above comparisons, we notice that our methods have much better results in terms of iteration numbers and CPU-time than classic interior-point-method IPM and FB-Algor. The TLCP method requires the fewest iteration numbers.

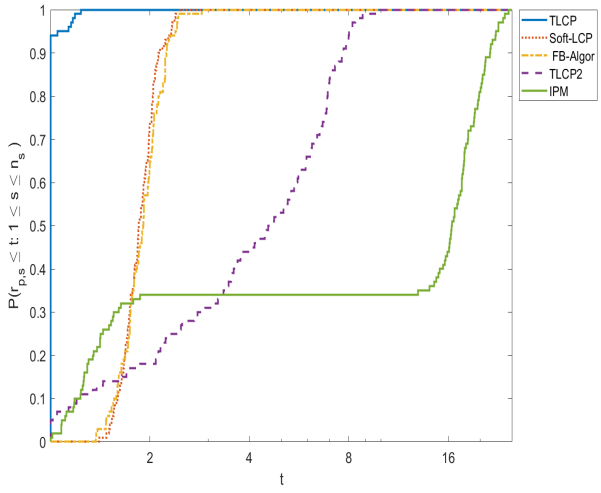


Figure 3. Performance profiles where $t_{p,s}$ represents the average computation time.

The figure above shows the performance profiles of five solvers where the performance measure is execution time. It is clear that the TLCP method captures our attention (admits the highest probability value). In fact, in the interval [0, 1], TLCP is able to solve 99% of the problems, while the other solvers do not reach 20% and require more time. We also notice that IPM is the slowest compared to others. However, for $t > 2$, the three algorithms TLCP, Soft-LCP and FB-Algor confirm their robustness. Figure 3 also indicates that, with respect to the computation time, with the same initial points and under the same stopping criterion, TLCP is the fastest solver, followed respectively by Soft-LCP, FB-Algor, TLCP2 and IPM.

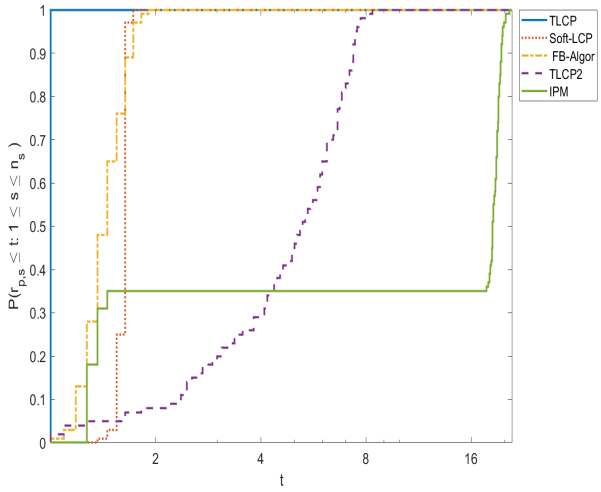
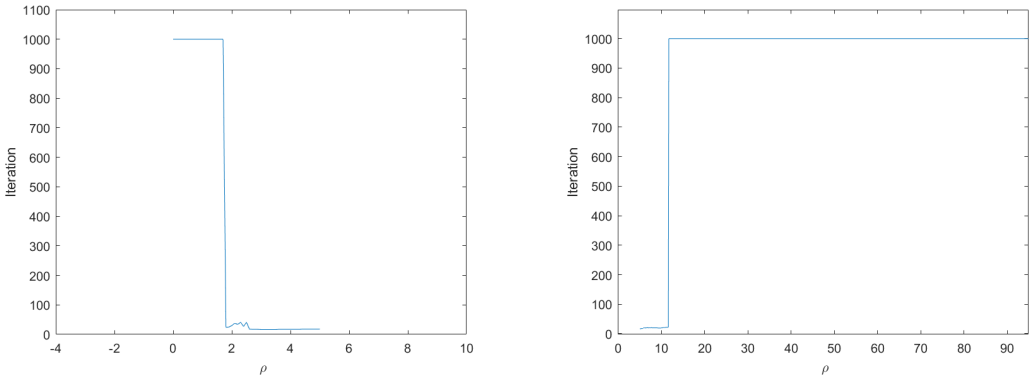


Figure 4. Performance profiles where $t_{p,s}$ represents the average number of iterations.

In Figure 5, we illustrate the performance profiles of five solvers considering the number of iterations required as a performance measure. We notice that TLCP is the winner (admits the highest probability value) followed by FB-Algor, Soft-LCP. We also note that IPM and TLCP2 need more iterations to resolve problems. The performance of Soft-LCP becomes interesting beyond $t = 2$.

6.1.1 *sensitivity of ρ*

We will now study the sensitivity of the parameter ρ. We solve our problem using Soft-LCP with different parameters ρ. We take $\rho = 1, 1.1, 1.2, ..., 100.$ and $n = 128$.



(a) Sensitivity Analysis $0 < \rho \leq 5$. (b) Sensitivity Analysis $5 \leq \rho \leq 100$.

Figure 5. Sensitivity Analysis of ρ.

We notice a loss of convergence between 0 and 2, however between 2 and 12 convergence is assured, then from 13, there is a divergence. We conclude that we cannot choose ρ as a random parameter. In all the cases where there is convergence, the number of iterations is almost the same however in the event of divergence the number of iterations exceeds the maximum number fixed in our algorithm. We have fixed $\rho = 3$ in our approach to ensure the convergence.

6.2 *An application for obstacle problem*

Let f and g two continuous functions defined in $[0, 1]$. We want to solve the following obstacle problem:
find $u : [0, 1] \to \mathbb{R}$ such that:

$$\begin{cases} -u^{"}(x) & \geq & f(x) \\ u(x) & \geq & g(x) \quad \text{on }]0,1[, \\ (-u^{"}(x) - f(x))(u(x) - g(x)) & = & 0 \end{cases}$$

and $u(0) = u(1) = 0$.
The first equation means a maximum concavity of the function u. In the second equation, we want the solution u to be above g. In the third equation, we have at least equality in one of the two previous equations. In order to get a linear complementarity problem, we set $z = u - g$ and we discretize by using the finite difference. We introduce a uniform subdivision $x_i = i * h, i = 0, \dots N + 1$ of $[0, 1]$, where $h = \frac{1}{N+1}$.
We use the second-order centered finite difference to approximate the second order derivatives $z^{''}(x)$ and $g^{''}(x)$. We then try to solve the following problem:

$$\begin{cases} \frac{-z_{i-1}+2z_i-z_{i+1}}{h^2} + \frac{-g_{i-1}+2g_i-g_{i+1}}{h^2} - f_i & \geq & 0 \\ z_i & \geq & 0 \\ \left(\frac{-z_{i-1}+2z_i+z_{i+1}}{h^2} + \frac{-g_{i-1}+2g_i-g_{i+1}}{h^2} - f_i\right)(z_i) & = & 0 \end{cases}, \quad \text{for } i = 1...N, \ u_0 = u_{N+1} = 0.$$

Where $g_i = g(x_i)$, $f_i = f(x_i)$, $z_i = z(x_i)$ and $u_i = u(x_i)$. We obtain the following complementarity problem:

$$\begin{aligned}(Mz+q)^T z &= 0\\ z &\geq 0\\ Mz+q &\geq 0\end{aligned}$$

where

$$M = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

and $q = Mg - f$.
If 1 is not an eigenvalue of M is equivalent to AVE, ([25], Prop. 2),

$$(M-I)^{-1}(M+I)x - |x| = (M-I)^{-1}q.$$

We present in the following figures, the results of our two methods and LPM method from [25]. The obstacle g is chosen here to be

$$g(x) = \max(0.8 - 20 * (x-0.2)^2, \max(1 - 20(x-0.75)^2, 1.2 - 30(x-0.41)^2))$$

$f(x) = 1$ and $N = 50$.

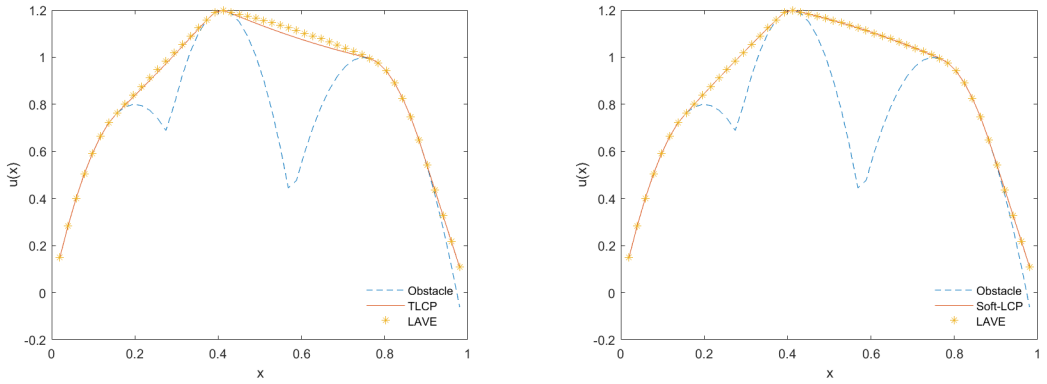


Figure 6. Numerical solution of the obstacle problem (6.2) with TCLP, Soft-LCP methods, and method from [25].

We remark that the both TLCP, Soft-LCP, and LPM method [25] have 19 common points on the curve g and none below g over 50 points. This example also confirms that our approach, TLCP and Soft-LCP method gives consistent results.

6.3 *An application on ordinary differential equation*

We consider the ordinary differential equation

$$x^{''}(t) - |x(t)| = -2 - t, \quad x(0) = -4, \quad x^{'}(0) = 5, \quad t \in [0,\ 5]. \tag{26}$$

First, we discretize the EDO equation by using the finite difference scheme. We use the second-order centred finite difference to approximate the second order derivative

$$\frac{x_{i-2} - 2x_{i-1} + x_i}{h^2} - |x_i| = (-2 - t)_i. \tag{27}$$

Equation (27) was derived with equispace gridpoints $t_i = ih,\ i = 1, ...N$. In order to approximate the Neumann boundary conditions we use a center difference

$$\frac{x_1 - x_{-1}}{2h} = x^{'}(0) = 1. \tag{28}$$

Using the classical decomposition of the absolute value [1] we reformulate (27) as follows

$$\begin{cases} N_1 x^+ - N_2 x^- = q, \\ 0 \leq x^+ \perp x^- \geq 0, \end{cases} \tag{29}$$

where

$$N_1 = \frac{1}{h^2} \begin{pmatrix} 2-h^2 & & & & \\ -2 & 1-h^2 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1-h^2 \end{pmatrix},$$

$$N_2 = \frac{1}{h^2} \begin{pmatrix} 2+h^2 & & & & \\ -2 & 1+h^2 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1+h^2 \end{pmatrix},$$

and $q = -\frac{1}{h^2} \begin{pmatrix} 8-10h \\ -4 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 2+h \\ 2+2h \\ \vdots \\ 2+Nh \end{pmatrix}$.

N_1 is invertible, then the problem (29) is reduced to a standard LCP.
We compare the obtained solution by Soft-LCP and TLCP to the predefined Runge-Kutta ode45 function in MATLAB [26].
The domain is $t \in [0, 5]$, initial conditions $x(0) = -4, x^{'}(0) = 5$ and $N = 100$.

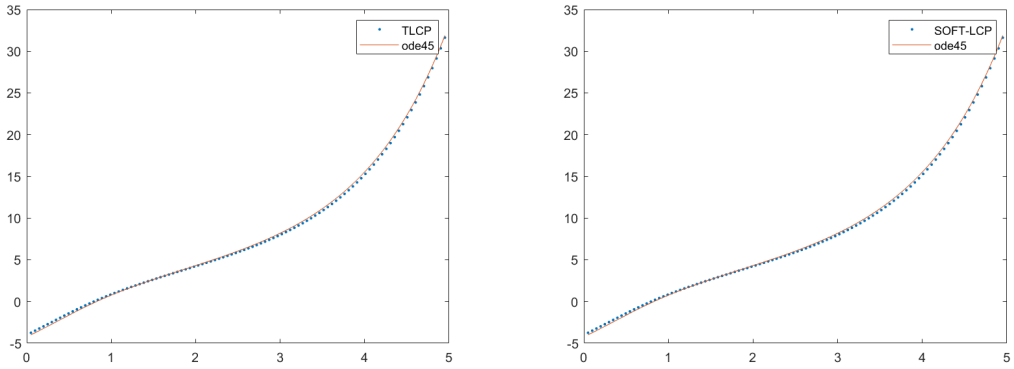


Figure 7. Numerical solution of (6.3) with ode45 and both methods.

Both methods solve the problem and gives consistent results.

6.4 *Extension to Absolute Value Equation*

We consider the absolute value equation AVE, defined as

$$Ax - |x| = b, \tag{30}$$

with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. We studied two cases where AVE has a unique solution and for general AVE. Using the same technique as in [1], (30) can be cast as the following complementarity problem

$$A(x^+ - x^-) - (x^+ + x^-) = b, \qquad 0 \leq x^+ \perp x^- \geq 0., \tag{31}$$

equivalent to

$$(A - I)x^+ = (A + I)x^- + b, \quad 0 \leq x^+ \perp x^- \geq 0, \tag{32}$$

where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. This decompsition guarantes that $|x| = x^+ + x^-$. So AVE can be cast as the following LCP

$$x^+ = Mx^- + q,\ 0 \leq x^+ \perp x^- \geq 0, \tag{33}$$

with $M = (A - I)^{-1}(A + I)$ and $q = (A - I)^{-1}b$.

6.4.1 Random uniquely solvable generated problem

We consider the special case where AVE is uniquely solvable, to guarantee the convergence of the Newton method. One way to generate such (AVE) is to generate a matrix A with singular values exceeding 1. We first chose a random A from a uniform distribution on $[-10, 10]$, then we chose a random x from a uniform distribution on $[-1, 1]$. Finally we computed $b = Ax - |x|$. We ensured that the singular values of each A exceeded 1 by actually computing the minimum singular value and rescaling A by dividing it by the minimum singular value multiplied by a random number in the interval $[0, 1]$.

We generate for the several values for $n = 32, 64, 128, 256, 512, 1024$, the data (A, b) by the following Matlab code in order to have a solution for AVE:

```
n=input('dimension of matrix A=');
rand('state',0);
A=10*(rand(n,n)-rand(n,n));
x=rand(n,1)-rand(n,1);
b=A*x-abs(x);
```

The required precision for solving AVE is 10^{-6}. For each n we consider 100 instances.

Now, we compare our methods Soft-LCP and TLCP to Generalized Newton method from [24], which is denoted GN. In this method, we solve each iteration a linear system:

$$(A - D(x^i))x^{i+1} = b \tag{34}$$

where $D(x^i) = \text{diag}(\text{sign}(x^i))$. Results are summarized in Table 6, which gives the number of iterations, the time required to solve all the 100 instances. Our methods solve all 100 AVEs to an accuracy of 10^{-6} and validate our approach. We notice that the GN method is the fastest because at each iteration it solves only one linear system, the TLCP method gives the fewest iterations to solve the 100 instances.

Table 6. Comparison of Soft-LCP and TLCP with GN method, in the case with singular values of A exceeds 1 for 100 randomly generated AVE of size n.

n	it-Soft-LCP	Time-Soft-LCP(s)	it-TLCP	Time-TLCP(s)	it-GN	Time-GN(s)
32	201	0.0394	104	0.0238	255	0.0071
64	201	0.1041	107	0.0646	274	0.0182
128	200	0.2844	111	0.1767	274	0.0641
256	212	1.6986	106	0.9727	290	0.2301
512	284	11.0947	110	5.0497	295	1.2925
1024	284	42.1565	111	45.3930	291	14.8541

6.4.2 Random generated problem

Now we present results for general AVE, which is the main interest of our algorithm. The data are generated like [25] for several n and for several values of the parameteres, in each situation we solve 100 instances of the problem. We choose a random A from a uniform distributin on $[-10, 10]$, then chose a random x from a uniform distribution on $[-1, 1]$ and set $b = Ax - |x|$. The data (A, b) are generated by Matlab script:

```
n=input('dimension of matrix A=');
R=10*(rand(n,n)-rand(n,n));
A=R/(min(svd(R))*rand(1));
x=rand(n,1)-rand(n,1);
b=A*x-abs(x).
```

We will compare 4 methods valid for general AVE:

- TLCP method from Algorithm 1;
- Soft-LCP method from Algorithm 2;
- Concave minimization method CMM from [25];
- Successive linear programming method LPM from [23];

In table 7-10, "nnztot" gives the number of violated expressions for all problems, "nnzx" gives the maximum violated expressions for one problem, "nb-iter" gives the number of iteration for all the problems. We also provide the time in seconds and the number of problems where we did not manage to solve AVE.

Table 7. Results from TLCP on with 100 consecutive random AVEs.

n	nnztot	nnzx	nb-iter	time(s)	nb-failure
32	3	1	1647	0.6274	3
64	5	1	1776	1.2548	5
128	5	1	2359	2.4182	7
256	8	1	2448	22.8817	8

Table 8. Results from Soft-LCP on with 100 consecutive random AVEs.

n	nnztot	nnzx	nb-iter	time(s)	nb-failure
32	1	1	960	0.4287	1
64	1	1	1032	0.8351	1
128	3	1	1478	1.6692	3
256	1	1	1996	18.3965	1

Table 9. Results from CMM on with 100 consecutive random AVEs.

n	nnztot	nnzx	nb-iter	time(s)	nb-failure
32	13	1	640	4.2832	13
64	11	1	588	7.0034	11
128	13	1	693	19.9940	13
256	15	1	753	143.6931	15

Table 10. Results from LPM on with 100 consecutive random AVEs.

n	nnztot	nnzx	nb-iter	time(s)	nb-failure
32	8	1	313	2.2422	8
64	19	4	411	6.0978	18
128	21	3	433	18.2642	20
256	29	5	606	156.4612	22

In every cases our methods manage to reduce the number of unsolved problem, which was our principal aim. In every case it gives the smallest number of unsolved problem in a very reasonable time.

7. Conclusion

In this paper, we propose two methods to solve the LCP. A complete analysis is provided to validate our approach. Furthermore, a numerical study shows that our approach is interesting. Numerical experiments on several LCP problems and a comparison with some existing methods proves the efficiency of our study.
We have presented an application of absolute value equation AVE and two examples (an obstacle problem and ODE) and show that our two methods are promising.

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