

# FREE PRE-LIE ALGEBRAS OF FINITE POSETS

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**ABSTRACT.** We first recall the construction of a twisted pre-Lie algebra structure on the species of finite connected topological spaces. Then, we construct a corresponding non-coassociative permutative (NAP) coproduct on the subspecies of finite connected  $T_0$  topological spaces, i.e., finite connected posets, and we prove that the vector space generated by isomorphism classes of finite posets is a free pre-Lie algebra and also a cofree NAP coalgebra. Further, we give an explicit duality between the non-associative permutative product and the proposed NAP coproduct. Finally, we prove that the results in this paper remain true for the finite connected topological spaces.

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## 1. INTRODUCTION

In this paper, we introduce a twisted non-associative permutative algebra structure on the species of finite connected posets. We recall the definition of a non-associative permutative algebra (in short NAP algebra), and the dual definition of a non-coassociative permutative coalgebra (in short NAP coalgebra). A (left) NAP algebra is a vector space  $V$  equipped with a bilinear product  $\cdot$  satisfying the relation:

$$(1.1) \quad a \cdot (b \cdot c) = b \cdot (a \cdot c), \text{ for all } a, b \text{ and } c \in V.$$

We note also that the notion of NAP algebra has emerged from the reference [15] by M. Livernet. It has also been previously defined in [11] by A. Dzhumadil'daev, and C. Löfwall under the name of "left commutative algebra". Dually a (left) NAP coalgebra is a vector space  $V$  equipped with

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a bilinear coproduct  $\delta: V \rightarrow V \otimes V$  satisfying the relation:

$$(1.2) \quad (Id \otimes \delta)\delta = \tau^{12}(Id \otimes \delta)\delta.$$

We recall the following rigidity theorem [15]: any pre-Lie algebra  $(V, \triangleright)$ , together with a non-associative permutative connected coproduct  $\delta$  satisfying the distributive law

$$(1.3) \quad \delta(a \triangleright b) = a \otimes b + (a \otimes \mathbf{1} + \mathbf{1} \otimes a) \triangleright \delta(b),$$

is a free pre-Lie algebra and a cofree NAP coalgebra.

We recall also in this paper the description of the free pre-Lie algebra in terms of rooted trees given in [10]. We define a new bilinear product  $\circ \rightarrow$  in the species of finite connected posets  $\mathbb{U}$  by: for all  $P \in \mathbb{U}_{X_1}$  and  $Q \in \mathbb{U}_{X_2}$ , where  $X_1$  and  $X_2$  are two finite sets

$$P \circ \rightarrow Q = \sum_{v \in \min(Q)} P \searrow_v Q,$$

where  $P \searrow_v Q$  is obtained from the Hasse graphs  $G_1$  and  $G_2$  of  $P$  and  $Q$  by adding an (oriented) edge from  $v$  in  $G_2$  to any minimal vertex of  $G_1$  [3].

We show that the bilinear product  $\circ \rightarrow$  verifies (1.1) in the monoidal category of species, and therefore endows the species  $\mathbb{U}$  with a NAP algebra structure.

For any finite set  $X$ , we define the coproduct  $\delta$  by:

$$\begin{aligned} \delta: \mathbb{U}_X &\longrightarrow (\mathbb{U} \otimes \mathbb{U})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{U}_Y \otimes \mathbb{U}_Z \\ P &\longmapsto \frac{1}{|\min(P)|} \sum_{I \odot P} I \otimes P \setminus I, \end{aligned}$$

where  $I \odot P$  means that  $I$  is a subset of  $P$  such that:

- $I_-$  is a singleton included in  $\min(P)$ ,
- and  $I$  is a connected component of the set  $\{x \in P, I_- <_P x\}$ .

with the set  $I_-$  is equal to the space  $\{x \notin I, \text{ there exists } y \in I \text{ such that } x \leq_P y\}$ .

We prove that the coproduct  $\delta$  is a twisted NAP coproduct, i.e., (1.2) is verified in the monoidal category of species.

M. Livernet in [15] proved the following rigidity theorem: for any pre-Lie algebra  $(V, \triangleright)$  together with a NAP connected coproduct  $\delta$  satisfying relation (1.3),  $V$  is the free pre-Lie algebra generated by  $\text{Prim}(V)$ .

Later in this paper, we prove a compatibility relation between the pre-Lie  $\searrow$  structure and the coproduct  $\delta$  by proving the twisted version of (1.3) namely:

$$\delta(P \searrow Q) = P \otimes Q + (P \otimes \mathbf{1} + \mathbf{1} \otimes P) \searrow \delta(Q),$$

for any pair  $(P, Q)$  of finite connected posets, where the unit  $\mathbf{1}$  is identified to the empty poset.

Applying Aguiar-Mahajan's bosonic Fock functor  $\overline{\mathcal{K}}$  [2] and M. Livernet's rigidity theorem [15] leads to the main result of the paper:  $(\overline{\mathcal{K}}(\mathbb{U}), \searrow)$  endowed with the coproduct  $\delta$  is a free pre-Lie algebra and a cofree NAP coalgebra.

We end up Section 3 by proving that the NAP product  $\circ\rightarrow$  and the NAP coproduct  $\delta$  are dual to each other, and we show that  $(\overline{\mathcal{K}(\mathbb{P})}, \Delta_{\searrow})$  is a coassociative cofree coalgebra, where  $\Delta_{\searrow}$  is the coassociative coproduct defined in [3].

In the last section, we prove that the results in this paper remain true for the finite connected topological spaces, with a small change on the definition of the coproduct  $\delta$ .

## 2. BASICS OF FINITE TOPOLOGIES

A partial order on a set  $X$  is a transitive, reflexive and antisymmetric relation on  $X$ . A finite poset is a finite set  $X$  endowed with a partial order  $\leq$ . Let  $P = (X, \leq_P)$  and  $Q = (X, \leq_Q)$  be two posets. We say that  $P$  is finer than  $Q$  if:  $x \leq_P y \implies x \leq_Q y$  for any  $x, y \in X$ . The Hasse diagram of poset  $P = (X, \leq_P)$  is obtained by representing any element of  $X$  by a vertex, and by drawing a directed edge from  $a$  to  $b$  if and only if  $a <_P b$ , and, for any  $c \in X$  such that  $a \leq_P c \leq_P b$ , one has  $a = c$  or  $b = c$ . We can say that  $I$  is an upper ideal of  $P$  ( $I \subset P$ ) if, for all  $x, y \in P$ ,  $(x \in I, x \leq_P y) \implies y \in I$ . We denote  $J(P)$  the set of upper ideals of  $P$ .

Recall [1, 12] that a finite topological space is a finite quasi-poset and vice versa. Any topology  $\mathcal{T}$  (hence any quasi-order on  $X$ ) gives rise to an equivalence relation:

$$(2.1) \quad x \sim_{\mathcal{T}} y \iff (x \leq_{\mathcal{T}} y \text{ and } y \leq_{\mathcal{T}} x).$$

Equivalence classes will be called bags here. This equivalence relation is trivial if and only if the quasi-order is a (partial) order, or equivalently, if the corresponding topology is  $T_0$ . Any topology  $\mathcal{T}$  on  $X$  defines a poset structure on the quotient  $X / \sim_{\mathcal{T}}$ , corresponding to the partial order induced by the quasi-order  $\leq_{\mathcal{T}}$ . More on finite topological spaces can be found in [4, 9, 12, 18].

Recall [2, 14] that a linear (tensor) species is a contravariant functor from the category of finite sets **Fin** with bijections into the category **Vect** of vector spaces (on some field  $\mathbf{k}$ ). Let  $\mathbb{E}$  be a linear species. We note  $\overline{\mathcal{K}}(\mathbb{E}) = \bigoplus_{n \geq 0} \mathbb{E}_n / S_n$ , where  $\mathbb{E}_n / S_n$  denotes the space of  $S_n$ -coinvariants of

$\mathbb{E}_n$ . The functor  $\overline{\mathcal{K}}$  from linear species to graded vector spaces is intensively studied in [2, chapter 15] under the name "bosonic Fock functor".

The species  $\mathbb{T}$  of finite topological spaces is defined as follows: for any finite set  $X$ ,  $\mathbb{T}_X$  is the vector space freely generated by the topologies on  $X$ . For any bijection  $\varphi : X \rightarrow X'$ , the isomorphism  $\mathbb{T}_{\varphi} : \mathbb{T}_{X'} \rightarrow \mathbb{T}_X$  is defined for any topology  $\mathcal{T}$  on  $X'$ , by the following relabelling:

$$\mathbb{T}_{\varphi}(\mathcal{T}) = \{\varphi^{-1}(Y), Y \in \mathcal{T}\}$$

It is well-known that the species  $\mathbb{T}$  of finite topological spaces is a twisted commutative Hopf algebra [12]: the product  $m$  is given by disjoint union, and the coproduct  $\Delta$  is a natural generalization of the Connes-Kremer coproduct on rooted forests. We defined another twisted Hopf

algebra structure in [3] by replacing  $\Delta$  with the coproduct

$$\begin{aligned} \Delta_{\searrow} : \mathbb{T}_X &\longrightarrow (\mathbb{T} \otimes \mathbb{T})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Z \\ \mathcal{T} &\longmapsto \sum_{Y \in \mathcal{T}} \mathcal{T}_{|Y} \otimes \mathcal{T}_{|X \setminus Y}. \end{aligned}$$

Where  $Y \in \mathcal{T}$ , stands for

- $Y \in \mathcal{T}$ ,
- $\mathcal{T}_{|Y} = \mathcal{T}_1 \dots \mathcal{T}_n$ , such that for all  $i \in \{1, \dots, n\}$ ,  $\mathcal{T}_i$  is connected and  $(\min \mathcal{T}_i = (\min \mathcal{T}) \cap \mathcal{T}_i$ , or there is a single common ancestor  $x_i \in \overline{X \setminus Y}$  to  $\min \mathcal{T}_i$ ), where  $\overline{X \setminus Y} = (X \setminus Y) / \sim_{\mathcal{T}_{|X \setminus Y}}$ .

If we apply the functor  $\overline{\mathcal{K}}$ , we therefore two commutative graded connected Hopf algebras  $\overline{\mathcal{K}}(\mathbb{T}, m, \Delta)$  and  $\overline{\mathcal{K}}(\mathbb{T}, m, \Delta_{\searrow})$ .

**Definition 2.1.** [8, 16] *A left pre-Lie algebra over a field  $\mathbf{k}$  is a  $\mathbf{k}$ -vector space  $A$  with a binary composition  $\triangleright$  that satisfies the left pre-Lie identity:*

$$(2.2) \quad (x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z),$$

for all  $x, y, z \in A$ . The left pre-Lie identity rewrites as:

$$(2.3) \quad L_{[x,y]} = [L_x, L_y],$$

where  $L_x : A \longrightarrow A$  is defined by  $L_x y = x \triangleright y$ , and where the bracket on the left-hand side is defined by  $[x, y] = x \triangleright y - y \triangleright x$ . As a consequence this bracket satisfies the Jacobi identity.

In fact, as mentioned in [3], the structure  $\searrow$  on the species of connected finite topological spaces  $\mathbb{V}$  defined by: for all  $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$  and  $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$  two finite connected topological spaces as:

$$\mathcal{T}_1 \searrow \mathcal{T}_2 = \sum_{v \in X_2} \mathcal{T}_1 \searrow_v \mathcal{T}_2,$$

is a pre-Lie structure, where  $\mathcal{T}_1 \searrow_v \mathcal{T}_2$  is obtained from the Hasse graphs  $G_1$  and  $G_2$  of  $\mathcal{T}_1$  and  $\mathcal{T}_2$  by adding an (oriented) edge from  $v$  in  $G_2$  to any minimal vertex of  $G_1$ .

We denote by  $\mathbb{P}$  the sub-species of  $\mathbb{T}$  consisting in partial order (i.e.,  $T_0$ -topologies). The binary product  $m$  (resp.  $\searrow$ ) restricts to finite posets (resp. finite connected posets), as well as both coproducts  $\Delta$  (resp.  $\Delta_{\searrow}$ ). Hence the triple  $(\mathbb{P}, m, \Delta)$  is a twisted Hopf subalgebra of  $(\mathbb{T}, m, \Delta)$ , the triple  $(\mathbb{P}, m, \Delta_{\searrow})$  is a twisted Hopf subalgebra of  $(\mathbb{T}, m, \Delta_{\searrow})$  and  $(\mathbb{P}, \searrow)$  is a twisted pre-Lie subalgebra of  $(\mathbb{T}, \searrow)$ .

### 3. FREE PRE-LIE ALGEBRAS AND COFREE COALGEBRAS

**3.1. Free pre-Lie algebras and rooted trees.** Let  $T$  be the vector space spanned by the set of isomorphism classes of rooted trees and  $H = S(T)$ . Grafting pre-Lie algebras of rooted trees were studied for the first time by F. Chapoton and M. Livernet [10], although the use of rooted trees can be traced back to A. Cayley [5]. The grafting product is given, for all  $t, s \in T$ , by:

$$(3.1) \quad t \rightarrow s = \sum_{s' \text{ vertex of } s} t \rightarrow_{s'} s,$$

where  $t \rightarrow_{s'} s$  is the tree obtained by grafting the root of  $t$  on the vertex  $s'$  of  $s$ . In other words, the operation  $t \rightarrow s$  consists of grafting the root of  $t$  on every vertex of  $s$  and summing up.

We note  $T(n)$  the linear span of rooted trees of degree  $n$ . For a vector space  $V$ , we denote by  $T(V)$  the space  $\bigoplus_n T(n) \otimes V^{\otimes n} / S_n$ . This is the linear span of rooted trees decorated by a basis of  $V$ . Here  $S_n$  is the symmetric group on  $n$  elements. Following the notation of Connes and Kreimer [6], any tree  $t$  writes  $t := B(v, t_1, t_2, \dots, t_n)$  that is decorated by  $v$ , and  $t_1, \dots, t_n$  are trees.

We notice that, if using the pre-Lie product  $\rightarrow$  in  $T(V)$ , one has:

$$B(v, t_1, \dots, t_n) = t_n \rightarrow B(v, t_1, \dots, t_{n-1}) - \sum_{0 < k < n} B(v, t_1, \dots, t_n \rightarrow t_k, \dots, t_{n-1})$$

**Proposition 3.1.** [10] *Equipped by  $\rightarrow$ , the space  $T$  is the free pre-Lie algebra with one generator.*

**Proposition 3.2.** [15] *Let  $V$  be a vector space. Then,  $T(V)$  together with the coproduct*

$$\delta(B(v, t_1, t_2, \dots, t_n)) = \sum_{0 < k < n+1} t_k \otimes B(v, t_1, \dots, \hat{t}_k, \dots, t_n),$$

*is the cofree NAP connected coalgebra generated by  $V$ .*

**3.2. Twisted NAP algebras of finite connected posets.** Let  $P$  be a finite connected poset, and  $I$  be a subset of  $P$ . We denote by  $I_-$  the set  $\{x \notin I, \text{ there exists } y \in I \text{ such as } x \leq_P y\}$ .

We note  $I \odot P$  whenever:

- $I_-$  is a singleton included in  $\min(P)$ ,
- and  $I$  is a connected component of the set  $\{x \in P, I_- <_P x\}$ .

**Lemma 3.1.** *For any finite connected poset  $P$ , and any subsets  $I, J$  of  $P$ , we have:*

- 1- *if  $I \odot P$  and  $J \odot P$ , then  $I = J$  or  $I \cap J = \emptyset$ .*
- 2- *If  $I \odot P$ , then  $\min(P \setminus I) = \min(P)$ .*
- 3- *If  $I \odot P$ , then  $P \setminus I$  is a connected poset.*

*Proof.* 1- Let  $P$  be a finite connected poset and  $I, J$  two subsets of  $P$ , such that  $I, J \odot P$ . So, we have two possible cases: either  $I_- \neq J_-$  or  $I_- = J_-$ .

- If  $I_- \neq J_-$ , with  $I_- = \{y\}$ , and  $J_- = \{z\}$ , then  $I$  is a connected component of the set  $E_1 = \{x \in P, y <_P x\}$  and  $J$  is a connected component of the set  $E_2 = \{x \in P, z <_P x\}$ . So  $I$  and  $J$  is a connected component of the set  $E_1 \sqcup E_2$ , then  $I = J$  or  $I \cap J = \emptyset$ . Gold  $I_- \neq J_-$ , then  $I \neq J$ . Hence  $I \cap J = \emptyset$ .

- If  $I_- = J_- = \{y\}$ , then  $I$  and  $J$  is two connected components of the set  $\{x \in P, y <_P x\}$ , then:  $I = J$  or  $I \cap J = \emptyset$ .

2- If there exists  $x \in \min(P) \cap I$ , then  $I_- <_P x$ , which is absurd. Thus,  $\min(P) \cap I = \emptyset$ . Then,  $\min(P \setminus I) = \min(P)$ .

3- If  $I \odot P$ , then  $I$  is a connected component of the set  $E = \{x \in P, I_- <_P x\}$ . We have  $P \setminus I$  is a connected poset if and only if, for all  $x, y \in P \setminus I$  there exist  $a_1, \dots, a_n \in P \setminus I$  such that  $x \mathcal{R} a_1 \dots \mathcal{R} a_n \mathcal{R} y$ , where  $\mathcal{R}$  is defined by:  $t_1 \mathcal{R} t_2 \iff (t_1 <_P t_2 \text{ or } t_2 <_P t_1)$ . We assume that  $x, a_1, \dots, a_n, y$  is the smallest chain that connects  $x$  to  $y$ .

If  $P \setminus I$  is not connected, there is  $k \in [n]$  such that  $a_k \in I$ . By choosing  $k$  appropriately we can also assume that  $a_{k-1}$  or  $a_{k+1}$  are not in  $I$ , then we have four possible cases:

- First case;  $a_{k-1} \leq_P a_k \leq_P a_{k+1}$ . Then this chain is not the smallest chain that connects  $x$  to  $y$ , which is absurd.
- Second case;  $a_{k-1} \geq_P a_k \geq_P a_{k+1}$ , the proof is similar.
- Third case;  $a_{k-1} \geq_P a_k \leq_P a_{k+1}$ . Since  $a_k \in I$ , then  $I_- <_P a_k$ . Moreover  $<_P$  is transitive, then  $I_- <_P a_{k-1}$ , so  $a_{k-1} \in E$ . Then we obtained:  $(a_{k-1}, a_k) \in E \times I$ ,  $I$  is a connected component of  $E$  and  $a_k <_P a_{k-1}$ . Hence  $a_{k-1} \in I$ , and similarly  $a_{k+1} \in I$ . Then  $a_{k-1} \in I$  and  $a_{k+1} \in I$ , which is in contradiction with the assumptions.
- Fourth case;  $a_{k-1} \leq_P a_k \geq_P a_{k+1}$ . In this case we have  $a_{k-1} \in I_-$  and  $a_{k+1} \in I_-$ , so  $a_{k-1} = a_{k+1}$ . Then this chain is not the smallest chain that connects  $x$  to  $y$ , which is absurd.

Hence,  $P \setminus I$  is a connected poset. □

**Definition 3.1.** *The bilinear product  $\circ \rightarrow$  is defined in the species of finite connected posets  $\mathbb{U}$  as follows: for all  $P \in \mathbb{U}_{X_1}$  and  $Q \in \mathbb{U}_{X_2}$ , where  $X_1$  and  $X_2$  are two finite sets:*

$$P \circ \rightarrow Q = \sum_{v \in \min(Q)} P \searrow_v Q.$$

**Proposition 3.3.** *The bilinear product  $\circ \rightarrow$  endows the species  $\mathbb{U}$  with a twisted NAP algebra structure i.e., the following identity is verified:*

$$P \circ \rightarrow (Q \circ \rightarrow R) = Q \circ \rightarrow (P \circ \rightarrow R)$$

for any three finite connected posets  $P, Q$  and  $R$ .

*Proof.* Let  $P, Q$  and  $R$  be three finite connected posets, we have:

$$\begin{aligned} P \circ \rightarrow (Q \circ \rightarrow R) &= \sum_{v \in \min(R)} P \circ \rightarrow (Q \searrow_v R) \\ &= \sum_{u \in \min(Q \searrow_v R)} \sum_{v \in \min(R)} P \searrow_u (Q \searrow_v R), \end{aligned}$$

and since  $\min(Q \searrow_v R) = \min(R)$ , for all  $v \in \min(R)$  then:

$$P \circ \rightarrow (Q \circ \rightarrow R) = \sum_{v, u \in \min(R)} P \searrow_u (Q \searrow_v R),$$

Which is symmetric on  $P$  and  $Q$ . Correspondingly, we obtain:

$$P \circ \rightarrow (Q \circ \rightarrow R) = Q \circ \rightarrow (P \circ \rightarrow R).$$

□

**Corollary 3.1.** [2, Chapter 17] *Applying the functor  $\overline{\mathcal{K}}$  gives that  $(\overline{\mathcal{K}}(\mathbb{U}), \circ \rightarrow)$  is a NAP algebra.*

**Definition 3.2.** [15] *Let  $(V, \delta)$  be a coalgebra, i.e. be a vector space  $V$  together with linear map  $\delta : V \longrightarrow V \otimes V$ . The following defines a filtration on  $V$ :*

- $\text{Prim}(V) = V_1 = \{x \in V, \delta(x) = 0\}$ ,
- $V_n = \{x \in V, \delta(x) \in \bigoplus_{0 < k < n} V_k \otimes V_{n-k}\}$ .

The coalgebra  $(V, \delta)$  is said to be connected if  $V = \bigcup_{k > 0} V_k$ .

**Proposition 3.4.** *The coproduct  $\delta : \mathbb{U} \longrightarrow \mathbb{U} \otimes \mathbb{U}$  defined for any finite set  $X$  by*

$$\begin{aligned} \delta : \mathbb{U}_X &\longrightarrow (\mathbb{U} \otimes \mathbb{U})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{U}_Y \otimes \mathbb{U}_Z \\ P &\longmapsto \frac{1}{|\min(P)|} \sum_{I \in P} I \otimes P \setminus I, \end{aligned}$$

*is a twisted connected NAP coproduct, i.e. the following identity is verified*

$$(Id \otimes \delta)\delta = \tau^{12}(Id \otimes \delta)\delta,$$

where  $\tau^{12} = \tau \otimes Id$ ,  $\tau$  is the flip .

*Proof.* Let  $P$  be a finite connected poset. We have

$$\begin{aligned} (Id \otimes \delta)\delta(P) &= \frac{1}{|\min(P)|} \sum_{I \in P} I \otimes \delta(P \setminus I) \\ &= \frac{1}{|\min(P)|} \sum_{\substack{I \in P \\ J \in P \setminus I}} \frac{1}{|\min(P \setminus I)|} I \otimes J \otimes ((P \setminus I) \setminus J) \\ &= \frac{1}{|\min(P)|^2} \sum_{\substack{I, J \in P \\ I \cap J = \emptyset}} I \otimes J \otimes (P \setminus (I \sqcup J)) \\ &= \tau^{12}(Id \otimes \delta)\delta(P). \end{aligned}$$

By the Definition 2.6 in [15] and the definition of the coproduct  $\delta$ , it is clear that the coalgebra  $(\mathbb{U}, \delta)$  is connected. Hence,  $(\mathbb{U}, \delta)$  is a twisted connected NAP coalgebra.  $\square$

**Example 3.1.**  $\delta(\wedge) = 0$

$\delta(\vee) = 2 \cdot \otimes \dagger$

$\delta(\bowtie) = \frac{1}{2} \cdot \otimes \wedge$

$\delta(\diamond) = \wedge \otimes \cdot$

**Corollary 3.2.** [2, Chapter 17] *Applying the functor  $\overline{\mathcal{K}}$  gives that  $(\overline{\mathcal{K}}(\mathbb{U}), \delta)$  is a NAP coalgebra.*

**3.3. Free twisted pre-Lie algebras of finite posets.** In this subsection, we prove the compatibility relation [15, Theorem] between the pre-Lie product  $\searrow$  and the NAP coproduct  $\delta$ .

**Theorem 3.1.** *Let  $P, Q$  be two finite connected posets. We have the following identity*

$$\delta(P \searrow Q) = P \otimes Q + (P \otimes \mathbf{1} + \mathbf{1} \otimes P) \searrow \delta(Q),$$

where the unit  $\mathbf{1}$  is identified to the empty poset.

*Proof.* Let  $P = (X_1, \leq_P)$ ,  $Q = (X_2, \leq_Q)$  be two finite connected posets, we have:

$$\begin{aligned}
\delta(P \searrow Q) &= \sum_{v \in X_2} \delta(P \searrow_v Q) \\
&= \sum_{\substack{v \in X_2 \\ I \otimes (P \searrow_v Q)}} \frac{1}{|\min(P \searrow_v Q)|} I \otimes ((P \searrow_v Q) \setminus I) \\
&= \sum_{\substack{v \in X_2 \\ I \otimes (P \searrow_v Q)}} \frac{1}{|\min(Q)|} I \otimes ((P \searrow_v Q) \setminus I) \\
&= \frac{1}{|\min(Q)|} \left( \sum_{\substack{v \in X_2, v \in \min(Q) \\ I \otimes (P \searrow_v Q)}} I \otimes ((P \searrow_v Q) \setminus I) + \sum_{\substack{v \in X_2, v \notin \min(Q) \\ I \otimes (P \searrow_v Q)}} I \otimes ((P \searrow_v Q) \setminus I) \right),
\end{aligned}$$

we notice that

- if  $v \in \min(Q)$ , then  $\{I, I \otimes (P \searrow_v Q)\} = \{P\} \cup \{I, I \otimes Q\}$ ,

and

- if  $v \notin \min(Q)$ , then  $\{I, I \otimes (P \searrow_v Q)\} = \{P \searrow_v J, J \otimes Q, v \in J\} \cup \{J, J \otimes Q, v \notin J\}$ .

Then

$$\begin{aligned}
\delta(P \searrow Q) &= \frac{1}{|\min(Q)|} \left( \sum_{v \in X_2, v \in \min(Q)} P \otimes Q + \sum_{\substack{v \in X_2, v \in \min(Q) \\ I \otimes Q}} I \otimes (P \searrow_v (Q \setminus I)) \right) \\
&\quad + \frac{1}{|\min(Q)|} \left( \sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \in J}} (P \searrow_v J) \otimes Q \setminus J + \sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \notin J}} J \otimes (P \searrow_v (Q \setminus J)) \right),
\end{aligned}$$

we notice that

$$\sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \in J}} (P \searrow_v J) \otimes Q \setminus J = \sum_{J \otimes Q} (P \otimes \mathbf{1}) \searrow (J \otimes Q \setminus J),$$

and

$$\sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \notin J}} J \otimes (P \searrow_v (Q \setminus J)) = \sum_{J \otimes Q} J \otimes (P \searrow_v (Q \setminus J)),$$

then

$$\sum_{\substack{v \in X_2, v \in \min(Q) \\ I \otimes Q}} I \otimes (P \searrow_v (Q \setminus I)) + \sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \in J}} J \otimes (P \searrow_v (Q \setminus J)) = \sum_{J \otimes Q} (\mathbf{1} \otimes P) \searrow (J \otimes Q \setminus J),$$

accordingly

$$\delta(P \searrow Q) = \frac{1}{|\min(Q)|} \left( \sum_{v \in X_2, v \in \min(Q)} P \otimes Q + \sum_{J \otimes Q} (P \otimes \mathbf{1}) \searrow (J \otimes Q \setminus J) + \sum_{J \otimes Q} (\mathbf{1} \otimes P) \searrow (J \otimes Q \setminus J) \right).$$

Hence,

$$\delta(P \searrow Q) = P \otimes Q + (P \otimes \mathbf{1}) \searrow \delta(Q) + (\mathbf{1} \otimes P) \searrow \delta(Q).$$

□



**Corollary 3.3.** *Applying the functor  $\overline{\mathcal{K}}$  gives that,  $(\overline{\mathcal{K}}(\mathbb{U}), \searrow)$  endowed with the coproduct  $\delta$  is a free pre-Lie algebra and a cofree NAP coalgebra.*

*Proof.* This is a direct consequence of Theorem 3.1 and M. Livernet's rigidity theorem [15, Theorem].  $\square$

**Corollary 3.4.**  *$\overline{\mathcal{K}}(\mathbb{U})$  is generated as a pre-Lie algebra by  $\text{Prim}(\overline{\mathcal{K}}(\mathbb{U}))$ .*

*Proof.* Let  $\overline{\mathcal{K}}(\mathbb{U})_1 = \text{Prim}(\overline{\mathcal{K}}(\mathbb{U})) = \{P \in \overline{\mathcal{K}}(\mathbb{U}), \delta(P) = 0\}$ , and let  $\overline{\mathcal{K}}(\mathbb{U})_n = \{P \in \overline{\mathcal{K}}(\mathbb{U}), \delta(P) \in \sum_{0 < k < n} \overline{\mathcal{K}}(\mathbb{U})_k \otimes \overline{\mathcal{K}}(\mathbb{U})_{n-k}\}$ , we notice that  $\overline{\mathcal{K}}(\mathbb{U}) = \bigcup_{k>0} \overline{\mathcal{K}}(\mathbb{U})_k$ , i.e. the vector space  $(\overline{\mathcal{K}}(\mathbb{U}), \delta)$  is connected. By applying Corollary 3.9 in [15], we obtain that  $\overline{\mathcal{K}}(\mathbb{U})$  is generated as a pre-Lie algebra by  $\text{Prim}(\overline{\mathcal{K}}(\mathbb{U}))$ .  $\square$

**Example 3.2.** *Here are the posets in  $\text{Prim}(\overline{\mathcal{K}}(\mathbb{U}))$  up to four vertices:*



**3.4. Duality relation between  $\circ \rightarrow$  and  $\delta$ .** In this subsection, we prove that the NAP product  $\circ \rightarrow$  and the NAP coproduct  $\delta$  are dual to each other modulo a symmetry factor.

**Definition 3.3.** *Let  $G$  be a group acting on  $X$ . For every  $x \in X$ :*

- we denote by  $G \cdot x := \{g \cdot x, g \in G\}$  the orbit of  $x$ ,
- we denote by  $G_x = \{g \in G, g \cdot x = x\}$  the stabilizer subgroup of  $G$  with respect to  $x$ .

**Proposition 3.5.** *Let  $G$  be a group acting on  $X$ , if  $G$  and  $X$  is finite, then the orbit-stabilizer theorem, together with Lagrange's theorem [19, Theorem 3.9], gives:*

$$|G \cdot x| = [G_x : G] = \frac{|G|}{|G_x|}.$$

*In particular, the cardinal of the orbit is a divisor of the group order.*

**Definition 3.4.** *For any poset  $P$  on a finite set  $X$ , we denote by  $\text{Aut}(P)$  the subgroup of permutations of  $X$  which are homeomorphisms with respect to  $P$ . The symmetry factor is defined by  $\sigma(P) = |\text{Aut}(P)|$ . Let  $X_1, X_2$  two finite sets, we define the linear map  $\langle \cdot, \cdot \rangle : \mathbb{U}_{X_1} \otimes \mathbb{U}_{X_2} \rightarrow \mathbb{K}$  by:*

$$\langle Q, R \rangle = \begin{cases} \sigma(Q) & \text{if } Q \approx R, \\ 0 & \text{otherwise.} \end{cases}$$

*In other terms,  $\langle Q, R \rangle$  is the number of isomorphisms between  $Q$  and  $R$ .*

**Theorem 3.2.** *Let  $P, Q$  and  $R$  three finite connected posets. We have the following identity*

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\text{min}(P)|} \langle P, Q \circ \rightarrow R \rangle.$$

*Proof.* Let  $P, Q$  and  $R$  three finite connected posets.

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\text{min}(P)|} \sum_{\substack{I \otimes P \\ I \approx Q \\ P \setminus I \approx R}} \sigma(Q)\sigma(R) = \frac{1}{|\text{min}(P)|} |A|$$

with

$$A = \{(I, \sigma_1, \sigma_2) \mid I \otimes P, \sigma_1 : I \xrightarrow{\sim} Q, \sigma_2 : P \setminus I \xrightarrow{\sim} R\}$$

and:

$$\langle P, Q \circlearrowright R \rangle = \sum_{\substack{v \in \min(R) \\ Q \setminus_{\sim v} R \approx P}} \sigma(P) = |B|$$

with

$$B = \{(v, \sigma) \mid v \in \min(R), \sigma : P \xrightarrow{\sim} Q \setminus_{\sim v} R\}.$$

Let us define now a map  $\phi : A \longrightarrow B$ . Let  $(I, \sigma_1, \sigma_2) \in A$ . We put  $w = I_-$ . As  $I \odot P$ ,  $w \in \min(P \setminus I)$ , so  $v = \sigma_2(w) \in \min(R)$ . As  $I \odot P$ ,  $P = I \setminus_{\sim w} (P \setminus I)$ , so we obtain an isomorphism  $\sigma : P \longrightarrow Q \setminus_{\sim v} R$  by taking

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in I, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

We then put  $\phi(I, \sigma_1, \sigma_2) = (v, \sigma)$ .

Now, we define a map  $\psi : B \longrightarrow A$ . If  $(v, \sigma) \in B$ , we put  $I = \sigma^{-1}(Q)$ . As  $Q \odot Q \setminus_{\sim v} R$ ,  $I \odot P$ . Moreover,  $\sigma_1 = \sigma|_I$  is a graph isomorphism from  $I$  to  $Q$  and  $\sigma_2 = \sigma|_{P \setminus I}$  is a graph isomorphism from  $P \setminus I$  to  $R$ . We put  $\psi(v, \sigma) = (I, \sigma_1, \sigma_2)$ .

Let  $(v, \sigma) \in B$ . We put  $\psi(v, \sigma) = (I, \sigma_1, \sigma_2)$  and  $\phi \circ \psi(v, \sigma) = (v', \sigma')$ . Then,  $I = \sigma^{-1}(Q)$  and  $w' = I_- = w$ , so  $v' = \sigma(w) = v$ . Moreover,

$$\begin{aligned} \sigma'_I &= \sigma_1 = \sigma|_I, \\ \sigma'_{P \setminus I} &= \sigma_2 = \sigma|_{P \setminus I}, \end{aligned}$$

so  $\sigma' = \sigma$ . Hence,  $\phi \circ \psi = \text{id}_B$ .

Let  $(I, \sigma_1, \sigma_2) \in A$ . We put  $\phi(I, \sigma_1, \sigma_2) = (v, \sigma)$  and  $\psi \circ \phi(I, \sigma_1, \sigma_2) = (I', \sigma'_1, \sigma'_2)$ . Then  $I' = \sigma^{-1}(Q) = I$ , by construction of  $\sigma$ . Even more,

$$\begin{aligned} \sigma'_1 &= \sigma|_I = \sigma_1, \\ \sigma'_2 &= \sigma|_{P \setminus I} = \sigma_2. \end{aligned}$$

So  $\psi \circ \phi = \text{id}_A$ . Finally,  $A$  and  $B$  are in bijection and we obtain:

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(P)|} \langle P, Q \circlearrowright R \rangle.$$

□

**Remark 3.1.** As mentioned in the paper [3], the binary product  $\nearrow$  on the species  $\mathbb{U}$  of finite connected posets, defined by  $P \nearrow Q := j(j(P) \searrow j(Q))$  for any pair  $(P, Q)$  of finite connected posets, is a pre-Lie product, where  $j$  is the involution which transforms  $\leq$  into  $\geq$ . Similarly,

$$P \circlearrowright Q := j(j(P) \circlearrowright j(Q)), \quad (\text{resp. } \tilde{\delta} = (j \otimes j)\delta \circ j)$$

endows the species  $\mathbb{U}$  with a twisted NAP algebra (resp. twisted connected NAP coalgebra) structure.

**Notation.** Let  $E$  be any finite set, and let  $\pi, \rho$  be two partitions of  $E$ . We denote by  $j(\pi, \rho)$  the rational number

$$j(\pi, \rho) := \frac{1}{|E|} \sum_{\substack{\alpha \text{ is a block of } \pi \\ \beta \text{ is a block of } \rho}} |\alpha \cap \beta| \frac{|\beta|}{|\alpha|}.$$

**Proposition 3.6.** *Let  $P, Q$  and  $R$  three finite connected posets. We denote by  $E = \{v, v \in \min(R) \mid P \approx Q \searrow_v R\}$ . Let  $\pi$  be the partition of  $E$  into  $\text{Aut}(P)$ -orbits, and let  $\rho$  be the partition of  $E$  into  $\text{Aut}(R)$ -orbits. We have the following identity*

$$j(\pi, \rho) = 1.$$

*Proof.* Let  $P, Q$  and  $R$  be three finite connected posets. In fact: firstly,

$$\begin{aligned} \langle P, Q \circ \rightarrow R \rangle &= \sum_{v \in \min(R)} \langle P, Q \searrow_v R \rangle \\ &= \sum_{\substack{v \in \min(R) \\ P \approx Q \searrow_v R}} \sigma(P) \\ &= \sigma(P)|E|. \end{aligned}$$

On the other hand

$$\begin{aligned} \langle \delta(P), Q \otimes R \rangle &= \frac{1}{|\min(P)|} \sum_{I \in P} \langle I, Q \rangle \langle P \setminus I, R \rangle \\ &= \frac{1}{|\min(P)|} \sum_{\substack{I \in P \\ I \approx Q, P \setminus I \approx R}} \sigma(Q) \sigma(R) \\ &= \frac{1}{|\min(R)|} \sum_{\substack{I \in P \\ I \approx Q, P \setminus I \approx R}} \sigma(Q) \sigma(R) \\ &= \frac{1}{|\min(R)|} \sum_{v \in \min(R)} \sum_{\substack{I \in P, L = \{v\} \\ I \approx Q, P \setminus I \approx R}} \sigma(Q) \sigma(R) \\ &= \frac{1}{|\min(R)|} \sum_{v \in \min(R)} N_v(P, Q, R) \sigma(Q) \sigma(R), \end{aligned}$$

where  $N_v(P, Q, R) :=$  the number of branches  $I$  of  $P$  above  $v$  isomorphic to  $Q$  such that  $P \setminus I$  isomorphic to  $R$ .

We notice that, for all  $v \in \min(R)$ :

- if  $P \not\approx Q \searrow_v R$ , then  $N_v(P, Q, R) = 0$ ,
- if  $P \approx Q \searrow_v R$ , then  $N_v(P, Q, R) = N_v(Q, R) :=$  the number of branches of  $Q \searrow_v R$  above  $v$  isomorphic to  $Q$ .

Accordingly

$$\begin{aligned} \langle \delta(P), Q \otimes R \rangle &= \frac{1}{|\min(R)|} \left( \sum_{\substack{v \in \min(R) \\ P \approx Q \searrow_v R}} N_v(P, Q, R) \sigma(Q) \sigma(R) + \sum_{\substack{v \in \min(R) \\ P \not\approx Q \searrow_v R}} N_v(P, Q, R) \sigma(Q) \sigma(R) \right) \\ &= \frac{1}{|\min(R)|} \sum_{\substack{v \in \min(R) \\ P \approx Q \searrow_v R}} N_v(P, Q, R) \sigma(Q) \sigma(R) \\ &= \frac{1}{|\min(R)|} \sum_{\substack{v \in \min(R) \\ P \approx Q \searrow_v R}} N_v(Q, R) \sigma(Q) \sigma(R). \end{aligned}$$

By using the orbit-stabilizer theorem, as well as Lagrange's theorem for the group  $(Aut(R), \circ)$  endowed with the law of composition, we therefore have:

$$|Aut(R) \cdot v| = \frac{|Aut(R)|}{|Aut(R)_v|}, \quad i.e. \quad |Aut(R) \cdot v| = \frac{\sigma(R)}{\sigma_v(R)}.$$

We notice that, for all  $v \in \min(R)$ :  $\sigma_v(Q \searrow_v R) = \sigma(Q)\sigma_v(R)N_v(Q, R)$ .

Then

$$\begin{aligned} \langle \delta(P), Q \otimes R \rangle &= \frac{1}{|\min(R)|} \sum_{v \in E} \sigma(Q)\sigma_v(R)|Aut(R) \cdot v|N_v(Q, R) \\ &= \frac{1}{|\min(R)|} \sum_{v \in E} \sigma_v(P)|Aut(R) \cdot v| \\ &= \frac{1}{|\min(R)|} \sum_{v \in E} \sigma(P) \frac{|Aut(R) \cdot v|}{|Aut(P) \cdot v|} \\ &= \frac{1}{|\min(R)|} \sigma(P)S, \end{aligned}$$

where

$$S = \sum_{v \in E} \frac{|Aut(R) \cdot v|}{|Aut(P) \cdot v|}.$$

Let  $\pi$  be the partition of  $E$  into  $Aut(P)$ -orbits, and let  $\rho$  be the partition of  $E$  into  $Aut(R)$ -orbits. We denote by  $\alpha \in \pi$  (respectively,  $\beta \in \rho$ ), if  $\alpha$  is a block of  $\pi$  (respectively, if  $\beta$  is a block of  $\rho$ ). So

$$\begin{aligned} S &= \sum_{v \in E} \frac{|Aut(R) \cdot v|}{|Aut(P) \cdot v|} \\ &= \sum_{v \in E} \sum_{\substack{\alpha \text{ is a } Aut(P)\text{-orbit of } v \\ \beta \text{ is a } Aut(R)\text{-orbit of } v}} \frac{|\beta|}{|\alpha|} \\ &= \sum_{\substack{\alpha \text{ is a } Aut(P)\text{-orbit} \\ \beta \text{ is a } Aut(R)\text{-orbit}}} |\alpha \cap \beta| \frac{|\beta|}{|\alpha|} \\ &= j(\pi, \rho), \end{aligned}$$

then  $\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(R)|} \sigma(P)j(\pi, \rho)$ . Correspondingly,

$$\langle \delta(P), Q \otimes R \rangle = \langle P, Q \circ \rightarrow R \rangle \frac{j_{P,R}(\pi, \rho)}{|\min(P)|}.$$

By applying Theorem 3.2, we obtain:

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(P)|} \langle P, Q \circ \rightarrow R \rangle = \langle P, Q \circ \rightarrow R \rangle \frac{j(\pi, \rho)}{|\min(P)|}.$$

Conclusion: for  $P, Q$  and  $R$  three finite connected posets, if we denote by  $E = \{v, v \in \min(R) \mid P \approx Q \searrow_v R\}$ , and for  $\pi$  (resp.  $\rho$ ) the partition of  $E$  into  $Aut(P)$ -orbits (resp. the partition of  $E$  into  $Aut(R)$ -orbits). Then we obtain

$$j(\pi, \rho) = 1.$$

□

**Remark 3.2.** We do not have  $j(\pi, \rho) = 1$  in general for two partitions of a finite set  $E$ . Indeed: let  $E$  be the finite set  $\{1, 2, 3, 4\}$ ,  $\pi = \{\{1, 2\}, \{3, 4\}\}$  and  $\rho = \{\{1, 2, 3\}, \{4\}\}$ . Then we obtain  $j(\pi, \rho) = \frac{5}{4}$  and  $j(\rho, \pi) = 1$ . This interesting index does not seem to be present in the literature.

**3.5. Free Lie algebras of finite posets.** F. Chapoton in [7] showed that free pre-Lie algebras are free as Lie algebras i.e., if  $(V, \triangleright)$  is a free pre-Lie algebra. Then it is a free Lie algebra for the Lie bracket  $[\cdot, \cdot]$  defined by  $[a, b] = a \triangleright b - b \triangleright a$ . This result has been obtained before with different methods by L. Foissy [13, Theorem 8.4].

**Proposition 3.7.** [7, 13] For any vector space  $V$ , the free pre-Lie algebra on  $V$  is isomorphic as a Lie algebra to the free Lie algebra on some set  $X(V)$  of generators.

**Application [17]** Let  $H$  be a Hopf algebra and  $H^0$  the graded dual of  $H$ . The primitive element algebra of the graded dual  $H^0$  with the bracket  $[\cdot, \cdot]$  is a Lie algebra. We denote by  $\star$  the Grossman-Larson product on the dual of  $H$  [3]. This product restricted to  $H^0$  is the graded dual of the coproduct  $\Delta_{\searrow}$  [3]. Applying the general setting above to the case  $H = \overline{\mathcal{K}}(\mathbb{P})$  we are studying, we obtain:

$(\overline{\mathcal{K}}(\mathbb{U}), \searrow) = (Prim(H^0), \searrow)$  is a free pre-Lie algebra, then it is also a free Lie algebra, therefore  $U(PrimH^0) = (S(PrimH^0), \star)$  is a free associative algebra. Then,  $(S(PrimH^0), \star)^0 = (H, \Delta_{\searrow})$  is a coassociative cofree coalgebra.

#### 4. GENERALIZATION TO FINITE TOPOLOGICAL SPACES

All the results in this paper remain true for the finite connected topological spaces, with a small change on the definition of the coproduct  $\delta$ . Indeed: let  $\mathcal{T} = (X, \leq_{\mathcal{T}})$  be a finite connected topological space. We define

$$\delta(\mathcal{T}) = \frac{1}{|\min(\mathcal{T})|} \sum_{Y \in \mathcal{T}} \mathcal{T}_Y \otimes \mathcal{T}_{|X \setminus Y},$$

where  $Y \in \mathcal{T}$  means that  $Y \in \mathcal{T}$  such that:

- $\overline{Y}_-$  is a singleton included in  $\min(\overline{\mathcal{T}})$ ,
- and  $Y$  is a branch of the set  $\{x \in X, \text{ such that for all } y \in Y_-, \text{ we have } y <_{\mathcal{T}} x \text{ and } y \not\leq x\}$ .

Here, the set  $Y_-$  equal to the space  $\{x \notin Y, \text{ there exists } y \in Y \text{ such that } x \leq y\}$ ,  $\overline{\mathcal{T}} = (X / \sim_{\mathcal{T}}, \leq)$  is the poset defined on  $X / \sim_{\mathcal{T}}$  corresponding to the partial order  $\leq$  induced by the quasi-order  $\leq_{\mathcal{T}}$ , and  $\overline{Y}_- = Y_- / \sim_{\mathcal{T}_{Y_-}}$ . Let  $Aut(\mathcal{T})$  be the subgroup of permutations of  $X$  which are homeomorphisms with respect to  $\mathcal{T}$ , and let  $G = \{g \in Aut(\mathcal{T}), g \cdot \overline{v} = \overline{v} \text{ for all } v \in X\}$ . We notice that  $G \subseteq Aut(\mathcal{T})$ ,  $G$  is a normal subgroup and  $Aut(\overline{\mathcal{T}}) = Aut(\mathcal{T})/G$ .

We notice that the results obtained in Lemma 3.1 also work for  $Y, Z \in \mathcal{T}$ . From  $Y, Z \in \mathcal{T}$  and Lemma 3.1, we get that both  $\overline{Y}, \overline{Z} \in \overline{\mathcal{T}}$ , then  $(\overline{Y} = \overline{Z} \text{ or } \overline{Y} \cap \overline{Z} = \emptyset)$ , and  $\min(\overline{\mathcal{T}}_{|X \setminus Y}) = \min(\overline{\mathcal{T}})$ . Then  $Y \cap Z = \emptyset$  and  $\min(\mathcal{T}_{|X \setminus Y}) = \min(\mathcal{T})$ .

Hence to show that the proposition 3.4 also works in topological cases, it suffices to change  $I, J \in P$  by  $Y, Z \in \mathcal{T}$  in the proof of proposition 3.6. To show that the result of Theorem 3.1 also works in connected finite topological spaces, we change  $I \in P$  by  $Y \in \mathcal{T}$  in the Proof 3.3.

Finally we prove the existence of a relation between  $\delta$  and  $\circlearrowright$  in the topological case. Exactly: for  $\mathcal{T}, \mathcal{T}'$  and  $\mathcal{T}''$  three finite connected topological spaces, we have

$$\langle \delta(\mathcal{T}), \mathcal{T}' \otimes \mathcal{T}'' \rangle = \frac{1}{n|\min(\mathcal{T})|} \langle \mathcal{T}, \mathcal{T}' \circlearrowright \mathcal{T}'' \rangle,$$

where  $n$  is the number of items in any bag  $\bar{v} \in \min(\overline{\mathcal{T}''})$ , such as  $\mathcal{T} \approx \mathcal{T}' \searrow_{\bar{v}} \mathcal{T}''$ . The proof is similar to the proof of Theorem 3.2. Indeed:

$$\langle \delta(\mathcal{T}), \mathcal{T}' \otimes \mathcal{T}'' \rangle = \frac{1}{|\min(\mathcal{T})|} \sum_{\substack{Y \otimes \mathcal{T} \\ \mathcal{T}|_Y \approx \mathcal{T}', \\ \mathcal{T}|_{X \setminus Y} \approx \mathcal{T}''}} \sigma(\mathcal{T}') \sigma(\mathcal{T}'') = \frac{1}{|\min(\mathcal{T})|} |A|$$

with

$$A = \{(Y, \sigma_1, \sigma_2) \mid Y \otimes \mathcal{T}, \sigma_1 : \mathcal{T}|_Y \xrightarrow{\sim} \mathcal{T}', \sigma_2 : \mathcal{T}|_{X \setminus Y} \xrightarrow{\sim} \mathcal{T}''\}.$$

On the other hand

$$\begin{aligned} \langle \mathcal{T}, \mathcal{T}' \circlearrowright \mathcal{T}'' \rangle &= \sum_{v \in \min(\mathcal{T}'')} \langle \mathcal{T}, \mathcal{T}' \searrow_v \mathcal{T}'' \rangle \\ &= \sum_{\substack{v \in \min(\mathcal{T}'') \\ \mathcal{T} \approx \mathcal{T}' \searrow_v \mathcal{T}''}} \sigma(\mathcal{T}) = |B|, \end{aligned}$$

with

$$B = \{(v, \sigma) \mid v \in \min(\mathcal{T}''), \sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}' \searrow_v \mathcal{T}''\}.$$

Let

$$C = \{(u, \sigma) \mid u = \bar{v}, v \in \min(\mathcal{T}''), \sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}' \searrow_v \mathcal{T}''\}.$$

We notice that for all  $u, u' \in C$ , we have  $|u| = |u'|$ . Hence  $\frac{|B|}{|C|}$  is the number of items in any bag  $\bar{v} \in \min(\overline{\mathcal{T}''})$ , such as  $\mathcal{T} \approx \mathcal{T}' \searrow_v \mathcal{T}''$ . We define

$$\begin{aligned} f : A &\longrightarrow C \\ (Y, \sigma_1, \sigma_2) &\longmapsto (u, \sigma), \end{aligned}$$

where  $u = \bar{v} = \overline{\sigma_2(Y_-)}$  and  $\sigma$  is a graph isomorphism from  $\mathcal{T}$  to  $\mathcal{T}' \searrow_v \mathcal{T}''$  defined by

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in Y, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

Now, we define a map

$$\begin{aligned} g : C &\longrightarrow A \\ (u, \sigma) &\longmapsto (Y, \sigma_1, \sigma_2), \end{aligned}$$

where  $Y = X_{\sigma^{-1}(\mathcal{T}')}$ ,  $\sigma_1 = \sigma|_{\mathcal{T}|_Y}$  is a graph isomorphism from  $\mathcal{T}|_Y$  to  $\mathcal{T}'$  and  $\sigma_2 = \sigma|_{\mathcal{T}|_{X \setminus Y}}$  is a graph isomorphism from  $\mathcal{T}|_{X \setminus Y}$  to  $\mathcal{T}''$ .

Let  $(u, \sigma) \in C$ , we put  $g(u, \sigma) = (Y, \sigma_1, \sigma_2)$ , where  $Y = X_{\sigma^{-1}(\mathcal{T}')}$ ,  $\sigma_1 = \sigma|_{\mathcal{T}'}$  and  $\sigma_2 = \sigma|_{\mathcal{T}''}$ . Then  $f \circ g(u, \sigma) = (u', \sigma')$ , with  $u' = \bar{w} = \overline{\sigma_2(Y_-)}$  and  $\sigma'$  is a graph isomorphism from  $\mathcal{T}$  to  $\mathcal{T}' \searrow_w \mathcal{T}''$  defined by

$$\sigma'(x) = \begin{cases} \sigma_1(x) & \text{if } x \in Y, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

Then,  $u' = \overline{\sigma_2(Y_-)} = u$  and  $\sigma' = \sigma$ . Hence  $f \circ g = Id_C$ .

Let  $(Y, \sigma_1, \sigma_2) \in A$ , we put  $f(Y, \sigma_1, \sigma_2) = (u, \sigma)$ , with  $u = \bar{v} = \overline{\sigma_2(Y_-)}$  and  $\sigma$  is a graph isomorphism from  $\mathcal{T}$  to  $\mathcal{T}' \searrow_{\mathcal{V}} \mathcal{T}''$  defined by

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in Y, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

Then,  $g \circ f(Y, \sigma_1, \sigma_2) = (Y', \sigma'_1, \sigma'_2)$ , with  $Y' = X_{\sigma^{-1}(\mathcal{T}')}$ ,  $\sigma'_1 = \sigma|_{\mathcal{T}_{|Y'}}$  is a graph isomorphism from  $\mathcal{T}_{|Y'}$  to  $\mathcal{T}'$  and  $\sigma'_2 = \sigma|_{\mathcal{T}_{|X \setminus Y'}}$  is a graph isomorphism from  $\mathcal{T}_{|X \setminus Y'}$  to  $\mathcal{T}''$ .

Then,  $Y' = X_{\sigma^{-1}(\mathcal{T}')} = Y$  and

$$\begin{cases} \sigma'_1 = \sigma|_{\mathcal{T}_{|Y'}} = \sigma_1, \\ \sigma'_2 = \sigma|_{\mathcal{T}_{|X \setminus Y'}} = \sigma_2. \end{cases}$$

So  $g \circ f = Id_A$ .

Finally,  $A$  and  $C$  are in bijection. Hence

$$\langle \delta(\mathcal{T}), \mathcal{T}' \otimes \mathcal{T}'' \rangle = \frac{1}{|\min(\mathcal{T})|} |A| = \frac{1}{|\min(\mathcal{T})|} |C| = \frac{1}{|\min(\mathcal{T})|} |C| \frac{|B|}{|B|} = \frac{|C|}{|B| |\min(\mathcal{T})|} \langle \mathcal{T}, \mathcal{T}' \circ \rightarrow \mathcal{T}'' \rangle.$$

**Remark.** All the results in this paper remain true for the posets (resp. topological spaces) decorated by  $F$ , that is to say pairs  $(\mathcal{A}, f)$ , where  $\mathcal{A} = (A, \leq_A)$  is a poset (resp. topology) and  $f : A \rightarrow F$  is a map.

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