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FREE PRE-LIE ALGEBRAS OF FINITE POSETS

M. AYADI

ABSTRACT. We first recall the construction of a twisted pre-Lie algebra structure on the species of finite connected topological spaces. Then, we construct a corresponding non-coassociative permutative (NAP) coproduct on the subspecies of finite connected T_0 topological spaces, i.e., finite connected posets, and we prove that the vector space generated by isomorphism classes of finite posets is a free pre-Lie algebra and also a cofree NAP coalgebra. Further, we give an explicit duality between the non-associative permutative product and the proposed NAP coproduct. Finally, we prove that the results in this paper remain true for the finite connected topological spaces.

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1. INTRODUCTION

In this paper, we introduce a twisted non-associative permutative algebra structure on the species of finite connected posets. We recall the definition of a non-associative permutative algebra (in short NAP algebra), and the dual definition of a non-coassociative permutative coalgebra (in short NAP coalgebra). A (left) NAP algebra is a vector space V equipped with a bilinear product \cdot satisfying the relation:

$$(1.1) \quad a \cdot (b \cdot c) = b \cdot (a \cdot c), \text{ for all } a, b \text{ and } c \in V.$$

We note also that the notion of NAP algebra has emerged from the reference [15] by M. Livernet. It has also been previously defined in [11] by A. Dzhumadil'daev, and C. Löfwall under the name of "left commutative algebra". Dually a (left) NAP coalgebra is a vector space V equipped with

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a bilinear coproduct $\delta: V \rightarrow V \otimes V$ satisfying the relation:

$$(1.2) \quad (Id \otimes \delta)\delta = \tau^{12}(Id \otimes \delta)\delta.$$

We recall the following rigidity theorem [15]: any pre-Lie algebra (V, \triangleright) , together with a non-associative permutative connected coproduct δ satisfying the distributive law

$$(1.3) \quad \delta(a \triangleright b) = a \otimes b + (a \otimes \mathbf{1} + \mathbf{1} \otimes a) \triangleright \delta(b),$$

is a free pre-Lie algebra and a cofree NAP coalgebra.

We recall also in this paper the description of the free pre-Lie algebra in terms of rooted trees given in [10]. We define a new bilinear product $\circ \rightarrow$ in the species of finite connected posets \mathbb{U} by: for all $P \in \mathbb{U}_{X_1}$ and $Q \in \mathbb{U}_{X_2}$, where X_1 and X_2 are two finite sets

$$P \circ \rightarrow Q = \sum_{v \in \min(Q)} P \searrow_v Q,$$

where $P \searrow_v Q$ is obtained from the Hasse graphs G_1 and G_2 of P and Q by adding an (oriented) edge from v in G_2 to any minimal vertex of G_1 [3].

We show that the bilinear product $\circ \rightarrow$ verifies (1.1) in the monoidal category of species, and therefore endows the species \mathbb{U} with a NAP algebra structure.

For any finite set X , we define the coproduct δ by:

$$\begin{aligned} \delta: \mathbb{U}_X &\longrightarrow (\mathbb{U} \otimes \mathbb{U})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{U}_Y \otimes \mathbb{U}_Z \\ P &\longmapsto \frac{1}{|\min(P)|} \sum_{I \odot P} I \otimes P \setminus I, \end{aligned}$$

where $I \odot P$ means that I is a subset of P such that:

- I_- is a singleton included in $\min(P)$,
- and I is a connected component of the set $\{x \in P, I_- <_P x\}$.

with the set I_- is equal to the space $\{x \notin I, \text{ there exists } y \in I \text{ such that } x \leq_P y\}$.

We prove that the coproduct δ is a twisted NAP coproduct, i.e., (1.2) is verified in the monoidal category of species.

M. Livernet in [15] proved the following rigidity theorem: for any pre-Lie algebra (V, \triangleright) together with a NAP connected coproduct δ satisfying relation (1.3), V is the free pre-Lie algebra generated by $\text{Prim}(V)$.

Later in this paper, we prove a compatibility relation between the pre-Lie \searrow structure and the coproduct δ by proving the twisted version of (1.3) namely:

$$\delta(P \searrow Q) = P \otimes Q + (P \otimes \mathbf{1} + \mathbf{1} \otimes P) \searrow \delta(Q),$$

for any pair (P, Q) of finite connected posets, where the unit $\mathbf{1}$ is identified to the empty poset.

Applying Aguiar-Mahajan's bosonic Fock functor $\overline{\mathcal{K}}$ [2] and M. Livernet's rigidity theorem [15] leads to the main result of the paper: $(\overline{\mathcal{K}}(\mathbb{U}), \searrow)$ endowed with the coproduct δ is a free pre-Lie algebra and a cofree NAP coalgebra.

We end up Section 3 by proving that the NAP product $\circ\rightarrow$ and the NAP coproduct δ are dual to each other, and we show that $(\overline{\mathcal{K}(\mathbb{P})}, \Delta_{\searrow})$ is a coassociative cofree coalgebra, where Δ_{\searrow} is the coassociative coproduct defined in [3].

In the last section, we prove that the results in this paper remain true for the finite connected topological spaces, with a small change on the definition of the coproduct δ .

2. BASICS OF FINITE TOPOLOGIES

A partial order on a set X is a transitive, reflexive and antisymmetric relation on X . A finite poset is a finite set X endowed with a partial order \leq . Let $P = (X, \leq_P)$ and $Q = (X, \leq_Q)$ be two posets. We say that P is finer than Q if: $x \leq_P y \implies x \leq_Q y$ for any $x, y \in X$. The Hasse diagram of poset $P = (X, \leq_P)$ is obtained by representing any element of X by a vertex, and by drawing a directed edge from a to b if and only if $a <_P b$, and, for any $c \in X$ such that $a \leq_P c \leq_P b$, one has $a = c$ or $b = c$. We can say that I is an upper ideal of P ($I \subset P$) if, for all $x, y \in P$, $(x \in I, x \leq_P y) \implies y \in I$. We denote $J(P)$ the set of upper ideals of P .

Recall [1, 12] that a finite topological space is a finite quasi-poset and vice versa. Any topology \mathcal{T} (hence any quasi-order on X) gives rise to an equivalence relation:

$$(2.1) \quad x \sim_{\mathcal{T}} y \iff (x \leq_{\mathcal{T}} y \text{ and } y \leq_{\mathcal{T}} x).$$

Equivalence classes will be called bags here. This equivalence relation is trivial if and only if the quasi-order is a (partial) order, or equivalently, if the corresponding topology is T_0 . Any topology \mathcal{T} on X defines a poset structure on the quotient $X / \sim_{\mathcal{T}}$, corresponding to the partial order induced by the quasi-order $\leq_{\mathcal{T}}$. More on finite topological spaces can be found in [4, 9, 12, 18].

Recall [2, 14] that a linear (tensor) species is a contravariant functor from the category of finite sets **Fin** with bijections into the category **Vect** of vector spaces (on some field \mathbf{k}). Let \mathbb{E} be a linear species. We note $\overline{\mathcal{K}(\mathbb{E})} = \bigoplus_{n \geq 0} \mathbb{E}_n / S_n$, where \mathbb{E}_n / S_n denotes the space of S_n -coinvariants of

\mathbb{E}_n . The functor $\overline{\mathcal{K}}$ from linear species to graded vector spaces is intensively studied in [2, chapter 15] under the name "bosonic Fock functor".

The species \mathbb{T} of finite topological spaces is defined as follows: for any finite set X , \mathbb{T}_X is the vector space freely generated by the topologies on X . For any bijection $\varphi : X \rightarrow X'$, the isomorphism $\mathbb{T}_{\varphi} : \mathbb{T}_{X'} \rightarrow \mathbb{T}_X$ is defined for any topology \mathcal{T} on X' , by the following relabelling:

$$\mathbb{T}_{\varphi}(\mathcal{T}) = \{\varphi^{-1}(Y), Y \in \mathcal{T}\}$$

It is well-known that the species \mathbb{T} of finite topological spaces is a twisted commutative Hopf algebra [12]: the product m is given by disjoint union, and the coproduct Δ is a natural generalization of the Connes-Kremer coproduct on rooted forests. We defined another twisted Hopf

algebra structure in [3] by replacing Δ with the coproduct

$$\begin{aligned} \Delta_{\searrow} : \mathbb{T}_X &\longrightarrow (\mathbb{T} \otimes \mathbb{T})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{T}_Y \otimes \mathbb{T}_Z \\ \mathcal{T} &\longmapsto \sum_{Y \in \mathcal{T}} \mathcal{T}_{|Y} \otimes \mathcal{T}_{|X \setminus Y}. \end{aligned}$$

Where $Y \in \overline{\mathcal{T}}$, stands for

- $Y \in \mathcal{T}$,
- $\mathcal{T}_{|Y} = \mathcal{T}_1 \dots \mathcal{T}_n$, such that for all $i \in \{1, \dots, n\}$, \mathcal{T}_i is connected and $(\min \mathcal{T}_i = (\min \mathcal{T}) \cap \mathcal{T}_i$, or there is a single common ancestor $x_i \in \overline{X \setminus Y}$ to $\min \mathcal{T}_i$), where $\overline{X \setminus Y} = (X \setminus Y) / \sim_{\mathcal{T}_{|X \setminus Y}}$.

If we apply the functor $\overline{\mathcal{K}}$, we therefore two commutative graded connected Hopf algebras $\overline{\mathcal{K}}(\mathbb{T}, m, \Delta)$ and $\overline{\mathcal{K}}(\mathbb{T}, m, \Delta_{\searrow})$.

Definition 2.1. [8, 16] *A left pre-Lie algebra over a field \mathbf{k} is a \mathbf{k} -vector space A with a binary composition \triangleright that satisfies the left pre-Lie identity:*

$$(2.2) \quad (x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z),$$

for all $x, y, z \in A$. The left pre-Lie identity rewrites as:

$$(2.3) \quad L_{[x,y]} = [L_x, L_y],$$

where $L_x : A \longrightarrow A$ is defined by $L_x y = x \triangleright y$, and where the bracket on the left-hand side is defined by $[x, y] = x \triangleright y - y \triangleright x$. As a consequence this bracket satisfies the Jacobi identity.

In fact, as mentioned in [3], the structure \searrow on the species of connected finite topological spaces \mathbb{V} defined by: for all $\mathcal{T}_1 = (X_1, \leq_{\mathcal{T}_1})$ and $\mathcal{T}_2 = (X_2, \leq_{\mathcal{T}_2})$ two finite connected topological spaces as:

$$\mathcal{T}_1 \searrow \mathcal{T}_2 = \sum_{v \in X_2} \mathcal{T}_1 \searrow_v \mathcal{T}_2,$$

is a pre-Lie structure, where $\mathcal{T}_1 \searrow_v \mathcal{T}_2$ is obtained from the Hasse graphs G_1 and G_2 of \mathcal{T}_1 and \mathcal{T}_2 by adding an (oriented) edge from v in G_2 to any minimal vertex of G_1 .

We denote by \mathbb{P} the sub-species of \mathbb{T} consisting in partial order (i.e., T_0 -topologies). The binary product m (resp. \searrow) restricts to finite posets (resp. finite connected posets), as well as both coproducts Δ (resp. Δ_{\searrow}). Hence the triple (\mathbb{P}, m, Δ) is a twisted Hopf subalgebra of (\mathbb{T}, m, Δ) , the triple $(\mathbb{P}, m, \Delta_{\searrow})$ is a twisted Hopf subalgebra of $(\mathbb{T}, m, \Delta_{\searrow})$ and (\mathbb{P}, \searrow) is a twisted pre-Lie subalgebra of (\mathbb{T}, \searrow) .

3. FREE PRE-LIE ALGEBRAS AND COFREE COALGEBRAS

3.1. Free pre-Lie algebras and rooted trees. Let T be the vector space spanned by the set of isomorphism classes of rooted trees and $H = S(T)$. Grafting pre-Lie algebras of rooted trees were studied for the first time by F. Chapoton and M. Livernet [10], although the use of rooted trees can be traced back to A. Cayley [5]. The grafting product is given, for all $t, s \in T$, by:

$$(3.1) \quad t \rightarrow s = \sum_{s' \text{ vertex of } s} t \rightarrow_{s'} s,$$

where $t \rightarrow_{s'} s$ is the tree obtained by grafting the root of t on the vertex s' of s . In other words, the operation $t \rightarrow s$ consists of grafting the root of t on every vertex of s and summing up.

We note $T(n)$ the linear span of rooted trees of degree n . For a vector space V , we denote by $T(V)$ the space $\bigoplus_n T(n) \otimes V^{\otimes n} / S_n$. This is the linear span of rooted trees decorated by a basis of V . Here S_n is the symmetric group on n elements. Following the notation of Connes and Kreimer [6], any tree t writes $t := B(v, t_1, t_2, \dots, t_n)$ that is decorated by v , and t_1, \dots, t_n are trees.

We notice that, if using the pre-Lie product \rightarrow in $T(V)$, one has:

$$B(v, t_1, \dots, t_n) = t_n \rightarrow B(v, t_1, \dots, t_{n-1}) - \sum_{0 < k < n} B(v, t_1, \dots, t_n \rightarrow t_k, \dots, t_{n-1})$$

Proposition 3.1. [10] *Equipped by \rightarrow , the space T is the free pre-Lie algebra with one generator.*

Proposition 3.2. [15] *Let V be a vector space. Then, $T(V)$ together with the coproduct*

$$\delta(B(v, t_1, t_2, \dots, t_n)) = \sum_{0 < k < n+1} t_k \otimes B(v, t_1, \dots, \hat{t}_k, \dots, t_n),$$

is the cofree NAP connected coalgebra generated by V .

3.2. Twisted NAP algebras of finite connected posets. Let P be a finite connected poset, and I be a subset of P . We denote by I_- the set $\{x \notin I, \text{ there exists } y \in I \text{ such as } x \leq_P y\}$.

We note $I \odot P$ whenever:

- I_- is a singleton included in $\min(P)$,
- and I is a connected component of the set $\{x \in P, I_- <_P x\}$.

Lemma 3.1. *For any finite connected poset P , and any subsets I, J of P , we have:*

- 1- *if $I \odot P$ and $J \odot P$, then $I = J$ or $I \cap J = \emptyset$.*
- 2- *If $I \odot P$, then $\min(P \setminus I) = \min(P)$.*
- 3- *If $I \odot P$, then $P \setminus I$ is a connected poset.*

Proof. 1- Let P be a finite connected poset and I, J two subsets of P , such that $I, J \odot P$. So, we have two possible cases: either $I_- \neq J_-$ or $I_- = J_-$.

- If $I_- \neq J_-$, with $I_- = \{y\}$, and $J_- = \{z\}$, then I is a connected component of the set $E_1 = \{x \in P, y <_P x\}$ and J is a connected component of the set $E_2 = \{x \in P, z <_P x\}$. So I and J is a connected component of the set $E_1 \sqcup E_2$, then $I = J$ or $I \cap J = \emptyset$. Gold $I_- \neq J_-$, then $I \neq J$. Hence $I \cap J = \emptyset$.

- If $I_- = J_- = \{y\}$, then I and J is two connected components of the set $\{x \in P, y <_P x\}$, then: $I = J$ or $I \cap J = \emptyset$.

2- If there exists $x \in \min(P) \cap I$, then $I_- <_P x$, which is absurd. Thus, $\min(P) \cap I = \emptyset$. Then, $\min(P \setminus I) = \min(P)$.

3- If $I \odot P$, then I is a connected component of the set $E = \{x \in P, I_- <_P x\}$. We have $P \setminus I$ is a connected poset if and only if, for all $x, y \in P \setminus I$ there exist $a_1, \dots, a_n \in P \setminus I$ such that $x \mathcal{R} a_1 \dots \mathcal{R} a_n \mathcal{R} y$, where \mathcal{R} is defined by: $t_1 \mathcal{R} t_2 \iff (t_1 <_P t_2 \text{ or } t_2 <_P t_1)$. We assume that x, a_1, \dots, a_n, y is the smallest chain that connects x to y .

If $P \setminus I$ is not connected, there is $k \in [n]$ such that $a_k \in I$. By choosing k appropriately we can also assume that a_{k-1} or a_{k+1} are not in I , then we have four possible cases:

- First case; $a_{k-1} \leq_P a_k \leq_P a_{k+1}$. Then this chain is not the smallest chain that connects x to y , which is absurd.
- Second case; $a_{k-1} \geq_P a_k \geq_P a_{k+1}$, the proof is similar.
- Third case; $a_{k-1} \geq_P a_k \leq_P a_{k+1}$. Since $a_k \in I$, then $I_- <_P a_k$. Moreover $<_P$ is transitive, then $I_- <_P a_{k-1}$, so $a_{k-1} \in E$. Then we obtained: $(a_{k-1}, a_k) \in E \times I$, I is a connected component of E and $a_k <_P a_{k-1}$. Hence $a_{k-1} \in I$, and similarly $a_{k+1} \in I$. Then $a_{k-1} \in I$ and $a_{k+1} \in I$, which is in contradiction with the assumptions.
- Fourth case; $a_{k-1} \leq_P a_k \geq_P a_{k+1}$. In this case we have $a_{k-1} \in I_-$ and $a_{k+1} \in I_-$, so $a_{k-1} = a_{k+1}$. Then this chain is not the smallest chain that connects x to y , which is absurd.

Hence, $P \setminus I$ is a connected poset. □

Definition 3.1. *The bilinear product $\circ \rightarrow$ is defined in the species of finite connected posets \mathbb{U} as follows: for all $P \in \mathbb{U}_{X_1}$ and $Q \in \mathbb{U}_{X_2}$, where X_1 and X_2 are two finite sets:*

$$P \circ \rightarrow Q = \sum_{v \in \min(Q)} P \searrow_v Q.$$

Proposition 3.3. *The bilinear product $\circ \rightarrow$ endows the species \mathbb{U} with a twisted NAP algebra structure i.e., the following identity is verified:*

$$P \circ \rightarrow (Q \circ \rightarrow R) = Q \circ \rightarrow (P \circ \rightarrow R)$$

for any three finite connected posets P, Q and R .

Proof. Let P, Q and R be three finite connected posets, we have:

$$\begin{aligned} P \circ \rightarrow (Q \circ \rightarrow R) &= \sum_{v \in \min(R)} P \circ \rightarrow (Q \searrow_v R) \\ &= \sum_{u \in \min(Q \searrow_v R)} \sum_{v \in \min(R)} P \searrow_u (Q \searrow_v R), \end{aligned}$$

and since $\min(Q \searrow_v R) = \min(R)$, for all $v \in \min(R)$ then:

$$P \circ \rightarrow (Q \circ \rightarrow R) = \sum_{v, u \in \min(R)} P \searrow_u (Q \searrow_v R),$$

Which is symmetric on P and Q . Correspondingly, we obtain:

$$P \circ \rightarrow (Q \circ \rightarrow R) = Q \circ \rightarrow (P \circ \rightarrow R).$$

□

Corollary 3.1. [2, Chapter 17] *Applying the functor $\overline{\mathcal{K}}$ gives that $(\overline{\mathcal{K}}(\mathbb{U}), \circ \rightarrow)$ is a NAP algebra.*

Definition 3.2. [15] *Let (V, δ) be a coalgebra, i.e. be a vector space V together with linear map $\delta : V \rightarrow V \otimes V$. The following defines a filtration on V :*

- $\text{Prim}(V) = V_1 = \{x \in V, \delta(x) = 0\}$,
- $V_n = \{x \in V, \delta(x) \in \bigoplus_{0 < k < n} V_k \otimes V_{n-k}\}$.

The coalgebra (V, δ) is said to be connected if $V = \bigcup_{k > 0} V_k$.

Proposition 3.4. *The coproduct $\delta : \mathbb{U} \longrightarrow \mathbb{U} \otimes \mathbb{U}$ defined for any finite set X by*

$$\begin{aligned} \delta : \mathbb{U}_X &\longrightarrow (\mathbb{U} \otimes \mathbb{U})_X = \bigoplus_{Y \sqcup Z = X} \mathbb{U}_Y \otimes \mathbb{U}_Z \\ P &\longmapsto \frac{1}{|\min(P)|} \sum_{I \in P} I \otimes P \setminus I, \end{aligned}$$

is a twisted connected NAP coproduct, i.e. the following identity is verified

$$(Id \otimes \delta)\delta = \tau^{12}(Id \otimes \delta)\delta,$$

where $\tau^{12} = \tau \otimes Id$, τ is the flip .

Proof. Let P be a finite connected poset. We have

$$\begin{aligned} (Id \otimes \delta)\delta(P) &= \frac{1}{|\min(P)|} \sum_{I \in P} I \otimes \delta(P \setminus I) \\ &= \frac{1}{|\min(P)|} \sum_{\substack{I \in P \\ J \in P \setminus I}} \frac{1}{|\min(P \setminus I)|} I \otimes J \otimes ((P \setminus I) \setminus J) \\ &= \frac{1}{|\min(P)|^2} \sum_{\substack{I, J \in P \\ I \cap J = \emptyset}} I \otimes J \otimes (P \setminus (I \sqcup J)) \\ &= \tau^{12}(Id \otimes \delta)\delta(P). \end{aligned}$$

By the Definition 2.6 in [15] and the definition of the coproduct δ , it is clear that the coalgebra (\mathbb{U}, δ) is connected. Hence, (\mathbb{U}, δ) is a twisted connected NAP coalgebra. \square

Example 3.1. $\delta(\wedge) = 0$

$\delta(\vee) = 2 \cdot \otimes \dagger$

$\delta(\bowtie) = \frac{1}{2} \cdot \otimes \wedge$

$\delta(\diamond) = \wedge \otimes \cdot$

Corollary 3.2. [2, Chapter 17] *Applying the functor $\overline{\mathcal{K}}$ gives that $(\overline{\mathcal{K}}(\mathbb{U}), \delta)$ is a NAP coalgebra.*

3.3. Free twisted pre-Lie algebras of finite posets. In this subsection, we prove the compatibility relation [15, Theorem] between the pre-Lie product \searrow and the NAP coproduct δ .

Theorem 3.1. *Let P, Q be two finite connected posets. We have the following identity*

$$\delta(P \searrow Q) = P \otimes Q + (P \otimes \mathbf{1} + \mathbf{1} \otimes P) \searrow \delta(Q),$$

where the unit $\mathbf{1}$ is identified to the empty poset.

Proof. Let $P = (X_1, \leq_P)$, $Q = (X_2, \leq_Q)$ be two finite connected posets, we have:

$$\begin{aligned}
\delta(P \searrow Q) &= \sum_{v \in X_2} \delta(P \searrow_v Q) \\
&= \sum_{\substack{v \in X_2 \\ I \otimes (P \searrow_v Q)}} \frac{1}{|\min(P \searrow_v Q)|} I \otimes ((P \searrow_v Q) \setminus I) \\
&= \sum_{\substack{v \in X_2 \\ I \otimes (P \searrow_v Q)}} \frac{1}{|\min(Q)|} I \otimes ((P \searrow_v Q) \setminus I) \\
&= \frac{1}{|\min(Q)|} \left(\sum_{\substack{v \in X_2, v \in \min(Q) \\ I \otimes (P \searrow_v Q)}} I \otimes ((P \searrow_v Q) \setminus I) + \sum_{\substack{v \in X_2, v \notin \min(Q) \\ I \otimes (P \searrow_v Q)}} I \otimes ((P \searrow_v Q) \setminus I) \right),
\end{aligned}$$

we notice that

- if $v \in \min(Q)$, then $\{I, I \otimes (P \searrow_v Q)\} = \{P\} \cup \{I, I \otimes Q\}$,

and

- if $v \notin \min(Q)$, then $\{I, I \otimes (P \searrow_v Q)\} = \{P \searrow_v J, J \otimes Q, v \in J\} \cup \{J, J \otimes Q, v \notin J\}$.

Then

$$\begin{aligned}
\delta(P \searrow Q) &= \frac{1}{|\min(Q)|} \left(\sum_{v \in X_2, v \in \min(Q)} P \otimes Q + \sum_{\substack{v \in X_2, v \in \min(Q) \\ I \otimes Q}} I \otimes (P \searrow_v (Q \setminus I)) \right) \\
&\quad + \frac{1}{|\min(Q)|} \left(\sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \in J}} (P \searrow_v J) \otimes Q \setminus J + \sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \notin J}} J \otimes (P \searrow_v (Q \setminus J)) \right),
\end{aligned}$$

we notice that

$$\sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \in J}} (P \searrow_v J) \otimes Q \setminus J = \sum_{J \otimes Q} (P \otimes \mathbf{1}) \searrow (J \otimes Q \setminus J),$$

and

$$\sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \notin J}} J \otimes (P \searrow_v (Q \setminus J)) = \sum_{J \otimes Q} J \otimes (P \searrow_v (Q \setminus J)),$$

then

$$\sum_{\substack{v \in X_2, v \in \min(Q) \\ I \otimes Q}} I \otimes (P \searrow_v (Q \setminus I)) + \sum_{\substack{v \in X_2, v \notin \min(Q) \\ J \otimes Q, v \in J}} J \otimes (P \searrow_v (Q \setminus J)) = \sum_{J \otimes Q} (\mathbf{1} \otimes P) \searrow (J \otimes Q \setminus J),$$

accordingly

$$\delta(P \searrow Q) = \frac{1}{|\min(Q)|} \left(\sum_{v \in X_2, v \in \min(Q)} P \otimes Q + \sum_{J \otimes Q} (P \otimes \mathbf{1}) \searrow (J \otimes Q \setminus J) + \sum_{J \otimes Q} (\mathbf{1} \otimes P) \searrow (J \otimes Q \setminus J) \right).$$

Hence,

$$\delta(P \searrow Q) = P \otimes Q + (P \otimes \mathbf{1}) \searrow \delta(Q) + (\mathbf{1} \otimes P) \searrow \delta(Q).$$

□

Corollary 3.3. *Applying the functor $\overline{\mathcal{K}}$ gives that, $(\overline{\mathcal{K}}(\mathbb{U}), \searrow)$ endowed with the coproduct δ is a free pre-Lie algebra and a cofree NAP coalgebra.*

Proof. This is a direct consequence of Theorem 3.1 and M. Livernet's rigidity theorem [15, Theorem]. \square

Corollary 3.4. *$\overline{\mathcal{K}}(\mathbb{U})$ is generated as a pre-Lie algebra by $\text{Prim}(\overline{\mathcal{K}}(\mathbb{U}))$.*

Proof. Let $\overline{\mathcal{K}}(\mathbb{U})_1 = \text{Prim}(\overline{\mathcal{K}}(\mathbb{U})) = \{P \in \overline{\mathcal{K}}(\mathbb{U}), \delta(P) = 0\}$, and let $\overline{\mathcal{K}}(\mathbb{U})_n = \{P \in \overline{\mathcal{K}}(\mathbb{U}), \delta(P) \in \sum_{0 < k < n} \overline{\mathcal{K}}(\mathbb{U})_k \otimes \overline{\mathcal{K}}(\mathbb{U})_{n-k}\}$, we notice that $\overline{\mathcal{K}}(\mathbb{U}) = \bigcup_{k > 0} \overline{\mathcal{K}}(\mathbb{U})_k$, i.e. the vector space $(\overline{\mathcal{K}}(\mathbb{U}), \delta)$ is connected. By applying Corollary 3.9 in [15], we obtain that $\overline{\mathcal{K}}(\mathbb{U})$ is generated as a pre-Lie algebra by $\text{Prim}(\overline{\mathcal{K}}(\mathbb{U}))$. \square

Example 3.2. *Here are the posets in $\text{Prim}(\overline{\mathcal{K}}(\mathbb{U}))$ up to four vertices:*



3.4. Duality relation between $\circ \rightarrow$ and δ . In this subsection, we prove that the NAP product $\circ \rightarrow$ and the NAP coproduct δ are dual to each other modulo a symmetry factor.

Definition 3.3. *Let G be a group acting on X . For every $x \in X$:*

- we denote by $G \cdot x := \{g \cdot x, g \in G\}$ the orbit of x ,
- we denote by $G_x = \{g \in G, g \cdot x = x\}$ the stabilizer subgroup of G with respect to x .

Proposition 3.5. *Let G be a group acting on X , if G and X is finite, then the orbit-stabilizer theorem, together with Lagrange's theorem [19, Theorem 3.9], gives:*

$$|G \cdot x| = [G_x : G] = \frac{|G|}{|G_x|}.$$

In particular, the cardinal of the orbit is a divisor of the group order.

Definition 3.4. *For any poset P on a finite set X , we denote by $\text{Aut}(P)$ the subgroup of permutations of X which are homeomorphisms with respect to P . The symmetry factor is defined by $\sigma(P) = |\text{Aut}(P)|$. Let X_1, X_2 two finite sets, we define the linear map $\langle \cdot, \cdot \rangle : \mathbb{U}_{X_1} \otimes \mathbb{U}_{X_2} \rightarrow \mathbb{K}$ by:*

$$\langle Q, R \rangle = \begin{cases} \sigma(Q) & \text{if } Q \approx R, \\ 0 & \text{otherwise.} \end{cases}$$

In other terms, $\langle Q, R \rangle$ is the number of isomorphisms between Q and R .

Theorem 3.2. *Let P, Q and R three finite connected posets. We have the following identity*

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(P)|} \langle P, Q \circ \rightarrow R \rangle.$$

Proof. Let P, Q and R three finite connected posets.

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(P)|} \sum_{\substack{I \otimes P \\ I \approx Q \\ P \setminus I \approx R}} \sigma(Q) \sigma(R) = \frac{1}{|\min(P)|} |A|$$

with

$$A = \{(I, \sigma_1, \sigma_2) \mid I \otimes P, \sigma_1 : I \xrightarrow{\sim} Q, \sigma_2 : P \setminus I \xrightarrow{\sim} R\}$$

and:

$$\langle P, Q \circlearrowright R \rangle = \sum_{\substack{v \in \min(R) \\ Q \setminus_{\downarrow v} R \approx P}} \sigma(P) = |B|$$

with

$$B = \{(v, \sigma) \mid v \in \min(R), \sigma : P \xrightarrow{\sim} Q \setminus_{\downarrow v} R\}.$$

Let us define now a map $\phi : A \rightarrow B$. Let $(I, \sigma_1, \sigma_2) \in A$. We put $w = I_-$. As $I \odot P$, $w \in \min(P \setminus I)$, so $v = \sigma_2(w) \in \min(R)$. As $I \odot P$, $P = I \setminus_{\downarrow w} (P \setminus I)$, so we obtain an isomorphism $\sigma : P \rightarrow Q \setminus_{\downarrow v} R$ by taking

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in I, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

We then put $\phi(I, \sigma_1, \sigma_2) = (v, \sigma)$.

Now, we define a map $\psi : B \rightarrow A$. If $(v, \sigma) \in B$, we put $I = \sigma^{-1}(Q)$. As $Q \odot Q \setminus_{\downarrow v} R$, $I \odot P$. Moreover, $\sigma_1 = \sigma|_I$ is a graph isomorphism from I to Q and $\sigma_2 = \sigma|_{P \setminus I}$ is a graph isomorphism from $P \setminus I$ to R . We put $\psi(v, \sigma) = (I, \sigma_1, \sigma_2)$.

Let $(v, \sigma) \in B$. We put $\psi(v, \sigma) = (I, \sigma_1, \sigma_2)$ and $\phi \circ \psi(v, \sigma) = (v', \sigma')$. Then, $I = \sigma^{-1}(Q)$ and $w' = I_- = w$, so $v' = \sigma(w) = v$. Moreover,

$$\begin{aligned} \sigma'_I &= \sigma_1 = \sigma|_I, \\ \sigma'_{P \setminus I} &= \sigma_2 = \sigma|_{P \setminus I}, \end{aligned}$$

so $\sigma' = \sigma$. Hence, $\phi \circ \psi = \text{id}_B$.

Let $(I, \sigma_1, \sigma_2) \in A$. We put $\phi(I, \sigma_1, \sigma_2) = (v, \sigma)$ and $\psi \circ \phi(I, \sigma_1, \sigma_2) = (I', \sigma'_1, \sigma'_2)$. Then $I' = \sigma^{-1}(Q) = I$, by construction of σ . Even more,

$$\begin{aligned} \sigma'_1 &= \sigma|_I = \sigma_1, \\ \sigma'_2 &= \sigma|_{P \setminus I} = \sigma_2. \end{aligned}$$

So $\psi \circ \phi = \text{id}_A$. Finally, A and B are in bijection and we obtain:

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(P)|} \langle P, Q \circlearrowright R \rangle.$$

□

Remark 3.1. As mentioned in the paper [3], the binary product \nearrow on the species \mathbb{U} of finite connected posets, defined by $P \nearrow Q := j(j(P) \searrow j(Q))$ for any pair (P, Q) of finite connected posets, is a pre-Lie product, where j is the involution which transforms \leq into \geq . Similarly,

$$P \circlearrowright Q := j(j(P) \circlearrowright j(Q)), \quad (\text{resp. } \tilde{\delta} = (j \otimes j)\delta \circ j)$$

endows the species \mathbb{U} with a twisted NAP algebra (resp. twisted connected NAP coalgebra) structure.

Notation. Let E be any finite set, and let π, ρ be two partitions of E . We denote by $j(\pi, \rho)$ the rational number

$$j(\pi, \rho) := \frac{1}{|E|} \sum_{\substack{\alpha \text{ is a block of } \pi \\ \beta \text{ is a block of } \rho}} |\alpha \cap \beta| \frac{|\beta|}{|\alpha|}.$$

Proposition 3.6. *Let P, Q and R three finite connected posets. We denote by $E = \{v, v \in \min(R) \mid P \approx Q \searrow_v R\}$. Let π be the partition of E into $\text{Aut}(P)$ -orbits, and let ρ be the partition of E into $\text{Aut}(R)$ -orbits. We have the following identity*

$$j(\pi, \rho) = 1.$$

Proof. Let P, Q and R be three finite connected posets. In fact: firstly,

$$\begin{aligned} \langle P, Q \circ \rightarrow R \rangle &= \sum_{v \in \min(R)} \langle P, Q \searrow_v R \rangle \\ &= \sum_{\substack{v \in \min(R) \\ P \approx Q \searrow_v R}} \sigma(P) \\ &= \sigma(P)|E|. \end{aligned}$$

On the other hand

$$\begin{aligned} \langle \delta(P), Q \otimes R \rangle &= \frac{1}{|\min(P)|} \sum_{I \in P} \langle I, Q \rangle \langle P \setminus I, R \rangle \\ &= \frac{1}{|\min(P)|} \sum_{\substack{I \in P \\ I \approx Q, P \setminus I \approx R}} \sigma(Q) \sigma(R) \\ &= \frac{1}{|\min(R)|} \sum_{\substack{I \in P \\ I \approx Q, P \setminus I \approx R}} \sigma(Q) \sigma(R) \\ &= \frac{1}{|\min(R)|} \sum_{v \in \min(R)} \sum_{\substack{I \in P, L = \{v\} \\ I \approx Q, P \setminus I \approx R}} \sigma(Q) \sigma(R) \\ &= \frac{1}{|\min(R)|} \sum_{v \in \min(R)} N_v(P, Q, R) \sigma(Q) \sigma(R), \end{aligned}$$

where $N_v(P, Q, R) :=$ the number of branches I of P above v isomorphic to Q such that $P \setminus I$ isomorphic to R .

We notice that, for all $v \in \min(R)$:

- if $P \not\approx Q \searrow_v R$, then $N_v(P, Q, R) = 0$,
- if $P \approx Q \searrow_v R$, then $N_v(P, Q, R) = N_v(Q, R) :=$ the number of branches of $Q \searrow_v R$ above v isomorphic to Q .

Accordingly

$$\begin{aligned} \langle \delta(P), Q \otimes R \rangle &= \frac{1}{|\min(R)|} \left(\sum_{\substack{v \in \min(R) \\ P \approx Q \searrow_v R}} N_v(P, Q, R) \sigma(Q) \sigma(R) + \sum_{\substack{v \in \min(R) \\ P \not\approx Q \searrow_v R}} N_v(P, Q, R) \sigma(Q) \sigma(R) \right) \\ &= \frac{1}{|\min(R)|} \sum_{\substack{v \in \min(R) \\ P \approx Q \searrow_v R}} N_v(P, Q, R) \sigma(Q) \sigma(R) \\ &= \frac{1}{|\min(R)|} \sum_{\substack{v \in \min(R) \\ P \approx Q \searrow_v R}} N_v(Q, R) \sigma(Q) \sigma(R). \end{aligned}$$

By using the orbit-stabilizer theorem, as well as Lagrange's theorem for the group $(Aut(R), \circ)$ endowed with the law of composition, we therefore have:

$$|Aut(R) \cdot v| = \frac{|Aut(R)|}{|Aut(R)_v|}, \quad i.e. \quad |Aut(R) \cdot v| = \frac{\sigma(R)}{\sigma_v(R)}.$$

We notice that, for all $v \in \min(R)$: $\sigma_v(Q \searrow_v R) = \sigma(Q)\sigma_v(R)N_v(Q, R)$.

Then

$$\begin{aligned} \langle \delta(P), Q \otimes R \rangle &= \frac{1}{|\min(R)|} \sum_{v \in E} \sigma(Q)\sigma_v(R)|Aut(R) \cdot v|N_v(Q, R) \\ &= \frac{1}{|\min(R)|} \sum_{v \in E} \sigma_v(P)|Aut(R) \cdot v| \\ &= \frac{1}{|\min(R)|} \sum_{v \in E} \sigma(P) \frac{|Aut(R) \cdot v|}{|Aut(P) \cdot v|} \\ &= \frac{1}{|\min(R)|} \sigma(P)S, \end{aligned}$$

where

$$S = \sum_{v \in E} \frac{|Aut(R) \cdot v|}{|Aut(P) \cdot v|}.$$

Let π be the partition of E into $Aut(P)$ -orbits, and let ρ be the partition of E into $Aut(R)$ -orbits. We denote by $\alpha \in \pi$ (respectively, $\beta \in \rho$), if α is a block of π (respectively, if β is a block of ρ). So

$$\begin{aligned} S &= \sum_{v \in E} \frac{|Aut(R) \cdot v|}{|Aut(P) \cdot v|} \\ &= \sum_{v \in E} \sum_{\substack{\alpha \text{ is a } Aut(P)\text{-orbit of } v \\ \beta \text{ is a } Aut(R)\text{-orbit of } v}} \frac{|\beta|}{|\alpha|} \\ &= \sum_{\substack{\alpha \text{ is a } Aut(P)\text{-orbit} \\ \beta \text{ is a } Aut(R)\text{-orbit}}} |\alpha \cap \beta| \frac{|\beta|}{|\alpha|} \\ &= j(\pi, \rho), \end{aligned}$$

then $\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(R)|} \sigma(P)j(\pi, \rho)$. Correspondingly,

$$\langle \delta(P), Q \otimes R \rangle = \langle P, Q \circ \rightarrow R \rangle \frac{j_{P,R}(\pi, \rho)}{|\min(P)|}.$$

By applying Theorem 3.2, we obtain:

$$\langle \delta(P), Q \otimes R \rangle = \frac{1}{|\min(P)|} \langle P, Q \circ \rightarrow R \rangle = \langle P, Q \circ \rightarrow R \rangle \frac{j(\pi, \rho)}{|\min(P)|}.$$

Conclusion: for P, Q and R three finite connected posets, if we denote by $E = \{v, v \in \min(R) \mid P \approx Q \searrow_v R\}$, and for π (resp. ρ) the partition of E into $Aut(P)$ -orbits (resp. the partition of E into $Aut(R)$ -orbits). Then we obtain

$$j(\pi, \rho) = 1.$$

□

Remark 3.2. We do not have $j(\pi, \rho) = 1$ in general for two partitions of a finite set E . Indeed: let E be the finite set $\{1, 2, 3, 4\}$, $\pi = \{\{1, 2\}, \{3, 4\}\}$ and $\rho = \{\{1, 2, 3\}, \{4\}\}$. Then we obtain $j(\pi, \rho) = \frac{5}{4}$ and $j(\rho, \pi) = 1$. This interesting index does not seem to be present in the literature.

3.5. Free Lie algebras of finite posets. F. Chapoton in [7] showed that free pre-Lie algebras are free as Lie algebras i.e., if (V, \triangleright) is a free pre-Lie algebra. Then it is a free Lie algebra for the Lie bracket $[\cdot, \cdot]$ defined by $[a, b] = a \triangleright b - b \triangleright a$. This result has been obtained before with different methods by L. Foissy [13, Theorem 8.4].

Proposition 3.7. [7, 13] For any vector space V , the free pre-Lie algebra on V is isomorphic as a Lie algebra to the free Lie algebra on some set $X(V)$ of generators.

Application [17] Let H be a Hopf algebra and H^0 the graded dual of H . The primitive element algebra of the graded dual H^0 with the bracket $[\cdot, \cdot]$ is a Lie algebra. We denote by \star the Grossman-Larson product on the dual of H [3]. This product restricted to H^0 is the graded dual of the coproduct Δ_{\searrow} [3]. Applying the general setting above to the case $H = \overline{\mathcal{K}}(\mathbb{P})$ we are studying, we obtain:

$(\overline{\mathcal{K}}(\mathbb{U}), \searrow) = (\text{Prim}(H^0), \searrow)$ is a free pre-Lie algebra, then it is also a free Lie algebra, therefore $U(\text{Prim}H^0) = (S(\text{Prim}H^0), \star)$ is a free associative algebra. Then, $(S(\text{Prim}H^0), \star)^0 = (H, \Delta_{\searrow})$ is a coassociative cofree coalgebra.

4. GENERALIZATION TO FINITE TOPOLOGICAL SPACES

All the results in this paper remain true for the finite connected topological spaces, with a small change on the definition of the coproduct δ . Indeed: let $\mathcal{T} = (X, \leq_{\mathcal{T}})$ be a finite connected topological space. We define

$$\delta(\mathcal{T}) = \frac{1}{|\min(\mathcal{T})|} \sum_{Y \in \mathcal{T}} \mathcal{T}_Y \otimes \mathcal{T}_{|X \setminus Y},$$

where $Y \in \mathcal{T}$ means that $Y \in \mathcal{T}$ such that:

- \overline{Y}_- is a singleton included in $\min(\overline{\mathcal{T}})$,
- and Y is a branch of the set $\{x \in X, \text{ such that for all } y \in Y_-, \text{ we have } y <_{\mathcal{T}} x \text{ and } y \not\leq x\}$.

Here, the set Y_- equal to the space $\{x \notin Y, \text{ there exists } y \in Y \text{ such that } x \leq y\}$, $\overline{\mathcal{T}} = (X / \sim_{\mathcal{T}}, \leq)$ is the poset defined on $X / \sim_{\mathcal{T}}$ corresponding to the partial order \leq induced by the quasi-order $\leq_{\mathcal{T}}$, and $\overline{Y}_- = Y_- / \sim_{\mathcal{T}_{Y_-}}$. Let $\text{Aut}(\mathcal{T})$ be the subgroup of permutations of X which are homeomorphisms with respect to \mathcal{T} , and let $G = \{g \in \text{Aut}(\mathcal{T}), g \cdot \overline{v} = \overline{v} \text{ for all } v \in X\}$. We notice that $G \subseteq \text{Aut}(\mathcal{T})$, G is a normal subgroup and $\text{Aut}(\overline{\mathcal{T}}) = \text{Aut}(\mathcal{T})/G$.

We notice that the results obtained in Lemma 3.1 also work for $Y, Z \in \mathcal{T}$. From $Y, Z \in \mathcal{T}$ and Lemma 3.1, we get that both $\overline{Y}, \overline{Z} \in \overline{\mathcal{T}}$, then $(\overline{Y} = \overline{Z} \text{ or } \overline{Y} \cap \overline{Z} = \emptyset)$, and $\min(\overline{\mathcal{T}}_{|X \setminus Y}) = \min(\overline{\mathcal{T}})$. Then $Y \cap Z = \emptyset$ and $\min(\mathcal{T}_{|X \setminus Y}) = \min(\mathcal{T})$.

Hence to show that the proposition 3.4 also works in topological cases, it suffices to change $I, J \in \mathcal{P}$ by $Y, Z \in \mathcal{T}$ in the proof of proposition 3.6. To show that the result of Theorem 3.1 also works in connected finite topological spaces, we change $I \in \mathcal{P}$ by $Y \in \mathcal{T}$ in the Proof 3.3.

Finally we prove the existence of a relation between δ and \circlearrowright in the topological case. Exactly: for $\mathcal{T}, \mathcal{T}'$ and \mathcal{T}'' three finite connected topological spaces, we have

$$\langle \delta(\mathcal{T}), \mathcal{T}' \otimes \mathcal{T}'' \rangle = \frac{1}{n|\min(\mathcal{T})|} \langle \mathcal{T}, \mathcal{T}' \circlearrowright \mathcal{T}'' \rangle,$$

where n is the number of items in any bag $\bar{v} \in \min(\overline{\mathcal{T}''})$, such as $\mathcal{T} \approx \mathcal{T}' \searrow_{\bar{v}} \mathcal{T}''$. The proof is similar to the proof of Theorem 3.2. Indeed:

$$\langle \delta(\mathcal{T}), \mathcal{T}' \otimes \mathcal{T}'' \rangle = \frac{1}{|\min(\mathcal{T})|} \sum_{\substack{Y \otimes \mathcal{T} \\ \mathcal{T}|_Y \approx \mathcal{T}', \\ \mathcal{T}|_{X \setminus Y} \approx \mathcal{T}''}} \sigma(\mathcal{T}') \sigma(\mathcal{T}'') = \frac{1}{|\min(\mathcal{T})|} |A|$$

with

$$A = \{(Y, \sigma_1, \sigma_2) \mid Y \otimes \mathcal{T}, \sigma_1 : \mathcal{T}|_Y \xrightarrow{\sim} \mathcal{T}', \sigma_2 : \mathcal{T}|_{X \setminus Y} \xrightarrow{\sim} \mathcal{T}''\}.$$

On the other hand

$$\begin{aligned} \langle \mathcal{T}, \mathcal{T}' \circlearrowright \mathcal{T}'' \rangle &= \sum_{v \in \min(\mathcal{T}'')} \langle \mathcal{T}, \mathcal{T}' \searrow_v \mathcal{T}'' \rangle \\ &= \sum_{\substack{v \in \min(\mathcal{T}'') \\ \mathcal{T} \approx \mathcal{T}' \searrow_v \mathcal{T}''}} \sigma(\mathcal{T}) = |B|, \end{aligned}$$

with

$$B = \{(v, \sigma) \mid v \in \min(\mathcal{T}''), \sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}' \searrow_v \mathcal{T}''\}.$$

Let

$$C = \{(u, \sigma) \mid u = \bar{v}, v \in \min(\mathcal{T}''), \sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}' \searrow_v \mathcal{T}''\}.$$

We notice that for all $u, u' \in C$, we have $|u| = |u'|$. Hence $\frac{|B|}{|C|}$ is the number of items in any bag $\bar{v} \in \min(\overline{\mathcal{T}''})$, such as $\mathcal{T} \approx \mathcal{T}' \searrow_v \mathcal{T}''$. We define

$$\begin{aligned} f : A &\longrightarrow C \\ (Y, \sigma_1, \sigma_2) &\longmapsto (u, \sigma), \end{aligned}$$

where $u = \bar{v} = \overline{\sigma_2(Y_-)}$ and σ is a graph isomorphism from \mathcal{T} to $\mathcal{T}' \searrow_v \mathcal{T}''$ defined by

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in Y, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

Now, we define a map

$$\begin{aligned} g : C &\longrightarrow A \\ (u, \sigma) &\longmapsto (Y, \sigma_1, \sigma_2), \end{aligned}$$

where $Y = X_{\sigma^{-1}(\mathcal{T}')}$, $\sigma_1 = \sigma|_{\mathcal{T}|_Y}$ is a graph isomorphism from $\mathcal{T}|_Y$ to \mathcal{T}' and $\sigma_2 = \sigma|_{\mathcal{T}|_{X \setminus Y}}$ is a graph isomorphism from $\mathcal{T}|_{X \setminus Y}$ to \mathcal{T}'' .

Let $(u, \sigma) \in C$, we put $g(u, \sigma) = (Y, \sigma_1, \sigma_2)$, where $Y = X_{\sigma^{-1}(\mathcal{T}')}$, $\sigma_1 = \sigma|_{\mathcal{T}'}$ and $\sigma_2 = \sigma|_{\mathcal{T}''}$. Then $f \circ g(u, \sigma) = (u', \sigma')$, with $u' = \bar{w} = \overline{\sigma_2(Y_-)}$ and σ' is a graph isomorphism from \mathcal{T} to $\mathcal{T}' \searrow_w \mathcal{T}''$ defined by

$$\sigma'(x) = \begin{cases} \sigma_1(x) & \text{if } x \in Y, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

Then, $u' = \overline{\sigma_2(Y_-)} = u$ and $\sigma' = \sigma$. Hence $f \circ g = Id_C$.

Let $(Y, \sigma_1, \sigma_2) \in A$, we put $f(Y, \sigma_1, \sigma_2) = (u, \sigma)$, with $u = \bar{v} = \overline{\sigma_2(Y_-)}$ and σ is a graph isomorphism from \mathcal{T} to $\mathcal{T}' \searrow_{\nu} \mathcal{T}''$ defined by

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x \in Y, \\ \sigma_2(x) & \text{otherwise.} \end{cases}$$

Then, $g \circ f(Y, \sigma_1, \sigma_2) = (Y', \sigma'_1, \sigma'_2)$, with $Y' = X_{\sigma^{-1}(\mathcal{T}')}$, $\sigma'_1 = \sigma|_{\mathcal{T}_{|Y'}}$ is a graph isomorphism from $\mathcal{T}_{|Y'}$ to \mathcal{T}' and $\sigma'_2 = \sigma|_{\mathcal{T}_{|X \setminus Y'}}$ is a graph isomorphism from $\mathcal{T}_{|X \setminus Y'}$ to \mathcal{T}'' .

Then, $Y' = X_{\sigma^{-1}(\mathcal{T}')} = Y$ and

$$\begin{cases} \sigma'_1 = \sigma|_{\mathcal{T}_{|Y'}} = \sigma_1, \\ \sigma'_2 = \sigma|_{\mathcal{T}_{|X \setminus Y'}} = \sigma_2. \end{cases}$$

So $g \circ f = Id_A$.

Finally, A and C are in bijection. Hence

$$\langle \delta(\mathcal{T}), \mathcal{T}' \otimes \mathcal{T}'' \rangle = \frac{1}{|\min(\mathcal{T})|} |A| = \frac{1}{|\min(\mathcal{T})|} |C| = \frac{1}{|\min(\mathcal{T})|} |C| \frac{|B|}{|B|} = \frac{|C|}{|B| |\min(\mathcal{T})|} \langle \mathcal{T}, \mathcal{T}' \circ \rightarrow \mathcal{T}'' \rangle.$$

Remark. All the results in this paper remain true for the posets (resp. topological spaces) decorated by F , that is to say pairs (\mathcal{A}, f) , where $\mathcal{A} = (A, \leq_A)$ is a poset (resp. topology) and $f : A \rightarrow F$ is a map.

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