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Abstract In this review, we focus on the works done by Yilmaz and by Drösler. They both proposed color vision models based on projective hyperbolic metric. We integrate those models in a unified mathematical framework, we called the Yilmaz-Drösler model. This model can be used to geometrically represent current views on color perception and processing. Because color vision is trichromatic and lights spectra are positive functions, the space of color perception is delimited by a convex cone in a three-dimensional metameric space. The space inside the cone can be decomposed into stacks of iso-intensity hyperboloid surfaces. Yilmaz postulated a Lorentz transformation to accounting for adaptation induced by change in illumination. As shown by Drösler, this transformation can be viewed as a projective transformation. We show that hyperbolic metric models with projection are well-suited for describing the relation between physiology and perception. This review aims at presenting those theories to a larger audience using a geometrical description.

Keywords Color Vision · Hyperbolic geometry · Yilmaz-Drösler model · Color Adaptation · Projective geometry

Introduction

Relations between physical and perceptual spaces have been studied for long time in psychophysics. Psychophysical laws have been proposed for weight, light intensity, length, or speed discrimination as few examples Bouguer (1729); Fechner (1907); Stevens (1957). However, the extension of those uni-dimensional laws to multidimensional perceptual systems, such as color vision, is challenging Helmholtz (1896, 1852); Schrödinger (1920); MacAdam (1970); Niall (2017). Due to trichromacy; color perception is a phenomenon that can be fully characterized in three dimensions. For a given observer under given viewing condition, a triplet of numbers (color coordinates in a color space) is necessary and sufficient to define a perceived color. Two triplets can correspond to either two observers in the same condition or to a single observer under two different conditions. But it is unclear how the values between the two triplets are related to each other within and between observers. For this reason, current color spaces can hardly account for human experience of color vision. We cannot predict the appearance of a physical stimulus for a given observer nor the shift in appearance when viewing conditions are changed.

Yilmaz (1961, 1962a,b) and Drösler (1994, 1995) both attempted to circumvent this difficulty by addressing the problems of adaptation, metric and transformation between color spaces. Yilmaz proposed that the perceptual color space is limited by a cone of maximal saturation. Color vision is restricted to that cone because lights spectra are positive functions of wavelength. In Yilmaz's color space, what counts for color perception is not the absolute coordinate of the physical stimulus but the relation between stimuli. Colors relative to the white remains invariant under change in illumination giving humans the impression that colors are constant physical properties of objects. The contribution of Drösler was to place color vision in the context of projective geometry. He considered the Weber ratio as a projective invariant. In doing so, he extended psychophysical principles to any multi-dimensional projective space. In the following, we will detail those studies and their geometrical formulations. We attempt to

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merge them into a modern mathematical framework embedding current models of color vision.

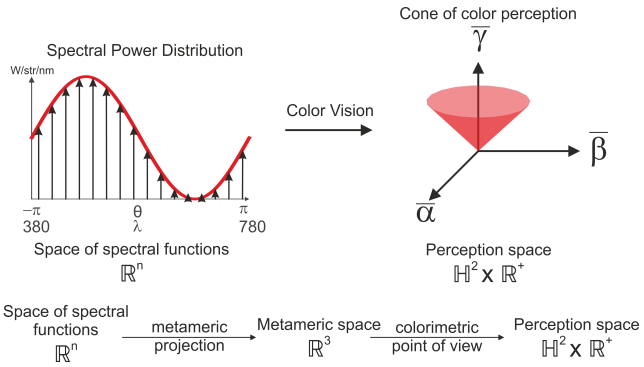


Fig. 1 Yilmaz-Drösler model of color vision. Our interpretation of the Yilmaz-Drösler color vision model is given by a transformation between the space of spectral functions, defined by light's spectral power distribution function, to a cone in a three-dimensional space where color perception takes place. Color perception is given by two alternative projections. First, the space of sampled spectral functions, isomorphic to $\mathbb{R}^n$, is projected orthogonally into a metameric space that corresponds to a span by the sensitivity functions of the three photoreceptors involved in color vision. Metameric space is three-dimensional isomorphic to $\mathbb{R}^3$. Because light's spectra are positive functions, light's domain is restricted to a cone shape volume into the three-dimensional metameric space. The visual system adopts a hyperbolic metric that is the natural metric inside a convex cone which defines color perception space isomorphic to $\mathbb{H}^2 \times \mathbb{R}^+$ (where $\mathbb{H}^2$ is the unit hyperboloid in the metameric coordinate system).

As illustrated in Figure 1, we define the correspondence between the space of spectral functions and perceptual space. This conception is common in Yilmaz and Drösler studies.

From space of light's spectral functions to perceptual color space

The metric of color vision space has been directly measured as contour of constant discrimination in physical color spaces. MacAdam (1942) showed that those contours are ellipses in the chromaticity diagram also called CIE1931-xy. Ellipses vary in size and orientation with their positions on the chromaticity diagram. This is a manifestation of a non Euclidean metric for color discrimination expressed in physical variables. It is indicative of local metric within the color space or otherwise said a local modification of the global metric. Discrimination contours were shown to vary quite a lot between and within observers Wyszecki and Fielder (1971); Brown and MacAdam (1949). This variation can be modeled as a consequence of a nonlinear and adaptive correspondence between physical and perceptual spaces Alleysson and Hérault (2001); Alleysson and Meary (2012).

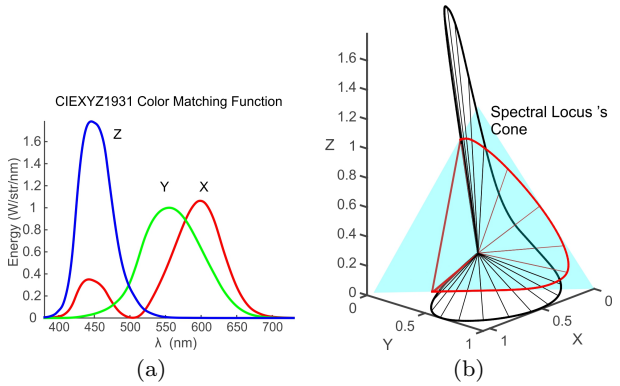
Many authors have tried to formalize the transformation from the space of light to the perceptual space as a metrical transform Riemann (1854); Helmholtz (1896); Schrödinger (1920); Niall (2017); Koenderink et al. (1972). Models are generally written as a Riemannian metric with an infinitesimal variation ds:

$$ds^2 = \frac{1}{(R+G+B)^{2-\alpha}} \left(\frac{dR^2}{R^\alpha} + \frac{dG^2}{G^\alpha} + \frac{dB^2}{B^\alpha} \right), \tag{1}$$

where R, G and B represent the components of the light in a color space and dR, dG, dB the variation around the components. For Helmholtz, Schrödinger and Koenderink models, α equal 2, 1, and 0, respectively Koenderink et al. (1972). Other recent approaches used hyperbolic models to represent the geometry of color space based on discrimination measurement Farup (2014); Lenz and Meer (1999). They maybe closer to a definition of the color space in term of psychological variable such as hue, saturation and brightness.

It was probably Grassmann Grassmann (1853); Krantz (1975) that firstly stated that color perception occurred in a space shaped by a cone. Using monochromatic light, several authors Guild (1931); Wright (1930) estimated color vision space using color matching experiment Maxwell (1860). For a color match the participant had to adjust the intensity of three monochromatic primaries (in the red, green and blue part of the spectrum) in order to visually equalize a reference monochromatic light. The level of intensities adjusted for the three primaries were taken as coordinates for the three color matching functions for all reference lights covering the visible spectrum.

Figure 2(a) shows the color matching functions X, Y and Z as normalized by the Commission International de L'Eclairage in 1931 (CIE1931-XYZ). Taking these functions as axis for a three-dimensional space one can construct a three-dimensional curve called spectrum locus. Each point of the curve corresponds to a wavelength and has coordinates given by the values of the functions XYZ at that wavelength (Figure 2(b)). The curve delimits a volume that could be considered as a deformed convex cone. If we project the curve on the chromaticity diagram we obtain the limit of the cone (red curve in Figure 2(b)). Whether or not we can define a transformation of the XYZ color matching functions that make the cone perfectly circular and symmetric around an axis, as assumed by the Yilmaz model, is largely documented in the book written by Koenderink (2010).



(a) (b)

Fig. 2 Cone in CIE color space. (a) Color matching experiment allows to measure three functions of wavelength X, Y and Z. Each value of these functions at a particular wavelength is the amount of intensity of three primaries needed to equalize a monochromatic light at that wavelength. (b) These functions are used to draw the curve of spectrum locus in a three-dimensional XYZ space. In this space the curve (black) is seen as being in the frontier of a deformed cone. This is illustrated by the projection of the spectrum locus on the plane orthogonal to the vector (1,1,1) using the CIE model (red curve).

Authors have also been looking for a direct correspondence between the space of light's spectral functions and the space of color perception. Weinberg (1976) proposed a model of the projection between physical to perceptual space using the formalism of Grassmannian space. Similarly, Ashtekar et al. (1999) were able to define a hyperbolic metric by assuming conic shape for the perceptual space. Suppes and Krantz (2007) proposed a review on the correspondence between physical and perceptual spaces for color vision.

Other models of the transformation from physical to perceptual spaces were derived using Lie algebra Hoffman (1966); Lenz et al. (2005). Resnikoff (1974) also used group transformation theory to justify a representation of the color vision space as a hyperbolic metric Provenzi (2017, 2016); Berthier and Provenzi (2019). Finally, we would like to recall the work by Benzécri (1981) who tried to make a systematic description from the ray of light to perception. We think that the work of Yilmaz and Drösler are the keys to unify these models and better understand the geometry of the color perception space.

Relativistic color vision: Yilmaz

Yilmaz (1961) proposed an analogy between color vision and space-time special relativity. He justified the analogy by considering that color experience is well described as hue, saturation and brightness which are cylindrical coordinates inside a cone of maximal saturation. Maximal saturation in color vision being analog to maximum speed (the speed of light in vacuum) in space-time relativity.

The envelop of the cone of maximal saturation Σ is defined as a null space $q(X) = 0$ of a quadratic form $q(X) = -\alpha^2 - \beta^2 + \Sigma^2\gamma^2$, in a three-dimensional color space equipped with an orthogonal basis $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$. With $X = (\alpha, \beta, \gamma)$, and α, β, γ are color coordinates of a point in the color space. The quadratic form is an invariant over color coordinate systems (with a common origin) and the transform between two color coordinate systems is given by the Lorentz transform. Lorentz's transformation leaves invariant the quadratic form $q(X)$ that link the coordinates. The equivalent Lorentz factor is written: $\Omega_{11} = \frac{1}{\sqrt{1-\frac{\sigma^2}{\Sigma^2}}}$ where σ is the saturation of the light and Σ is the maximum possible saturation (see section The two rooms experiment: Yilmaz for further explanations). The invariant, based on a quadratic form, can be similarly constructed from a Lorentzian inner product Ratcliffe (2013). Therefore, the space of color perception, restricted to a cone, is parametrized by a stack of hyperboloid sheets. (Figure 3).

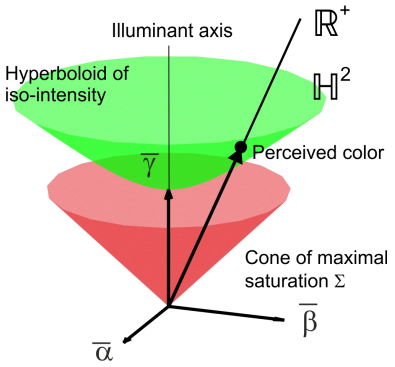


Fig. 3 Yilmaz color perception space. According to Yilmaz, color vision space is delimited by a convex cone of maximum saturation Σ defined in an orthogonal basis $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$. The axis $\bar{\gamma}$ represent the direction of the illuminant. In this space, a perceived color is represented by a point inside the cone or, equivalently, a vector from the origin to the point. The point corresponds to the metameric projection of a light's spectrum. The angular departure of this vector from $\bar{\gamma}$ cannot be larger than the limits imposed by the vertices of the cone. The interior of the cone can be parametrized by a stack of hyperboloid sheets oriented along $\bar{\gamma}$ (to each point inside the cone it corresponds an hyperboloid that passes through it). A particular revolution hyperboloid symmetric around $\bar{\gamma}$ defines the surface of constant intensity $k = 1$. All points that belong to this surface correspond to colors that are of same perceived intensity.

Yilmaz gave a particular interpretation for the Lorentz transformation. The cone of maximal saturation is oriented along the axis $\bar{\gamma}$. This axis represents the illumi-

nant of the visual scene. When the illuminant change, the orientation of the cone of perceivable colors change. The perceptual invariant is the white color which is always located on the axis representing the illuminant. White, defined as the property of a diffuse surface that perfectly reflects illumination, remains perceptually constant. The other perceived colors are defined by their positions relative to $\bar{\gamma}$ and their perceptual distances are preserved. Yilmaz (1961)[p. 242] wrote: *From one illuminant to another the light reflected from each object changes. Under this change object-color identities and their interrelationships must not change. Therefore the transformation must be such that the change in the physical characteristics due to the change in the illuminant is compensated by the transformations so as to leave object-colors and their interrelationships invariant.*

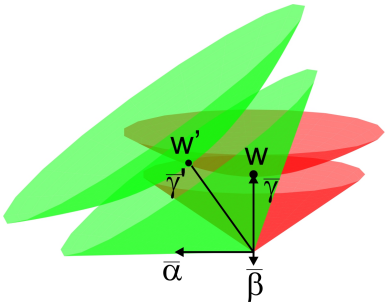


Fig. 4 Adaptation to illumination change. When illuminant changes, the cone of perceived color and the unit hyperboloid change accordingly. If $\bar{\gamma}'$ is the vector for the new illuminant then the new cone and the new hyperboloid were aligned to it. The transformation conserves the perceived white point. If the white point is called W under the former illuminant, it is located on the $\bar{\gamma}$ axis because that axis represents achromatic colors. Under the new illuminant, the white point is transformed into W' placed on the $\bar{\gamma}'$ axis. The transformation between the two coordinate systems is given by a Lorentz transformation because the white point slides along the unit hyperboloid and the transformation is a hyperbolic rotation.

Yilmaz's interpretation of Lorentz's transformation is different from its interpretation in special relativity. In special relativity, a Lorentz transformation gives a new oblique coordinate system in which the new space-time has the properties of length contraction and time dilation. For color vision those properties are not reproduced. The Lorentz transformation is a change of the symmetry axis of the cone of perceived colors according to the vector representing the new illuminant.

There are several physiological and psychological evidences for a change of perceptual space with adaptation. Photoreceptors are equipped with photopigments that participate for the transformation of photons into neural signals. Those photopigments are subject to bleaching under moderate and high light intensity Rushton and Henry (1968). Bleaching reduces the sensitivity of the photoreceptors and potentially modulates differentially the axis of the metameric coordinate system. From a psychological point of view, two surfaces that are metameric under one illuminant are not necessary metameric under another one Foster et al. (2006). This implies that the change in the position of the cone of perceived colors is certainly due to a change in the metameric sub-space.

Trichromacy is a strong limitation for light encoding but by adapting its measurement process to environmental conditions, the visual system is able to extend its discrimination power between lights under illumination change. This transformation of the cone of perceived colors allows color constancy, the well-known observation that humans are able to perceive constant color despite changes in illumination D'Zmura and Lennie (1986); Foster (2003, 2011).

The two rooms experiment: Yilmaz

Yilmaz (1961, 1962b) proposed an experiment[1] to show how the hyperbolic model of color vision may behave when illuminant is changed. His analytic formulation showed a Lorentz transformation between the white points under two illuminants.

In his model, Yilmaz considered that color vision space belongs to a cone of maximal saturation Σ into a three-dimensional Euclidean space with coordinate system $\{0, \bar{\alpha}, \bar{\beta}, \bar{\gamma}\}$ (Figure 6(a)). Any physical light F corresponds to a point in that Euclidean space of coordinate $F(\alpha, \beta, \gamma) = \alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma}$. Saturation of the light is given by: $\sigma = \frac{\sqrt{\alpha^2+\beta^2}}{\gamma}$.

Yilmaz considered general change of illuminant from illuminant I_1 to illuminant I_2. Under I_1, the observer has basis $B_1 = \{0, \bar{\alpha}_1, \bar{\beta}_1, \bar{\gamma}_1\}$ and $(\alpha_1, \beta_1, \gamma_1)$ are the coordinates of a light. Under I_2 the observer basis is $B_2 = \{0, \bar{\alpha}_2, \bar{\beta}_2, \bar{\gamma}_2\}$ with coordinate of the light being $(\alpha_2, \beta_2, \gamma_2)$. Yilmaz used the projection onto a chromaticity diagram, which is defined by the disk of radius Σ situated at $\gamma = 1$, to represent the different illuminants and their respective cone envelops as shown in Figure 6(b).

His goal was to find the transformation Ω of the basis from B_1 to B_2. Yilmaz made two assumptions: (1)

[1] This experiment is not a proper psychology experiment because no experimental results based on observers have been reported as far as the authors known. One can consider the experiment as a proof of concept.

The transformation is linear. (2) The transformation leaves invariant the β axis.

The constraint of condition (2) for the vectors representing the two illuminants I_1 and I_2 is that they lie on a line at β constant, even $\beta = 0$. This kind of simplification is usual in space-time relativity where one considers the motion between two Galilean coordinate systems on the x direction only to simplify the writing of the Lorentz transformation. So he wrote the transformation as:

$$\begin{bmatrix}\alpha_2\\ \beta_2\\ \gamma_2\end{bmatrix} = \Omega \begin{bmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{bmatrix}, \quad \Omega = \begin{bmatrix}\Omega_{11} & 0 & \Omega_{13}\\ 0 & 1 & 0\\ \Omega_{31} & 0 & \Omega_{33}\end{bmatrix}. \tag{2}$$

The experimental set-up consists of two rooms. Two halves of a white paper, approximating perfect reflectors, are placed in each room. A small hole between the two rooms allowed the observer to see the paper in room 2 or 1 while standing in room 1 or 2, respectively. Each room can be illuminated by a different illuminant. Three experiments have been proposed with the set-up. They are illustrated in Figure 5.

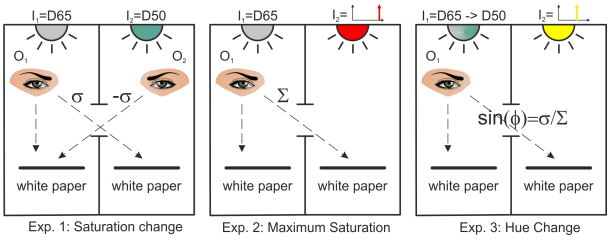


Fig. 5 The two rooms experiment. Yilmaz proposed three experiments: (1) Change of perceived saturation with illuminant changes; (2) The judgment of the appearance of a fully saturated color; (3) Change in hue due to observer's adaptation induced by varying illuminant. All perceived changes being estimated using a color atlas under illuminant I_1.

In the mathematical development given by Yilmaz, the coordinates $(\alpha_1, \beta_1, \gamma_1)$ for illuminant I_1 in room 1 and $(\alpha_2, \beta_2, \gamma_2)$ for illuminant I_2 in room 2 serve for several light sources depending on experiments. Figure 5 indicates to which illuminant the coordinates correspond to.

Exp 1

The first room was illuminated with illuminant I_1 and the second room with illuminant I_2. Illuminants were chosen to be *two different illuminants of near daylight chromaticity* Yilmaz (1962b). We choose D65 and D50 for illustration. When the observer is placed in the first room he sees the white paper as white and through the hole he sees the white paper in the other room *chromatically colored with a shift in saturation* σ (assessed by matching with a patch on the OSA atlas under illuminant I_1). The same observer move to the second room, all other things being identical, he sees the white paper in room 1 with a slight shift in saturation compared to the white paper in its current room. However, the shift in saturation goes in opposite hue direction. Yilmaz denotes it as $-\sigma$, indicating that the *hue is complementary to the former hue*. This should be understood as $\alpha = \sigma\cos(\pi)$ in the coordinate system (α, β) of the chromaticity diagram for which $\gamma = 1$. Indeed, let us suppose that the illuminant I_2 appear a bit reddish compared to illuminant I_1. In this case, σ (or $\alpha = \sigma\cos(0)$) represents a shift toward the red whereas $-\sigma$ (or $\alpha = \sigma\cos(\pi)$) represents a shift toward the green.

In this first experiment, the white paper in room 1 seen from room 1 is perceived as white. So, $(\alpha_1, \beta_1, \gamma_1)$ is proportional to $(0, 0, 1)$ because the white is on the $\bar{\gamma}_1$ axis by definition. Similarly, the white paper in room 2 seen from room 2 is white too. So, $(\alpha_2, \beta_2, \gamma_2)$ is also proportional to $(0, 0, 1)$.

From Equation 2, we have $\alpha_2 = \Omega_{11}\alpha_1 + \Omega_{13}\gamma_1$. But because the white paper appear white under I_2 in room 2, $\alpha_2 = 0$, so:

$$\alpha_2 = \Omega_{11}\alpha_1 + \Omega_{13}\gamma_1 = 0,$$
$$\Omega_{13} = -\frac{\Omega_{11}\alpha_1}{\gamma_1}.$$

The cone of color vision under I_1 is defined by: $\alpha_1^2 + \beta_1^2 = \Sigma^2\gamma_1^2$. Any light of saturation σ inside the cone has coordinates that follow the equation $\alpha_1^2 + \beta_1^2 = \sigma^2\gamma_1^2$, thus $\sigma = \sqrt{\alpha_1^2 + \beta_1^2}/\gamma_1$. But, because the change in coordinate leaves β invariant, the β component of σ is null, so we have $\sigma = \alpha_1/\gamma_1$ which implies:

$$\boxed{\Omega_{13} = -\sigma\Omega_{11}}. \tag{3}$$

So, α_2 can be expressed as:

$$\alpha_2 = \Omega_{11}\left(\alpha_1 - \sigma\gamma_1\right).$$

Expressing $-\sigma$ in term of α_2 and γ_2 and expressing γ_2 using Equation 2 when the observer move to room 2 gives:

$$-\sigma = \frac{\alpha_2}{\gamma_2} = \frac{\Omega_{11}(\alpha_1 - \sigma\gamma_1)}{\Omega_{31}\alpha_1 + \Omega_{33}\gamma_1}.$$

But, because the white paper is seen white in room 1 under I_1, we have $\alpha_1 = 0$. So, $-\sigma = -\sigma\Omega_{11}/\Omega_{33}$ thus:

$$\boxed{\Omega_{11} = \Omega_{33}}. \tag{4}$$

Exp 2

The observer is placed in room 1 illuminated with the illuminant I_1. Room 2 is then illuminated with a very narrow red spectrum (point R in Figure 6(b)). From room 1, the observer has to match the color reflected by the white paper in room 2 using the OSA color atlas. Yilmaz predicted that observer would choose the most saturated color of the atlas because it would have the feeling that the paper is still more saturated than the available atlas color.

Maximal saturation in either cone for I_1 or I_2 illuminant is given by $\sigma = \Sigma$, which gives:

$$\Sigma = \frac{\alpha_1}{\gamma_1} \Rightarrow \alpha_1 = \Sigma\gamma_1$$
$$\Sigma = \frac{\alpha_2}{\gamma_2} = \frac{\Omega_{11}(\alpha_1 - \sigma\gamma_1)}{\Omega_{31}\alpha_1 + \Omega_{11}\gamma_1} = \frac{\Omega_{11}(\Sigma - \sigma)}{\Omega_{31}\Sigma + \Omega_{11}},$$
$$\boxed{\Omega_{31} = -\Omega_{11}\frac{\sigma}{\Sigma^2}}. \quad (5)$$

Exp 3

During the experiment, the illuminant in room 1 is changed from I_1 to I_2. The white paper in room 2 is illuminated with a monochromatic yellow (point Y_1 or Y_2 in Figure 6(b) depending on the illuminant I_1 or I_2 in room 1). Because of the variation in illuminant in room 1, the observer placed in room 1 experiences a shift of hue of the white paper in room 2 seen through the hole. This shift is a consequence of a move toward a new adaptation state.

For the modeling of the experiment 3, Yilmaz used the coordinate (α, β) in the chromaticity diagram which is a projection into the plane at $\gamma_1 = 1$ in basis B_1 (Figure 6(b)). The monochromatic yellow is considered placed in an orthogonal direction to the $\bar{\alpha}$ axis and fully saturated so the coordinates of Y_1 in B_1 are $\alpha_1 = 0$ and $\beta_1 = \gamma_1\Sigma$ (Figure 6(b)). Considering now the coordinate system B_2 for I_2, we see on Figure 6(b) that Y_1 transposes into Y_2 to belong to the limit of saturation in B_2. According to the Pythagorean theorem in the right angled triangle $I_1I_2Y_2$, for expressing $\cos(\phi)$ on Figure 6(b), we can estimate:

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = -\frac{\sigma}{\sqrt{\Sigma^2 - \sigma^2}} = \frac{-\sigma}{\Sigma\sqrt{1 - \frac{\sigma^2}{\Sigma^2}}}.$$

In B_2 the tangent can also be expressed as the ratio of the coordinate of Y_2 (because the angle between $\bar{\beta}_2$ and $\overrightarrow{I_2Y_2}$ is also ϕ). Using Equation 2 and the coordinate of Y_1 in B_1 we have:

$$\tan\phi = \frac{\alpha_2}{\beta_2} = \frac{\Omega_{11}(\alpha_1 - \sigma\gamma_1)}{\beta_1} = -\frac{\Omega_{11}\sigma}{\Sigma}.$$

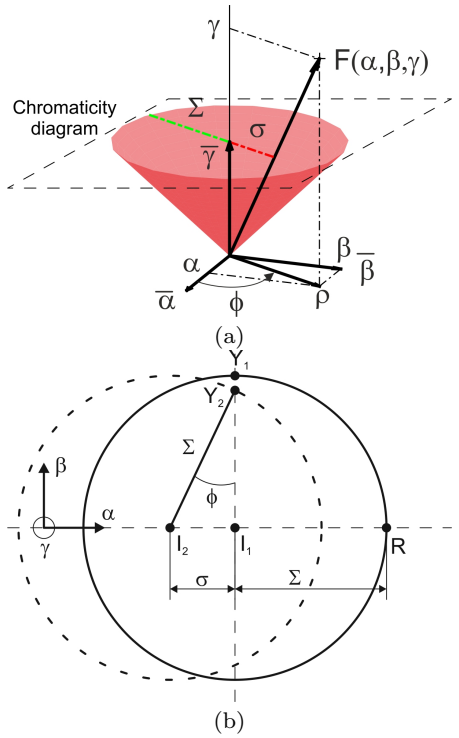


Fig. 6 Yilmaz two rooms experiment modeling. (a) Yilmaz considers that color perception space belong to a cone in a three-dimensional space. Any light represented by its spectral distribution function has three coordinates α, β and γ on the corresponding axis $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$. Another way of representing a point in that space is the use of cylindrical coordinates ρ, ϕ and γ. Saturation $\sigma = \rho/\gamma$ is defined as being the amount of deviation from the $\bar{\gamma}$ axis. This deviation is always less than Σ which define the interior of the cone as $\sigma \leq \Sigma$. Yilmaz proposed that the chromaticity diagram should be a disk defined by the intersection of the plane $\bar{\gamma} = 1$ and the cone, so it is a disk of radius Σ placed at $\gamma = 1$. (b) Yilmaz assumes that the two cones for the two adaptation states of the observer under two illuminants I_1 and I_2 respectively are represented as perfect circles in the chromaticity diagram. The two circles represent the rotation of the two cones respectively symmetric to I_1 and I_2 for the point of view given by the axis $\bar{\gamma}$. In this representation the monochromatic red (R) and the two yellows (seen under I_1 (Y_1) and under I_2 (Y_2)) allow the calculation of the shift in hue as angle ϕ in Experiment 3.

By identification of the two expressions of the tangent we have:

$$\boxed{\Omega_{11} = \frac{1}{\sqrt{1 - \left(\frac{\sigma}{\Sigma}\right)^2}}}, \quad (6)$$

which could be consider as the Lorentz factor for color vision.

Over the three experiments, the transform from illuminant I_1 to I_2 is given by the following formula:

$$\alpha_2 = \frac{\alpha_1 - \Delta\sigma\gamma_1}{\sqrt{1 - \left(\frac{\Delta\sigma}{\Sigma}\right)^2}},\ \beta_2 = \beta_1,\ \gamma_2 = \frac{\gamma_1 - \frac{\Delta\sigma}{\Sigma^2}\alpha_1}{\sqrt{1 - \left(\frac{\Delta\sigma}{\Sigma}\right)^2}}, \quad (7)$$

with $\Delta\sigma$ is the relative saturation of the illuminant I_2 compare to I_1. The transformation in matrix form writes:

$$\begin{bmatrix}\alpha_2\\ \beta_2\\ \gamma_2\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{1-\left(\frac{\Delta\sigma}{\Sigma}\right)^2}} & 0 & \frac{-\Delta\sigma}{\sqrt{1-\left(\frac{\Delta\sigma}{\Sigma}\right)^2}}\\ 0 & 1 & 0\\ \frac{-\frac{\Delta\sigma}{\Sigma^2}}{\sqrt{1-\left(\frac{\Delta\sigma}{\Sigma}\right)^2}} & 0 & \frac{1}{\sqrt{1-\left(\frac{\Delta\sigma}{\Sigma}\right)^2}}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{bmatrix}. \quad (8)$$

Posing $\Delta s = \operatorname{arctanh}(\frac{\Delta\sigma}{\Sigma})$ as being the relative saturability (in analogy to rapidity in space-time) of the illuminant I_2 compare to illuminant I_1, the transformation writes:

$$\Omega = \begin{bmatrix}\cosh(\Delta s) & 0 & -\Sigma\sinh(\Delta s)\\ 0 & 1 & 0\\ -\frac{1}{\Sigma}\sinh(\Delta s) & 0 & \cosh(\Delta s)\end{bmatrix}, \quad (9)$$

which is an hyperbolic rotation of boost Δs as illustrated in Figure 4 for which $W' = \Omega W$.

Using change in observer's adaptation state induced by the change in illuminant along with a carefully designed thought experiment, Yilmaz (1961, 1962a,b) did a *tour de force* by showing a transformation similar to the Lorentz transformation for special relativity. As we show earlier (Relativistic color vision: Yilmaz), Lorentz transformation is justified by a hyperbolic rotation of the white point with adaptation change. The work of Drösler in psychophysics extended this model in the context of projective geometry.

Projective hyperbolic metric color space: Drösler

The law of psychophysics proposed by Bouguer (1729) and Weber (1846) for the perception of light intensity and the perception of weight, respectively, could be stated as follow. Let's suppose an initial condition of intensity I that corresponds either to the perception of intensity of a light or of the weight of an object. The Bouguer-Weber law stipulates that the minimum variation ΔI of the physical variable that elicit a perceived difference for the observer is given by $\frac{\Delta I}{I} = k$. So the perceived minimal variation of intensity ΔI depends on the reference intensity I such that the ratio between them remain constant.

The Bouguer-Weber ratio defines a direct correspondence between physical and perceptual variables. Let's call S the sensation of the observer. Suppose this sensation is a function f of the physical intensity I, $S = f(I)$. From the definition of f we can write $\Delta S = f'(I)\Delta I$ where $f'(I) = dS/dI$ is the derivative of f, dS and dI are the infinitesimal variation of the variables. We consider ΔS and ΔI commensurate because the variation are supposed smalls. Let's define the minimal perceived variation of sensation being constant and equal to 1, $\Delta S = 1$ for a participant to report a perceived difference in sensation. According to Bouguer-Weber ratio, we may write $\Delta S = 1 = \frac{1}{I}\Delta I$ which in turn define $f'(I) = 1/I$. Thus f is defined as $f(I) = \int_{I_0}^{I}\frac{1}{x}dx$. As Fechner (1907) states, we have $f(I) = \log\frac{I}{I_0}$ with I_0 being the absolute detection threshold for the physical variable. According to the Weber-Fechner formulation, physical and perceptual variables are related non linearly by a log function.

Psychophysical laws are well formalized by contrast measurements. Indeed if you ask an observer to describe feeling when seeing a light or holding a weight in hand, the observer would probably respond something that you can hardly put on a scale. But if you ask to report when a difference is perceived in between two different intensities of light (or weight), you can plot these two intensities as points on a scale. Figure 7 shows the relationship between the physical and perceptual scales in case of a Weber-Fechner law of kind $S = 4\log(I)$ with 4 being an arbitrary value for illustration and $I_0 = 1$ for referring to the previous notation. Following this law, a difference in sensation depends on a geometrical increase of this physical variable I. Let's define $(\Delta I)_i = I_{i+1} - I_i$. We can check by the points defined on the figure that $(\Delta I)_i/I_i = k$. We have $(I_1 - I_0)/I_0 = (e^{1/4} - 1)/1 = 0.2840$, $(I_2 - I_1)/I_1 = (e^{1/2} - e^{1/4})/e^{1/4} = 0.2840$, and so on. Thus, the logarithmic law, allow to transform the geometrical scale of stimulus intensity I into an arithmetical scale of sensation S with the property that the Weber ratio is constant.

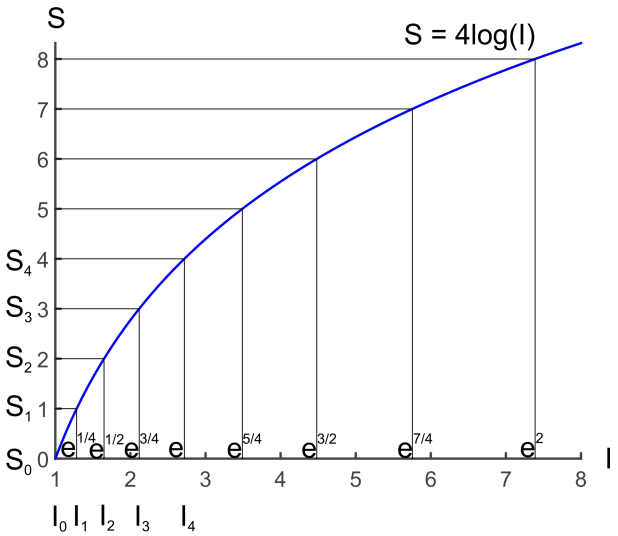


Fig. 7 Weber-Fechner psychophysical law. The psychophysical law associates a geometric interval scale of intensity I to a linear interval scale of sensation S.

Psychophysical laws have inspired numerous articles in the literature related to the neural origin of the non linear relation between physical and perceptual spaces Billock and Tsou (2011). Baird and Noma (1978) have reviewed the literature on psychophysical laws focusing on the scaling fundamentals. Scaling was also considered by Yilmaz (1967) who showed that psychophysical laws provide scale invariant properties that could be useful for living systems. He showed that the law of Stevens (1957) made a scale invariant by change of the origin, scale and inverse. The same idea is also presented in the work of Benzécri (1979). Scale invariance was also central in Resnikoff's work. In his book Resnikoff (1989) proposed an interpretation of the invariance caused by the logarithmic law in terms of information theory.

Drösler (1995, 1994) proposed a different interpretation of the invariance phenomenon. He formalized psychophysical laws in term of projective geometry. Projective geometry is a mathematical formalism born from the need to represent three-dimensional spatial relationship between objects by their perspective into a two-dimensional drawing on a sheet of paper. Drösler showed that the Weber ratio could be seen as a projective invariant of the projective line. It is invariant over any projective transformation. This allowed to understand visual non linearity as a projection and also to extend Weber ratio and more generally psychophysical laws into multi-dimensional models (Figure 8).

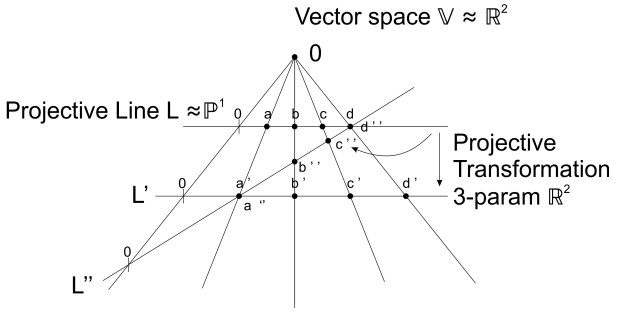


Fig. 8 Invariance of the cross-ratio. Consider lines a, b, c, d and a", b", c", d" as two projective lines. Because the points are related by the projection originating from 0, they are related by a projective transform. The transform generates a non linear scale between the point a", b", c" and d" from the arithmetic scale given by a, b, c, d. Nevertheless, the cross-ratio $[a, b, c, d]$ is identical to the cross-ratio $[a'', b'', c'', d'']$.

In projective geometry, cross-ratio is invariant under any linear projective transformation. Consider four points a, b, c, d on a real projective line $\mathbb{P}^1 = \mathbb{R} \cup \{\infty\}$. The cross-ratio is defined as:

$$\rho = [a, b, c, d] = \frac{\overline{ca}\,\overline{db}}{\overline{cb}\,\overline{da}} = \frac{(a-c)(b-d)}{(b-c)(a-d)}. \tag{10}$$

From Figure 8 we can check that the two cross-ratios $[a, b, c, d]$ and $[a', b', c', d']$ are equal because $\overline{ab} = k\overline{a'b'}$. The conservation of the cross ratio for the line L'' is less intuitive. The scale defined by a, b, c, d is a linear/arithmetic scale whereas a'', b'', c'', d'' define a non-linear scale. However, the two scales are related through a projective transform because the points a, a' and a'' are related through the line originating at 0. Thus, there is a projective transformation that transform a, b, c and d into a'', b'', c'' and d''. If two scales are related each other through a projective transform, they could be projectively equivalent even if one of the scale is linear with an arithmetic progression and the other is a non linear progression.

However, if two scales are related through a log function (Fechner (1907)), or through a power law (Stevens (1957)), they cannot originate from a projectivity. Instead, it was shown by Poincaré (1906, 1887) that the general transform between projective scales is analytically given by: $y = \frac{ax+b}{cx+d}$ where x is a variable on a projective line $L = \mathbb{P}^1$ mapped to another projective line $L' = \mathbb{P}^1$ with variable y. The parameters a, b, c and d define the transformation from L to L' in the homogeneous coordinate in $\mathbb{R}^2$ by:

$$\begin{bmatrix} sy \\ s \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \Rightarrow y = \frac{sy}{s} = \frac{ax+b}{cx+d}. \tag{11}$$

Figure 9(a) shows that if we draw lines from points on the abscissa and the ordinate for the psychophysics model of Figure 7 these lines do not cross on a single point. Rather the intersection of two lines seems to shift back along a curve. This is due to the $\log(x)$ function which tends to infinity when x increases. In opposite, if we draw the function that relates two orthogonal axis (the abscissa and the ordinate) with lines originating from a single point (Figure 9(b)) the function is neither a log nor a power law. In that case the function has an horizontal asymptote and does not tends to infinity when x increases.

In vision study, the kinetic of the photoreceptor response to light has been shown to follow the Naka and Rushton (1966) function which can be written $y = k\frac{x^n}{x^n+x_0^n} \approx \frac{ax+b}{cx+d}$ if $n = 1$. Where x is the light intensity received by the photoreceptor, y the current it delivered through its polarization, and x_0 the factor of adaptation. From a neurobiological point of view, this function is more plausible than a log (because it saturates). It shows its usefulness for chromatic adaptation and color discrimination Meylan et al. (2007); Alleysson and Hérault (2001); Alleysson and Meary (2012). However, the drawback is that the value of the derivative is

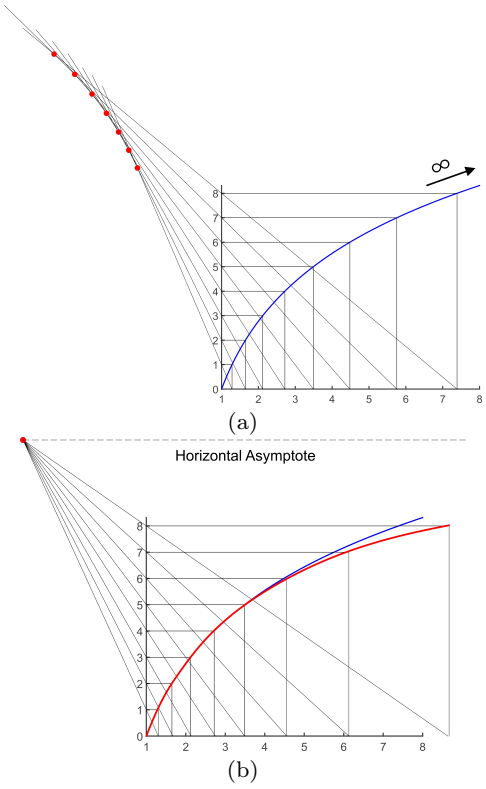


Fig. 9 Log is not projective. (a) If two scales are related to each other through a logarithm function, they are not projectively related. This means that one cannot find a projective point from which lines relate the two scales. Instead, the intersection of lines passing through the position of the two scales is not fixed. (b) If one chooses to relate the two scales with a projective transform by connecting them through lines from a projective origin point, the function that is drawn is of the form $y = \frac{ax+b}{cx+d}$

given by:

$$\frac{df(x)}{dx} = \frac{ad-bc}{(cx+d)^2} \approx \frac{1}{x^2}, \tag{12}$$

instead of $1/x$ in Weber-Fechner law. This implies constant $k = \Delta I/I^2$, instead of $k = \Delta I/I$.

Drösler discussed this point of view and proposed, like Yilmaz did, that the quadratic form for color vision should have the envelop of a cone as null space. There is a natural distance between two points a and b that can be defined in a space equipped with a quadratic form (Figure 10). This distance $d(a,b)$ is the logarithm of the cross-ratio, $[p,q,a,b]$ of four points a, b, p, q. Where p, q are the points of intersection of the line (ab) with the isotropic cone. This formulation was introduced by Laguerre-Verly (1853) and has been extended by what is called today the Cayley-Klein metric Klein (1873, 1897) or Hilbert (1895) metric.

From this definition of the distance, we infer an intuitive justification for the *log* function found in psychophysics. In the case of an one dimensional experiment, the independent variable goes from 0, absence of sensation, to infinity or saturation. Thus $p = 0$ and $q = \infty$. In that case, the distance formula leads to $d(a,b) = k \log \frac{b}{a}$ due to the properties of the cross-ratio, $[0,\infty,a,b] = b/a$. In multidimensional cases, such as color, the line (ab) do not necessarily originate from 0. For color discrimination for example you could have two points a and b such that the line (ab) does not cross zero.

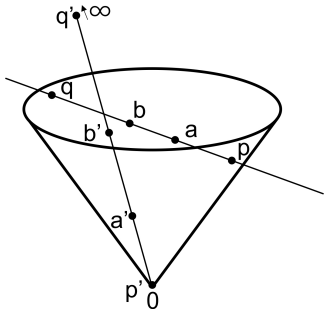


Fig. 10 Cayley-Klein metric. Inside a quadric, such as the isotropic cone in three-dimensional space, there is a natural metric that could be set up. Considering two points a and b inside the quadric. The distance between a and b is given by $d(a,b) = k \log([p,q,a,b])$ where k is a constant, p and q are the intersection of the line (ab) with the quadric and $[p,q,a,b]$ is the cross-ratio between points p, q, a and b. This formalism can explain why psychophysicists have observed a logarithmic relation. Suppose that the Weber ratio is measured along a line going from $p' = 0$ to $q' = \infty$, then the distance resumes to $d(a',b') = k \log \frac{b'}{a'}$ which is the Fechner law.

In our interpretation of Drösler's work, visual adaptation is the consequence of the visual system's state given the physical space.

The Yilmaz-Drösler color vision model

In this part, we give our interpretation of the Yilmaz-Drösler model of color perception. The model involves two steps. The first step is an orthogonal projection of the space of spectral functions onto a subspace of metamerism related to the measurement of spectral functions by the photoreceptors (Figure 1). The second step is a representation of the light on a space restricted by a convex cone and endowed with a hyperbolic metric (Figure 3). The direction of the cone and the associated hyperbolic metric change according to the adaptation of the observer induced by the illumination (Figure 4). The axis of the cone is aligned with the vector that represents the illuminant in the metameric space. This is what we call the colorimetric point of view adopted by the visual system.

We define the projection of the points in the convex cone as a central projection on the unit hyperboloid

(Figure 13). A point in the convex cone can be decomposed into its color components (*i.e.*, the position of its projection on the unit hyperboloid), and its intensity, given by the affix on the projective line. This decomposition could be assimilated to a chrominance-luminance decomposition. This decomposition is not unique because it depends on the orientation of the cone and its associated hyperbolic metric on the metameric space accordingly to the adaptation state of the observer.

Space of spectral functions $\mathcal{L}$

Let's consider a function of wavelength $f(\lambda)$ that can represent the spectrum of a light. Yilmaz proposed a change of variable from the wavelength variable $\lambda = [\lambda_m, \lambda_M]$ to a phase variable $\phi = [-\pi, \pi]$ given by[2]:

$$\phi = \frac{2\pi(\lambda - \lambda_m)}{\lambda_M - \lambda_m} - \pi. \tag{13}$$

It is common to associate the space of spectral functions with a real vector space Schrödinger (1920); Weinberg (1976); Dubois (2009). Yilmaz (1961) talk about a Hilbert vector space for the space of spectral functions. Even if the formalism of Hilbert space allows to consider a function as being an infinite dimensional vector, here we restrict functions to their discrete approximation for simplification. We consider then ϕ as a discrete variable being sampled into n different values along its range. Any functions $\ell(\phi)$ are then represented as a row vector $\boldsymbol{\ell} = [\ell_1, \ell_2, ..., \ell_n]^t$, with $\ell_i = \ell(\phi_i)$, $\phi_i = -\pi + 2\pi(i-1)/n$, with $i = 1..n$. This vector of n components corresponds to a sampling of a spectrum function. We called $\mathcal{L}$ the space of all spectral functions given by their sampled points on the n value of ϕ and represented by a vector of n components.

It is usual in spectroscopy to use the scalar product between spectral functions. The scalar product is written:

$$(f, g) = \int_{\lambda_m}^{\lambda_M} f(\lambda) g(\lambda) d\lambda, \tag{14}$$

This scalar product is transposed in $\mathcal{L}$ as a scalar product between vectors and can be written:

$$\forall \boldsymbol{\ell}, \boldsymbol{m} \in \mathcal{L}, \quad (\boldsymbol{\ell}, \boldsymbol{m}) = \frac{2\pi}{n} \boldsymbol{\ell}^t \boldsymbol{m}. \tag{15}$$

The norm of a vector is given by:

$$\|\boldsymbol{\ell}\| = \sqrt{(\boldsymbol{\ell}, \boldsymbol{\ell})} = \sqrt{\frac{2\pi}{n} \boldsymbol{\ell}^t \boldsymbol{\ell}} \tag{16}$$

With this definition of the scalar product, we defined $\mathcal{L}$ to be isomorphic to $\mathbb{R}^n$.

[2] ϕ is rather defined in the interval $[0, 2\pi]$ in Yilmaz's papers.

Space of metamerism $\mathcal{M} \subset \mathcal{L}$

Following Schrödinger (1920) views, Yilmaz proposed that if color vision belongs to a three-dimensional space it should be a projection from the space of spectrum functions to the three-dimensional vector space in which color perception arise.

Yilmaz proposed three functions $\alpha(\phi)$, $\beta(\phi)$ and $\gamma(\phi)$ that served as a basis for a three-dimensional subspace of the space of spectral functions $\mathcal{L}$[3]. He called this subspace an idealized color space. But, one can call this subspace a space of metamerism because two spectra that are represented the same in this subspace are not distinguishable thus perceived the same. Even if their spectral distribution functions differ. We interpret then these functions as a basis for an idealized space of metamerism $\mathcal{M}$. These functions are written:

$$\begin{aligned} \alpha(\phi) &= \frac{1}{\sqrt{\pi}} \cos(\phi), \\ \beta(\phi) &= \frac{1}{\sqrt{\pi}} \sin(\phi), \\ \gamma(\phi) &= \frac{1}{\sqrt{2\pi}}. \end{aligned} \tag{17}$$

It is easy to show that those functions are of norm one and are orthogonal, $(\alpha, \alpha) = 1$, $(\beta, \beta) = 1$, $(\gamma, \gamma) = 1$, $(\alpha, \beta) = 0$, $(\alpha, \gamma) = 0$ and $(\beta, \gamma) = 0$. They form an orthonormal basis of a three-dimensional vector subspace of the space of spectral functions.

The three functions (Figure 11(a)) correspond to the first three components of the Fourier basis of $L^2([-\pi, \pi])$ (the space of square integral functions on the range $[-\pi, \pi]$). This choice of the basis functions for the metameric space is not related to mathematical properties of the space of spectral functions nor it corresponds to the measured spectral sensitivities of the photoreceptors in the human eye or color matching functions. It is driven by the orthogonality of the basis function according to the scalar product defined in Equation 14 and by a parametrization of the function by ϕ that corresponds to hue[4].

Let's call $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ the row vectors corresponding to the sampled basis functions of $\mathcal{M}$. The metameric space $\mathcal{M}$ is a subspace of $\mathcal{L}$ defined through an orthogonal projection $\boldsymbol{R}$. The projection operator $\boldsymbol{R}$ is

[3] This model is called Grassmann model in Koenderink (2010) book

[4] Yilmaz (1961)[p. 243 before e)] justify further the use of lower term of the Fourier expansion: *Higer terms in the expansion do not have the property of isotropy and are therefore unacceptable for the invariance of high saturation.*

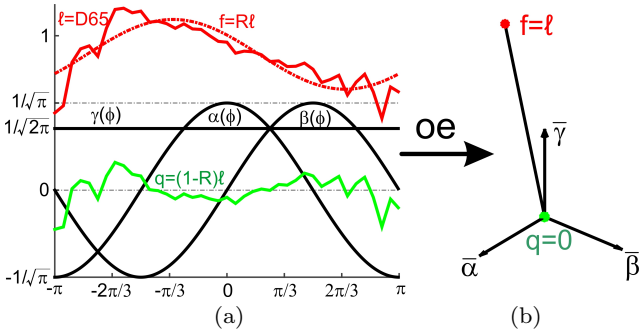
an operator of $\mathcal{L}$ onto itself as follow:

$$\begin{aligned} &\boldsymbol{R}\colon \mathcal{L} \to \mathcal{L} \supset \mathcal{M} \\ &\quad \boldsymbol{\ell} \mapsto \boldsymbol{m} = \boldsymbol{R}\boldsymbol{\ell}, \\ &\boldsymbol{R} = \frac{2\pi}{n}\left[\boldsymbol{\alpha}\,\boldsymbol{\beta}\boldsymbol{\gamma}\right]\left[\boldsymbol{\alpha}\,\boldsymbol{\beta}\,\boldsymbol{\gamma}\right]^t = \boldsymbol{B}\boldsymbol{B}^t, \\ &\boldsymbol{B} = \sqrt{\frac{2\pi}{n}}\left[\boldsymbol{\alpha}\,\boldsymbol{\beta}\,\boldsymbol{\gamma}\right]. \end{aligned} \tag{18}$$

The $\boldsymbol{R}$ matrix of size $n \times n$ is what is called the Cohen matrix R Cohen and Kappauf (1982); Cohen (2001) and represents the spectral operator that project any light's spectral power distribution vector onto its metameric fundamental component. Formally, Cohen R matrix is rather defined as $\boldsymbol{R} = \boldsymbol{B}(\boldsymbol{B}^t\boldsymbol{B})^{-1}\boldsymbol{B}^t$ but because $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are orthogonal vectors in $\mathcal{L}$, the product $\boldsymbol{B}^t\boldsymbol{B}$ is the identity matrix. $\boldsymbol{R}$ is a projection operator, we can check easily that it is idempotent ($\boldsymbol{R}^2 = \boldsymbol{R}\boldsymbol{R} = \boldsymbol{R}$) because $\boldsymbol{B}^t\boldsymbol{B}$ is the identity matrix. $\boldsymbol{R}$ is an orthogonal projector that splits the space of spectral functions $\mathcal{L}$ into two orthogonal subspaces. The two orthogonal subspaces are called the fundamental space of metamerism of dimension 3 (the image of the linear projector $\boldsymbol{R}$) and the black space (the kernel of $\boldsymbol{R}$) of dimension $n-3$ (Koenderink, 2010). Any spectrum function can be written as a sum of its fundamental plus black space components:

$$\boldsymbol{\ell} = \boldsymbol{R}\boldsymbol{\ell} + (\mathbf{1} - \boldsymbol{R})\boldsymbol{\ell}, \tag{19}$$

where $\mathbf{1}$ is the identity matrix of dimension n. Figure 11 shows an example of the projection operator.



(a) (b)

Fig. 11 Metamerism projection. (a) Yilmaz defined three spectral functions $\alpha(\phi)$, $\beta(\phi)$ and $\gamma(\phi)$ that served as a basis functions for the three-dimensional subspace $\mathcal{M}$ of metamerism. The projector $\boldsymbol{R}$ project any vector $\boldsymbol{\ell}$ on its fundamental $\boldsymbol{f}$. The difference vector $\boldsymbol{q} = \boldsymbol{\ell} - \boldsymbol{f}$ is invisible for the metameric subspace. (b) The linear application œ provides a three-dimensional representation of the metameric space, isomorphic to $\mathbb{R}^3$. The images of the three functions through œ form the standard basis $\{\bar{\alpha} = \text{œ}(\boldsymbol{\alpha}) = (1,0,0), \bar{\beta} = (0,1,0), \bar{\gamma} = (0,0,1)\}$. The images of $\boldsymbol{\ell}$ and $\boldsymbol{f} = \boldsymbol{R}\boldsymbol{\ell}$ give the same point because they are metameric. œ($\boldsymbol{q}$) is equal to zero because it is invisible for the metamerism space.

We can define a linear application to project any spectrum vector onto a three-dimensional space as follow:

$$\begin{aligned} &\text{œ}\colon \mathcal{L} \to \mathcal{M} \\ &\quad \boldsymbol{\ell} \mapsto x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \text{œ}(\boldsymbol{\ell}) = \sqrt{\frac{2\pi}{n}}\boldsymbol{B}^t\boldsymbol{\ell} \end{aligned} \tag{20}$$

So, every vector $\boldsymbol{\ell} \in \mathcal{L}$ could be represented by $x = \text{œ}(\boldsymbol{\ell})$ in the metameric space $\mathcal{M}$. $\mathcal{M}$ is then equal to the Euclidean space $\mathbb{R}^3$ because its basis vectors are given by $\{(1,0,0) = \text{œ}(\boldsymbol{\alpha}), (0,1,0) = \text{œ}(\boldsymbol{\beta}), (0,0,1) = \text{œ}(\boldsymbol{\gamma})\}$ and the scalar product is $x.y = \sum_i x_i y_i = \text{œ}(\boldsymbol{x})^t\text{œ}(\boldsymbol{y})$.

Metamerism between lights is defined by equal coordinates in the metameric space $\mathcal{M}$. The projection of a spectrum vector into the metameric space define equivalence classes between spectra. Two vectors $\boldsymbol{\ell}$ and $\boldsymbol{m}$ are metameric if they project on the same point on the metameric space. This can be written as:

$$\boldsymbol{\ell} \sim \boldsymbol{m} \implies \text{œ}(\boldsymbol{\ell}) = \text{œ}(\boldsymbol{m}). \tag{21}$$

By defining three spectral functions as a basis for the idealized space of metamerism, Yilmaz was able to represent metameric colors as equivalent classes of light spectra. It should be noted that those functions are not the only one that define an orthogonal basis for a three-dimensional metamerism subspace of $\mathcal{L}$. Again, those functions are idealized compared to metamerism in human vision because they are not issued from the CIE measurement (Figure 2). However, those functions allow the definition of the color vision cone in a simple way as we will see now.

The cone C of perceived color

The locus of spectral functions corresponding to lights in the metameric space is a restricted domain because light's spectral functions are positive functions. We call $\mathcal{L}^+$ the restriction of the space of functions to positive functions. To compute the domain of positive functions in the metameric space, let consider the canonical basis $\mathcal{B}$ of $\mathbb{R}^n$ which is the set of vectors e_i:

$$\begin{aligned} &\mathcal{B} = \{e_1, e_2, ..., e_i, ..., e_n\}, \\ &e_i = [0,0,...,0,1,0,..,0]^t, \end{aligned} \tag{22}$$

where the value 1 is at the position i. A sampled light spectral function defined by its spectral power distribution vector is a positive sum of the basis vectors:

$$\boldsymbol{\ell} = \sum_i \ell_i e_i \quad with \quad \ell_i \geq 0. \tag{23}$$

The limit of the representation of lights in the metameric space is given by the image of the basis $\mathcal{B}$ of $\mathbb{R}^n$

(scaled by $n/2\sqrt{\pi}$ for illustration purpose) through the linear application œ, which can be written:

$$\text{œ}\left(\frac{n}{2\sqrt{\pi}}\left[e_1 \dots e_n\right]\right) = \sqrt{\pi}\begin{bmatrix}\alpha_1 & \dots & \alpha_n\\ \beta_1 & \dots & \beta_n\\ \gamma_1 & \dots & \gamma_n\end{bmatrix}. \tag{24}$$

The locus of the basis vectors of $\mathbb{R}^n$ are the vertices of the envelop $\delta C = \text{œ}(\mathcal{B})$ of a cone C in the metameric space $\mathbb{R}^3$. The cone has an aperture of $\Sigma = \sqrt{2}$, because of the definition of the basis functions $\alpha(\phi)$, $\beta(\phi)$ and $\gamma(\phi)$ (Figure 12(b)).

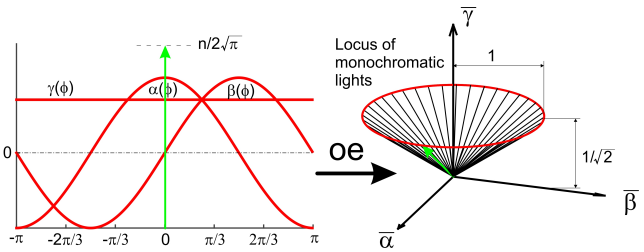


Fig. 12 The cone of perceived color (a) The image of the basis $\mathcal{B}$ of the spectral functions (scaled at $n/(2\sqrt{\pi})$), through the application œ, are the vertices of the envelop of the cone materialized by the red circle in (b). Because a light spectral function is a positive function it is a positive linear combination of the vertex, therefore it is located in the interior of the cone. So the image of light spectra is inscribed into the cone $C = \text{œ}(\mathcal{L}^+)$. Vertices also correspond to monochromatic lights, thus the envelop of the cone is the image of monochromatic lights which are maximal saturated lights. So the cone envelop is the locus of maximal saturation lights.

The vectors of the basis $\mathcal{B}$ can also be seen as the sampled monochromatic lights. Thus, the envelop of the cone also defines the locus of monochromatic lights, the most saturated lights, in the metameric space as predicted by Yilmaz (1961). The locus of monochromatic light forms the vertices of a cone having an aperture $\Sigma = \sqrt{2}$. Figure 12(b) show the three-dimensional space $\mathcal{M}$ along with the locus of monochromatic lights that form the vertices of the cone. Any light can be written as a positive weighted sum of monochromatic lights. Thus, lights are inscribed into the volume delimited by the cone $C = \text{œ}(\mathcal{L}^+)$.

The equation of the envelop of the cone is given in Cartesian and parametric form by:

$$\delta C = \{\mathbf{x} = (\alpha, \beta, \gamma) \mid \mathbf{x}^t \mathcal{J} \mathbf{x} = 0\}, \tag{25}$$

$$\delta C(k, \phi) = k\begin{bmatrix}\cos(\phi)\\ \sin(\phi)\\ 1/\Sigma\end{bmatrix},$$

$$\mathcal{J} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & \Sigma^2\end{bmatrix}, \Sigma = \sqrt{2}$$

where $\mathbf{x}^t \mathcal{J} \mathbf{x} = -\alpha^2 - \beta^2 + \Sigma^2\gamma^2$ is the quadratic form that remains invariant in the Yilmaz's idealized model of color perception space.

Functions chosen by Yilmaz imply a cone of perceived colors with aperture $\Sigma = \sqrt{2}$. Because no natural lights can be represented outside the cone, there is no need for the visual system to adopt a metric for the whole metameric space. Instead it is more efficient for the visual system to develop a metric associated to the cone domain of the encoded light.

A natural parametrization of the points inside the cone is given by a foliation with hyperboloid sheets. Consider the set of hyperboloid surfaces parametrized by $k > 0$ as follows:

$$\mathbb{H}^2_k = \{\mathbf{X} = (\alpha, \beta, \gamma) \mid \mathbf{X}^t \mathcal{J} \mathbf{X} = k^2\}, \tag{26}$$

$$\mathbb{H}^2_k(s, \phi) = k\begin{bmatrix}\sinh(s)\cos(\phi)\\ \sinh(s)\sin(\phi)\\ \frac{1}{\Sigma}\cosh(s)\end{bmatrix}.$$

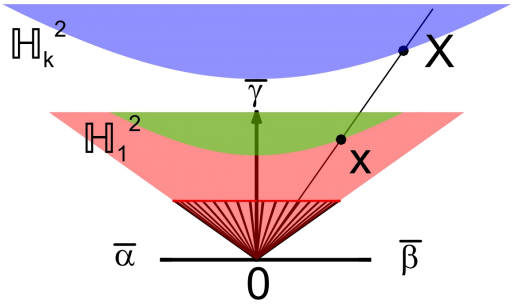


Fig. 13 Projection in Yilmaz-Drösler model. Inside the cone, a natural parametrization is given by a set of hyperboloids of revolution around $\bar{\gamma}$. Any point $\mathbf{X}$ inside the cone belong to a particular hyperboloid of parameter $k = \sqrt{\mathbf{X}^t J \mathbf{X}}$. This point could be projected onto a point $\mathbf{x} = \mathbf{X}/(\Sigma k)$ in the unit hyperboloid. The coordinates (s, ϕ) in the unit hyperboloid can be considered as chrominance value whereas the affixe k of X on the line (Ox) is the luminance value of a color.

This parametrization highlights the projective nature of the cone. If $\mathbf{X} = (\alpha, \beta, \gamma)$ is a point inside the cone, it has an intensity k given by $k = \sqrt{\mathbf{X}^t \mathcal{J} \mathbf{X}}$. Using Drösler's work we can project that point on the unit hyperboloid by defining $\mathbf{x} = \mathbf{X}/(\Sigma k)$. Figure 13 shows the projection $\mathbf{x}$ of a point $\mathbf{X}$ onto the unit hyperboloid. Any point inside the space delimited by the cone can be fully defined by its coordinates on the unit hyperboloid, s and ϕ, and an affix, k, on the projective line. This representation is a chrominance-luminance decomposition on the hyperbolic metric.

Such projection could explain why the appearance of colors is the same while the overall level of light increase or decrease. For a broad range of light level, before receptor saturation, the visual system is able to

build a representation of the color that is independent of this level. Only k changes when the overall level of light changes along the illuminant axis. The parameters s and ϕ, that define the perceived color, are constants.

But, this decomposition is not unique because it depends on the direction of the axis $\bar{\gamma}$ in the metameric space. Under two different illuminations, the axis of the cone of perceived color is changing. We assimilate that transformation as a involuntary change in colorimetric point of view adopted by the observer on the visual scene.

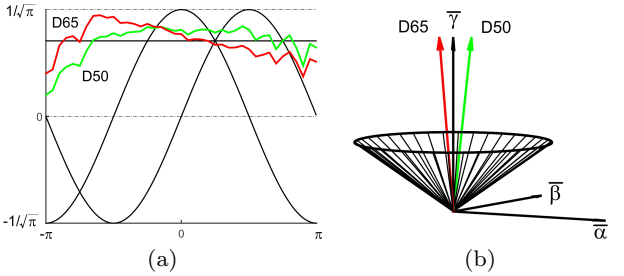


Fig. 14 D65 and D50 in Yilmaz-Drösler model. (a) Spectral power distribution of the two illuminants D65 (red), D50 (green) plotted together with Yilmaz's basis spectral functions. (b) Corresponding vectors in the three-dimensional metameric space according to the basis functions of the Yilmaz-Drösler model.

Color adaptation induced by change in illuminant

We can model Yilmaz's two rooms experiment using the perceptual space defined above. It is implicit in Yilmaz's modeling of the metameric space using $\alpha(\phi)$, $\beta(\phi)$ and $\gamma(\phi)$ that the observer is adapted to an isoenergetic illuminant. The cone in the metameric space is aligned along $\bar{\gamma}$, which corresponds to a constant function of wavelength, $\bar{\gamma} \rightarrow \gamma(\phi) \rightarrow \gamma(\lambda)$. Yilmaz represents the two conditions of adaptation (I_1 or I_2) in the chromaticity diagram associated to an observer adapted to $\bar{\gamma}$ (with I_1 corresponds to the axis $\bar{\gamma}$). This is why he drew the two illuminants in a common representation in Figure 6(b). In doing so he did an approximation because he represented the two cones in the chromaticity diagram using circles (saying: *From one illuminant to another the limiting saturation remains almost unchanged*) instead of ellipses (Compare Figure 6(b) with Figure 15).

For I_1 and I_2 in experiment 1, Yilmaz (1962b) talked about *two different illuminants of near daylight chromaticity*, that we choose previously to be $D65$ and $D50$. We then take the point of view given by $\bar{\gamma}$ to represent the two cones under the two illuminants. Because the quadratic form is invariant in any coordinate systems, the two cones have a fixed shape (same aperture). They differs only by their orientation relative to the illuminant axis. Knowing the spectral power distribution functions for the two illuminants D50 and D65 Hunt and Pointer (2011) we can draw the illuminants as vectors starting from 0 (Figure 14).

It should be noted that the length of the vectors representing illuminants is arbitrary because of the projective nature of color vision. Whatever the level of the illuminant, lights are represented by the unit hyperboloid and an affix on the projective line. Only the affix will be changed by the level of illuminant. With this in mind, the coordinates of the illuminants D65 and D50 have been normalized to unit norm according to scalar product in $\mathbb{R}^3$.

$$\gamma_1 = \begin{bmatrix} 0.099 \\ -0.15 \\ 0.98 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0.16 \\ 0.03 \\ 0.99 \end{bmatrix}. \tag{27}$$

Those vectors define two novel basis for the metameric space in which equations of the cones are identical apart from a change of variable. The conservation of the quadratic form $q(X)$ between the two coordinate systems for the two cones gives: $-\alpha_1^2 - \beta_1^2 + \Sigma^2\gamma_1^2 = 0$ and $-\alpha_2^2 - \beta_2^2 + \Sigma^2\gamma_2^2 = 0$. Where γ_1 and γ_2 are the coordinates on the axis corresponding to the two illuminants in the metameric space of the observer adapted to $\bar{\gamma}$.

Figure 15(a) shows the rotation of the two unit hyperboloids for the two illuminants. The chromaticity diagram from the point of view of the Yilmaz-Drösler model is given by the so-called Klein disk through the stereographic projection on the plane orthogonal to the achromatic axis $\bar{\gamma}$ at level $\gamma = 1$. In the chromaticity diagram the two hyperboloid surfaces project onto elliptic surfaces (Figure 15(b)). The corresponding hyperboloid models are aligned along and symmetric around either $\overline{\gamma_1}$ or $\overline{\gamma_2}$ following illumination by I_1 or I_2.

To account for change in illumination from $\overline{\gamma_1}$ to $\overline{\gamma_2}$ from the point of view of $\bar{\gamma}$, we can define two transformations T_1 and T_2 that change $\bar{\gamma}$ axis into $\overline{\gamma_1}$ and $\overline{\gamma_2}$. The transformation consists of finding an orthogonal basis for which the third vector of the basis is the axis of the illuminant. The two transforms are defined as follows:

$$\begin{aligned} T_1 &= \left[\overline{\alpha_1}\ \overline{\beta_1}\ \overline{\gamma_1}\right], \\ \beta_1 = \left[1\ 0\ 0\right] \wedge \overline{\gamma_1},\ \overline{\beta_1} &= \beta_1 / \parallel \beta_1 \parallel, \\ \overline{\alpha_1} &= \overline{\gamma_1} \wedge \overline{\beta_1}, \\ T_2 &= \left[\overline{\alpha_2}\ \overline{\beta_2}\ \overline{\gamma_2}\right], \\ \beta_2 = \left[1\ 0\ 0\right] \wedge \overline{\gamma_2},\ \overline{\beta_2} &= \beta_2 / \parallel \beta_2 \parallel, \\ \overline{\alpha_2} &= \overline{\gamma_2} \wedge \overline{\beta_2}, \end{aligned} \tag{28}$$

where $\wedge$ is the vector product in $\mathbb{R}^3$.

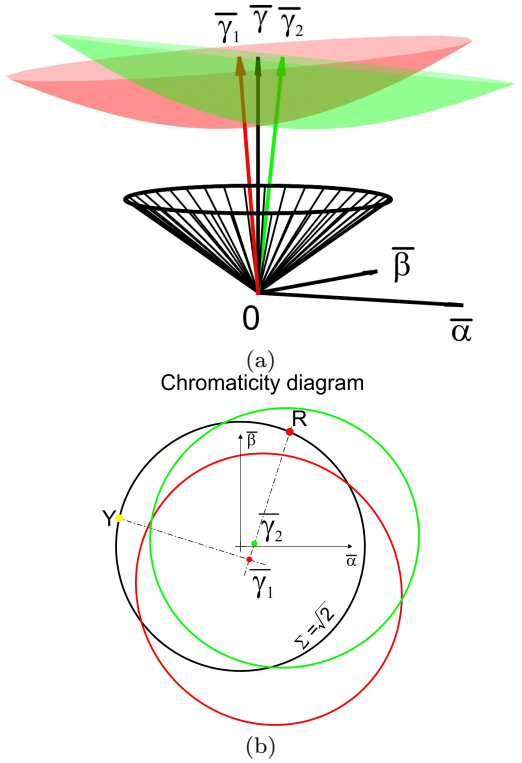


Fig. 15 Transformation from D65 to D50. (a) When the cone is moving with illuminant, the change of metrics corresponds to two different unit hyperboloids centered on each illuminant axis. (b) Representation of the two cones in the chromaticity diagram from the point of view of observer adapted to $\bar{\gamma}$. The two disks are deformed into ellipses because of the projection in the chromaticity diagram. Yilmaz approximated these ellipses as identical circles (see Figure 6(b)).

The transformation from $\overline{\gamma_1}$ to $\overline{\gamma_2}$ is given by $\overline{\gamma_2} = T_2 T_1^{-1} \overline{\gamma_1}$ thus $\Omega = T_2 T_1^{-1}$. In this case, Ω is not a Lorentz transformation, it is rather a rotation of the achromatic axis. This is due to the additional projection onto the unit hyperboloid after the Lorentz transformation as suggested by Drösler. Suppose I_1 and I_2 were aligned along the $\bar{\beta}$ axis (Yilmaz simplification) and suppose the perceived intensity of the two illuminants are equal for an observer adapted to $\overline{\gamma_1}$ (they belong to the same hyperboloid under $\overline{\gamma_1}$ axis as in Figure 4), then we retrieve the Lorentz transformation Ω given by Yilmaz.

Having a more sophisticated model does not change the conclusion of Yilmaz. Adaptive transformations in the perceptive color space should imply relativity because the quadratic form is invariant. Drölser provided a projective reading that is the key for understanding its application in psychophysics.

Conclusion

Projective hyperbolic model provides an invariant representation for a broad range of intensity and chromaticity of illumination. This could be a way by which the brain represents light. Yilmaz introduced the concept when proposing that change in illuminant could induce a transformation in perceptive space that behaves like the Lorentz transformation. The analogy between color adaptation and special relativity rests on the invariance of a quadratic form that defines a limiting cone. In this conception, the saturation limit in color vision is the analog of limiting speed of light in vacuum of Einstein's special relativity. Saturability is the analog to rapidity. But change of illuminant causes a change of the coordinate system in which the hyperbolic metric is defined. In this interpretation, interrelation between colors remains fixed and provides a constant color perception despite global change in the physics of the lights reflected by objects. Because of the decomposition of the cone of perceived colors $C \subset \mathbb{R}^3$ into luminance $\mathbb{R}^+$ and chrominance $\mathbb{H}^2$ through a central projection, the representation of color is invariant for the overall level of the illumination. Considering color perception as a relativistic phenomenon is, in our view, the best way to formalize adaptation in color vision.

Relativity models have been considered by several authors as model for describing perception. Considering the perception of motion, for example, Caelli et al. (1978) assumed a limit in the encoding of the speed of motion. They justified theoretically and experimentally the use of Lorentz's equations (with a velocity limit imposed by neural coding is $c' \ll c$) to explain why the perceived velocity of an object does not correspond to its physical velocity. Binocular vision had also been stated in term of hyperbolic metric based in the limit imposed by the visual field Luneburg (1950); Indow (1967). For most of the cognitive dimension of human perception, one can imagine a limiting variable that defines a limiting cone in which perception takes place. For color we have seen that this is the positiveness of the light spectra that delimit the cone of maximal saturation for which the visual system adopts an appropriate metric.

Those color vision models date from the early seventies but their advantages are still underestimated by the color community. This is probably because the experiments of Yilmaz are not easy to set up and experimental results have never been published. To be fully applicable, this model must clearly state its relation with neural activity. If the visual system represents the world on a hyperbolic manifold, we should be able to empirically test the existence of this hyperbolic representation. To-

day neurosciences use elaborate brain imaging apparatus to visualize brain functioning in vivo. Still, the operations achieved by the brain are difficult to understand on the basis of neural activity. Finding evidence of hyperbolic transformation for color vision would be a fundamental result for modeling how neurons could encode color information. The next step in the model would be to establish a clear relation between geometry and neural physiology for perception.

Beyond that, it is very exciting to consider non linearities in general cognitive process as a projective transform on a representation that defines metamerism. This approach has the advantage of considering physiology as a limit for cognitive representation or as a cone of maximum cognitive ability (perceived saturation for instance). This could help fill the gap between psychophysics and neurosciences.

As stated by Poincaré (1895) about the geometry of space, there is not a preferred geometry *per se*. For color vision, projective hyperbolic metric is a powerful tool for describing mechanism of color perception and its adaptation.

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