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Homology of depth-graded motivic Lie algebras and koszulity

par Benjamin ENRIQUEZ et Pierre LOCHAK

Résumé. *Homologies d'algèbres de Lie graduées associées aux algèbres de Lie motiviques et koszulité.* La conjecture de Broadhurst-Kreimer (BK) décrit la série de Hilbert d'une algèbre de Lie bigraduée $\mathfrak{a}$ reliée aux nombres multizétas. Brown a conjecturé une description de l'homologie de $\mathfrak{a}$ (conjecture homologique (CH)), et montré qu'elle implique la conjecture BK. Nous montrons qu'une partie de CH est équivalente à une présentation de $\mathfrak{a}$, et que la partie restante de CH est équivalente à un énoncé plus faible. Nous montrons enfin qu'en admettant la première partie de CH, la partie restante de CH est équivalente à l'un des énoncés suivants : (a) annulation du troisième groupe d'homologie d'une algèbre de Lie avec présentation quadratique, construite à partir des polynomes de périodes des formes modulaires ; (b) koszulité de l'algèbre enveloppante de cette algèbre de Lie.

Abstract. The Broadhurst-Kreimer (BK) conjecture describes the Hilbert series of a bigraded Lie algebra $\mathfrak{a}$ related to the multizeta values. Brown proposed a conjectural description of the homology of this Lie algebra (homological conjecture (HC)), and showed it implies the BK conjecture. We show that a part of HC is equivalent to a presentation of $\mathfrak{a}$, and that the remaining part of HC is equivalent to a weaker statement. Finally, we prove that granted the first part of HC, the remaining part of HC is equivalent to either of the following equivalent statements : (a) the vanishing of the third homology group of a Lie algebra with quadratic presentation, constructed out of the period polynomials of modular forms ; (b) the koszulity of the enveloping algebra of this Lie algebra.

Introduction

0.1. The background. The multizeta values (MZVs) are a family of real numbers $\zeta(n_1, .., n_s)$, where $s \geq 1$ and $n_1, \ldots, n_{s-1} \geq 1$, $n_s \geq 2$ ([Z]).

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Denote by $\mathcal{Z} \subset \mathbb{R}$ the $\mathbb{Q}$-vector subspace spanned by these numbers ; it is a subring of $\mathbb{R}$. A wealth of algebraic relations between these numbers are known ; in particular either of the following two types of relations are expected to provide a complete set of relations between the MZV's :

• *associator relations* arise from the fact that the MZVs are the coefficients of an analytic object called the Knizhnik-Zamolodchikov associator, and that this object satisfies the "pentagon and duality relations" ; additional relations relating the MZVs and the complex number $2\pi\mathrm{i}$ can be derived in the same way from the "hexagon relation" ([Dr, LM]) ;

• the (regularized) *double shuffle relations* are derived from the representations of the MZVs both as (discrete) sums and as iterated integrals ([Rac]).

The connection between the various associator relations has been elucidated in [Fur1]. It can be explained as follows. Let $\mathcal{Z}_{assoc}$ (resp., $\tilde{\mathcal{Z}}_{assoc}$) be the $\mathbb{Q}$-ring generated by formal analogues $\zeta^f(n_1, \dots, n_s)$ of the genuine MZVs (resp., these analogues and a formal analogue $(2\pi\mathrm{i})^f$ of $2\pi\mathrm{i}$), subject to the pentagon and duality relations (resp., to these relations and the hexagon relation). Then $\tilde{\mathcal{Z}}_{assoc}$ is isomorphic to the quadratic extension of $\mathcal{Z}_{assoc}$ generated by $(2\pi\mathrm{i})^f$ subject to the relation $((2\pi\mathrm{i})^f)^2 = -24\zeta^f(2)$, so $\tilde{\mathcal{Z}}_{assoc}$ is a free $\mathcal{Z}_{assoc}$-module with basis $(1, (2\pi\mathrm{i})^f)$.

It has also been shown ([Fur2]) that the associator relations on MZVs actually imply the double shuffle relations. More precisely, let $\mathcal{Z}_{ds}$ be the $\mathbb{Q}$-ring generated by the $\zeta^f(n_1, \dots, n_s)$, subject to the double shuffle and regularization relations. Then there is a surjective ring morphism $\mathcal{Z}_{ds} \to \mathcal{Z}_{assoc}$, taking each generator to the one with the same label.

A ring $\mathcal{Z}_{mot}$ of motivic analogues of the MZVs has been constructed ([B1], with the notation $\mathcal{H}$) ; it is linearly spanned by motivic analogues $\zeta^{\mathfrak{m}}(n_1, \dots, n_s)$ of the MZVs. The motivic MZVs satisfy the associator relations (see [B3], end of Section 2.2) so there is a ring morphism $\mathcal{Z}_{assoc} \to \mathcal{Z}_{mot}$, given by $\zeta^f(n_1, \dots, n_s) \mapsto \zeta^{\mathfrak{m}}(n_1, \dots, n_s)$. Finally, there is a (period) evaluation morphism $\mathcal{Z}_{mot} \to \mathcal{Z}$, given by $\zeta^{\mathfrak{m}}(n_1, \dots, n_s) \mapsto \zeta(n_1, \dots, n_s)$. All this gives rise to a sequence of surjective ring morphisms

$$\mathcal{Z}_{ds} \to \mathcal{Z}_{assoc} \to \mathcal{Z}_{mot} \to \mathcal{Z}. \tag{0.1}$$

The rings $\mathcal{Z}_{mot}$, $\mathcal{Z}_{assoc}$, $\mathcal{Z}_{ds}$ are equipped with a grading, called the *weight grading,* for which $\zeta^{\mathfrak{m}}(n_1, \dots, n_s)$ and $\zeta^f(n_1, \dots, n_s)$ have weight $n_1 + \dots + n_s$; the (very strong) *direct sum conjecture* asserts that this also defines a grading on the ring $\mathcal{Z}$. The rings $\mathcal{Z}, \mathcal{Z}_{mot}, \mathcal{Z}_{assoc}, \mathcal{Z}_{ds}$ are also equipped with an increasing filtration, the *depth filtration,* whose dth part is the linear span of all the elements corresponding to $(n_1, \dots, n_s)$, with $s \leq d$. This filtration is compatible with the grading, unconditionally in the case of $\mathcal{Z}_{mot}, \mathcal{Z}_{assoc}, \mathcal{Z}_{ds}$, and under the direct sum conjecture in the case of $\mathcal{Z}$. The Hilbert series of $\mathcal{Z}_{mot}$ with respect to the weight grading has been

computed in [B1]. It is generally conjectured that the maps from (0.1) are isomorphisms compatible with the gradings and filtrations. Under the direct sum conjecture, the depth-graded of $\mathcal{Z}$ is a bigraded algebra ; a formula for its double Hilbert series is conjectured in [BK] ; we will call this the Broadhurst-Kreimer (BK) conjecture. Combining the BK conjecture with that of the isomorphism of $\mathcal{Z}$, $\mathcal{Z}_{mot}$, $\mathcal{Z}_{assoc}$ and $\mathcal{Z}_{ds}$, one obtains three variants of the BK conjecture predicting the double Hilbert series of the depth-graded of $\mathcal{Z}_{mot}$, $\mathcal{Z}_{assoc}$ and $\mathcal{Z}_{ds}$.

The graded and filtered algebras $\mathcal{Z}_{mot}$, $\mathcal{Z}_{assoc}$ and $\mathcal{Z}_{ds}$ are related to prounipotent Lie groups as follows. Set $\mathcal{Z}^0_{mot} := \mathcal{Z}_{mot}/(\zeta^{\mathfrak{m}}(2))$, then there is an isomorphism of graded and filtered algebras $\mathcal{Z}_{mot} \simeq \mathcal{Z}^0_{mot} \otimes \mathbb{Q}[X]$, where X has weight 2 and all the powers $X, X^2, \ldots$ have depth 1 ([B1], (2.13), with the notation $\mathcal{H} = \mathcal{Z}_{mot}$, $\mathcal{A} = \mathcal{Z}^0_{mot}$; X corresponds to $\zeta^{\mathfrak{m}}(2)$). Moreover, $\mathcal{Z}^0_{mot}$ identifies with the function algebra of a prounipotent group scheme $U^{\mathfrak{m}}$ arising in the theory of mixed Tate motives ([B1], Section 2.1 and main result) and is the space in which Goncharov's motivic MZV's are defined ([G1, G2]). Its Lie algebra $\mathfrak{u}^{\mathfrak{m}}$ is graded and equipped with a decreasing filtration, which induces an isomorphism $\mathcal{Z}^0_{mot} \simeq U(\mathfrak{u}^{\mathfrak{m}})^\vee$ of graded and filtered algebras. In the same way, the fact that the rings $\mathcal{Z}_{assoc}, \mathcal{Z}_{ds}$ are polynomial in infinitely many variables ([And], Remark 25.9.3.2) implies that there are isomorphisms of graded and filtered algebras $\mathcal{Z}_{assoc} \simeq \mathcal{Z}^0_{assoc} \otimes \mathbb{Q}[X]$, $\mathcal{Z}_{ds} \simeq \mathcal{Z}^0_{ds} \otimes \mathbb{Q}[X]$, where X is a formal variable of weight 2 with $X, X^2, \ldots$ of depth 1, and $\mathcal{Z}^0_{assoc} := \mathcal{Z}_{assoc}/(\zeta^f(2))$, $\mathcal{Z}^0_{ds} := \mathcal{Z}_{ds}/(\zeta^f(2))$; moreover, there are isomorphisms of graded and filtered algebras $\mathcal{Z}^0_{assoc} \simeq U(\mathfrak{grt}_1)^\vee$ ([Dr], Prop. 5.9), $\mathcal{Z}^0_{ds} \simeq U(\mathfrak{ds}_0)^\vee$ ([Rac], Thm. I), where $\mathfrak{grt}_1$ is the "graded version of the Grothendieck-Teichmüller Lie algebra" arising in the study of associators, and $\mathfrak{ds}_0$ is the "double shuffle Lie algebra" arising in the study of combinatorial relations between MZVs. Here and below we consider, as is usual in this context, *graded* duals, the graded pieces of the various algebras being always finite dimensional. The above facts can be expressed as the following compatible isomorphisms of graded and filtered algebras

$$\mathcal{Z}_{mot} \simeq U(\mathfrak{u}^{\mathfrak{m}})^\vee \otimes \mathbb{Q}[X], \quad \mathcal{Z}_{assoc} \simeq U(\mathfrak{grt}_1)^\vee \otimes \mathbb{Q}[X], \quad \mathcal{Z}_{ds} \simeq U(\mathfrak{ds}_0)^\vee \otimes \mathbb{Q}[X],$$

which in their turn yield compatible isomorphisms of bigraded algebras

$$\mathrm{gr}_{dpth}(\mathcal{Z}_{mot}) \simeq \mathrm{gr}_{dpth}(U(\mathfrak{u}^{\mathfrak{m}}))^\vee \otimes \mathbb{Q}[X],$$

$$\mathrm{gr}_{dpth}(\mathcal{Z}_{assoc}) \simeq \mathrm{gr}_{dpth}(U(\mathfrak{grt}_1))^\vee \otimes \mathbb{Q}[X],$$

$$\mathrm{gr}_{dpth}(\mathcal{Z}_{ds}) \simeq \mathrm{gr}_{dpth}(U(\mathfrak{ds}_0))^\vee \otimes \mathbb{Q}[X].$$

The variants of the BK conjecture predicting the double Hilbert series of $\mathrm{gr}_{dpth}(\mathcal{Z}_{assoc})$ and its analogues therefore translate into conjectural formulas for the double Hilbert series of $\mathrm{gr}_{dpth}(U(\mathfrak{grt}_1))$ and its analogues, which will be recalled in Section 3 (Conjecture 3.1).

0.2. Contents of the paper. Recently, F. Brown proposed a conjecture describing the homology of these bigraded Lie algebras ([B2]) and implying these conjectural formulas. The main purpose of this paper is to partially reduce and derive some consequences of Brown's homological conjecture.

In Section 1, we introduce several objects : the Lie algebras $\mathfrak{u}^{\mathfrak{m}}, \mathfrak{grt}_1, \mathfrak{ds}_0$; the sequence of morphisms $\mathfrak{u}^{\mathfrak{m}} \to \mathfrak{grt}_1 \to \mathfrak{ds}_0$ of Lie algebras dual to the sequence of morphisms $\mathcal{Z}^0_{ds} \to \mathcal{Z}^0_{assoc} \to \mathcal{Z}^0_{mot}$ induced by (0.1) ; the depth-graded version of these Lie algebras and of this sequence of morphisms ; and an upper bound Lie algebra $\mathfrak{ls}$ of $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$. In Section 2, we present some known results on these Lie algebras. In Section 3, we present the BK conjectures and its variants, as well as Brown's homological conjecture, which presents itself in four versions relative to the Lie algebras $\mathfrak{u}^{\mathfrak{m}}$, $\mathfrak{grt}_1$, $\mathfrak{ds}_0$, $\mathfrak{ls}$; we prove that the version relative to $\mathfrak{ls}$ implies all the other versions.

The main results of the paper are contained in the remaining sections. In Section 4, we show that a part of the homological conjecture for one of these Lie algebras (more precisely, the part predicting the values of the first and second homology groups) is equivalent to a presentation of the same Lie algebra. The proof of this result is close to the proof that any positively graded Lie algebra L has a presentation with generating space $H_1(L)$ and relation space $H_2(L)$, where $H_i(L)$ denotes the ith Lie algebra homology group of L with trivial coefficients ([H], Section 3) ; the arguments from [H] are themselves analogues of those of [S], Chap. 2 in the pro-p group situation. In Section 5, we show that the remaining part of the homological conjecture, more precisely, the vanishing of the homology groups of order ≥ 3, is equivalent to a weaker statement, namely the vanishing of the third homology group. Again, the technical result used here is an analogue of a homological result for pro-p groups from [S]. In Section 6, we relate the homological conjecture with the structure of a Lie algebra $\mathfrak{M}_0$ with quadratic presentation, constructed out of the period polynomials of cuspidal modular forms. We show that granted the first part of the homological conjecture relative to any of the Lie algebras $\mathfrak{u}^{\mathfrak{m}}$, $\mathfrak{grt}_1$, $\mathfrak{ds}_0$, $\mathfrak{ls}$, the remaining part of this conjecture is equivalent to either of the following equivalent statements : (a) the vanishing of the Lie algebra homology group with trivial coefficients $H_3(\mathfrak{M}_0, \mathbb{Q})$; (b) the koszulity of the enveloping algebra $U(\mathfrak{M}_0)$.

1. Motivic and related Lie algebras

In this section, we recall a few facts about the Lie algebra $\mathfrak{g}^{\mathfrak{m}}$ attached to the category of mixed Tate motives (§1.1) ; the realization of its Lie subalgebra $\mathfrak{u}^{\mathfrak{m}}$ by derivations of a free Lie algebra in two generators (§§1.2, 1.3) ; and the construction of Lie algebras related to $\mathfrak{u}^{\mathfrak{m}}$ (§§1.4, 1.5, 1.6). These are all defined over $\mathbb{Q}$.

1.1. Motivic background. The Tannakian category $\mathrm{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ can be defined unconditionally (see [DG] ; for a survey see [And]). It is equipped with the de Rham fiber functor. Let $\mathcal{G}_{\mathrm{MT}(\mathbb{Z})}$ be the automorphism group of this functor ; this is a group scheme over $\mathbb{Q}$. It decomposes as a semidirect product $\mathcal{G}_{\mathrm{MT}(\mathbb{Z})} = \mathcal{U}_{\mathrm{MT}(\mathbb{Z})} \rtimes \mathbb{G}_m$, where $\mathcal{U}_{\mathrm{MT}(\mathbb{Z})}$ is a prounipotent $\mathbb{Q}$-group scheme. Its Lie algebra decomposes as $\mathfrak{g}^{\mathfrak{m}} = \mathfrak{u}^{\mathfrak{m}} \rtimes \mathbb{Q}$. It follows from work of Borel and Beilinson that there exist elements s_{2i+1}, $i \geq 1$ of $\mathfrak{u}^{\mathfrak{m}}$, freely generating $\mathfrak{u}^{\mathfrak{m}}$, and a right inverse $\mathbb{G}_m \to \mathcal{G}_{\mathrm{MT}(\mathbb{Z})}$ of the projection $\mathcal{G}_{\mathrm{MT}(\mathbb{Z})} \to \mathbb{G}_m$, for which each s_{2i+1} has weight $2i+1$. We will henceforth view $\mathfrak{u}^{\mathfrak{m}}$ as graded by this grading.

1.2. The Poisson-Ihara Lie algebra $(\mathfrak{g}, \langle , \rangle)$. Let e_0, e_1 be free non-commutative variables of weight 1, and let $\mathfrak{g} := \mathbb{L}(e_0, e_1)$ be the free $\mathbb{Q}$-Lie algebra generated by these variables ; its bracket will be denoted $[,]$. There is a linear map $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$, from $\mathfrak{g}$ to the Lie algebra of its derivations, given by $f \mapsto D_f$, where $D_f : e_0 \mapsto 0$, $e_1 \mapsto [e_1, f]$. When equipped with the *Poisson-Ihara Lie bracket* $\langle , \rangle$ given by

$$\langle f, g \rangle := [f, g] + D_f(g) - D_g(f),$$

$\mathfrak{g}$ is a Lie algebra, and the map $f \mapsto D_f$ is a Lie algebra morphism. Moreover, $(\mathfrak{g}, \langle , \rangle)$ is graded for the weight grading. The Lie algebra $(\mathfrak{g}, [,])$ is also equipped with a grading, for which e_1 has degree 1 and e_0 has degree 0. This equips $(\mathfrak{g}, \langle , \rangle)$ with a grading, the *depth grading*. The *depth filtration* on $(\mathfrak{g}, \langle , \rangle)$ is the decreasing Lie algebra filtration given by $F^i(\mathfrak{g}) := \oplus_{j|j \geq i}\{\text{part of } \mathfrak{g} \text{ of depth } j\}$.

1.3. The Lie algebra morphism $\mathfrak{u}^{\mathfrak{m}} \to (\mathfrak{g}, \langle , \rangle)$. The motivic Galois group $\mathcal{G}_{\mathrm{MT}(\mathbb{Z})}$ acts naturally on the fundamental groupoids of certain geometric objects of $\mathrm{MT}(\mathbb{Z})$. In the case of $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$, this induces a weight-graded Lie algebra morphism $\mathfrak{u}^{\mathfrak{m}} \to \mathrm{Der}(\mathfrak{g})$. It can be shown that this morphism factors through the inclusion $(\mathfrak{g}, \langle , \rangle) \subset \mathrm{Der}(\mathfrak{g})$ and therefore gives rise to a graded Lie algebra morphism $\mathfrak{u}^{\mathfrak{m}} \to (\mathfrak{g}, \langle , \rangle)$. Moreover, it follows from [B1] that this morphism is *injective.*

1.4. Lie algebras related to $\mathfrak{u}^{\mathfrak{m}}$. In [Dr, Rac], explicit weight-graded Lie subalgebras $\mathfrak{grt}_1$ and $\mathfrak{ds}_0$ of $(\mathfrak{g}, \langle , \rangle)$ are introduced.

There is a sequence of Lie algebra morphisms

$$\mathfrak{u}^{\mathfrak{m}} \simeq \mathrm{im}(\mathfrak{u}^{\mathfrak{m}} \to \mathfrak{g}) \hookrightarrow \mathfrak{grt}_1 \hookrightarrow \mathfrak{ds}_0 \subset (\mathfrak{g}, \langle , \rangle) \tag{1.1}$$

where, as mentioned above, the initial isomorphism follows from [B1], and the next injections follow from work of Drinfeld and Ihara, and from [Fur2], respectively.

1.5. The Lie algebra $\mathfrak{ls}$. The Lie algebra $\mathfrak{ds}_0$ is defined to be the set of elements of the free algebra generated by e_0, e_1 which are primitive for two coproducts, the shuffle and the stuffle coproduct. The associated graded of the stuffle coproduct for the depth filtration can be computed explicitly. This gives rise to an explicit upper bound Lie algebra for the depth-graded of the double shuffle Lie algebra $\mathfrak{ds}_0$, called the "linearized shuffle" Lie algebra and denoted $\mathfrak{ls}$. We have therefore a double inclusion $\mathrm{gr}_{dpth}(\mathfrak{ds}_0) \subset \mathfrak{ls} \subset (\mathfrak{g}, \langle , \rangle)$ (see [B2]).

Remark. This space was introduced earlier in [IKZ] and, in dual form, in [G1, G2] ; moreover, the Lie algebras $\mathfrak{ls}$ and $\mathfrak{ds}_0$ are respectively isomorphic to the Lie algebras introduced in [E1, E2] under the notation $\mathsf{ARI}^{\mathrm{al/al}}_{\mathrm{ent}}$ and $\mathsf{ARI}^{\mathrm{al/il}}_{\mathrm{ent}}$.

1.6. Bigraded Lie algebras. The depth filtration of the Lie algebra $(\mathfrak{g}, \langle , \rangle)$ induces a filtration on each of its weight-graded Lie subalgebras ; the associated graded Lie algebra is then a (depth,weight)-bigraded Lie subalgebra of $\mathrm{gr}_{dpth}(\mathfrak{g}, \langle , \rangle) = (\mathfrak{g}, \langle , \rangle)$. The sequence (1.1) therefore gives rise to a sequence

$$\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}}) \hookrightarrow \mathrm{gr}_{dpth}(\mathfrak{grt}_1) \hookrightarrow \mathrm{gr}_{dpth}(\mathfrak{ds}_0) \hookrightarrow \mathfrak{ls} \subset (\mathfrak{g}, \langle , \rangle). \tag{1.2}$$

Numerical explorations of these algebras (unpublished) have been conducted by various authors since the early nineties, reaching weights roughly between 20 and 30. They seem to confirm that the first two injections of (1.1), and the first three injections of (1.2) could in fact be isomorphisms.

2. Known results on depth-graded Lie algebras

In this section, we recall some basic results on the Lie algebras $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$ and $\mathfrak{ls}$. We first introduce the space P of even period polynomials (§2.1) and then, using it, we make explicit a system of generators and relations for the above Lie algebras (§§2.2, 2.3). Finally we draw some consequences for $\mathfrak{ls}$ (§2.4) and as well as for the other Lie algebras $\mathfrak{a}$ (§2.5).

2.1. The spaces S and P. Let S denote the (complex) vector space of cusp forms for the full modular group $\mathrm{PSL}_2(\mathbb{Z})$, which decomposes as $\mathsf{S} = \bigoplus_n \mathsf{S}_{2n}$ where S_{2n} denotes the space of forms of weight $2n$ $(n \geq 0)$; one sets $\mathbb{S}(s) := \sum_{n\geq 0} \dim(\mathsf{S}_{2n}) s^{2n}$. Then

$$\mathbb{S}(s) = \frac{s^{12}}{(1-s^4)(1-s^6)}. \tag{2.1}$$

Let $\tilde{\mathsf{P}}_{2n}$ denote the $\mathbb{Q}$-vector space of all even period polynomials of degree $2n-2$, which are the polynomials in $\mathbb{Q}[X, Y]$ satisfying certain symmetry

conditions : evenness in X and Y, antisymmetry with respect to the exchange of X and Y and a functional equation. The additional condition of divisibility by X^2Y^2 defines a hyperplane $\mathsf{P}_{2n} \subset \tilde{\mathsf{P}}_{2n}$. The Eichler-Shimura correspondence sets up a linear isomorphism $\mathsf{S}_{2n} \simeq \mathsf{P}_{2n} \otimes_{\mathbb{Q}} \mathbb{C}$ for any $n \geq 0$. We set $\mathsf{P} := \bigoplus_n \mathsf{P}_{2n}$.

2.2. Generators and relations in $\mathfrak{l}\mathfrak{s}$.

2.2.1. *Generators.* The depth one subspace $\mathfrak{l}\mathfrak{s}_1$ of $\mathfrak{l}\mathfrak{s}$ is 1-dimensional in each odd weight ≥ 3 and 0-dimensional in all the other weights, therefore if $\mathbb{O}(s) := \sum_{n\geq 0} \dim(\mathfrak{l}\mathfrak{s}_1[n])s^n$, then

$$\mathbb{O}(s) = \frac{s^3}{1-s^2} \tag{2.2}$$

(where $[n]$ means the part of weight n). On the other hand, there exists an injective linear map $\mathsf{e} : \mathsf{P} \to \mathfrak{l}\mathfrak{s}_4$, compatible with the weight gradings on both sides (see [B2]).

2.2.2. *Relations.* As mentioned above, there is an isomorphism of graded vector spaces $\mathfrak{l}\mathfrak{s}_1 \simeq X^3\mathbb{C}[X^2]$, where the degree on the right-hand side is the degree in X. On the other hand, there is an injection $\mathsf{P} \hookrightarrow X^2Y^2\mathbb{C}[X^2, Y^2]^{as}$, where *as* means antisymmetry in X^2, Y^2, and where P_{2n} maps to the part of total degree $2n-2$. The composition

$$\mathsf{P} \hookrightarrow X^2Y^2\mathbb{C}[X^2, Y^2]^{as} \overset{XY\times -}{\to} X^3Y^3\mathbb{C}[X^2, Y^2]^{as} \simeq \Lambda^2(X^3\mathbb{C}[X^2]) \simeq \Lambda^2(\mathfrak{l}\mathfrak{s}_1)$$

is then an injective graded linear map $\mathsf{P} \to \Lambda^2(\mathfrak{l}\mathfrak{s}_1)$. On the other hand, the Lie bracket of $\mathfrak{l}\mathfrak{s}$ induces a linear map $\Lambda^2(\mathfrak{l}\mathfrak{s}_1) \to \mathfrak{l}\mathfrak{s}_2$. It follows from [GKZ, Sch] that these two maps combine into an exact sequence

$$0 \to \mathsf{P} \to \Lambda^2(\mathfrak{l}\mathfrak{s}_1) \to \mathfrak{l}\mathfrak{s}_2 \to 0.$$

2.3. Generators and relations in $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{l}\mathfrak{s}$.

Let $\mathfrak{a}$ be one of the Lie algebras $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{l}\mathfrak{s}$. There are isomorphisms $\mathfrak{a}_i \simeq \mathfrak{l}\mathfrak{s}_i$ for $i = 1, 2$ between the depth $1, 2$ subspaces of $\mathfrak{a}$ and $\mathfrak{l}\mathfrak{s}$. Therefore there is an exact sequence

$$0 \to \mathsf{P} \to \Lambda^2(\mathfrak{a}_1) \to \mathfrak{a}_2 \to 0.$$

isomorphic to the analogous exact sequence for the Lie algebra $\mathfrak{l}\mathfrak{s}$.

2.4. Consequences for $\mathfrak{ls}$.

2.4.1. *A model Lie algebra* $\mathfrak{M}$. We define a (weight, depth)-bigraded Lie algebra $\mathfrak{M}$, which will be called the "model" Lie algebra, by the following presentation. The space of generators is the direct sum of :

• a copy of the (weight, depth)-bigraded vector space $\mathfrak{ls}_1$;

• a copy $\mathsf{P}\{4\}$ of the space P, where $\{4\}$ means that for each n, the weight n component of P is placed in bidegree $(n, 4)$.

The map $\mathsf{P} \to \Lambda^2(\mathfrak{ls}_1)$ constructed in Subsection 2.2.2 induces a bigraded linear map $\mathsf{P}\{2\} \hookrightarrow \Lambda^2(\mathfrak{ls}_1)$, where $\{2\}$ has the same meaning as above. We then obtain a composed map

$$\mathsf{P}\{2\} \hookrightarrow \Lambda^2(\mathfrak{ls}_1) \simeq \mathbb{L}_2(\mathfrak{ls}_1) \hookrightarrow \mathbb{L}_2(\mathfrak{ls}_1 \oplus \mathsf{P}\{4\}) \hookrightarrow \mathbb{L}(\mathfrak{ls}_1 \oplus \mathsf{P}\{4\}),$$

where $\mathbb{L}(V)$ is the free Lie algebra generated by a (possibly graded) vector space V, and $\mathbb{L}_i(V)$ is its degree i part (with respect to V). The space of defining relations of $\mathfrak{M}$ is defined to be the image of this map. We have therefore

$$\mathfrak{M} := \mathbb{L}(\mathfrak{ls}_1 \oplus \mathsf{P}\{4\})/(\mathsf{P}\{2\}),$$

where $(\mathsf{P}\{2\})$ is the ideal generated by the image of the map $\mathsf{P}\{2\} \hookrightarrow \mathbb{L}(\mathfrak{ls}_1 \oplus \mathsf{P}\{4\})$.

2.4.2. *A morphism* $\mathfrak{M} \to \mathfrak{ls}$. It follows from Subsections 2.2.1 and 2.2.2 that there exists a morphism of bigraded Lie algebras

$$\varphi : \mathfrak{M} \to \mathfrak{ls}, \tag{2.3}$$

defined by restriction to the space of generators of $\mathfrak{M}$ as follows : the restriction of φ to $\mathfrak{ls}_1$ is the canonical injection $\mathfrak{ls}_1 \hookrightarrow \mathfrak{ls}$; its restriction to $\mathsf{P}\{4\}$ is the composed map $\mathsf{P}\{4\} \simeq \mathsf{P} \xrightarrow{\mathsf{e}} \mathfrak{ls}_4 \hookrightarrow \mathfrak{ls}$.

2.4.3. *The homology of* $\mathfrak{ls}$. If $\mathfrak{a}$ is a Lie algebra, we denote by $H_\cdot(\mathfrak{a})$ its homology with coefficients in the trivial module $\mathbf{k}$, given by the homology of the complex $\ldots \to \Lambda^3(\mathfrak{a}) \xrightarrow{[,]\otimes \mathrm{id}} \Lambda^2(\mathfrak{a}) \xrightarrow{[,]} \mathfrak{a} \to 0$. This is a graded cocommutative coalgebra, depending functorially on $\mathfrak{a}$.

It follows from the definition of $\mathfrak{M}$ that

$$H_1(\mathfrak{M}) \simeq \mathfrak{ls}_1 \oplus \mathsf{P}\{4\}.$$

Moreover, the composed map $\mathsf{P}\{2\} \to \Lambda^2(\mathfrak{ls}_1) \to \Lambda^2(\mathfrak{M}) \xrightarrow{[,]} \mathfrak{M}$ is zero. This induces a map

$$\mathsf{P}\{2\} \to H_2(\mathfrak{M}).$$

One then checks that these maps are compatible with coproducts, so that the diagram

$$\begin{tikzcd} \mathsf{P}\{2\} \arrow[r] \arrow[d] & \Lambda^2(\mathfrak{ls}_1) \arrow[r] & \Lambda^2(\mathfrak{ls}_1 \oplus \mathsf{P}\{4\}) \arrow[d] \\ H_2(\mathfrak{M}) \arrow[rr] & & \Lambda^2(H_1(\mathfrak{M})) \end{tikzcd} \tag{2.4}$$

commutes. Using the morphism $H_.(\mathfrak{M}) \to H_.(\mathfrak{ls})$ induced by the morphism (2.3), one obtains :

Proposition 2.1. i) *There exists a bigraded linear map* $\mathfrak{ls}_1 \oplus \mathsf{P}\{4\} \to H_1(\mathfrak{ls})$ *;*

ii) *there exists a bigraded linear map* $\mathsf{P}\{2\} \to H_2(\mathfrak{ls})$ *;*

iii) *the diagram (2.4) commutes.*

2.5. Consequences for depth-graded Lie algebras. Let $\mathfrak{a}$ be one of the Lie algebras $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{ls}$.

2.5.1. *The Lie algebra* $\mathfrak{M}_0$*.* We set

$$\mathfrak{M}_0 := \mathbb{L}(\mathfrak{ls}_1)/(\mathsf{P}\{2\}).$$

Then $\mathfrak{M}_0$ is a (weight, depth)-bigraded Lie algebra. Then $\mathfrak{M}$ can be identified with the free product of Lie algebras $\mathfrak{M}_0 * \mathbb{L}(\mathsf{P}\{4\})$.

2.5.2. *A morphism* $\mathfrak{M}_0 \to \mathfrak{a}$*.* It follows from Subsection 2.3 that there exists a morphism of bigraded Lie algebras

$$\varphi_0 : \mathfrak{M}_0 \to \mathfrak{a},$$

defined by restriction to the space of generators of $\mathfrak{M}_0$ as follows : the restriction of φ_0 to $\mathfrak{ls}_1$ is the canonical injection $\mathfrak{ls}_1 \simeq \mathfrak{a}_1 \hookrightarrow \mathfrak{a}$.

2.5.3. *The homology of* $\mathfrak{a}$*.* The map $\mathfrak{ls}_1 \to \mathfrak{M}_0$ induces a linear map $\mathfrak{ls}_1 \to H_1(\mathfrak{M}_0)$. The linear map $\mathsf{P}\{2\} \to \Lambda^2(\mathfrak{ls}_1)$ is such that the composite map $\mathsf{P}\{2\} \to \Lambda^2(\mathfrak{ls}_1) \hookrightarrow \Lambda^2(\mathfrak{M}_0) \xrightarrow{[.]} \mathfrak{M}_0$ is zero. One derives from there a linear map $\mathsf{P}\{2\} \to H_2(\mathfrak{M}_0)$. One checks that the diagram

$$\begin{tikzcd} \mathsf{P}\{2\} \arrow[r] \arrow[d] & \Lambda^2(\mathfrak{ls}_1) \arrow[d] \\ H_2(\mathfrak{M}_0) \arrow[r] & \Lambda^2(H_1(\mathfrak{M}_0)) \end{tikzcd} \tag{2.5}$$

commutes. Using functoriality of Lie algebra homology with respect to the morphism $\varphi_0 : \mathfrak{M}_0 \to \mathfrak{a}$, and the fact that the morphism $\mathfrak{M}_0 \to \mathfrak{a}$ is an isomorphism in depths 1 and 2, one obtains :

Proposition 2.2. i) *There exists a bigraded linear map* $\mathfrak{ls}_1 \to H_1(\mathfrak{a})$, *which is an isomorphism in depth* 1 *;*

ii) *there exists a bigraded linear map* $\mathsf{P}\{2\} \to H_2(\mathfrak{a})$, *which is an isomorphism in depth* 2 *;*

iii) *the diagram (2.5) commutes.*

3. Conjectures on depth-graded Lie algebras

We present the motivic/combinatorial analogue of the Broadhurst-Kreimer conjecture, which is a statement about Hilbert series (§3.1). We then present the homological counterpart of this conjecture, due to Brown, in §3.2. We then discuss the relations between both conjectures (§3.3).

3.1. Hilbert series. Based on numerical experimentation, the authors of [BK] formulated a conjecture on dimensions of the depth-graded spaces of (real) multizeta values. The authors of [B2, CGS] then proposed motivic/combinatorial analogues of this conjecture, which we now recall.

Conjecture 3.1. *(*[BK, B2, CGS]*) Let* $\mathfrak{a}$ *be one of the Lie algebras* $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{ls}$. *Let* $S(\mathfrak{a})$ *be its symmetric algebra and denote by* $S(\mathfrak{a})[n,d]$ *its piece of weight* n *and depth* d. *Then there is an identity of formal series in two variables* s,t *:*

$$\sum_{n,d\geq 0} \dim S(\mathfrak{a})[n,d]\cdot s^n t^d = \frac{1}{1-\mathbb{O}(s)t+\mathbb{S}(s)t^2-\mathbb{S}(s)t^4},$$

where $\mathbb{O}(s)$, $\mathbb{S}(s)$ *are defined in (2.2), (2.1).*

Conjecture 3.1 for the Lie algebra $\mathfrak{a}$ will be denoted $\mathrm{HS}(\mathfrak{a})$ ("Hilbert series statement for $\mathfrak{a}$"). Then :

Proposition 3.1. *There holds the sequence of implications*

$$\mathrm{HS}(\mathfrak{ls}) \Rightarrow \mathrm{HS}(\mathrm{gr}_{dpth}(\mathfrak{ds}_0)) \Rightarrow \mathrm{HS}(\mathrm{gr}_{dpth}(\mathfrak{grt}_1)) \Rightarrow \mathrm{HS}(\mathrm{gr}_{dpth}(\mathfrak{g}^{\mathfrak{m}})).$$

Proof. Set $\mathfrak{a}_0 := \mathfrak{g}^{\mathfrak{m}}$, $\mathfrak{a}_1 := \mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathfrak{a}_2 := \mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{a}_3 := \mathfrak{ls}$. For $i = 0,\dots,3$, let $f_i(s,t) := \sum_{n,d\geq 0} \dim S(\mathfrak{a}_i)[n,d]\cdot s^n t^d \in \mathbb{Z}_{\geq 0}[t][[s]]$. The inclusions $\mathfrak{a}_0 \subset \cdots \subset \mathfrak{a}_3$ imply $f_0(s,t) \leq \cdots \leq f_3(s,t)$, where $\leq$ means that the difference belongs to $\mathbb{Z}_{\geq 0}[t][[s]]$. The structure result on $\mathfrak{g}^{\mathfrak{m}}$ (see Subsection 1.1) implies that $f_0(s,1) = 1/(1-\mathbb{O}(s))$.

Fix $i \in \{0,1,2\}$. The statement $\mathrm{HS}(\mathfrak{a}_{i+1})$ implies that $f_{i+1}(s,t) = 1/(1-\mathbb{O}(s)t+\mathbb{S}(s)t^2-\mathbb{S}(s)t^4) =: g(s,t)$. It follows that

$$f_i(s,t) \leq g(s,t). \tag{3.1}$$

On the other hand,

$$f_i(s,1) \geq f_0(s,1) = g(s,1). \tag{3.2}$$

If f, g belong to $\mathbb{Z}_{\geq 0}[t]$ and are such that $f(t) \leq g(t)$ and $f(1) \geq g(1)$, then $f = g$. It follows that if F, G belong to $\mathbb{Z}_{\geq 0}[t][[s]]$ and are such that $F(s,t) \leq G(s,t)$ and $F(s,1) \geq G(s,1)$, then $F = G$. (3.1) and (3.2) then imply $f_i(s,t) = g(s,t)$, therefore $\mathrm{HS}(\mathfrak{a}_i)$. $\square$

3.2. Homology.

Conjecture 3.2. *(*[B2]*) Let* $\mathfrak{a}$ *be one of* $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{ls}$.

i) *There exists a bigraded linear isomorphism* $\mathfrak{ls}_1 \oplus \mathsf{P}\{4\} \simeq H_1(\mathfrak{a})$, *such that the diagram*

$$\begin{tikzcd} \mathfrak{ls}_1 \oplus \mathsf{P}\{4\} \arrow[r] \arrow[dr] & H_1(\mathfrak{a}) \arrow[d] \\ & H_1(\mathfrak{ls}) \end{tikzcd}$$

commutes ;

ii) *the map* $\mathsf{P}\{2\} \to H_2(\mathfrak{a})$ *is a linear isomorphism ;*

iii) *for any* $k \geq 3$, $H_k(\mathfrak{a}) = 0$.

Conjecture 3.2 for the Lie algebra $\mathfrak{a}$ will be denoted $\mathrm{HC}(\mathfrak{a})$ ("homology conjecture for Lie algebra $\mathfrak{a}$"). Taking into account Proposition 2.2, ii), $\mathrm{HC}(\mathfrak{a})$, ii), says that $H_2(\mathfrak{a})$ is concentrated in depth 2. It also follows from Proposition 2.2 that for any Lie algebra $\mathfrak{a}$ in sequence (1.2), the following diagram commutes :

$$\begin{tikzcd} \mathsf{P}\{2\} \arrow[r] \arrow[d] & \Lambda^2(\mathfrak{ls}_1) \arrow[d] \\ H_2(\mathfrak{a}) \arrow[r] & \Lambda^2(H_1(\mathfrak{a})) \end{tikzcd}$$

3.3. Relation between the Hilbert series and the homological conjectures. It has been proved in [B2] that for $\mathfrak{a}$ one of the Lie algebras $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{ls}$, $\mathrm{HC}(\mathfrak{a})$ implies $\mathrm{HS}(\mathfrak{a})$.

4. Homological conjecture and presentation

In this section we establish the equivalence between part of the homological Conjecture 3.2 and presentation results for the corresponding Lie algebras. In §4.1, we prove one implication, and in §4.2, we show the converse implication. As mentioned in the Introduction, the results of this section are close to those of [H] and [S], Chap. 2.

4.1. From presentation to homology. If V is a vector space and $I \subset \mathbb{L}(V)$ is a Lie ideal, then the Lie analogue of Hopf's formula (see [H], Proposition 5.6) gives

$$H_2(\mathbb{L}(V)/I) = I/[\mathbb{L}(V), I];$$

it can be derived from the definition of the homology of $\mathbb{L}(V)/I$ using a long exact sequence.

If now R is a vector subspace of $\mathbb{L}_2(V)$ and $I := (R) \subset \mathbb{L}(V)$ is the ideal generated by R, then I decomposes as $I = R \oplus [V, R] \oplus [V, [V, R]] \oplus \cdots$ according to the degree with respect to V, while $[\mathbb{L}(V), I] = [V, R] \oplus [V, [V, R]] \oplus \cdots$; it follows that

$$H_2(\mathbb{L}(V)/(R)) = R. \tag{4.1}$$

One derives from there

$$H_2(\mathfrak{M}) = \mathsf{P}\{2\}. \tag{4.2}$$

Also recall from §2.4.3 that

$$H_1(\mathfrak{M}) = \mathfrak{M}/[\mathfrak{M}, \mathfrak{M}] = \mathfrak{ls}_1 \oplus \mathsf{P}\{4\}. \tag{4.3}$$

(4.3) and (4.2) then imply :

Lemma 4.1. *Let $\mathfrak{a}$ be one of the Lie algebras $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{ls}$. If there is a bigraded Lie algebra isomorphism $\mathfrak{M} \simeq \mathfrak{a}$, such that the diagram*

$$\begin{tikzcd} \mathfrak{M} \arrow[r] \arrow[dr] & \mathfrak{a} \arrow[d] \\ & \mathfrak{ls} \end{tikzcd}$$

commutes, then $\mathrm{HC}(\mathfrak{a})$ *i) and ii) hold.*

4.2. From homology to presentation. Let $\mathfrak{g}$ be a positively graded Lie algebra, so $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots$, with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. The grading of $\mathfrak{g}$ then induces a positive grading on each of the homology groups of $\mathfrak{g}$. There is a quotient map $\mathfrak{g} \to H_1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, which is surjective and graded. Set $H := H_1(\mathfrak{g})$ and let $H = H_1 \oplus H_2 \oplus \ldots$ be the degree decomposition of H. Let us choose a graded right inverse $H \to \mathfrak{g}$ of the projection map $\mathfrak{g} \to H$. It gives rise to a surjective, graded Lie algebra morphism $\mathbb{L}(H) \to \mathfrak{g}$. Let I be the kernel of this morphism. Then I is a graded ideal of $\mathbb{L}(H)$; let $I = I_2 \oplus I_3 \oplus \ldots$ be the degree decomposition of I. Then there is a graded Lie algebra isomorphism $\mathbb{L}(H)/I \simeq \mathfrak{g}$. Equation (4.1) then implies that $H_2(\mathfrak{g}) = I/[\mathbb{L}(H), I]$. *Assume that $H_2(\mathfrak{g})$ is concentrated in degree 2.* Then

$$I_3 = [H_1, I_2], \quad I_4 = [H_2 + [H_1, H_1], I_2] + [H_1, I_3],$$

$$I_5 = [H_3 + [H_1, H_2] + [H_1, [H_1, H_1]], I_2] + [H_2 + [H_1, H_1], I_3] + [H_1, I_4], \quad \text{etc.}$$

It follows that, for any $n > 0$,

$$I_{n+2} \subset \sum_{k \geq 1} \sum_{\substack{n_1, \ldots, n_k > 0, \\ n_1 + \cdots + n_k = n}} [H_{n_1}, [H_{n_2}, \ldots, [H_{n_k}, I_2]]]. \tag{4.4}$$

Since, conversely, the right-hand side of (4.4) is contained in its left-hand side, one has, for any $n > 0$,

$$I_{n+2} = \sum_{k\geq 1} \sum_{\substack{n_1,\dots,n_k>0,\\ n_1+\cdots+n_k=n}} [H_{n_1}, [H_{n_2}, \dots, [H_{n_k}, I_2]]].$$

It follows that $I = (I_2)$. We also have $H_2(\mathfrak{g}) \simeq I_2$, and the injective composed map $H_2(\mathfrak{g}) \simeq I_2 \subset \mathbb{L}_2(H_1) \subset \mathbb{L}_2(H) \simeq \Lambda^2(H_1(\mathfrak{g}))$ can be identified with the coproduct map $H_2(\mathfrak{g}) \to \Lambda^2(H_1(\mathfrak{g}))$. It follows that this coproduct map is injective.

All this implies :

Lemma 4.2. *If $\mathfrak{g}$ is a positively graded Lie algebra, such that $H_2(\mathfrak{g})$ is concentrated in degree 2, then the coproduct map $H_2(\mathfrak{g}) \to \Lambda^2(H_1(\mathfrak{g}))$ is injective, and there exists an isomorphism of graded Lie algebras*

$$\mathfrak{g} \simeq \mathbb{L}(H_1(\mathfrak{g}))/(H_2(\mathfrak{g})),$$

where $H_2(\mathfrak{g})$ is viewed as a subspace of $\mathbb{L}_2(H_1(\mathfrak{g}))$ via the sequence of maps $H_2(\mathfrak{g}) \to \Lambda^2(H_1(\mathfrak{g})) \simeq \mathbb{L}_2(H_1(\mathfrak{g})))$.

This lemma can be adapted to the case when $\mathfrak{g}$ is also equipped with a second grading, compatible with the first one. Let $\mathfrak{a}$ be one of the Lie algebras $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{ls}$, and assume that it satisfies HC($\mathfrak{a}$), i) and ii). Then applying the graded version of Lemma 4.2 to $\mathfrak{a}$ equipped with the depth grading as the first grading, and the weight grading as the second grading, we obtain a bigraded isomorphism $\mathfrak{M} \simeq \mathfrak{a}$.

4.3. Combining Lemma 4.2 with Lemma 4.1, we obtain :

Theorem 4.1. *For $\mathfrak{a}$ one of the Lie algebras*

$$\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}}),\ \mathrm{gr}_{dpth}(\mathfrak{grt}_1),\ \mathrm{gr}_{dpth}(\mathfrak{ds}_0),\ \mathfrak{ls},$$

the conjunction of HC($\mathfrak{a}$) *i) and ii) is equivalent to the existence of a bigraded Lie algebra isomorphism $\mathfrak{M} \simeq \mathfrak{a}$, such that the diagram in Lemma 4.1 commutes.*

Remark. Under HC($\mathfrak{a}$) i) and ii), Theorem 4.1 gives a presentation of $\mathfrak{a}$. When $\mathfrak{a} = \mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, this presentation was first conjectured in [E1], §§17, 18 and [E2], §§7, 8.

5. Reduction of the homological conjecture

Let $\mathfrak{g} = \oplus_{i>0}\mathfrak{g}_i$ be a positively graded Lie algebra over a field $\mathbf{k}$ of characteristic 0. We assume that the graded pieces $\mathfrak{g}_i$ are finite dimensional.

Let $\mathcal{M}od$ be the category whose objects are the graded $\mathfrak{g}$-modules V of the form $V = \bigoplus_{i\in\mathbb{Z}} V_i$, where each graded piece V_i is finite dimensional

and $V_i = 0$ for $i \ll 0$, and the morphisms are the graded $\mathfrak{g}$-module morphisms. Let $\mathcal{V}ec$ denote the category of graded vector spaces with the same conditions.

Define the *valuation* v as the map from the set of objects of any of these categories to $\mathbb{Z} \cup \{\infty\}$ given by $v(V) = \min\{i \in \mathbb{Z}, V_i \neq 0\}$ for V a nonzero object and $v(0) = \infty$. For a an integer, define the *shift operator by* a, denoted $V \mapsto V[a]$, as the self-map of the set of objects of any of the categories $\mathcal{M}od$ or $\mathcal{V}ec$, given by $V[a]_i = V_{i+a}$ for any $i \in \mathbb{Z}$.

For $k \geq 0$, there is a functor $H_k : \mathcal{M}od \to \mathcal{V}ec$ given by $H_k(V) := H_k(\mathfrak{g}, V)$ (Lie algebra homology group) ; we call it the *k-th homology functor.*

Let $n > 0$ be an integer. Denote again by $\mathbf{k}$ the trivial $\mathfrak{g}$-module (one-dimensional, concentrated in degree 0), and *assume that* $H_n(\mathfrak{g}, \mathbf{k}) = 0$.

Let V be a nonzero object in $\mathcal{M}od$; set $v := v(V)$. For $s \geq 0$, define the *truncated module* $V\{s\}$ as the object of $\mathcal{M}od$ given by $V\{s\}_i = V_i$ for $i \geq v + s$, $V\{s\}_i = 0$ otherwise (in particular, $V\{0\} = V$). Then for any $s \geq 0$, there is an exact sequence in $\mathcal{M}od$

$$0 \to V\{s+1\} \to V\{s\} \to \mathbf{k}[s+v]^{\dim(V_{s+v})} \to 0,$$

where the last module is a direct sum of copies of shifts of the trivial module $\mathbf{k}$. The assumption $H_n(\mathbf{k}) = 0$ implies $H_n(\mathbf{k}[s+v]) = 0$, therefore the homology long exact sequence implies that, for any $s \geq 0$, the map $H_n(V\{s+1\}) \to H_n(V\{s\})$ is *onto.* For any $s \geq 0$, each of the maps of the sequence $H_n(V\{s\}) \to H_n(V\{s-1\}) \to \cdots \to H_n(V\{0\}) = H_n(V)$ is therefore onto, so that

$$\text{the composed map } H_n(V\{s\}) \to H_n(V) \text{ is onto.} \tag{5.1}$$

On the other hand, $H_n(V\{s\})$ is the homology group of a complex constructed out of the graded vector space $\Lambda^{\cdot}(\mathfrak{g}) \otimes V\{s\}$, which is nonzero only in degrees $\geq v + s$, therefore

$$H_n(V\{s\}) \text{ is nonzero only in degrees } \geq v + s. \tag{5.2}$$

For any integer $k \in \mathbb{Z}$ and any $s \geq \max(0, k - v + 1)$, (5.2) implies that (degree k part of $H_n(V\{s\})$)=0 and (5.1) implies that the map (degree k part of $H_n(V\{s\})$) $\to$(degree k part of $H_n(V)$) is onto, so that (degree k part of $H_n(V)) = 0$. It follows that $H_n(V) = 0$. Therefore, if $H_n(\mathfrak{g}, \mathbf{k}) = 0$, then

$$\text{for any object } V \text{ of } \mathcal{M}od, \text{ one has } H_n(V) = 0. \tag{5.3}$$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. When equipped with the grading induced by $\mathfrak{g}$ and with the action of $\mathfrak{g}$ by left multiplication, $U(\mathfrak{g})$ is an object of $\mathcal{M}od$. The truncation $U(\mathfrak{g})\{1\}$ can be identified with the

kernel $U(\mathfrak{g})_+$ of the counit map $U(\mathfrak{g}) \to \mathbf{k}$. Then there is an exact sequence in $\mathcal{M}od$

$$0 \to U(\mathfrak{g})_+ \to U(\mathfrak{g}) \to \mathbf{k} \to 0.$$

Shapiro's lemma implies that $H_i(\mathfrak{g}, U(\mathfrak{g})) = 0$ for any integer $i > 0$, whereas $H_0(\mathfrak{g}, U(\mathfrak{g})) \simeq \mathbf{k} \simeq H_0(\mathfrak{g}, \mathbf{k})$. The homology long exact sequence then implies an isomorphism $H_{i+1}(\mathfrak{g}, \mathbf{k}) \simeq H_i(\mathfrak{g}, U(\mathfrak{g})_+)$ for any integer i. Combining this equality for $i = n$ with the specialization of (5.3) for $V = U(\mathfrak{g})_+$, we obtain $H_{n+1}(\mathbf{k}) = 0$.

We summarize these results as follows :

Proposition 5.1. *Let $\mathfrak{g} = \oplus_{i>0}\mathfrak{g}_i$ be a positively graded Lie algebra over a field $\mathbf{k}$ of characteristic 0, such that all the graded pieces $\mathfrak{g}_i$ are finite dimensional. Let n be an integer ≥ 0. If $H_n(\mathfrak{g}, \mathbf{k}) = 0$, then for any $m \geq n$, $H_m(\mathfrak{g}, \mathbf{k}) = 0$.*

Remark. Proposition 5.1 illustrates a general principle in homological algebra for (pro)nilpotent objects ; compare in particular Proposition 21 in [S] (§4) for pro-p groups. □

An immediate corollary of Proposition 5.1 is the following result.

Theorem 5.1. *For $\mathfrak{a}$ one of the Lie algebras*

$$\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}}),\ \mathrm{gr}_{dpth}(\mathfrak{grt}_1),\ \mathrm{gr}_{dpth}(\mathfrak{ds}_0),\ \mathfrak{ls},$$

item iii) of the Homological Conjecture 3.2 is equivalent to $H_3(\mathfrak{a}) = 0$.

6. Homological conjecture and koszulity

In this section, we show that for $\mathfrak{a}$ one of the Lie algebras $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$ or $\mathfrak{ls}$, the conjecture HC($\mathfrak{a}$) implies two equivalent statements, namely the vanishing of the homology group $H_3(\mathfrak{M}_0, \mathbb{Q})$, and the koszulity of the algebra $U(\mathfrak{M}_0)$, equipped with the depth grading. This section is organized as follows. In §6.1 we compute the homology of free products of algebras ; then in §6.2, we use this result to show that HC($\mathfrak{a}$) implies the vanishing of $H_3(\mathfrak{M}_0, \mathbb{Q})$. In §6.3 we use a result of Goncharov ([G1]) to prove a vanishing $(\mathfrak{ls}_1 \otimes \mathsf{P}) \cap (\mathsf{P} \otimes \mathfrak{ls}_1) = 0$ on the generators and relations of $\mathfrak{M}_0$. After recalling some results on Koszul algebras (§6.4), we study Lie algebras with a quadratic presentation in §6.5 : we prove that for such a Lie algebra with space of generators V and space of relations $R \subset \mathbb{L}_2(V)$, such that $(V \otimes R) \cap (R \otimes V) = 0$, the vanishing of the third Lie algebra homology group is equivalent to the koszulity of its enveloping algebra. In §6.6 we gather all these results in order to prove the main Theorem 6.1.

6.1. Homology of free products of algebras. Let $\mathbf{k}$ be a field. The category of $\mathbf{k}$-algebras with unit is equipped with a coproduct operation (see [Bbk], Chap. III, p. 195, exercice 6). For A, B two $\mathbf{k}$-algebras with unit, we denote by $A * B$ their coproduct in this category (also called the *free product* of A and B).

The coproduct property implies that if A, B admit presentations of the form $A = T(V)/(R)$, $B = T(W)/(S)$, where V, W are vector spaces and R, S are vector subspaces of $T(V), T(W)$, then a presentation of $A * B$ is

$$A * B \simeq T(V \oplus W)/(R \oplus S), \tag{6.1}$$

where we use the canonical injections $R \subset T(V) \hookrightarrow T(V \oplus W)$, $S \subset T(W) \hookrightarrow T(V \oplus W)$.

It is proved in *loc. cit.* that $A * B$ is equipped with a complete increasing filtration $P_0 \subset P_1 \subset \cdots \subset A * B$, where P_n is the image of the part of tensor degree $\leq n$ of the tensor algebra $T(A \oplus B)$ under the algebra morphism $T(A \oplus B) \to A * B$ induced by the linear map $A \oplus B \to A * B$, the direct sum of the canonical maps $A \to A * B$, $B \to A * B$, and that there exist isomorphisms

$$P_0 \simeq \mathbf{k}, \quad P_{2n}/P_{2n-1} \simeq ((A/\mathbf{k}) \otimes (B/\mathbf{k}))^{\otimes n} \oplus ((B/\mathbf{k}) \otimes (A/\mathbf{k}))^{\otimes n} \text{ if } n > 0,$$

$$P_{2n+1}/P_{2n} \simeq ((A/\mathbf{k}) \otimes (B/\mathbf{k}))^{\otimes n} \otimes (A/\mathbf{k}) \oplus ((B/\mathbf{k}) \otimes (A/\mathbf{k}))^{\otimes n} \otimes (B/\mathbf{k})$$

if $n \geq 0$. One derives from there that if A, B are equipped with augmentation morphisms $A \to \mathbf{k}$, $B \to \mathbf{k}$ with kernels I, J, then injection followed by product induces an isomorphism

$$\mathbf{k} \oplus \Big(\bigoplus_{n>0} (I \otimes J)^{\otimes n} \oplus (J \otimes I)^{\otimes n} \Big) \oplus \Big(\bigoplus_{n \geq 0} (I \otimes J)^{\otimes n} \otimes I \oplus (J \otimes I)^{\otimes n} \otimes J \Big) \xrightarrow{\sim} A * B. \tag{6.2}$$

Moreover, the augmentation morphisms $A \to \mathbf{k}$, $B \to \mathbf{k}$ induce an augmentation morphism $A * B \to \mathbf{k}$, whose kernel $\mathcal{I}$ can be identified with the image of the sum of all the summands of (6.2) except $\mathbf{k}$. Together with the fact that (6.2) is an isomorphism, this implies :

Lemma 6.1. *The map $(A * B) \otimes_A I \oplus (A * B) \otimes_B J \to \mathcal{I}$ induced by the product is an isomorphism of left $A * B$-modules.*

Let $C \to \mathbf{k}$ be an augmented algebra and let N be a C-module. Recall that $\mathbf{k}$ is a C-module through the augmentation map. Let $K \subset C$ be the kernel of this map ; there is an exact sequence of C-modules

$$0 \to K \to C \to \mathbf{k} \to 0.$$

As C is a free C-module, one has $\mathrm{Tor}_i^C(C, \mathbf{k}) = 0$ for any $i \geq 1$. The long exact sequence for Tor then yields the isomorphism

$$\mathrm{Tor}_i^C(\mathbf{k}, N) = \mathrm{Tor}_{i-1}^C(K, N) \quad \text{for any } i \geq 1. \tag{6.3}$$

Let now $A \to \mathbf{k}$, $B \to \mathbf{k}$ be augmented algebras and let M be an $A * B$-module. The isomorphism (6.3) implies that, for any $i > 1$, there is an isomorphism

$$\mathrm{Tor}_i^{A*B}(\mathbf{k}, M) \simeq \mathrm{Tor}_{i-1}^{A*B}(\mathcal{I}, M). \tag{6.4}$$

Lemma 6.1 then implies that

$$\mathrm{Tor}_i^{A*B}(\mathcal{I}, M) \simeq \mathrm{Tor}_{i-1}^{A*B}((A*B)\otimes_A I, M) \oplus \mathrm{Tor}_{i-1}^{A*B}((A*B)\otimes_B J, M) \tag{6.5}$$

According to (6.2), inclusion followed by product induces an isomorphism of A-modules

$$A \otimes X \xrightarrow{\sim} A * B, \quad \text{where} \quad X = \mathbf{k} \oplus \big(\bigoplus_{n>0} (J \otimes I)^{\otimes n}\big) \oplus \big(\bigoplus_{n>0} (J \otimes I)^{\otimes n} \otimes J\big)$$

where $A \otimes X$ is viewed as a free A-module and $A * B$ is viewed as an A-module under left multiplication. It follows that $A * B$ is free as an A-module, and therefore is flat over A. Likewise, $A * B$ is flat over B. One derives from there isomorphisms $\mathrm{Tor}_{i-1}^{A*B}((A * B) \otimes_A I, M) \simeq \mathrm{Tor}_{i-1}^{A}(I, M)$ and likewise exchanging A and B, therefore

$$\begin{aligned} &\mathrm{Tor}_{i-1}^{A*B}((A * B) \otimes_A I, M) \oplus \mathrm{Tor}_{i-1}^{A*B}((A * B) \otimes_B J, M) \\ &\simeq \mathrm{Tor}_{i-1}^{A}(I, M) \oplus \mathrm{Tor}_{i-1}^{B}(J, M). \end{aligned} \tag{6.6}$$

Using again (6.3), we obtain

$$\mathrm{Tor}_{i-1}^{A}(I, M) \oplus \mathrm{Tor}_{i-1}^{B}(J, M) \simeq \mathrm{Tor}_i^{A}(\mathbf{k}, M) \oplus \mathrm{Tor}_i^{B}(\mathbf{k}, M). \tag{6.7}$$

Combining (6.4), (6.5), (6.6) and (6.7), we obtain :

Proposition 6.1. *If A, B are augmented algebras and M is any $A * B$-module, then*

$$\mathrm{Tor}_i^{A*B}(\mathbf{k}, M) = \mathrm{Tor}_i^{A}(\mathbf{k}, M) \oplus \mathrm{Tor}_i^{B}(\mathbf{k}, M)$$

for any $i \geq 2$.

6.2. Homological conjecture and vanishing of $H_3(\mathfrak{M}_0, \mathbf{k})$. Let $\mathfrak{a}$ be one of the Lie algebras $\mathrm{gr}_{dpth}(\mathfrak{d}\mathfrak{s}_0), \ldots, \mathfrak{l}\mathfrak{s}$ in sequence (1.2). Assume that HC($\mathfrak{a}$) i) and ii) hold. By Theorem 4.1, we then have a Lie algebra isomorphism $\mathfrak{a} \simeq \mathfrak{M}$. According to Theorem 5.1, HC($\mathfrak{a}$), iii) says that $H_3(\mathfrak{a}, \mathbb{Q}) = 0$. Given the isomorphism $\mathfrak{a} \simeq \mathfrak{M}$, this last statement is equivalent to

$$H_3(\mathfrak{M}, \mathbb{Q}) = 0. \tag{6.8}$$

Set $A := U(\mathfrak{M}_0)$ and $B := T(\mathsf{P}\{4\})$. Identity (6.1) implies that $A * B$ identifies with $U(\mathfrak{M})$ as an augmented algebra. Proposition 6.1 and the identity $H_i(\mathfrak{g}, M) = \mathrm{Tor}_i^{U(\mathfrak{g})}(\mathbb{Q}, M)$ relating the homology of a Lie algebra $\mathfrak{g}$ with values in a $\mathfrak{g}$-module M with the Tor groups of its enveloping algebra ([W], Cor. 7.3.6) imply the equality $H_i(\mathfrak{M}, \mathbb{Q}) = H_i(\mathfrak{M}_0, \mathbb{Q}) \oplus H_i(\mathbb{L}(\mathsf{P}\{4\}), \mathbb{Q})$ for any $i \geq 2$. The equality $H_i(\mathbb{L}(\mathsf{P}\{4\}), \mathbb{Q}) = 0$ for $i \geq 2$ then implies that

$H_i(\mathfrak{M},\mathbb{Q}) = H_i(\mathfrak{M}_0,\mathbb{Q})$ for $i \geq 2$, and therefore $H_3(\mathfrak{M},\mathbb{Q}) = H_3(\mathfrak{M}_0,\mathbb{Q})$. Equality (6.8) is therefore equivalent to $H_3(\mathfrak{M}_0,\mathbb{Q}) = 0$.

We have proved :

Proposition 6.2. *Assume that for* $\mathfrak{a}$ *one of the Lie algebras* $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$, $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$, $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$, $\mathfrak{ls}$, $\mathrm{HC}(\mathfrak{a})$ *i) and ii) hold. Then* $\mathrm{HC}(\mathfrak{a})$ *iii) is equivalent to* $H_3(\mathfrak{M}_0,\mathbb{Q}) = 0$.

6.3. The equality $(\mathfrak{ls}_1 \otimes \mathsf{P}) \cap (\mathsf{P} \otimes \mathfrak{ls}_1) = 0$**.** The three last lines of [G1] contain the computation of the homology of a complex denoted there $\mathcal{D}_{\bullet,3} \to \mathcal{D}_{\bullet,2} \otimes \mathcal{D}_{\bullet,1} \to \Lambda^3(\mathcal{D}_{\bullet,1})$: this complex is acyclic except at $\mathcal{D}_{\bullet,3}$. The dual complex is $\Lambda^3(\mathfrak{ls}_1) \stackrel{[,]\otimes \mathrm{id}}{\to} \mathfrak{ls}_2 \otimes \mathfrak{ls}_1 \stackrel{[,]}{\to} \mathfrak{ls}_3$; the result of [G1] then implies that the map

$$\Lambda^3(\mathfrak{ls}_1) \stackrel{[,]\otimes \mathrm{id}}{\to} \mathfrak{ls}_2 \otimes \mathfrak{ls}_1 \tag{6.9}$$

is injective.

Since the kernel of the map $[,] : \Lambda^2(\mathfrak{ls}_1) \to \mathfrak{ls}_2$ is equal to P, the kernel of the map (6.9) is equal to $(\mathsf{P} \otimes \mathfrak{ls}_1) \cap \Lambda^3(\mathfrak{ls}_1)$. It then follows form the injectivity of (6.9) that

$$(\mathsf{P} \otimes \mathfrak{ls}_1) \cap \Lambda^3(\mathfrak{ls}_1) = 0. \tag{6.10}$$

The subspace $(\mathfrak{ls}_1 \otimes \mathsf{P}) \cap (\mathsf{P} \otimes \mathfrak{ls}_1)$ of $\mathfrak{ls}_1^{\otimes 3}$ is contained in $\mathfrak{ls}_1 \otimes \Lambda^2(\mathfrak{ls}_1)$ (as P consists of antisymmetric tensors) and in $\Lambda^2(\mathfrak{ls}_1) \otimes \mathfrak{ls}_1$ (for the same reason), therefore it is contained in the intersection of these spaces, namely $\Lambda^3(\mathfrak{ls}_1)$. It follows that $(\mathfrak{ls}_1 \otimes \mathsf{P}) \cap (\mathsf{P} \otimes \mathfrak{ls}_1)$ is contained in the intersection of $\Lambda^3(\mathfrak{ls}_1)$ with $\mathsf{P} \otimes \mathfrak{ls}_1$, which by (6.10) is zero. We have proved :

Proposition 6.3. *The equality* $(\mathfrak{ls}_1 \otimes \mathsf{P}) \cap (\mathsf{P} \otimes \mathfrak{ls}_1) = 0$ *holds (equality of subspaces of* $\mathfrak{ls}_1^{\otimes 3}$*).*

6.4. The notion of koszulity. Let $A = \oplus_{i \geq 0} A_i$ be a graded connected algebra over a field $\mathbf{k}$, with finite dimensional graded pieces. Set $A_+ := \oplus_{i>0} A_i$; equip the space

$$T_+(A_+) := \oplus_{s>0} A_+^{\otimes s}$$

with the bidegree (syzygy degree, weight) by assigning to the piece $A_{n_1} \otimes \cdots \otimes A_{n_s}$ the bidegree $(s, n_1 + \cdots + n_s)$. The space $T_+(A_+)$ is equipped with a differential of bidegree $(-1,0)$ given by $a_1 \otimes \cdots \otimes a_s \mapsto \sum_{i=1}^{s-1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_s$. The corresponding homology group $H(T_+(A_+), d)$ inherits a bigrading, so it decomposes as

$$H(T_+(A_+), d) = \oplus_{i,j>0} \mathrm{Tor}_{ij}^A(\mathbf{k}, \mathbf{k})$$

with a double subscript (ij is shorthand for i, j). Observe that the group $\mathrm{Tor}_{ij}^A(\mathbf{k}, \mathbf{k})$ vanishes unless $i \leq j$. The graded algebra A is called *Koszul* iff

the groups $\mathrm{Tor}_{ij}^A(\mathbf{k},\mathbf{k})$ vanish unless $i=j$ ([PP], Def. 1, p. 19, and *loc. cit.*, identity relating Tor and Ext, end of p. 3).

When A is quadratically presented, i.e., $A=T(V)/(R)$, where V is a finite dimensional vector space and $R\subset V^{\otimes 2}$ is a vector subspace, and V has degree 1, the *diagonal* homology $\oplus_{i>0}\mathrm{Tor}_{ii}^A(\mathbf{k},\mathbf{k})$ identifies as a graded vector space with the positive part of the graded coalgebra dual to A, say $C(A):=\oplus_{i\geq 0}C_i(A)$, with ([LV], §3.1.3 or [PP], Chap. 1, §3)

$$C_i(A)=\cap_{j=0}^{i-2}V^{\otimes j}\otimes R\otimes V^{\otimes i-2-j}$$

6.5. Koszulity of enveloping algebras. Let V be a finite dimensional vector space over a field $\mathbf{k}$ and $R\subset\mathbb{L}_2(V)$ be a vector subspace. Let $\mathfrak{g}:=\mathbb{L}(V)/(R)$ be the corresponding Lie algebra. Its enveloping algebra will be denoted by A, so $A=U(\mathfrak{g})=T(V)/(R)$. If we assign to V the degree 1, this is a quadratic algebra.

There are canonical isomorphisms $H_i(\mathfrak{g})\simeq C_i(A)$ for $i=1,2$. Together with the equality $H_i(\mathfrak{g})=\oplus_j\mathrm{Tor}_{ij}^A(\mathbf{k},\mathbf{k})$ (see e.g. [W], Cor. 7.3.6) and the isomorphism $C_i(A)\simeq\mathrm{Tor}_{ii}^A(\mathbf{k},\mathbf{k})$ (see §6.4 above), this implies

$$\mathrm{Tor}_{ij}^A(\mathbf{k},\mathbf{k})=0 \text{ for } i\neq j \text{ and } i=1,2. \tag{6.11}$$

Assume now that $(V\otimes R)\cap(R\otimes V)=0$. Then $C_i(A)=0$ for any $i\geq 3$, so that the higher diagonal homology vanishes :

$$\mathrm{Tor}_{ii}^A(\mathbf{k},\mathbf{k})=0 \text{ for } i\geq 3. \tag{6.12}$$

Granted (6.11), the koszulity of A is equivalent to $\mathrm{Tor}_{ij}^A(\mathbf{k},\mathbf{k})=0$ for $i\neq j$, $i\geq 3$. In view of (6.12) this is the same as

$$\mathrm{Tor}_{ij}^A(\mathbf{k},\mathbf{k})=0 \text{ for } i\geq 3 \text{ and any } j>0.$$

In turn, this statement is equivalent to

$$\oplus_{j>0}\mathrm{Tor}_{ij}^A(\mathbf{k},\mathbf{k})=0 \text{ for } i\geq 3,$$

which finally translates into

$$H_i(\mathfrak{g},\mathbf{k})=0 \text{ for } i\geq 3.$$

Taking into account Proposition 5.1, this is equivalent to the vanishing of $H_3(\mathfrak{g},\mathbf{k})$.

We have thus shown

Proposition 6.4. *Let V be a finite dimensional vector space and $R\subset\mathbb{L}_2(V)$ be a vector subspace. Assume that $(V\otimes R)\cap(R\otimes V)=0$. Let $\mathfrak{g}:=\mathbb{L}(V)/(R)$ be the Lie algebra with space of generators V and space of relations R. The following conditions are equivalent :*

i) the algebra $U(\mathfrak{g})$, equipped with the grading for which V has degree 1, is Koszul;

ii) $H_3(\mathfrak{g},\mathbf{k})=0$.

Note that, in this proposition, the category of finite dimensional vector spaces can be replaced by that of $\mathbb{Z}_{\geq 0}$-graded vector spaces with finite dimensional graded pieces.

Combining Proposition 6.3 with this variant of Proposition 6.4 applied to $V = \mathfrak{ls}_1$, $R = \mathsf{P}$ and $\mathfrak{g} = \mathfrak{M}_0$ and $\mathbf{k} = \mathbb{Q}$, we obtain

Proposition 6.5. *The following statements are equivalent :*
i) $H_3(\mathfrak{M}_0, \mathbb{Q}) = 0$ *;*
ii) the algebra $U(\mathfrak{M}_0)$*, equipped with the depth grading, is Koszul.*

Remark. As $\mathfrak{M}_0$ is graded by depth, so is its homology group $H_3(\mathfrak{M}_0, \mathbb{Q})$. If $H_3(\mathfrak{M}_0, \mathbb{Q})_d$ denotes the depth d part of this group, then $H_3(\mathfrak{M}_0, \mathbb{Q})_d = 0$ for $d = 1, 2$ for obvious reasons. Proposition 6.3 implies that $H_3(\mathfrak{M}_0, \mathbb{Q})_3 = 0$.

6.6. Main theorem. Combining Theorem 4.1, Proposition 6.2 and Proposition 6.5, we obtain :

Theorem 6.1. *Let* $\mathfrak{a}$ *be one of the Lie algebras* $\mathrm{gr}_{dpth}(\mathfrak{u}^{\mathfrak{m}})$*,* $\mathrm{gr}_{dpth}(\mathfrak{grt}_1)$*,* $\mathrm{gr}_{dpth}(\mathfrak{ds}_0)$*,* $\mathfrak{ls}$*. Then :*

(a) *The conjunction of* $\mathrm{HC}(\mathfrak{a})$ *i) and ii) is equivalent to the isomorphism* $\mathfrak{a} \simeq \mathfrak{M}$*.*

(b) *Granted the conjunction of* $\mathrm{HC}(\mathfrak{a})$ *i) and ii),* $\mathrm{HC}(\mathfrak{a})$ *iii) is equivalent to either of the following statements : (1) the vanishing of* $H_3(\mathfrak{M}_0, \mathbb{Q})$ *; (2) the koszulity of the algebra* $U(\mathfrak{M}_0)$*, equipped with the depth grading (for which* $\mathfrak{ls}_1$ *has depth* 1*).*

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