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# On the length of the shortest path in a sparse Barak-Erdős graph

Bastien Mallein\*      Pavel Tesemnikov†

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## Abstract

We consider an inhomogeneous version of the Barak-Erdős graph, i.e. a directed Erdős-Rényi random graph on  $\{1, \dots, n\}$  with no loop. Given  $f$  a Riemann-integrable non-negative function on  $[0, 1]^2$  and  $\gamma > 0$ , we define  $G(n, f, \gamma)$  as the random graph with vertex set  $\{1, \dots, n\}$  such that for each  $i < j$  the directed edge  $(i, j)$  is present with probability  $p_{i,j}^{(n)} = \frac{f(i/n, j/n)}{n^\gamma}$ , independently of any other edge. We denote by  $L_n$  the length of the shortest path between vertices 1 and  $n$ , and take interest in the asymptotic behaviour of  $L_n$  as  $n \rightarrow \infty$ .

## 1 Introduction

The Barak-Erdős graph, introduced by Barak and Erdős in [1], is a random directed graph with no loop constructed as follows. Given  $n \in \mathbb{N}$  and  $p \in (0, 1)$ , the Barak-Erdős graph  $G(n, p)$  is a graph with vertex set  $\{1, \dots, n\}$  such that for each  $i < j$ , the edge  $(i, j)$  from vertex  $i$  to vertex  $j$  is present with probability  $p$ , independently of any other directed edge. This graph is a directed acyclic version of the well-known Erdős-Rényi graph, introduced by Erdős and Rényi in [4]. It can be used to model community food webs in ecology [14], or the task graph for parallel processing in computer sciences [8].

In particular, the length (number of edges) of the longest (directed) path, denoted  $M_n$ , has been the subject of multiple studies, as  $M_n + 1$  is the number of steps needed to complete the task graph assuming maximal parallelization. Newman [15] proved that  $\frac{M_n}{n}$  converges in law to a deterministic function  $p \mapsto C(p)$ . Increasingly precise bounds were obtained on this function and its generalizations by [5, 3, 6, 12, 13, 7].

In the present article, we take interest in the length  $L_n$  of the *shortest* path between vertices 1 and  $n$  in this graph, which has been much less studied. It is worth noting that for fixed value of  $p$ , one has

$$\mathbf{P}(L_n = 1) = p \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbf{P}(L_n = 2) = 1 - p,$$

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\*LAGA UMR 7539, Université Sorbonne Paris Nord, and DMA, UMR 8553, École Normale Supérieure, Email: mallein@math.univ-paris13.fr

†Novosibirsk State University, Sobolev Institute of Mathematics and MCA, Email: tesemnikov.p@gmail.com

as with probability  $1 - (1 - p^2)^{n-2}$ , there is a vertex  $j \in \{2, \dots, n-1\}$  connected to both 1 and  $n$ , hence  $L_n$  is equal to 1 or 2 with high probability. In particular, the length of the shortest path in dense graphs remains tight.

This fact is mentioned in [17], which takes interest in the asymptotic behaviour of  $L_n^{(\gamma)}$ , the length of the shortest path between vertices 1 and  $n$  in a graph with connexion constant  $p_n = n^{-\gamma}$ , i.e. in the limit of sparse graphs. There, it is shown that for all  $k \geq 2$ ,

$$\lim_{n \rightarrow \infty} \mathbf{P}(L_n^{(\gamma)} \leq k) = 0 \quad \text{if } 1 - \frac{1}{k} < \gamma.$$

We extend this result in the present article by obtaining the convergence in distribution of  $L_n^{(\gamma)}$  for all  $\gamma \in (0, 1)$ .

We consider here the asymptotic behaviour of the length of the shortest path between 1 and  $n$  in a space-inhomogeneous version of the Barak-Erdős graph, defined as follows. Let  $f$  be a Riemann-integrable positive function on  $[0, 1]^2$  and  $\gamma \in (0, 1)$ . For each  $n \in \mathbb{N}$  and  $i < j$ , we set  $p_{i,j}^{(n)} = \frac{f(i/n, j/n)}{n^\gamma}$ . The space-inhomogeneous sparse Barak-Erdős graph  $G(n, f, \gamma)$  is defined as a graph with vertex set  $\{1, \dots, n\}$  such that for each  $i < j$ , the directed edge  $(i, j)$  is present with probability  $p_{i,j}^{(n)}$ .

We remark that for  $\gamma = 0$ , the sequence  $G(n, f, 0)$  is a sequence of graphs converging to a *graphon* (a scaling limit of random graphs introduced by Lovász and Szegedy [11]). More generally, putting a mass  $n^\gamma$  to each edge of the graph  $G(n, f, \gamma)$ , the sequence of weighted graphs converges to the graphon  $f$ . We refer to [10] for definitions and results linked to graphons, and to [9] for a short self-contained introduction.

The main result of the article is the following estimate on the asymptotic behaviour of  $L_n$  for  $\gamma = 1 - \frac{1}{k}$ .

**Theorem 1.1.** *Let  $f$  be a Riemann-integrable non-negative function on  $[0, 1]^2$  and  $k \in \mathbb{N}$ . We fix  $\gamma = 1 - \frac{1}{k}$  and we set*

$$c_k(f) = \int_{0 < u_1 < \dots < u_{k-1} < 1} \prod_{j=0}^{k-1} f(u_j, u_{j+1}) du_1 \cdots du_{k-1} \in [0, \infty],$$

with  $u_0 = 0$  and  $u_k = 1$ . Writing  $L_n$  for the length of the shortest path between vertices 1 and  $n$  in  $G(n, f, \gamma)$ , we have

$$\lim_{n \rightarrow \infty} \mathbf{P}(L_n = k + 1) = 1 - \mathbf{P}(L_n = k) = \exp(-c_k(f)).$$

By coupling arguments, Theorem 1.1 can be extended to describe the convergence in distribution of  $L_n$  as  $n \rightarrow \infty$  for any space-inhomogeneous Barak-Erdős graph.

**Corollary 1.2.** *Let  $f$  be a Riemann-integrable positive function on  $[0, 1]^2$  and  $\gamma > 0$ . For  $k \geq 2$ , we have*

$$\lim_{n \rightarrow \infty} \mathbf{P}(L_n \leq k) = \begin{cases} 0 & \text{if } k < \frac{1}{1-\gamma}, \\ 1 - e^{-c_k(f)} & \text{if } k = \frac{1}{1-\gamma}, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that for  $\gamma = 1$ , the Barak-Erdős graph becomes unconnected, so that  $L_n = \infty$  with positive probability. In the present article, we do not treat the case  $np_n \rightarrow \infty$  with  $n^{1-\varepsilon}p_n \rightarrow 0$  for all  $\varepsilon > 0$ . However, a phase transition should be observed for the asymptotic behaviour of  $L_n$  when  $p_n \approx \frac{\log n}{n}$ , as the graph becomes disconnected.

## 1.1 Some examples and applications

A class of inhomogeneous Barak-Erdős graphs previously studied are strongly inhomogeneous graphs. In this class of graphs, the probability of presence of the edge  $(i, j)$  is given by  $\theta \frac{(j-i)^\alpha}{n^\beta}$ , with  $\theta > 0$ ,  $\beta > 0$  and  $\alpha \in (-1, \beta)$ . This model can be constructed as an inhomogeneous Barak-Erdős graph, setting  $f(x, y) = \theta(y-x)^\alpha$  and  $\gamma = \beta - \alpha$ . Applying Corollary 1.2, we observe that for any  $k \geq 2$ , if  $1 - \frac{1}{k-1} < \beta - \alpha < 1 - \frac{1}{k}$ , we have  $L_n \rightarrow k$  in probability. Similarly, if  $\beta - \alpha = 1 - \frac{1}{k}$ , we set

$$c_k = \theta^k \int_{S_k} \prod_{j=1}^k t_j^\alpha dt_1 \cdots dt_{k-1} = \frac{\theta^k \Gamma(1 + \alpha)^k}{\Gamma(k(1 + \alpha))},$$

where  $S_k = \{(t_1, \dots, t_k) \in [0, 1]^k : t_1 + \dots + t_k = 1\}$ . We conclude by Theorem 1.1 that  $L_n$  converges in distribution to  $e^{-c_k} \delta_k + (1 - e^{-c_k}) \delta_{k+1}$  as  $n \rightarrow \infty$ .

Remark that using the coupling given in Proposition 2.3, for a similar model with  $\alpha \leq -1$ , we can obtain

$$\lim_{n \rightarrow \infty} L_n = k \text{ in probability if } \beta - \alpha = \left[ 1 - \frac{1}{k-1}, 1 - \frac{1}{k} \right).$$

This result is an extension of Tesemnikov's [17] estimates on the length of the shortest path in the inhomogeneous Barak-Erdős graph, setting  $\beta = 0$ . Outside of the boundary cases, the convergence in probability of  $L_n$  to  $k \in \mathbb{N}$  can be obtained through first- and second-moment methods, with using Lemma 2.1. We handle the boundary cases by using the Chen-Stein method, showing in Lemma 2.2 that the law of the number of paths of length  $k$  is close to a Poisson distribution for  $n$  large enough.

Another example of interest in the case of Barak-Erdős graphs with exponential density of connexion. Setting  $\lambda, \mu \in \mathbb{R}$  and  $\gamma > 0$ , we consider a Barak-Erdős graph with  $p_{i,j}^{(n)} = \frac{e^{\lambda(j-i)/n + \mu}}{n^\gamma}$ . In this case, we have

$$c_k(f) = \int_{0 < u_1 < \dots < u_{k-1} < 1} \prod_{j=0}^{k-1} e^{\lambda(u_{j+1} - u_j) + \mu} du_1 \cdots du_{k-1} = \frac{e^{\lambda + k\mu}}{(k-1)!},$$

and Theorem 1.1 and Corollary 1.2 apply.

## 2 Proof of the main result

For each  $k \leq n$ , we denote by

$$\Gamma_k(n) = \{\rho \in \mathbb{N}^{k+1} : \rho_0 = 1 < \rho_1 < \dots < \rho_{k-1} < \rho_k = n\}$$

the set of increasing paths of length  $k$  from 1 to  $n$ . As a first step towards estimating the length of the shortest path in a space-inhomogeneous Barak-Erdős graph, we compute the mean number of paths of length  $k$ .

**Lemma 2.1.** *Let  $\gamma > 0$  and  $f$  a Riemann-integrable non-negative function. For  $n \in \mathbb{N}$ , we write  $Z_n(k)$  for the number of paths of length  $k$  between 1 and  $n$  in  $G(n, f, \gamma)$ , we have*

$$\mathbf{E}(Z_n(k)) \sim n^{(k-1)-k\gamma} c_k(f) \quad \text{as } n \rightarrow \infty.$$

*Proof.* By linearity of the expectation, we have

$$\begin{aligned} \mathbf{E}(Z_n(k)) &= \sum_{\rho \in \Gamma_k(n)} \prod_{j=0}^{k-1} p_{\rho_j, \rho_{j+1}}^{(n)} \\ &= n^{(k-1)-k\gamma} \frac{1}{n^{k-1}} \sum_{\rho \in \Gamma_k(n)} \prod_{j=0}^{k-1} f\left(\frac{\rho_j}{n}, \frac{\rho_{j+1}}{n}\right). \end{aligned}$$

Then  $\lim_{n \rightarrow \infty} \frac{1}{n^{k-1}} \sum_{\rho \in \Gamma_k(n)} \prod_{j=0}^{k-1} f\left(\frac{\rho_j}{n}, \frac{\rho_{j+1}}{n}\right) = c_k(f)$ , by Riemann integration, which completes the proof.  $\square$

In particular, we remark that under the assumptions of Theorem 1.1, the mean number of paths of length  $k$  in  $G(n, f, \gamma)$  converges to  $c_k(f)$ . Using this observation, we now prove that the number of paths of length  $k$  converges to a Poisson-distributed random variable.

**Lemma 2.2.** *With the notation and assumptions of Theorem 1.1, we have*

$$\lim_{n \rightarrow \infty} Z_n(k) = \mathcal{P}(c_k(f)) \quad \text{in distribution.}$$

*Proof.* We use the Chen-Stein method [2, 16] to prove the convergence in distribution of  $Z_n(k)$ . More precisely, we show that for all  $j \in \mathbb{N}$  we have

$$\lim_{n \rightarrow \infty} j \mathbf{P}(Z_n(k) = j) - \mathbf{E}(Z_n(k)) \mathbf{P}(Z_n(k) = j - 1) = 0. \quad (2.1)$$

Together with a tightness argument (due to the fact that  $\mathbf{E}(Z_n(k))$  converges), it proves that  $Z_n(k)$  converges in distribution to a Poisson variable with parameter  $\lim_{n \rightarrow \infty} \mathbf{E}(Z_n(k)) = c_k(f)$ .

Let  $j \in \mathbb{N}$ , we rewrite

$$j \mathbf{P}(Z_n(k) = j) = \mathbf{E} \left( \sum_{\rho \in \Gamma_k(n)} \mathbf{1}_{\{\rho \text{ open}\}} \mathbf{1}_{\{Z_n(k)=j\}} \right), \quad (2.2)$$

where  $\rho$  is said to be open if all edges  $(\rho_i, \rho_{i+1})$  are present in the graph. Moreover for all  $\rho \in \Gamma_k(n)$ , we have

$$\begin{aligned} &|\mathbf{P}(Z_n(k) = j | \rho \text{ open}) - \mathbf{P}(Z_n(k) = j - 1)| \\ &\leq \mathbf{P} \left( \begin{array}{l} \text{there exists a path of length } k \text{ between } 1 \text{ and } n \\ \text{distinct from } \rho \text{ that shares an edge with } \rho \end{array} \right). \end{aligned}$$

Indeed, to construct a graph with same law as  $G(n, f, \gamma)$  conditionally on  $\rho$  being open, it is enough to add to the graph  $G(n, f, \gamma)$  the edges  $(\rho_i, \rho_{i+1})$  for all  $1 \leq i \leq k$  if these are not already present. If opening these edges creates new paths, then these paths would have to share at least one edge with  $\rho$ .

We remark that if there exists a path of length  $k$  between 1 and  $n$  distinct from  $\rho$  and that shares an edge with  $\rho$ , then there exists  $0 \leq i_1 < i_2 \leq k$  and  $2 \leq \ell < k$  such that there exists a path of length  $\ell$  between  $\rho_{i_1}$  and  $\rho_{i_2}$  that does not intersect  $\rho$ . Writing  $Y_{i_1, i_2, \ell}$  the number of such paths, with the same method as in Lemma 2.1, we compute

$$\begin{aligned} \mathbf{E}(Y_{i_1, i_2, \ell}) &= \sum_{\rho_{i_1} < \bar{\rho}_1 < \dots < \bar{\rho}_{\ell-1} < \rho_{i_2}} \prod_{q=0}^{\ell-1} p_{\bar{\rho}_q, \bar{\rho}_{q+1}}^{(n)} \\ &\leq n^{-\gamma \ell} \sum_{\rho_{i_1} < \bar{\rho}_1 < \dots < \bar{\rho}_{\ell-1} < \rho_{i_2}} \prod_{q=0}^{\ell-1} f\left(\frac{\bar{\rho}_q}{n}, \frac{\bar{\rho}_{q+1}}{n}\right) \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

using that there are at most  $n^{\ell-1}$  paths of length  $\ell$  between  $\rho_{i_1}$  and  $\rho_{i_2}$ . Therefore, by union bound, we deduce that

$$\lim_{n \rightarrow \infty} \mathbf{P}(Z_n(k) = j | \rho \text{ open}) - \mathbf{P}(Z_n(k) = j - 1) = 0,$$

which implies, by (2.2), that

$$\sum_{\rho \in \Gamma_k(n)} \mathbf{P}(Z_n(k) = j \text{ and } \rho \text{ open}) - \mathbf{P}(Z_n(k) = j - 1) \mathbf{E}(Z_n(k)) = o(\mathbf{E}(Z_n(k))).$$

As  $\mathbf{E}(Z_n(k))$  is bounded, we obtain (2.1).

We remark that  $\sup_{n \in \mathbb{N}} \mathbf{E}(Z_n(k)) < \infty$ , hence  $(Z_n(k))$  is tight. Consider any subsequence  $(n_j)$  so that  $Z_{n_j}(k)$  converges in distribution as  $j \rightarrow \infty$ . Writing  $Y$  a random variable with this distribution, we have for all  $j \in \mathbb{N}$ :

$$j \mathbf{P}(Y = j) = c_k(f) \mathbf{P}(Y = j - 1),$$

using that  $\mathbf{E}(Z_{n_j}(k)) \rightarrow c_k(f)$ . Hence  $\mathbf{P}(Y = j) = \frac{c_k(f)^j}{j!} \mathbf{P}(Y = 0)$ , with  $\mathbf{P}(Y > n) \rightarrow 0$  as  $n \rightarrow \infty$ . We conclude that  $Y$  is a  $\mathcal{P}(c_k(f))$  random variable.

As any converging subsequence of  $(Z_n(k))$  is converging to  $\mathcal{P}(c_k(f))$  in law, we conclude that  $Z_n(k)$  converges to  $\mathcal{P}(c_k(f))$  in law as  $n \rightarrow \infty$ .  $\square$

Before turning to the corollary, we introduce the following coupling estimate, which loosely states that a more connected graph will have a shorter shortest path between 1 and  $n$ .

**Proposition 2.3.** *Let  $G_n, \bar{G}_n$  be two inhomogeneous Barak-Erdős graphs such that an edge between  $i$  and  $j$  is present with probability  $p_{i,j}^{(n)}$  and  $\bar{p}_{i,j}^{(n)}$  respectively. If  $p_{i,j}^{(n)} \leq \bar{p}_{i,j}^{(n)}$  for any  $i$  and  $j$ , then there exists a coupling between  $G_n$  and  $\bar{G}_n$  such that  $L_n \geq \bar{L}_n$ .*

*Proof.* We assume  $\bar{G}_n$  to be constructed on some probability space. Take any existing edge  $(i, j)$  of  $\bar{G}_n$  and do the following procedure: chosen edge is kept in graph with probability  $p_{i,j}^{(n)}/\bar{p}_{i,j}^{(n)}$  and deleted with remained probability. This procedure creates a random graph distributed exactly as  $G_n$  and is a subgraph of  $\bar{G}_n$ . Therefore, as no new edge was added, the length of the shortest path cannot have decreased.  $\square$

*Proof of Corollary 1.2.* We assume first that  $k < \frac{1}{1-\gamma}$ . Then, by Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \sum_{j=1}^k \mathbf{E}(Z_n(j)) = 0,$$

therefore  $\mathbf{P}(L_n \leq k) \rightarrow 0$  by Markov inequality.

The case  $k = \frac{1}{1-\gamma}$  is covered by Theorem 1.1.

Finally, if  $k > \frac{1}{1-\gamma}$ , then for all  $A > 0$ , the Barak-Erdős graph  $G(n, f, \gamma)$  can be coupled with  $G(n, Af, \frac{k-1}{k})$  for  $n$  large enough, using Proposition 2.3. Therefore

$$\liminf_{n \rightarrow \infty} \mathbf{P}(L_n \leq k) \geq 1 - e^{-A^k c_k(f)},$$

using Theorem 1.1 and that  $c_k(Af) = A^k c_k(f)$ . As  $f$  is positive,  $c_k(f)$  is positive, and letting  $A \rightarrow \infty$  we conclude that  $\mathbf{P}(L_n \leq k) \rightarrow 1$ .  $\square$

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