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On the length of the shortest path in a sparse Barak-Erdős graph

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Abstract

We consider an inhomogeneous version of the Barak-Erdős graph, i.e. a directed Erős-Rényi random graph on $\{1, \ldots, n\}$ with no loop. Given f a Riemann-integrable non-negative function on $[0,1]^2$ and $\gamma > 0$, we define $G(n, f, \gamma)$ as the random graph with vertex set $\{1, \ldots, n\}$ such that for each i < j the directed edge (i, j) is present with probability $p_{i,j}^{(n)} = \frac{f(i/n,j/n)}{n^{\gamma}}$, independently of any other edge. We denote by L_n the length of the shortest path between vertices 1 and n, and take interest in the asymptotic behaviour of L_n as $n \to \infty$.

1 Introduction

The Barak-Erdős graph is a random directed graph with no loop constructed in the following fashion. Given $n \in \mathbb{N}$ and $p \in (0, 1)$, the Barak-Erős graph G(n, p)is a graph with vertex set $\{1, \ldots, n\}$ such that for each i < j, the edge (i, j)from vertex i to vertex j is present with probability p, independently of any other directed edge. This graph is a directed acyclic version of the well-known Erdős-Rényi graph. It can be used to model community food webs in ecology [10], or the task graph for parallel processing in computer sciences [6].

In particular, the length (number of edges) of the longest (directed) path, denoted M_n , has been the subject of multiple studies, as $M_n + 1$ is the number of steps needed to complete the task graph assuming maximal parallelization. Newman [9] proved that $\frac{M_n}{n}$ converges in law to a deterministic function $p \mapsto C(p)$. Increasingly precise bounds were obtained on this function and its generalizations by [3, 2, 4, 7, 8, 5].

In the present article, we take interest in the length L_n of the *shortest* path between vertices 1 and n in this graph, which has been much less studied. It is worth noting that for fixed value of p, one has

$$\mathbf{P}(L_n = 1) = p$$
 and $\lim_{n \to \infty} \mathbf{P}(L_n = 2) = 1 - p$,

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as with probability $1 - (1 - p^2)^{n-2}$, there is a vertex $j \in \{2, \dots, n-1\}$ connect to both 1 and n, hence L_n is equal to 1 or 2 with high probability. In particular, the length of the shortest path in dense graphs remains tight.

This fact is mentioned in [12], which takes interest in the asymptotic behaviour of $L_n^{(\gamma)}$, the length of the shortest path between vertices 1 and n in a graph with connexion constant $p_n = n^{-\gamma}$, i.e. in the limit of sparse graphs. There, it is shown that for all $k \geq 2$,

$$\lim_{n \to \infty} \mathbf{P}(L_n^{(\gamma)} \le k) = 0 \quad \text{if } 1 - \frac{1}{k} < \gamma.$$

We extend this result in the present article by obtaining the convergence in distribution of $L_n^{(\gamma)}$ for all $\gamma \in (0, 1)$.

We consider here the asymptotic behaviour of the length of the shortest path between 1 and n in a time-inhomogeneous version of the Barak-Erdős graph, defined as follows. Let f be a Riemann-integrable positive function on $[0,1]^2$ and $\gamma \in (0,1)$. For each $n \in \mathbb{N}$ and i < j, we set $p_{i,j}^{(n)} = \frac{f(i/n,j/n)}{n^{\gamma}}$. The time-inhomogeneous sparse Barak-Erdős graph $G(n, f, \gamma)$ is defined as a graph with vertex set $\{1, \ldots, n\}$ such that for each i < j, the directed edge (i, j) is present with probability $p_{i,j}^{(n)}$. The main result of the article is the following estimate on the asymptotic

behaviour of L_n for $\gamma = 1 - \frac{1}{k}$.

Theorem 1.1. Let f be a Riemann-integrable non-negative function on $[0,1]^2$ and $k \in \mathbb{N}$. We fix $\gamma = 1 - \frac{1}{k}$ and we set

$$c_k(f) = \int_{0 < u_1 < \dots < u_{k-1} < 1} \prod_{j=0}^{k-1} f(u_j, u_{j+1}) \mathrm{d}u_1 \cdots \mathrm{d}u_{k-1} \in [0, \infty],$$

with $u_0 = 0$ and $u_1 = 1$. Writing L_n for the length of the shortest path in $G(n, f, \gamma)$, we have

$$\lim_{n \to \infty} \mathbf{P}(L_n = k + 1) = 1 - \mathbf{P}(L_n = k) = \exp(-c_k(f)).$$

By coupling arguments, Theorem 1.1 can be extended to describe the convergence in distribution of L_n as $n \to \infty$ for any time-inhomogeneous Barak-Erdős graph.

Corollary 1.2. Let f be a Riemann-integrable positive function on $[0, 1]^2$ and $\gamma > 0$. For $k \geq 2$, we have

$$\lim_{n \to \infty} \mathbf{P}(L_n \le k) = \begin{cases} 0 & \text{if } k < \frac{1}{1 - \gamma}, \\ e^{-c_k(f)} & \text{if } k = \frac{1}{1 - \gamma}, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that for $\gamma = 1$, the Barak-Erdős graph becomes unconnected, so that $L_n = \infty$ with positive probability. In the present article, we do not treat the case $np_n \to \infty$ with $n^{1-\varepsilon}p_n \to 0$ for all $\varepsilon > 0$. However, a phase transition should be observed for the asymptotic behaviour of L_n when $p_n \approx \frac{\log n}{n}$, as the graph becomes disconnected.

1.1 Some examples and applications

A class of inhomogeneous Barak-Erdős graphs previously studied strongly inhomogeneous graph. In this class of graphs, the probability of presence of the edge (i, j) is given by $\theta \frac{(j-i)^{\alpha}}{n^{\beta}}$, with $\theta > 0$, $\beta > 0$ and $\alpha \in (-1, \beta)$. This model can be constructed as an inhomogeneous Barak-Erdős graph, setting $f(x, y) = \theta(y - x)^{\alpha}$ and $\gamma = \beta - \alpha$. Applying Corollary 1.2, we observe that for any $k \ge 2$, if $1 - \frac{1}{k-1} < \beta - \alpha < 1 - \frac{1}{k}$, we have $L_n \to k$ in probability. Additionally, if $\beta - \alpha = 1 - \frac{1}{k}$, we set

$$c = \theta^k \int_{S_k} \prod_{j=1}^k t_j^{\alpha} \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1} = \frac{\theta^k \Gamma(1+\alpha)^k}{\Gamma(k(1+\alpha))}$$

where $S_k = \{(t_1, \ldots, t_k) : t_1 + \cdots + t_k = 1\}$. We conclude by Theorem 1.1 that L_n converges in distribution to $e^{-c}\delta_k + (1 - e^{-c})\delta_{k+1}$ as $n \to \infty$.

Remark that using the coupling given in Proposition 2.3, for a similar model with $\alpha \leq -1$, we can obtain

$$\lim_{n \to \infty} L_n = k \text{ in probability if } \beta - \alpha = \left[1 - \frac{1}{k-1}, 1 - \frac{1}{k}\right).$$

This result is an extension of Tesemnikov's [12] estimates on the length of the shortest path in the inhomogeneous Barak-Erdős graph, setting $\beta = 0$. Outside of the boundary cases, the convergence in probability of L_n to $k \in \mathbb{N}$ can be obtained through first- and second-moment methods, using see Lemma 2.1. We handle the boundary cases by using the Chen-Stein method, showing in Lemma 2.2 that the law of the number of paths of length k is close to a Poisson distribution for n large enough.

An other example of interest in the case of Barak-Erdős graphs with exponential density of connexion. Setting $\lambda, \mu \in \mathbb{R}$ and $\gamma > 0$, we consider a Barak-Erdős graph with $p_{i,j}^{(n)} = \frac{e^{\lambda(j-i)/n+\mu}}{n^{\gamma}}$. In this case, we have

$$c_k(f) = \int_{0 < u_1 < \dots < u_{k-1} < 1} \prod_{j=0}^{k-1} e^{\lambda(u_{j+1} - u_j) + \mu} \mathrm{d}u_1 \cdots \mathrm{d}u_{k-1} = \frac{e^{\lambda + k\mu}}{k!}$$

and Theorem 1.1 and Corollary 1.2 apply.

2 Proof of the main result

For each $k \leq n$, we denote by $\Gamma_k(n) = \{\rho \in \mathbb{N}^{n+1} : \rho_0 = 1 < \rho_1 < \cdots < \rho_{k-1} < \rho_k = n\}$ the set of increasing paths from 1 to n. As a first step towards estimating the length of the shortest in a time-inhomogeneous Barak-Erdős graph, we compute the mean number of paths of length k.

Lemma 2.1. Let $\gamma > 0$ and f a Riemann-integrable non-negative function. For $n \in \mathbb{N}$, we write $Z_n(k)$ for the number of paths of length k between 1 and n in $G(n, f, \gamma)$, we have

$$\mathbf{E}(Z_n(k)) \sim n^{(k-1)-k\gamma} c_k(f)$$

Proof. By linearity of the expectation, we have

$$\mathbf{E}(Z_n(k)) = \sum_{\rho \in \Gamma_k(n)} \prod_{j=0}^{k-1} p_{\rho_j, \rho_{j+1}}^{(n)}$$

= $n^{(k-1)-k\gamma} \frac{1}{n^{k-1}} \sum_{\rho \in \Gamma_k(n)} \prod_{j=0}^{k-1} f\left(\frac{\rho_j}{n}, \frac{\rho_{j+1}}{n}\right).$

Then $\lim_{n\to\infty} \frac{1}{n^{k-1}} \sum_{\rho\in\Gamma_k(n)} \prod_{j=0}^{k-1} f\left(\frac{\rho_j}{n}, \frac{\rho_{j+1}}{n}\right) = c_k(f)$, by Riemann integration, which completes the proof.

In particular, we remark that under the assumptions of Theorem 1.1, the mean number of paths of length k in $G(n, f, \gamma)$ converges to $c_k(f)$. Using this observation, we now prove that the number of paths of length k converges to a Poisson variable.

Lemma 2.2. With the notation and assumptions of Theorem 1.1, we have

$$\lim_{k \to \infty} Z_n(k) = \mathcal{P}(c_k(f)) \quad in \ distribution.$$

Proof. We use the Chen-Stein method [1, 11] to prove the convergence in distribution of $Z_n(k)$. More precisely, we show that for all $j \in \mathbb{N}$ we have

$$\lim_{n \to \infty} j \mathbf{P}(Z_n(k) = j) - \mathbf{E}(Z_n(k)) \mathbf{P}(Z_n(k) = j - 1) = 0.$$
(2.1)

Together with a tightness argument (due to the fact that $\mathbf{E}(Z_n(k))$ converges), it proves that $Z_n(k)$ converges in distribution to a Poisson variable with parameter $\lim_{n\to\infty} \mathbf{E}(Z_n(k)) = c_k(f)$.

Let $j \in \mathbb{N}$, we rewrite

$$j\mathbf{P}(Z_n(k)=j) = \mathbf{E}\left(\sum_{\rho\in\Gamma_k(n)} \mathbf{1}_{\{\rho \text{ open}\}} \mathbf{1}_{\{Z_n(k)=j\}}\right),$$
(2.2)

where ρ is said to be open if all edges (ρ_i, ρ_{i+1}) are present in the graph. Moreover for all $\rho \in \Gamma_k(n)$, we have

$$|\mathbf{P}(Z_n(k) = j|\rho \text{ open}) - \mathbf{P}(Z_n(k) = j-1)|$$

 $\leq \mathbf{P}$ (exists a path of length k between 1 and n sharing an edge with ρ).

Indeed, to construct a graph with same law as $G(n, f, \gamma)$ conditionally on ρ being open, it is enough to add to the graph $G(n, f, \gamma)$ the edges $(\rho_j, \dots, \rho_{j+1})$ for all $1 \leq j \leq n$ if these are not already present. If opening these edges creates new paths, then these path would have to share at least one edge with ρ .

We remark that if there exists a path of length k between 1 and n, there exists $0 \leq i < j \leq k$ and $2 \leq \ell < k$ such that there exists a path of length ℓ between ρ_i and ρ_j that does not intersect ρ . Writing $Y_{i,j,\ell}$ the number of such paths, with the same method as in Lemma 2.1, we compute

$$\mathbf{E}\left(Y_{i,j,\ell}\right) = \sum_{\substack{\rho_i < \overline{\rho}_1 < \dots < \overline{\rho}_{\ell-1} < \rho_j \\ \overline{\rho} \in \Gamma_k(\ell)}} \prod_{q=0}^{\ell-1} p_{\overline{\rho}_q,\overline{\rho}_{q+1}}^{(n)}$$
$$\leq n^{-\gamma\ell} \sum_{\overline{\rho} \in \Gamma_k(\ell)} \prod_{j=0}^{\ell-1} f\left(\frac{\overline{\rho}_q}{n}, \frac{\overline{\rho}_{q+1}}{n}\right) \to 0 \text{ as } n \to \infty.$$

Therefore, by union bound, we deduce that

$$\lim_{n \to \infty} \mathbf{P}(Z_n(k) = j | \rho \text{ open}) - \mathbf{P}(Z_n(k) = j - 1) = 0.$$

which then yields by (2.2)

$$\sum_{\rho \in \Gamma_k(n)} \mathbf{P}(Z_n(k) = j \text{ and } \rho \text{ open}) - \mathbf{P}(Z_n(k) = j-1) \mathbf{E}(Z_n(k)) = o(\mathbf{E}(Z_n(k))).$$

As $\mathbf{E}(Z_n(k))$ is bounded, we obtain (2.1).

We remark that $\sup_{n \in \mathbb{N}} \mathbf{E}(Z_n(k)) < \infty$, hence $(Z_n(k))$ is tight. Consider any subsequence (n_j) so that $Z_{n_j}(k)$ converges in distribution as $j \to \infty$. Writing Y a random variable with this distribution, we have for all $j \in \mathbb{N}$:

$$j\mathbf{P}(Y=j) = c_k(f)\mathbf{P}(Y=j-1),$$

using that $\mathbf{E}(Z_{n_j}(k)) \to c_k(f)$. Hence $\mathbf{P}(Y = j) = \frac{c_k(f)^j}{j!}\mathbf{P}(Y = 0)$, with $\mathbf{P}(Y > n) \to 0$ as $n \to \infty$. We conclude that Y is a $\mathcal{P}(c_k(f))$ random variable.

As any converging subsequence of $(Z_n(k))$ is converging to $\mathcal{P}(c_k(f))$ in law, we conclude that $Z_n(k)$ converges to $\mathcal{P}(c_k(f))$ in law as $n \to \infty$.

Before turning to the corollary, we introduce the following coupling estimate, which loosely states that a more connected graph will have a shorter shortest path between 1 and n.

Proposition 2.3. Let G_n , \overline{G}_n be two inhomogeneous Barak-Erdős graphs such that an edge between *i* and *j* is present with probability $p_{i,j}^{(n)}$ and $\overline{p}_{i,j}^{(n)}$ respectively. If $p_{i,j}^{(n)} \leq \overline{p}_{i,j}^{(n)}$ for any *i* and *j*, then there exists a coupling between G_n and \overline{G}_n such that $L_n \geq \overline{L}_n$.

Proof. We assume \overline{G}_n to be constructed on some probability space. Take any existing edge (i, j) of \overline{G}_n and do the following procedure: chosen edge is stayed in graph with probability $p_{i,j}^{(n)}/\overline{p}_{i,j}^{(n)}$ and deleted with remained probability. This procedure creates a random graph distributed exactly as G_n and is a subgraph of G_n . Therefore, as no new edge was added, the length of the shortest path cannot have decrease.

Proof of Corollary 1.2. We assume first that $k < \frac{1}{1-\gamma}$. Then, by Lemma 2.1, we have

$$\lim_{n \to \infty} \sum_{j=1}^{k} \mathbf{E}(Z_n(j)) = 0,$$

therefore $\mathbf{P}(L_n \leq k) \to 0$ by Markov inequality.

The case $k = \frac{1}{1-\gamma}$ is covered by Theorem 1.1. Finally, if $k > \frac{1}{1-\gamma}$, then for all A > 0, the Barak-Erdős graph $G(n, f, \gamma)$ can be coupled with $G(n, Af, \frac{k-1}{k})$ for n large enough, using Proposition 2.3. Therefore

$$\liminf_{n \to \infty} \mathbf{P}(L_n \le k) \ge 1 - e^{-A^{\kappa} c_k(f)},$$

using Theorem 1.1 and that $c_k(Af) = A^k c_k(f)$. As f is positive, $c_k(f)$ is positive, and letting $A \to \infty$ we conclude that $\mathbf{P}(L_n \leq k) \to 1$.

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