

RATIONAL VS TRANSCENDENTAL POINTS ON ANALYTIC RIEMANN SURFACES

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ABSTRACT. Let (X, L) be a polarized variety over a number field K . We suppose that L is an hermitian line bundle. Let M be a non compact Riemann Surface and $U \subset M$ be a relatively compact open set. Let $\varphi : M \rightarrow X(\mathbf{C})$ be a holomorphic map. For every positive real number T , let $A_U(T)$ be the cardinality of the set of $z \in U$ such that $\varphi(z) \in X(K)$ and $h_L(\varphi(z)) \leq T$. After a revisitation of the proof of the sub exponential bound for $A_U(T)$, obtained by Bombieri and Pila, we show that there are intervals of the reals such that for T in these intervals, $A_U(T)$ is upper bounded by a polynomial in T . We then introduce subsets of type S with respect of φ . These are compact subsets of M for which an inequality similar to Liouville inequality on algebraic points holds. We show that, if M contains a subset of type S , then, for every value of T the number $A_U(T)$ is bounded by a polynomial in T . As a consequence, we show that if M is a smooth leaf of an algebraic foliation in curves defined over K then $A_U(T)$ is bounded by a polynomial in T . Let $S(X)$ be the subset (full for the Lebesgue measure) of points which verify some kind of Liouville inequalities. In the second part we prove that $\varphi^{-1}(S(X)) \neq \emptyset$ if and only if $\varphi^{-1}(S(X))$ is full for the Lebesgue measure on M .

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1. INTRODUCTION

In the recent years there have been an interest in the following problem: Let X be a projective variety defined over \mathbf{Q} and $M \subset X(\mathbf{C})$ be an analytic subset. Can we estimate, as a function of T , the number of *rational* points with height less or equal then T which are contained in M ? A cornerstone in this topic is the paper by Bombieri and Pila [4] which essentially give a satisfactory answer in the case when M is a real analytic curve in \mathbf{R}^2 . After Bombieri and Pila's Theorem, a big amount of work on related problems has been done, culminating with the theorem by Pila and Wilkie [13] who could understand the problem in the case when M is a definable subset in a suitable σ -minimal structure. In many researches on this topic, the Bombieri and Pila results have been taken as a "black box" and applied to study more general cases.

In this paper we would like to analyze again the problem of estimating the number of points of bounded height in a Riemann surface which is contained, and Zariski dense, in a projective variety.

Suppose that X is a projective variety of dimension $n > 1$ defined over \mathbf{Q} (in order to simplify notations, in this introduction we will work over \mathbf{Q} , the case of a general number field will be treated in the paper). Fix an ample line bundle L over X and a height function $h_L(\cdot)$ associated to it.

Let M be a non compact Riemann surface and U a relatively compact open set of it.

Let $\varphi : M \rightarrow X(\mathbf{C})$ be a holomorphic map with Zariski dense image.

In this paper we are mostly interested in estimating from above, as a function of T , the cardinality $A_U(T)$ of the set $S_U(T) := \{z \in U / \varphi(z) \in X(\mathbf{Q}) \text{ and } h_L(\varphi(z)) \leq T\}$.

The starting point of the paper is a revisitation of the Bombieri and Pila Theorem in the complex analytic setting. We give a self contained proof of their theorem which is more inspired to technics from analytic and diophantine geometry. We prove the following:

Theorem 1.1. *(Bombieri - Pila, cf. Theorem 4.1) For every positive number ϵ , we have*

$$(1.1) \quad A_U(T) \ll \exp(\epsilon T)$$

where the involved constants depend on U , φ , ϵ and L but not on T .

The proof follows only partially the strategy of Bombieri and Pila. Actually it follows more the pattern of classical proofs in transcendence: one covers U with $O(\exp(\epsilon T))$ open sets of small area. Then we show that on each of these open set the cardinality of the points of height less or equal then T is bounded by a constant. This is done by constructing, via a form of Siegel Lemma and Liouville inequality, a section of degree *depending only on ϵ but independent on T* , vanishing on each of these points.

There are examples in the literature (cf. for instance [15]) which show that Theorem 1.1 is optimal. Never the less one may wonder for how many T 's, the number $A_U(T)$ is indeed big when compared to T . We discovered the following interesting fact: there are intervals I as big as we want such that, for every $T \in I$, the number $A_U(T)$ is smaller than a polynomial in T . More precisely we proved:

Theorem 1.2. (cf. Theorem 4.8) *Let $A > 1$, $\gamma > \frac{n}{n-1}$ and $\epsilon > 0$. Then we can find a unbounded subset R of $]1; +\infty[$, such that, for every $t \in R$ and $T \in [t, At]$, one has $A_U(T) \leq \epsilon T^\gamma$.*

The proof uses again a form of Siegel Lemma and of Liouville inequality and an argument by contradiction (which actually prevents to control the sets I_t). The reason why we cannot deduce a stronger version of Theorem 1.1 from the strategy of proof of Theorem 1.2 relies on the use of the Liouville inequality: in order to obtain an upper bound for the number of zeroes of an analytic function on U , we need an upper bound of the norm of it and a lower bound of its norm on a specific point where it do not vanish. This principle is explained in section 3.

Actually, in order to obtain a good upper bound for $A_U(T)$, it suffices to know a lower bound of the norm of a section of a line bundle, not just on a point (where a priori it could vanish) but only over a suitable subset. For this reason we introduce the following definition: we suppose that the holomorphic line bundle L over $X(\mathbf{C})$ is equipped with a smooth hermitian metric $\|\cdot\|$ and, for $s \in H^0(X(\mathbf{C}), L^d)$, we denote by $\|s\| := \sup_{z \in X(\mathbf{C})} \{\|\varphi^*(s)\|(z)\}$.

Definition 1.3. (cf. Definition 5.4) *Let $B \subset U$ be a compact set and a be a real number. We will say that B is a subset of type S_a of M with respect to φ if, we can find a positive constant $A > 1$ such that, for every positive integer d and $s \in H^0(\mathcal{X}, \mathcal{L}^d) \setminus \{0\}$ we have that*

$$(1.2) \quad \log \|s\|_B \geq -A(\log^+ \|s\| + d)^a.$$

Where, if B is a subset of U , then $\|s\|_B := \sup_{z \in B} \{\|\varphi^*(s)\|(z)\}$.

And we prove

Theorem 1.4. (cf. Theorem 5.5) *Suppose that we can find a subset $B \subset U$ of type S_a with respect to φ . Then*

$$(1.3) \quad A_U(T) \ll T^{2a}.$$

Observe that if $B \subset B_1$ then B_1 is of type S_a if B is. Thus, a priori, one could always replace a subset B with U (or its closure) in the definition. Nevertheless, the conclusion of Theorem 1.4 holds even if B is a single point. In a previous paper [9] and in this paper, we show that points of type S_a are particularly interesting in these kind of questions. At the moment it seems to us that the two extremal cases, B is a single point or B is the closure of U , are interesting to consider, for this reason, we prefer to introduce the notion of of type S_a for arbitrary set and not just for U .

Of course, in general it will be not easy to guarantee the existence of a subset of type S_a with respect to φ . Never the less there is an important case of maps where we can guarantee

the existence of such sets: the leaves of foliations. Suppose that X is equipped with an algebraic foliation by curves \mathcal{F} (a priori not smooth). We can prove that every relatively compact open neighborhood of a rational point of a leaf is of type S_a for a suitable a (cf. after) and consequently we find:

Theorem 1.5. *(cf. Theorem 5.9) Let \mathcal{F} be an algebraic foliation on a smooth quasi projective variety Z defined over a number field K . Let $p \in Z(K)$ be a rational point and $h : \Delta_1 \rightarrow Z_\sigma(\mathbf{C})$ be a disk through p tangent to \mathcal{F} . Suppose that the dimension of the Zariski closure of $h(\Delta)$ is $\ell > 1$. Let $0 < r < 1$ be a real number, then, for every $\epsilon > 0$ we have*

$$(1.4) \quad A_{\Delta_r}(T) \ll_\epsilon T^{2\ell+\epsilon}.$$

Particular cases of Theorem 1.5 have been proved in [2] and [6].

Here and after Δ_r will be the disk $\{|z| < r\}$ in \mathbf{C} . In order to prove this, we require a Zero Lemma which have been proved by Nesterenko and generalized by Binyamini which holds for smooth points of foliations and for some kind of singularities of them. For this reason, the method could be generalized to the case some kind of singular points of the foliation but we do not think that it would be the case for a general singularity. In particular, the exponent of T in Theorem 1.5 is related to this Zero Lemma.

The bigger the subset B is and the easier should be to find it. But, in order to deduce the consequence on the number of rational points, it suffices that B is non empty. In particular it can be a single point. In a previous paper [9], we proved that the set $S_a(X)$ of points of type S_a in $X(\mathbf{C})$ (definition similar to Definition 1.3, cf. Definition 5.1) are full in $X(\mathbf{C})$ for the Lebesgue measure as soon as $a \geq n + 2$. In the last part of this paper we prove the following:

Theorem 1.6. *(cf. Theorem 6.1) For every $a > 0$, $\varphi^{-1}(S_a(X)) \neq \emptyset$ if and only if $\varphi^{-1}(S_a(X))$ is full for the Lebesgue measure on M .*

Which means that, as soon as the image of φ touches $S_a(X)$, it is almost totally contained in it, and consequently it contains few rational points.

Another interesting consequence of Theorem 1.6 is that, as soon as $\varphi^{-1}(S_a(X)) \neq \emptyset$, every relatively compact open subset of M is of type S_a for some a . What we actually prove is a bit more general than Theorem 1.6: in order to obtain the fullness of the set $\varphi^{-1}(S_a(X))$, it suffices the existence of a relatively compact subset of M of type S_a with respect to φ . Thus, an interesting corollary of Theorem 1.6 is:

Theorem 1.7. *(cf. Theorem 6.3) Let \mathcal{F} be a foliation on a smooth quasi projective variety Z defined over a number field K . Let $p \in Z(K)$ be a rational point and $h : \Delta_1 \rightarrow Z_\sigma(\mathbf{C})$ be the analytic leaf of \mathcal{F} through p . Then, if $h(\Delta_1)$ is Zariski dense in Z , we have that $h^{-1}(S(Z_K))$ is full in Δ_1 .*

The proof of Theorem 1.6 is quite long and relies on the standard Borel–Cantelli Lemma 2.4, and an estimate of the area of the set where the norm of a section of a hermitian line bundle on a Riemann surface is small. This estimate relies on some tools in complex analysis and the classical Bloch–Cartan estimate (cf. the beginning of sub section 6.1).

In the last section 7 we address some questions which arise naturally and that, we hope, will be clarified in a future.

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2. NOTATIONS AND BASIC FACTS FROM ARITHMETIC GEOMETRY, COMPLEX ANALYSIS AND MEASURE THEORY.

2.1. Tools and notations from arithmetic geometry and Arakelov theory. Let K be a number field and O_K be its ring of integers. We will denote by M_K^∞ the set of infinite places of K . We fix a place $\sigma_0 \in M_K^\infty$.

Let X_K be a projective variety of dimension N defined over K

If $\tau \in M_K^\infty$ and F is an object over X_K (F may be a sheaf, a divisor, a cycle...), we will denote by X_τ the complex variety $X_K \otimes_\tau \mathbf{C}$ and by F_τ the restriction of F to X_τ .

A model $\mathcal{X} \rightarrow \text{Spec}(O_K)$ of X_K is a flat projective O_K scheme whose generic fiber is isomorphic to X_K . Suppose that L_K and \mathcal{X} are respectively a line bundle over X_K and a model of it; We will say that a line bundle \mathcal{L} over \mathcal{X} is a model of L_K if its restriction to the generic fiber is isomorphic to L_K .

Let \mathcal{X} be a model of X_K . A hermitian line bundle $\overline{\mathcal{L}}$ is a couple $(\mathcal{L}, \langle \cdot; \cdot \rangle_\sigma)_{\sigma \in M_K^\infty}$, where \mathcal{L} is a line bundle over \mathcal{X} and, for every $\sigma \in M_K^\infty$, $\langle \cdot; \cdot \rangle_\sigma$ is a continuous hermitian product on L_σ , with the condition that, if $\sigma = \bar{\tau}$, then $\langle \cdot; \cdot \rangle_\sigma = \overline{\langle \cdot; \cdot \rangle_\tau}$. An hermitian vector bundle is defined similarly.

If s is a local section of L_σ defined in a neighborhood of a point $z \in X_\sigma(\mathbf{C})$, then we will denote by $\|s\|_\sigma$ the real number $\langle s, s \rangle_\sigma^{1/2}$.

If X_K is a projective variety, it is easy to see that for every line bundle L_K on X_K , we can find an embedding $\iota : X_K \hookrightarrow P_K$, where P_K is a smooth projective variety and $L = \iota^*(M)$ with M line bundle on P_K . A metric on L_K is said to be smooth if it is the restriction of a smooth metric on M .

Let $(\mathcal{L}, \langle \cdot; \cdot \rangle_\sigma)_{\sigma \in M_K^\infty}$ be a hermitian line bundle on a model \mathcal{X} of X_K . If $s \in H^0(X_K, L_K^d)$ is a non zero section, we will denote by $\log^+ \|s\|$ the real number $\sup_{\tau \in M_K^\infty} \{0, \log \|s_\tau\|_\tau\}$. More generally, of a is a real number, we will denote by a^+ the real number $\sup\{a, 0\}$ and by a_+ the real number $\sup\{1, a\}$.

We recall some properties of hermitian line bundles on projective varieties:

- Let L be a hermitian ample line bundle on a projective variety Z equipped with a smooth metric ω . Over $H^0(Z, L^d)$ we can define two natural norms:

$$(2.1) \quad \|s\|_{\text{sup}} := \sup_{z \in Z} \{\|s\|(z)\} \quad \text{and} \quad \|s\|_{L^2} := \sqrt{\int_Z \|s\|^2 \omega^n}.$$

These norms are comparable: we can find constants C_i such that

$$(2.2) \quad C_1 \|s\|_{L^2} \leq \|s\|_{\text{sup}} \leq C_2^d \|s\|_{L^2}.$$

This statement (due to Gromov) is proved for instance in [14] Lemma 2 p. 166 when Z is smooth. The general statement can be deduced by taking a resolution of singularities (remark that the proof of [14] Lemma 2 p. 166 do not require that L is ample).

- Suppose that L is a hermitian ample line bundle on a smooth projective variety X defined over \mathbf{C} . Then we can find a constant C for which the following holds: for every couple of positive integers d_1 and d_2 and non vanishing global sections $s_1 \in H^0(X, L^{d_1})$ and $s_2 \in H^0(X, L^{d_2})$ we have

$$(2.3) \quad \log \|s_1\|_{\text{sup}} + \log \|s_2\|_{\text{sup}} \geq \log \|s_1 \cdot s_2\|_{\text{sup}} \geq (d_1 + d_2)C + \log \|s_1\|_{\text{sup}} + \log \|s_2\|_{\text{sup}}.$$

This is an easy consequence of [1] Proposition 1.4.2.

We recall the following standard facts of Arakelov theory:

- If L is a hermitian line bundle over $\text{Spec}(O_K)$ and $s \in L$ is a non vanishing section, we define

$$(2.4) \quad \widehat{\deg}(L) := \log(\text{Card}(L/sO_K)) - \sum_{\sigma \in M_K^\infty} \log \|s\|_\sigma.$$

If E is a hermitian vector bundle of rank r on $\text{Spec}(O_K)$, we define $\widehat{\deg}(E) := \widehat{\deg}(\wedge^r E)$ and the slope of E is $\widehat{\mu}(E) = \frac{\widehat{\deg}(E)}{r}$.

- Within all the sub bundles of E there is one whose slope is maximal, we denote by $\widehat{\mu}_{\text{max}}(E)$ its slope. It is easy to verify that $\widehat{\mu}_{\text{max}}(E_1 \oplus E_2) = \max\{\widehat{\mu}_{\text{max}}(E_1), \widehat{\mu}_{\text{max}}(E_2)\}$.

We will need the following version of the Siegel Lemma:

Lemma 2.1. (*Siegel Lemma*) *Let E_1 and E_2 be hermitian vector bundles over O_K . Let $f : E_1 \rightarrow E_2$ be a non injective linear map. Denote by $m = \text{rk}(E_1)$ and $n = \text{rk}(Ker(f))$. Suppose that there exists a positive real constant C such that:*

- E_1 is generated by elements of sup norm less or equal than C .
- For every infinite place σ we have $\|f\|_\sigma \leq C$

Then there exists a non zero element $v \in Ker(f)$ such that

$$(2.5) \quad \sup_{\sigma \in M_K^\infty} \{\log \|v\|_\sigma\} \leq \frac{m}{n} \log(C^2) + \left(\frac{m}{n} - 1\right) \widehat{\mu}_{\text{max}}(E_2) + 3 \log(n) + A$$

where A is a constant depending only on K .

A proof of this version of Siegel Lemma can be found in [7].

Let $\mathcal{X} \rightarrow \text{Spec}(O_K)$ be a projective arithmetic variety and $\overline{\mathcal{L}}$ be an hermitian line bundle over it. We will say that $\overline{\mathcal{L}}$ is *arithmetically very ample* if:

- i) the involved line bundle \mathcal{L} is relatively ample over \mathcal{X} ;
- ii) the O_K -algebra $\bigoplus_{d \geq 0} H^0(\mathcal{X}, \mathcal{L}^d)$ is generated by $H^0(\mathcal{X}, \mathcal{L})$;
- iii) the normed O_K -module $H^0(\mathcal{X}, \mathcal{L})$ is generated by sections of sup norm less or equal then one.

Observe that, if $\overline{\mathcal{L}}$ is arithmetically very ample, then for every $d \geq 0$ the normed O_K -module $H^0(\mathcal{X}, \mathcal{L}^d)$ is generated by sections of sup norm less or equal then one.

Definition 2.2. An arithmetic polarization $(\mathcal{X}, \overline{\mathcal{L}})$ of X_K is the choice of the following data:

- a) An ample line bundle L_K over X_K ;
- b) A projective model $\mathcal{X} \rightarrow \text{Spec}(O_K)$ of X_K over O_K .
- c) A arithmetically very ample line bundle hermitian line bundle $\overline{\mathcal{L}}$ over \mathcal{X} which is a model of $L_K^{\otimes r}$ for a suitable positive interger r .
- d) For every $\tau \in M_K^\infty$, we suppose that the metric on L_τ is smooth and positive.

Remark that, if L_K is an ample line bundle over X_K , we can find a positive integer d and an arithmetic polarization $(\mathcal{X}, \overline{\mathcal{L}}_1)$ such that $\overline{\mathcal{L}}_1$ is a model of L_K^d .

- If $(\mathcal{X}, \overline{\mathcal{L}})$ is an arithmetic polarization of X_K , then we we can find constants C_1 and C_2 such that

$$(2.6) \quad C_1^{d^{N+1}} T^{d^N} \leq \text{Card} \left(\left\{ s \in H^0(\mathcal{X}, \mathcal{L}^d) \mid \sup_{\tau \in M_K^\infty} \{ \|s\|_\tau \} \leq T \right\} \right) \leq C_2^{d^{N+1}} T^{d^N}.$$

This is a consequence of [17], Theorem 1.4 , [10] Theorem 2 and the comparison above.

- Let L/K be a finite extension and O_L the ring of integers of L . An L -point of X_K is a K -morphism $P_L : \text{Spec}(L) \rightarrow X_K$. The set of L points of X_K is noted $X_K(L)$. If $(\mathcal{X}, \overline{\mathcal{L}})$ is an arithmetic polarization of X_K , by the valuative criterion of properness, every L -point $P_L : \text{Spec}(L) \rightarrow X_K$ extends uniquely to an O_K -morphism $P_{O_L} : \text{Spec}(O_L) \rightarrow \mathcal{X}$. In this case, $P_{O_L}^*(\overline{\mathcal{L}})$ is a hermitian line bundle on $\text{Spec}(O_L)$. We define the height of P_L with respect to $\overline{\mathcal{L}}$ to be the real number $h_{\overline{\mathcal{L}}}(P_L) := \frac{\widehat{\text{deg}}(P_{O_L}^*(\overline{\mathcal{L}}))}{[L:\mathbf{Q}]}$.
- *Liouville inequality:* Let $p \in X_K(K)$ be a rational point. Let $p_0 \in X_{\sigma_0}$ be its image. Then, for every positive integer d and global section $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ such that $s(p) \neq 0$ we have

$$(2.7) \quad \log \|s\|_{\sigma_0}(p) \geq -[K:\mathbf{Q}] h_{\overline{\mathcal{L}}}(p) \cdot (\log \|s\| + d).$$

For a proof of this form of Liouville inequality cf. [9] Theorem 3.1.

2.2. Tools and basic facts from Nevanlinna theory. . We will denote by Δ_r the disk $\{|z| < r\}$. Let M be a non compact Riemann surface and $U \subset M$ be a relatively compact open set whose border is a smooth Jordan curve which we denote by ∂U . We denote by \overline{U} the closure of U in M .

Denote by d the standard differential operator on functions on M and write it as $d = \partial + \bar{\partial}$. We denote by d^c the operator $d^c := \frac{1}{4\pi i} \cdot (\partial - \bar{\partial})$. The operator $dd^c : C^\infty(M) \rightarrow A^{(1,1)}(M)$ is called the Laplace operator. A function f such that $dd^c(f) = 0$ is said to be harmonic.

The Laplace operator can be extended to an operator from the vector space generated by smooth and sub harmonic functions.

– The Green function on U is a function $g_U(z, w) : U \times U \setminus D \rightarrow [0, +\infty[$ (here D is the diagonal) such that:

(i) For every $p \in U$ the function $g_U(z; \cdot)$ is C^∞ and harmonic in $U \setminus \{p\}$. Moreover $g_U(p, \cdot)|_{\partial U} = 0$.

(ii) For every $p \in U$, if $\iota : \Delta_1 \rightarrow U$ is a holomorphic embedding such that $\iota(0) = p$, then $\iota^*(g(z, \cdot)) + \ln|z|$ extends to an harmonic function on Δ_1 .

– Fix a point $z_0 \in U$. If we extend by zero outside \bar{U} the Green function $g_U(z_0, z)$, the following equation holds: $dd^c g_U(z_0, z) = \delta_{z_0} - d\mu_{U, z_0}$; where $d\mu_{U, z_0}$ is a measure of total mass one supported on ∂U (the current δ_{z_0} is the Dirac measure with support on z_0).

Suppose that L is a hermitian line bundle on M . Suppose that s is a meromorphic section of L .

– If s is a meromorphic section of L ; we denote by $div(s)$ the formal sum $div(s) := \sum_{z \in M} v_z(s)[z]$; where $v_z(s)$ is the multiplicity of s in z (order at z of the zero or the pole of a/any local meromorphic function defining s). This sum may be infinite but its restriction to every relatively compact open set is finite.

– For every $z_0 \in M$ denote by Ω_{z_0} the cotangent space of M in z_0 ; If we denote by \mathcal{I}_{z_0} the ideal sheaf of z_0 in M , for every positive integer n , we have an exact sequence

$$(2.8) \quad 0 \longrightarrow L_{z_0} \otimes \Omega_{z_0}^{\otimes n} \longrightarrow L \otimes \mathcal{O}_{M, z_0} / \mathcal{I}_{z_0}^{n+1} \longrightarrow L \otimes \mathcal{O}_{M, z_0} / \mathcal{I}_{z_0}^n \longrightarrow 0.$$

Let $s \in H^0(M, L) \setminus \{0\}$ be a non zero global section. The fact that $v_{z_0}(s) = n$ means that the class of s in $L \otimes \mathcal{O}_{M, z_0} / \mathcal{I}_{z_0}^n$ is zero but its class in $L \otimes \mathcal{O}_{M, z_0} / \mathcal{I}_{z_0}^{n+1}$ is not. Consequently, by the exact sequence above, s defines an element $j^n(s)_{z_0} \in L_{z_0} \otimes \Omega_{z_0}^{\otimes n}$ called the n -th jet of s at z_0 .

– The current $\delta_{div(s)} - dd^c \ln \|s\|$ (the current δ_D is the dirac measure with support on the divisor D) extends to a C^∞ (1, 1)-form $c_1(L)$ called *the first Chern Class of L* .

– The function

$$(2.9) \quad T(z, L, r) := \int_0^r \frac{dt}{t} \cdot \int_{g_U(z; \cdot) < \ln(t)} c_1(L)$$

is called *the Nevanlinna Characteristic function* of L with respect to U and base point z . It is linear as a function of L .

– The Nevanlinna First Main Theorem holds: if s is a meromorphic section of L such that $v_z(s) = 0$ then

$$(2.10) \quad T(z, L, r) + \int_{\partial U} \ln \|s\| d\mu_{U, z} = \sum_{w \in U} v_w(s) \cdot g_U(z, w) + \ln \|s\|(z).$$

More generally, if we fix an hermitian metric on Ω_z , $s \in H^0(M, L) \setminus \{0\}$ and $v_z(s) = n$, then

$$(2.11) \quad T(z, L, r) + \int_{\partial U} \ln \|s\| d\mu_{U,z} = \sum_{w \in U} v_w(s) \cdot g_U(z, w) + \ln \|j^n(s)_z\| + n \cdot C.$$

Where C is a constant depending only on the metric on Ω_z .

Remark 2.3. Strictly speaking, one could weaken the hypothesis on the border of U . The regularity of it is assumed to guarantee the existence of the Green functions and to prove the Nevanlinna First Main Theorem. With some work, one could weaken this hypothesis, but since at the moment we do not see any particular improvements on the results proposed in this paper, we prefer to keep this hypothesis as it is.

2.3. Tools from complex analysis and measure theory. Let X be a smooth projective variety of dimension N defined over \mathbf{C} . A positive $(1, 1)$ form ω on X induces a volume form ω^N and consequently a measure $\mu_\omega(\cdot)$ on X . Let $A \subset X$ be a subset. We will say that A is *full in X* if the measure of $\mu_\omega(X \setminus A) = 0$. If ω_1 is another positive $(1, 1)$ form on X , then by compactness of X it is easy to see that $\mu_\omega(X \setminus A) = 0$ if and only if $\mu_{\omega_1}(X \setminus A) = 0$; thus the "fullness" of A is independent on the chosen metric.

We recall the classical Theorem of Borel–Cantelli, which can be found in any standard book in measure theory:

Proposition 2.4. *Let X be a variety equipped with the Lebesgue measure μ . Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of measurable sets of X such that*

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty$$

then

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k\right) = 0.$$

That means that almost all $x \in X$ belong only to finitely many A_n .

Of course, the Borel–Cantelli Theorem above holds under less restrictive hypotheses; to avoid the introduction of further notations, we stated here just in the form we need in this paper.

3. DEGREE OF A DIVISOR ON A BOUNDED DOMAIN.

In this section we will work over \mathbf{C} .

In the paper [16], the author proves an estimate (Lemma 1 of op. cit) for the number of zeros of a holomorphic function on a disk in terms of the maximum of it over the disk itself and over a smaller disk. In this section we reinterpretate it in terms of Nevanlinna Theory and

generalize it to an estimate of the number of zeroes (counted with multiplicity) of a section of a line bundle over a relatively compact open set of a Riemann surface.

Let (X, L) be a projective variety with an ample line bundle equipped with a positive metric.

Let M be a Riemann surface and $U \subset M$ be a relatively compact open domain. Let $\varphi : M \rightarrow X$.

Fix a positive integer d . If $s \in H^0(X, L^d) \setminus \{0\}$, we may write $\text{div}(\varphi^*(s)) = \sum_{z \in M} n_z(s) \cdot z$ where $n_z(s)$ is the multiplicity of $\varphi^*(s)$ at z and it is a positive integral number which is zero for every z up to a (at most) countable set.

More generally, we write $\text{div}_U(\varphi^*(s)) = \sum_{z \in U} n_z(s) \cdot z$. This is a finite sum because U is relatively compact.

We will denote by $\text{deg}_U(s)$ the positive integer $\text{deg}(\text{div}_U(\varphi^*(s))) = \sum_{z \in U} n_z(s)$. Observe that, again, this degree is finite because U is relatively compact. It is the number of zeros of $\varphi^*(s)$ restricted to U counted with multiplicities.

Let $W \subset U$ be a compact set.

Let $s \in H^0(X, L^d)$ be a global section and suppose that $\varphi^*(s)|_W$ does not vanish identically.

Denote by $\|s\|_W$ the real number $\sup_{z \in W} \{\|\varphi^*(s)\|(z)\}$.

The norm of a global section and the number $\text{deg}_U(s)$ are related by the following theorem:

Theorem 3.1. *Under the hypotheses above we can find constants A and B , depending on L , U , φ and W such that*

$$(3.1) \quad A \cdot d + B \cdot (\log \|s\|_{\text{sup}} - \log \|s\|_W) \geq \text{deg}_U(s).$$

Proof. We fix a relatively compact open neighborhood V of \bar{U} (closure of U) with smooth border.

We denote by $g_V(z, w)$ the Green function of V .

Since $g_V(z, w)$ is bigger or equal then zero on the closure of V and it is subharmonic there, by the standard mean inequality for subharmonic functions, one sees that $g_V(z, w) > 0$ on V (and $g_V(z, w) = 0$ on the border ∂V). Consequently there is a constant $a > 0$ (depending only on U and V) such that $g_V(z, w) > a$ for every z and w in U .

Lemma 3.2. *We can find a constant B_1 , which depends only on V , φ and L , such that, for every $w_0 \in V$ we have*

$$(3.2) \quad \int_0^1 \frac{dt}{t} \int_{g_V(w_0, \cdot) \leq \log(t)} \varphi^*(c_1(L)) \leq B_1.$$

Proof. Since every line bundle is trivial on V , we can choose a trivialization $\varphi^*(L) = \mathcal{O}_V \cdot e$. Moreover the norm $\|e\|(z)$ is bounded on \bar{V} . Fix a constants A_i such that $A_1 \leq \|e\|(z) \leq A_2$ for every $z \in \bar{V}$.

Let $w_0 \in V$. The first main theorem with base point w_0 applied to the section e gives

$$(3.3) \quad \int_0^1 \frac{dt}{t} \int_{g_V(w_0, \cdot) \leq \log(t)} \varphi^*(c_1(L)) + \int_{\partial V} \log \|e\| d\mu_{V, w_0} = \log \|e\|(w_0)$$

The conclusion of the Lemma follows \square

Let $s \in H^0(X, L^d) \setminus \{0\}$. Let $w_1 \in W$ be a point such that $\|\varphi^*(s)\|(w_1) = \|s\|_W$.

We apply the first main theorem to s , taking as base point w_1 and using the linearity of the characteristic function, we find:

$$(3.4) \quad d \cdot \int_0^1 \frac{dt}{t} \int_{g_V(w_1, \cdot) \leq \log(t)} \varphi^*(c_1(L)) + \int_{\partial V} \log \|\varphi^*(s)\| d\mu_{V, w_1} \geq \sum_{z \in U} n_z(s)(g_V(w_1, z)) + \log \|s\|_W.$$

Thus

$$(3.5) \quad B_1 \cdot d + \log \|s\|_{\text{sup}} \geq \sum_{z \in U} n_z(s) \cdot a + \log \|s\|_W$$

where B_1 is the constant of Lemma 3.2. The conclusion of the Theorem follows. \square

If we apply Theorem 3.1 to the case when X is the projective line and M is the unit disk, we find a geometric reformulation of the key Lemma 1 of [16]:

Over the vector space $H^0(X, L^d)$ we have two natural norms: Let $s \in H^0(X, L^d) \setminus \{0\}$:

- the sup on X norm : $\|s\|_{\text{sup}} := \sup\{\|s\|(x) / x \in X\}$.
- The sup on W norm: $\|s\|_W := \sup\{\|\varphi^*(s)\|(z) / z \in W\}$.

In general we will have $\|s\|_W \leq \|s\|_{\text{sup}}$ and these two norms may be quite different.

We will denote by $\|\varphi\|(L, W, d)$ the real number

$$(3.6) \quad \|\varphi\|(L, W, d) := \sup\left\{\frac{\|s\|_{\text{sup}}}{\|s\|_W} / s \in H^0(X, L^d) \setminus \{0\}\right\}.$$

Thus we find

Proposition 3.3. *Under the hypotheses above, we can find a positive number A depending only on U and M and a positive number B depending only on U , L and φ such that, for every section $s \in H^0(X, L^d) \setminus \{0\}$, we have*

$$(3.7) \quad A \cdot \log \|\varphi\|(L, U, d) + B \cdot d \geq \deg_U(s)$$

In order to prove the Proposition, it suffices to suppose that $\|s\|_{\text{sup}} = 1$ and consequently $\log \|\varphi^*(s)\|(w_1) \geq -\log \|\varphi\|(L, U, d)$.

4. BOMBIERI–PILA REVISITED

In the seminal paper [4] the authors proved that, on the graph of a real analytic function, there are at most $O(\exp(\epsilon T))$ points of (logarithmic) height at most T . In this section we will prove a similar result but more in the spirit of the geometric transcendental theory. One can remark that the proof given is more in the spirit of classical transcendental theory.

Let M be a Riemann surface and Let $U \subset M$ be a relatively compact open set. Let (X, L) be a polarized projective variety defined over a number field K . We fix an embedding $\sigma_0 : K \hookrightarrow \mathbf{C}$. In this section we will denote by $X(\mathbf{C})$ the complex points of the variety X_{σ_0} (notations as in section 2).

We fix a holomorphic map $\varphi : M \rightarrow X(\mathbf{C})$ with Zariski dense image.

For every positive number T let

$$(4.1) \quad S_U(T) : \{z \in U / \varphi(z) \in X(K) \text{ and } h_L(\varphi(z)) \leq T\}$$

and Let $A_U(T)$ be the cardinality of it.

In this section we prove the following (slight) generalization of the Bombieri–Pila Theorem in this contest:

Theorem 4.1. *For every positive number ϵ , we have*

$$(4.2) \quad A_U(T) \ll \exp(\epsilon T)$$

where the involved constants depend on U , φ , ϵ and L but not on T .

Remark 4.2. Strictly speaking, Theorem 4.1 is of different nature with respect to the original Theorem of Bombieri and Pila. In [4] the authors prove a theorem for the rational points of *real curves* and here we prove a similar statement for Riemann surfaces (thus real varieties of dimension two). Nevertheless we think that this result is not new but just a revisitation of [4].

Before giving the proof of the theorem, we prove some lemmas which will be useful in the sequel.

We fix a relatively compact open set $V \subset M$ which contains \overline{U} . Let $g_V(z, w)$ be the Green function of V and we fix a smooth metric ω with induced distance $d_V(z, w)$ on V .

Lemma 4.3. *With the notations above, the function $\log(d_V(z, w)) + g_V(z, w)$ extends to a continuous function on $V \times V$.*

Remark 4.4. Besides the fact that this Lemma seems to be very natural, we did not find any statement of this kind in the literature. For this reason we propose a proof of it.

Proof. Fix a coordinate disk D in V with coordinate z .

We claim that the function $g_V(z, w) + \log(|z - w|)$ is continuous on $D \times D$. Indeed, denote by $h_V(z, w) := g_V(z, w) + \log(|z - w|)$. By properties of the Green function, for every z_0 and w_0 in D , the functions $h_V(z_0, w)$ and $h_V(z, w_0)$ are harmonic and bounded on D . Fix $z_0 \in D$. We have that $|h_V(z, w) - h_V(z_0, z_0)| \leq |h_V(z, w) - h_V(z_0, w)| + |h_V(z_0, w) - h_V(z_0, w_0)|$. By Harnack's inequality, the two terms of this sum are bounded by $\epsilon(h_V(z_0, w_0) + A)$ as soon as (z, w) is sufficiently near to (z_0, w_0) . The claim follows.

On the other side, we claim that, on the same disk, also the function $\frac{d_V(z; w)}{|z - w|}$ extends to a positive continuous function on $D \times D$.

We may suppose that, on D , the hermitian metric is give by a $(1, 1)$ form $\omega = iF(z)dz \wedge d\bar{z}$ where $F(z)$ is a positive smooth function on D .

In order to prove the claim, we will prove that, given $z_0 \in D$, then $\lim_{(z;w) \rightarrow (z_0, z_0)} \frac{d_V(z;w)}{|z-w|} = F(z_0)$. Fix $\epsilon > 0$. We can choose two concentric disks $U_\epsilon \subset V_\epsilon$ centered in z_0 such that, for every $z \in U_\epsilon$ we have $F(z_0) - \epsilon \leq F(z) \leq F(z_0) + \epsilon$ and, for every couple z and w in V_ϵ the geodesic curve between z and w with respect to the metric ω is entirely contained in V_ϵ (in order to obtain this, it suffices that, for every $z \in U_\epsilon$, the set of points with ω -distance less or equal to 2ϵ from z is entirely contained in V_ϵ).

Let z and w in U_ϵ . We denote by $\alpha_\omega(z, w)$ and $\alpha_s(z, w)$ the geodesic paths between z and w with respect to the metric ω and the standard euclidean metric on the disk respectively. If β is a path, we denote by $\ell_\omega(\beta)$ and $\ell_s(\beta)$ the length of it with respect to the metric ω and the standard euclidean metric.

We recall that the distance between z and w may be defined in two ways: either it is the length of the geodesic path between z and w or it is the minimum between the lengths of all the paths between them.

If z and w are points in U_ϵ we have that $d_V(z, w) \leq \ell_\omega(\alpha_s(z, w)) (\leq F(z_0) + \epsilon)|z - w|$ and $(F(z_0) - \epsilon)|z - w| \leq (F(z_0) - \epsilon)\ell_s(\alpha_\omega(z, w)) \leq \ell_\omega(\alpha_\omega(z, w)) = d_V(z, w)$. Since ϵ is arbitrarily small, the claim follows. Thus $\log(d_V(z, w)) - \log(|z - w|)$ extends to a continuous function on D .

Consequently $g_V(z, w) + \log(d_V(z, w))$, being the sum of two continuous functions is continuous. \square

As a corollary of the Lemma above, we find that there are constants A_i , depending only on U, V and the chosen metric, such that, for every $(z, w) \in U \times U$, we have

$$(4.3) \quad A_1 \leq \log(d_V(z, w)) + g_V(z, w) \leq A_2$$

The main lemma we need is the following:

Lemma 4.5. *Let d_0 a sufficiently big integer. There exist constants C_i , depending only on d_0, U, X and φ , with the following property: let d be a positive integer, W be a open set in U and r_W be its diameter. If $d \geq d_0$ and $r_W \leq C_1 \exp(-\frac{C_2 T}{d^{n-1}})$, then there exists a section $s \in H^0(X, L^d) \setminus \{0\}$ such that $\varphi^*(s)$ vanishes on every point of $S_W(T)$.*

Proof. Denote by $h^0(X, L^d)$ the rank of $H^0(\mathcal{X}, \mathcal{L}^d)$, it is well known that can find a positive constant B_1 such that, for every $\epsilon_1 > 0$ and d sufficiently big, we have

$$(4.4) \quad B_1(1 - \epsilon_1)d^n \leq h^0(X, L^d) \leq B_1(1 + \epsilon_1)d^n.$$

We also recall that, by definition of arithmetic polarization, we may suppose that $H^0(\mathcal{X}, \mathcal{L}^d)$ is generated by sections of norm less or equal than one.

Fix an $\epsilon > 0$ and we suppose that d is big enough to have that $\epsilon h^0(X, L^d) > 1$. Choose an integer A such that $(1 - 2\epsilon)h^0(X, L^d) \leq A \leq (1 - \epsilon)h^0(X, L^d)$ and a subset $H_W(d) \subset S_W(T)$ of cardinality A (if the cardinality of $S_W(T)$ is smaller then the Lemma easily follows from linear algebra).

Denote by $E(T)$ the O_K module $\bigoplus_{z \in H(T)} \mathcal{L}^d|_{f(z)}$. The rank of E is A and $\mu_{\max}(E(T)) \leq dT$.

We have a natural restriction map

$$(4.5) \quad \delta_T : H^0(\mathcal{X}, \mathcal{L}^d) \longrightarrow E(T).$$

By Gromov theorem 2.2, if we put on $H^0(\mathcal{X}, \mathcal{L}^d)$ the L_2 hermitian structure and on $E(T)$ the direct sum hermitian structure, the norm of δ is bounded by $C_0 d$ for a suitable constant C_0 .

Denote by $K(T)$ the kernel of δ_T and by $k(T)$ its rank. By construction we have that

$$(4.6) \quad \frac{h^0(X, L^d)}{k(T)} \leq \frac{h^0(X, L^d)}{h^0(X, L^d) - (1 - \epsilon)h^0(X, L^d)} = \frac{1}{\epsilon}.$$

We may then apply Siegel Lemma 2.1 and we obtain that there is a non vanishing section $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ such that $\varphi^*(s)$ vanishes on every point of $E(T)$ and $\log \|s\|_{\text{sup}} \leq C_3 dT$ for a suitable constant C_3 independent on T .

Let $w \in S_W(T)$ we will now prove that $\varphi^*(s)$ vanishes also on w . Suppose that $\varphi^*(s)$ do not vanish on w .

We apply Nevanlinna First Main Theorem to the section $\varphi^*(s)$ over the open set V with base point w and obtain a constant A_2 independent on w such that

$$(4.7) \quad dA_2 + \int_{\partial V} \log \|\varphi^*(s)\| d\mu_{V,w} \geq \sum_{z \in S_W(T)} g_w(w, z) + \log \|\varphi^*(s)\|(w).$$

Remark that A_2 is independent on w_0 because of Lemma 3.2. From the upper bound of the norm of $\varphi^*(s)$, the estimate 4.3, the hypothesis on the diameter of W and Liouville inequality 2.7 we obtain

$$(4.8) \quad dA_1 + C_3 dT \geq A(A_1 - \log(r_W)) - dT - A_3$$

for suitable constants A_i independent on w (and on W). Since $A \geq B_1 \cdot d^n$ (for a suitable constant depending only on L and d), we obtain, constants $B_i > 0$ (independent on w) such that

$$(4.9) \quad \log(r_w) \geq B_2 - B_3 \frac{T}{d^{n-1}}.$$

And this contradicts the hypothesis of the Lemma as soon as $C_1 < B_1$ and $C_2 > B_2$. \square

Proof. (of Theorem 4.1) Now the proof is immediate. We fix the constants C_i as in the Lemma 4.5 and d a sufficiently big integer such that $\frac{C_2}{d^{n-1}} < \epsilon$.

For every T , we may write $U = \bigcup W_i$, were W_i are open set of diameter at most $C_1 \exp(-\frac{C_2}{d^{n-1}} T)$.

The number of these W_i is at most $2C_1 \exp(\frac{2C_2}{d^{n-1}} T)$.

By Lemma 4.5, for every W there is a non vanishing section $s_W \in H^0(X; L^d)$, such that $S_W(T) \subset \{z \in U / \varphi^*(s_W) = 0\}$. By Theorem 3.3, for each s_W , the cardinality of the set $\{z \in U / \varphi^*(s_W) = 0\}$ is bounded by a constant (depending on d). Thus the conclusion of the Theorem follows. \square

We would like to remark that, strictly speaking, one could give a proof of Theorem 4.1 simply by reduction to the case when U is a unit disk (with a coordinate z). This would allow, essentially to avoid Lemma 4.3 (or more specifically, use an explicit and much simpler version of it on a disk). Never the less, the interest of this proof is to point out the main analytic tool which is needed which is Lemma 4.3.

4.1. On distribution of rational points on a Riemann Surface. Examples provided by Surroca [15], show that that the bound of Bombieri and Pila, Theorem 4.1, cannot be improved. Never the less, given an analytic map as above, we may ask for which value of T the number $A_U(T)$ is big and for which it is not. We will show now that, for many values of T , the number $A_U(T)$ is actually bounded by a polynomial in T . In order to quantify this we give the following definitions:

Definition 4.6. Let $\varphi : M \rightarrow X$, and U as in Theorem 4.1. Let $\gamma \in \mathbb{R} > 0$ and $\epsilon > 0$. denote by $L(\varphi, \gamma, \epsilon) \subset \mathbb{R}$ the following set:

$$(4.10) \quad L(\varphi, \gamma, \epsilon) := \{T \in \mathbb{R} / A_U(T) \leq \epsilon T^\gamma\}.$$

In this section we will prove that, even if the Bombieri Pila estimate is optimal, for big values of γ the set $L(\varphi, \gamma, \epsilon)$ is very big. Actually it is almost as big as we want.

Definition 4.7. Let $A > 1$ be a real number. We will say that an interval $I \subset \mathbb{R}$ is geometrically wider than A if there is $t \in I \cap]1, +\infty[$ such that the interval $[t, At]$ is contained in I .

Theorem 4.8. Let $\varphi : M \rightarrow X$, and U as in Theorem 4.1. Let $\gamma > \frac{n}{n-1}$, $A > 1$ and $\epsilon > 0$. Then the set $L(\varphi, \gamma, \epsilon)$ is unbounded and contains infinitely many intervals $I \subset \mathbb{R}$ which are geometrically wider than A .

This theorem generalize Theorem 1.3 of [15]. The Theorem means that there are infinitely many intervals of the form $[r, Ar]$ with $\sup\{r\} = +\infty$ and such that, for every T in one of these intervals, the number $A_U(T)$ is "small".

Proof. As in the proof of Theorem 4.1, we fix a relatively compact open set V containing \bar{U} . We use notations of the proof of Lemma 4.5.

We suppose that the conclusion of the Theorem is false. Consequently we may find a strictly increasing sequence $(T_n)_{n \in \mathbb{N}}$ such that:

- For every n we have $T_{n+1} \leq A \cdot T_n$;
- $A_U(T_n) \geq \epsilon T_n^\gamma$;
- $\lim_{n \rightarrow \infty} T_n = +\infty$.

We may also suppose that T_1 is very big. In particular we suppose that $(\epsilon T_1)^{\gamma/n}$ is very big.

Fix an integer d_1 such that $(\epsilon T_1)^{\gamma/n} \leq d_1 \leq (\epsilon T_1)^{\gamma/n} + 1$ and a subset $H_U(T_1) \subseteq S_U(T_1)$ of cardinality A_1 , where A_1 is an integer which verify $(1 - 2\epsilon)h^0(X, L^d) \leq A_1 \leq (1 - \epsilon)h^0(X, L^d)$.

Following the same strategy in the proof of Lemma 4.5, an application of Siegel Lemma 2.1 allows to construct a non vanishing global section $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ such that:

- There exists a constant C_1 independent on T_1 , on d_1 such that $\log \|s\| \leq C_1 T_1^{\frac{\gamma}{n}+1}$;
- $\varphi^*(s)$ vanishes on every point of $H_U(d_1)$.

Again, as in the proof of Lemma 4.5, we will now prove that the constructed section s actually vanishes on every point of $S_U(T_1)$.

Let w_0 be a point of $S_U(T_1)$ and suppose that $\varphi^*(s)$ do not vanish on it. We apply Nevanlinna First Main theorem, Lemma 3.2 and Liouville inequality to the section $\varphi^*(s)$ and the domain V with base point w_0 and we obtain the existence of constants C_i , independent on w_0 and s such that

$$(4.11) \quad C_2 T^{\gamma/n} + C_1 T_1^{\frac{\gamma}{n}+1} \geq \sum_{z \in H_U(d_1)} g_V(w_0, z) + \log \|\varphi^*(s)\|(w_0).$$

By Lemma 4.3 and the fact that \bar{U} is compact and contained in V , we may find a constant $C_3 > 0$ such that, for every $z \in U$ we have $g_V(z, w_0) \geq C_3$.

We observe that that the cardinality $H_U(d_1)$ is lower bounded by $C_4 d^n$ (with C_4 depending only on X and L). We apply Liouville Inequality to $\varphi^*(s)$ and w_0 and we find that, for T_0 sufficiently big, we have:

$$(4.12) \quad C_5 T_1^{\frac{\gamma}{n}+1} \geq C_6 T_1^\gamma.$$

This is impossible for our choice of γ and for T_1 big enough. Thus $\varphi^*(s)$ vanishes on every point of $S_U(T_1)$.

By induction, we may suppose that $\varphi^*(s)$ vanishes on every points of $S_U(T_n)$ and we will now prove that it vanishes on every point of $S_U(T_{n+1})$.

Suppose that w_{n+1} is an element of $S_U(T_{n+1})$ such that $\varphi^*(s)(w_{n+1}) \neq 0$.

Once again, we apply Nevanlinna First Main Theorem and Liouville inequality to $\varphi^*(s)$ and V with base point w_{n+1} , using the estimates above on the Green functions we obtain the existence of constants C_i independent on w_{n+1} :

$$(4.13) \quad C_2 T_1^{\gamma/n} + C_1 T_1^{\frac{\gamma}{n}+1} \geq C_6 T_n^\gamma - C_7 T_1^{\gamma/n} T_{n+1} - C_8 T_1^{\frac{\gamma}{n}+1}.$$

Which gives, since $T_{n+1} \leq AT_n$,

$$(4.14) \quad C_1 T_n^{\frac{\gamma}{n}+1} \geq C_6 T_n^\gamma.$$

And this is impossible as soon as T_n is sufficiently big.

The conclusion of the Theorem follows because $\varphi^*(s)$ is a section of an holomorphic line bundle thus it can vanish only on finitely many points of U . \square

5. RIEMANN SURFACES CONTAINING SUBSETS OF TYPE S_a

In the paper [9], inspired by the work of Chudnovski [5], we introduced a class of points on a projective variety which verify some inequalities of Liouville type: the points of type S_a . Let's recall (and slightly modify) this definition:

Let (X_K, \mathcal{L}) be an arithmetically polarized projective variety defined over a number field K of dimension at least two. Again, we fix an embedding $\sigma_0 : K \hookrightarrow \mathbf{C}$ and we denote X_{σ_0} the base change of X_K to \mathbf{C} .

Definition 5.1. *Let $z \in X_{\sigma_0}(\mathbf{C})$. We will say that z is of type S (or that $z \in S(\mathcal{X})$) if we can find positive constants $a = a(z, \mathcal{L})$, $A = A(z, \mathcal{L})$ depending on z, \mathcal{L} such that, for every positive integer d and every non zero global section $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ we have that*

$$\log \|s_{\sigma_0}\|_{\sigma_0}(z) \geq -A(\log^+ \|s\| + d)^a.$$

Moreover we will denote by $S(X_K)$ the subset of $X_{\sigma_0}(\mathbf{C})$ of points of type S . If a_0 is fixed, we will denote by $S_{a_0}(X_K)$ the set of S -points of X_{σ_0} for which the involved constant a is a_0 .

Observe that $S(X_K) = \cup_{a \geq 0} S_a(X_K)$.

We proved in [9] the following theorem:

Theorem 5.2. *Let X_K as before. The following properties hold:*

- a) *If $a < \dim(X_K) + 1$ then $S_a(X_K) = \emptyset$.*
- b) *If $a \geq \dim(X_K) + 2$, the set $S_a(X_K)$ is full in $X_{\sigma_0}(\mathbf{C})$.*
- c) *Let $Y \subset X_{\sigma_0}(\mathbf{C})$ be a compact Riemann surface and $a \geq \dim(X_K) + 2$, then $Y \cap S_a(X_K) \neq \emptyset$ if and only if $Y \cap S_a(X_K)$ is full in Y .*

Let M be a Riemann surface and $U \subset M$ be a relatively compact open set. Let $\varphi : M \rightarrow X_{\sigma}(\mathbf{C})$ be an holomorphic map with Zariski dense image. The interest of points of type S_a is the following theorem, proved in [9] Theorem 10.1:

Theorem 5.3. *We keep notations as in the previous section. If, for some real number a we have $\varphi^{-1}(S_a) \cap U \neq \emptyset$, then*

$$(5.1) \quad A_U(T) \ll T^a.$$

Theorem 5.3 can be weakened a bit: instead of supposing that the norm is "big" on a point, we may suppose that it is "big" just on a compact subset of U .

Definition 5.4. *Let $\varphi : M \rightarrow X_{\sigma}(\mathbf{C})$ and U as before. Let $B \subset U$ be a compact set and a be a real number. We will say that B is a subset of type S_a of M with respect to φ if, we can find a positive constant $A > 1$ such that, for every positive integer d and $s \in H^0(\mathcal{X}, \mathcal{L}^d) \setminus \{0\}$ we have that*

$$(5.2) \quad \log \|s\|_B \geq -A(\log^+ \|s\| + d)^a.$$

It is possible to prove that, in this case, we must have $a \geq \dim(X_K) + 1$.

We can then generalize Theorem 5.3 to subsets of type S_a :

Theorem 5.5. *Let $\varphi : M \rightarrow X_{\sigma}(\mathbf{C})$ and U as before. Suppose that we can find a subset $B \subset U$ of type S_a with respect to φ . Then*

$$(5.3) \quad A_U(T) \ll T^{2a}.$$

Proof. The proof follows the same paths of the proof of Theorem 4.8 or the proof of Theorem 5.2. For reader's convenience, we give here a short sketch of it. Fix $\gamma > \dim(X_K)$.

Let T be sufficiently big. As in the proof of 4.8, we can find an integer d with $T^{\gamma/n} \leq d \leq T^{\gamma/n} + 1$ and a global section $s \in H^0(\mathcal{X}, \mathcal{L}^d) \setminus \{0\}$ such that:

- $\log \|s\|_{\text{sup}} \leq C_1 T^{\gamma/n+1}$ for a constant C_1 independent on T ;
- for every $w \in S_U(T)$, we have $\varphi^*(s)(w) = 0$.

We may now apply Theorem 3.1 and conclude. \square

Of course, Theorem 5.3 is just a particular case of Theorem 5.5. We will now see another application of Theorem 5.5:

5.1. Rational points on leaves of one dimensional algebraic foliations. Let Z be a smooth quasi projective variety of dimension N defined over K . Let T_Z its tangent bundle. A one dimensional algebraic foliation \mathcal{F} over Z is an algebraic sub line bundle $H \hookrightarrow T_Z$ (the quotient is locally free).

Fix an algebraic one dimensional foliation \mathcal{F} and a rational point $p \in Z(K)$. Denote by \widehat{Z}_p the formal completion of Z at p .

The *formal leaf* of the foliation \mathcal{F} at p is a formal morphism $\iota : \text{Spf}(K[[X]]) := \widehat{\mathbb{A}}_0^1 \rightarrow \widehat{Z}_p$ such that the natural differential morphism $\iota^*(\widehat{\Omega}_{Z_p/K}^1) \rightarrow \widehat{\Omega}_{\widehat{\mathbb{A}}_0^1/K}^1$ factorizes through the natural surjection $\iota^*(\widehat{\Omega}_{Z_p/K}^1) \rightarrow \iota^*(H^\wedge)$. Since K is of characteristic zero, the formal leaf exists and it is unique (up to reparametrization).

An holomorphic morphism $h : \Delta_1 \rightarrow Z_\sigma(\mathbf{C})$ is said to be *tangent to the foliation in p* if $h(0) = p$ and the natural morphism $h^*(\Omega_{Z_\sigma}^1) \rightarrow \Omega_{\Delta_1}^1$ factorizes through the natural surjection $h^*(\Omega_{Z_\sigma}^1) \rightarrow h^*(H^\wedge)$. By Frobenius theorem over \mathbf{C} , we can always find morphisms tangent to the foliation at p .

By unicity of the formal leaf, the formal completion of a morphism tangent to the \mathcal{F} at p coincides with its formal leaf of it.

Denote by \mathbb{A}_n^1 the scheme $\text{Spec}(K[X]/(X^{n+1}))$. It is called the n -th formal neighborhood of 0 in $\widehat{\mathbb{A}}_0^1$ and there is a natural inclusion $j_n : \mathbb{A}_n^1 \hookrightarrow \widehat{\mathbb{A}}_0^1$. Denote by Ω_0 the fibre at 0 of the cotangent bundle of $\widehat{\mathbb{A}}_0^1$, we have a canonical exact sequence

$$(5.4) \quad 0 \longrightarrow \Omega_0^{\otimes n} \longrightarrow \mathcal{O}_{\mathbb{A}_{n+1}^1} \longrightarrow \mathcal{O}_{\mathbb{A}_n^1} \longrightarrow 0.$$

By construction we have natural inclusions $\iota_n := \iota \circ j_n : \mathbb{A}_n^1 \rightarrow Z$ which factorize through ι .

Let \overline{Z} be a (smooth) projective compactification of Z and L be a ample line bundle over it. A global section $s \in H^0(\overline{Z}, L^d)$ is said to be *vanishing at order n over the formal leaf ι at p* if $\iota_n^*(s) = 0$ but $\iota_{n+1}^*(s) \neq 0$. In this case, via the exact sequence 5.4, the section s canonically defines a section $j^n(s) \in \Omega_0^n \otimes L_p$ called the *n -jet of s at the formal leaf ι* . In this case we will denote the integer n by $\text{ord}_{\mathcal{F}, p}(s)$.

If we fix an arithmetic polarization $(\overline{\mathcal{Z}}, \overline{\mathcal{L}})$ of the couple (\overline{Z}, L) with \overline{Z} normal. The point p extends to a section $P : \text{Spec}(O_K) \rightarrow \overline{\mathcal{Z}}$. It is possible to fix an integral structure $\widetilde{\Omega}_0$ of Ω_0 for which the following holds:

– There exists a constant C_p such that, if $s \in H^0(\overline{\mathcal{Z}}, \mathcal{L}^d)$ vanishes at the order n over the formal leaf ι , then $C_p^n \cdot n! \cdot j^n(s) \in \widetilde{\Omega}_0^{\otimes n} \otimes \mathcal{L}^d|_P$.

The proof of this fact is the main topic of section 3 of [8].

If we fix an hermitian structure on the O_K -module $\widetilde{\Omega}_0$, a direct application of Schwartz inequality gives the following:

Proposition 5.6. *We can find a constant C , independent on s for which the following holds: Let $s \in H^0(\overline{\mathcal{Z}}, \mathcal{L}^d)$ a section vanishing at order n on the formal leaf ι , then*

$$(5.5) \quad \log \|j^n(s)\|_\sigma \geq -C(n \log(n) + d + \log^+ \|s\|_{\text{sup}}).$$

As a consequence we find that, in order to find a lower bound for the norm of the jet of section vanishing at p , we need to bound the order of vanishing. The order of vanishing can be bounded by the following proposition:

Proposition 5.7. *Suppose that the (formal) leaf at p is Zariski dense. We can find a constant C_p for which the following holds: For every positive integer d and every non vanishing global section $s \in H^0(\overline{Z}, L^d)$ we have*

$$(5.6) \quad \text{ord}_{\mathcal{F}, p}(s) \leq C_p d^{\dim(Z)}.$$

The proof of the proposition leans on the following theorem, which, a priori, is just the case when Z is the projective space:

Proposition 5.8. *Let $D := \sum_{i=1}^N P_i(x_1, \dots, x_N) \frac{\partial}{\partial x_i}$ be a differential operator on the affine space \mathbb{A}^N with $P_i(x_1, \dots, x_N) \in \mathbf{C}[x_1, \dots, x_N]$. Suppose that the variety $V(P_1, \dots, P_N)$ do not contain the origin. Let $V_{\mathcal{F}, 0}$ be the formal leaf through the origin of the foliation defined by D . Let ℓ be dimesion of the Zariski closure of $V_{\mathcal{F}, 0}$. Then we can find a constant C such that, for every polynomial $Q(x_1, \dots, x_N) \in \mathbf{C}[x_1, \dots, x_N]$ of degree less or equal than d not vanishing identically on $V_{\mathcal{F}, 0}$, we have*

$$(5.7) \quad \text{ord}_{\mathcal{F}, 0}(Q) \leq C d^\ell.$$

The proof of the Proposition above can be found in [12] or [3].

In order to deduce Proposition 5.7 from Proposition 5.8 it suffices to remark that we can suppose that the involved line bundle H^{-1} of the foliation is very ample and consequently the foliation comes from the restriction of a foliation on \mathbf{P}^N .

We can deduce from the proposition above the fact that the number of rational points of height less or equal than T on a disk which is the analytic leaf of a foliation, grows polynomially with T :

Theorem 5.9. *Let \mathcal{F} be a foliation on a smooth quasi projective variety Z defined over a number field K . Let $p \in Z(K)$ be a rational point and $h : \Delta_1 \rightarrow Z_\sigma(\mathbf{C})$ be a holomorphic morphism tangent to \mathcal{F} at p . Suppose that the dimension of the Zariski closure of $h(\Delta_1)$ is $\ell > 1$. Let $0 < r < 1$ be a real number, then, for every $\epsilon > 0$ we have*

$$(5.8) \quad A_{\Delta_r}(T) \ll_\epsilon T^{2\ell+\epsilon}.$$

Proof. By Theorem 5.5, it suffices to prove that such a Δ_r contains a compact subset of type $S_{\ell+\epsilon}$.

Replacing Z by the Zariski closure of the leaf and taking a resolution of singularities, we may suppose that the leaf is Zariski dense in Z .

Choose a disk $\Delta_{r_0} \subset \Delta_r$ (with $r_0 < r$). Let $d > 0$ be a positive integer and $s \in H^0(\overline{\mathcal{L}}, \mathcal{L}^d) \setminus \{0\}$.

Let $n = \text{ord}_{\mathcal{F},p}(s)$. By Nevanlinna First Main Theorem applied to the disk Δ_{r_0} and the line bundle \mathcal{L} we may find a constant A for which

$$(5.9) \quad A \cdot d + \int_0^{2\pi} \log \|s\|_{\sigma}(r_0 \cdot \exp(2\pi i\theta)) \frac{d\theta}{2\pi} \geq \log \|j^n(s)\|_{\sigma}.$$

Thus there is a point $w_0 \in \partial\Delta_{r_0}$ such that

$$(5.10) \quad \log \|s\|(w_0) \geq \log \|j^n(s)\|_{\sigma} - A \cdot d.$$

The conclusion follows from Proposition 5.7 and Proposition 5.6. \square

6. RIEMANN SURFACES CONTAINING POINTS OF TYPE S_a

We saw in the previous section that if a Zariski dense Riemann surface in a projective variety contains a point of type S then it must contain "few" rational points. We also remarked that, in order to contain few rational points, it suffices that it contains a subset of type S .

In this section we will show that if a Riemann surface contains a subset of type S which is relatively compact, then the set of points of type S contained in it is full for the Lebesgue measure.

As a corollary, as in the case of compact Riemann Surfaces, (cf. Theorem 5.2 (c)), we find that the fact that the intersection of the Riemann surface with points of type S is non empty is equivalent with the fact that the points of type S in it form a full set.

We fix a Riemann surface M and an holomorphic map $\varphi : M \rightarrow X_{\sigma}(\mathbf{C})$ with Zariski dense image. Essentially, in this section we will prove the following general principle:

Either the image of M via φ do not intersect the set of points of type S of X_K ; in this case it *might* contain many rational points, or it does intersect the set of points of type S ; in this case, essentially every point of the image is of type S and there are very few rational points contained in it.

A non compact Riemann surface M may be equipped with a canonical metric of constant curvature (up to a scalar factor). Let $\alpha : \widetilde{M} \rightarrow M$ be the universal covering of M . The Riemann surface \widetilde{M} is either \mathbf{C} or the disk Δ_1 .

The main theorem of this section is the following:

Theorem 6.1. *Suppose that $\varphi : M \rightarrow X_{\sigma}(\mathbf{C})$ is a holomorphic map as above. Suppose that M contains a compact subset $B \subset M$ of type S with respect to φ . Then the set $\varphi^{-1}(S(X_K))$ is full in M .*

As a corollary we find:

Theorem 6.2. *With the notations as above, then $\varphi^{-1}(S(X_K)) \neq \emptyset$ if and only if $\varphi^{-1}(S(X_K))$ is full in M for the Lebesgue measure.*

Another important corollary of Theorem 6.1 and of the proof of Theorem 5.9 is the following:

Theorem 6.3. *Let \mathcal{F} be a foliation on a smooth quasi projective variety Z defined over a number field K . Let $p \in Z(K)$ be a rational point and $h : \Delta_1 \rightarrow Z_\sigma(\mathbf{C})$ be the analytic leaf of \mathcal{F} through p . Then, if $h(\Delta_1)$ is Zariski dense in Z , we have that $h^{-1}(S(Z_K))$ is full in Δ_1 .*

For the proof, it suffices to remark that, in the proof of Theorem 5.9 we proved that every neighborhood of p in Δ_1 is of type S with respect to h .

6.1. Area of the set of points where a global section is small. Theorem 6.1 will be consequence of the following estimate, which will be interesting in its own.

Let M be a Riemann Surface. We fix an hermitian line bundle L on M . We fix a point $z_0 \in U$. For every positive integer d , we will denote by $H_B^0(M; L^d)$ the subspace of section $s \in H^0(M, L^d)$ for which $\sup_{z \in M} \{\|s\|\} \leq C(s)$, where $C(s)$ is a positive constant, depending on s . For $s \in H_B^0(M; L^d)$, we will denote by $\|s\|_M$ the number $\sup_{z \in M} \{\|s\|\}$. More generally, if $B \subset M$, we will denote by $\|s\|_B$ the number $\sup_{z \in B} \{\|s\|\}$. We fix a universal covering $\alpha : \widetilde{M} \rightarrow M$ and consequently we can fix a metric $\mu(\cdot)$ with constant curvature on M . The distance defined by the metric will be denoted by $d_M(\cdot, \cdot)$.

Let $U \subset M$ be a non dense relatively open set. The main theorem of this sub section is:

Theorem 6.4. *Let M be a Riemann surface, L be an hermitian line bundle over it, U be a non dense relatively compact open subset of M and $W \subset U$ any subset. We can find positive real constants C_i , depending only on M, L, U and W for which the following holds: Let $0 < \eta < 1$ be a positive constant. For every positive integer d and global section $s \in H_B^0(M, L^d) \setminus \{0\}$ such that $s(z_0) \neq 0$, define*

$$(6.1) \quad B(s) := \left\{ z \in U / \ln \|s\|(z) \leq -C_1 \left(\ln \left(\frac{1}{\eta} \right) + 1 \right) (\log^+ \|s\|_M + d) + 3 \ln \|s\|_W \right\}.$$

Then

$$(6.2) \quad \mu(B(s)) \leq C_2 \eta^2.$$

In order to prove Theorem 6.4, we need some classical results from complex analysis. The first one is the classical estimate by Bloch and Cartan ([11] Theorem 3.1 page 236):

Proposition 6.5. *Let a_1, \dots, a_n be n complex numbers (which may not be distinct). Let H be a positive real number. Then the numbers $z \in \mathbf{C}$ for which one has the inequality*

$$(6.3) \quad \prod_{i=1}^n |z - a_i| \leq \left(\frac{H}{2e} \right)^n$$

is contained in the union of at most n disks, such that the sum of the radii is bounded by H .

We will use this proposition in the following form:

Proposition 6.6. *Let a_1, \dots, a_n and H as in Proposition 6.5. Then*

$$(6.4) \quad \mu(\{z \in \mathbf{C} / \prod_{i=1}^n |z - a_i| \leq \left(\frac{H}{2e}\right)^n\}) \leq \pi H^2.$$

Another standard result of complex analysis is the following formula due to Poisson: Let $f(z)$ be a holomorphic function in the disk $\{|z| \leq R\}$. Write $f(z) = u(z) + iv(z)$ then, for every z such that $|z| < R$, we have:

$$(6.5) \quad f(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \cdot d\theta + iv(0).$$

We will first prove Theorem 6.4 in the case when U is the disk $\Delta_r := \{|z| < r\}$ inside the disk $\Delta_1 := \{|z| < 1\}$ and then a topological argument will allow to deduce the general case. In this case we may suppose that $z_0 = 0$.

We would like to remark that, strictly speaking (modulo some adaptations), only the case of disks is necessary for the proofs of this paper, but for sake of completeness, and for future reference, we prefer giving the proof for a general Riemann Surface.

The first three lemmas are from complex analysis:

Lemma 6.7. *Let $f(z)$ be an holomorphic function on Δ_1 . Let $r < R < 1$ be a real number. Write $f(z) = u(z) + iv(z)$ and denote by $A_R(f) := \sup\{u(z); / |z| < R\}$. Then the following inequality holds:*

$$(6.6) \quad \sup_{\Delta_r} \{|f(z)|\} \leq [A_R(f) - u(0)] \frac{2r}{R-r} + |f(0)|.$$

Proof. Since $u(z)$ is harmonic, we have $u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) d\theta$. From this we obtain

$$(6.7) \quad f(z) = \frac{1}{\pi} \int_0^{2\pi} u(Re^{i\theta}) \frac{z}{Re^{i\theta} - z} d\theta + f(0).$$

In particular, applying this to $f(z) = 1$ we obtain $\frac{1}{\pi} \int_0^{2\pi} \frac{z}{Re^{i\theta} - z} d\theta = 0$. From this we obtain

$$(6.8) \quad -f(z) = \frac{1}{\pi} \int_0^{2\pi} [A_R(f) - u(Re^{i\theta})] \frac{z}{Re^{i\theta} - z} d\theta + f(0).$$

Since, in the integrant, $A_R(f) - u(Re^{i\theta}) \geq 0$ the conclusion follows. \square

As a consequence, we obtain this non trivial estimate for non vanishing holomorphic functions on a disk:

Lemma 6.8. *Let $0 < r < R < 1$ real numbers. Let $f(z)$ be a holomorphic function in Δ_1 non vanishing in $\bar{\Delta}_R$. Then, for every $z \in \Delta_r$ we have*

$$(6.9) \quad \ln |f(x)| \geq -\frac{2r}{R-r} \sup_{\Delta_R} \{\ln |f(z)|\} + \frac{R+r}{R-r} \ln |f(0)|.$$

Proof. We first remark that we can suppose that $f(0) = 1$.

Since f is non vanishing in $\overline{\Delta}_R$ we may find a neighborhood of it and a holomorphic function $g(z) = u(z) + iv(z)$ over this neighborhood, such that $f(z) = e^{g(z)}$. Moreover, since $f(0) = 1$ then $g(0) = 0$. It is easy to see that $\ln |f(z)| = u(z)$ and consequently $\sup_{\Delta_R} \{\ln |f(z)|\} = A_R(g)$ (notation as in the previous lemma). If we apply Lemma 6.7 to the function $g(z)$ we obtain

$$(6.10) \quad -u(z) \leq \frac{2r}{R-r} A_R(g).$$

From this the conclusion follows. \square

We generalize now Lemma 6.8 to a general holomorphic function on the disk. This can be seen as a simplified version of Theorem 6.4 to the case when M is a disk, U is a smaller disk and L is the trivial line bundle with the trivial metric.

Lemma 6.9. *Let $0 < r < R < 1$ be two real numbers. Then we can find three explicit positive constants B_i ($i = 1, 2, 3$) depending only on r and R for which the following holds: Let $f(z)$ be a holomorphic function on Δ_1 such that $f(0) \neq 0$. Let η be a positive real number less or equal than 1. Then:*

$$(6.11) \quad \mu \left\{ z \in \Delta_r / \ln |f(z)| < -(B_1 + B_2 \ln \left(\frac{1}{\eta} \right)) \sup_{w \in \Delta_R} \{\ln |f(w)|\} + \ln |f(0)| \right\} \leq B_3 \cdot \eta^2.$$

Proof. Once again it suffices to treat the case when $f(0) = 1$. We fix an arbitrary r_1 such that $r < r_1 < R$. Let a_1, \dots, a_n be the set of zeros of $f(z)$ inside the closed disk of radius r_1 , counted with multiplicity. By Jensen formula we can find a positive constant A such that

$$(6.12) \quad n \leq A_0 \cdot \sup_{w \in \Delta_R} \{\ln |f(w)|\}.$$

(The constant A_0 is $1/\ln(R/r_1)$). The dependence on r , r_1 and R of each of the constants appearing in this proof can be explicitly written, but, since we do not need the precise form of them, we will denote them by capital A 's).

Let

$$(6.13) \quad \varphi(z) := \frac{(-R)^n}{a_1 \cdots a_n} \cdot \prod_{i=1}^n \frac{R(z - a_i)}{R^2 - z\overline{a_i}}.$$

Observe that $\varphi(0) = 1$. If $f(0) \neq 1$, change $\varphi(z)$ with $\varphi(z) \cdot f(0)$. The function $g(z) := \frac{f(z)}{\varphi(z)}$ is holomorphic on Δ_{r_1} and has no zeros on the closed disk $\overline{\Delta}_r$. Consequently we can apply Lemma 6.8 to it and find that, for every $z \in \Delta_r$ we have:

$$(6.14) \quad \ln |g(z)| \geq -A_1 \sup_{w \in \Delta_{r_1}} \{\ln |g(w)|\}.$$

Where A_1 is a positive constant depending only on r , r_1 and R . Since by hypothesis, $|\varphi(z)| > 1$ if $|z| = R$, then for the maximum modulus principle, for every $z \in \Delta_r$,

$$(6.15) \quad \ln |f(z)| \geq -A_1 \sup_{w \in \Delta_R} \{\ln |f(w)|\} + \ln |\varphi(z)|$$

We can find a constant positive A_3 such that, for every $z \in \Delta_r$ we have $\frac{R}{|a_i||R^2 - z\bar{a}_i|} \geq A_3$. Thus $|\varphi(z)| \geq \prod_{i=1}^n |z - a_i| \cdot A_3^n$

Consequently, for every $z \in \Delta_r$ we have

$$(6.16) \quad \ln |f(z)| \geq -A_1 \sup_{w \in \Delta_R} \{\ln |f(w)|\} + \ln \prod_{i=1}^n |z - a_i| + n \ln(A_3).$$

Let $z \in \Delta_r$ such that

$$(6.17) \quad \ln |f(z)| \leq - \left(A_1 + A_0 \ln \left(\frac{1}{\eta} \right) \right) \sup_{w \in \Delta_R} \{\ln |f(w)|\};$$

then, since $\eta < 1$, formula 6.12 implies

$$(6.18) \quad \ln |f(z)| \leq - \left(A_1 \cdot \sup_{w \in \Delta_R} \{\ln |f(w)|\} + n \ln \left(\frac{1}{\eta} \right) \right);$$

As a consequence of 6.16 and 6.18 we find that, if $z \in \Delta_r$ satisfy the estimate 6.17 then z is contained in the set

$$(6.19) \quad \left\{ w \in \Delta_r / \prod_{i=1}^n |z - a_i| \leq \left(\frac{\eta}{A_3} \right)^n \right\}.$$

Since, by Proposition 6.6, the area of this last set is bounded above by $\left(\frac{4e^2\pi}{A_3^2} \right) \cdot \eta^2$, the Lemma follows if we put $B_1 = A_1$, $B_2 = A_0$ and $B_3 = \left(\frac{4e^2\pi}{A_3^2} \right)$. \square

Following the same proof, Lemma 6.9 can be improved to an arbitrary holomorphic function on the unit disk: the non vanishing at the origin condition can be removed: if $f(z)$ is an holomorphic function on Δ_1 , we can write it as $f(z) = z^i h(z)$ with $h(0) \neq 0$.

Lemma 6.10. *Let $0 < r < R < 1$ be as in Lemma 6.9. Then we can find constants B_i for which the following holds. Let $f(z) = z^i h(z)$ be a holomorphic function on Δ_1 (with $h(0) \neq 0$). Let $\eta > 0$ be a positive number less than one. Then:*

$$(6.20) \quad \mu \left\{ z \in \Delta_r / \ln |f(z)| < - \left(B_1 + B_2 \cdot \ln \left(\frac{1}{\eta} \right) \right) \sup_{w \in \Delta_R} \{\ln |f(w)|\} + (\ln |h(0)| + i \cdot \ln(R)) \right\} \leq B_3 \cdot \eta^2.$$

Proof. It suffices to change the $\varphi(z)$ of the proof of Lemma 6.9 with the function $\psi(z) := \varphi(z) \cdot \frac{z^i}{(R)^i}$ and proceed as in that proof. \square

We can now give the proof of Theorem 6.4.

Proof. (of Theorem 6.4). We deal first with the case when the universal covering is Δ_1 . We can choose a universal covering $\alpha : \Delta_1 \rightarrow M$ such that $\alpha(0) \in U$.

Since U is relatively compact, We can find a compact closed set $B \in \alpha^{-1}(\overline{U})$ having diameter (in the hyperbolic metric) strictly smaller than some real number r_0 and such that $\alpha|_B : B \rightarrow \overline{U}$ is surjective. In particular we can find $r < 1$ such that $B \subset \Delta_r$. Denote by W_0 the subset $\alpha|_B^{-1}(W) \subset B$. r can be chosen big enough to have that, every automorphism ι of Δ_1 which sends an element of B in 0 has the property that $\iota(B) \subset \Delta_r$.

Fix a real number R such that $1 > R > r$.

Observe that $\alpha^*(L)$ is isomorphic to the trivial line bundle on Δ_1 . Thus $\alpha^*(L) = e \cdot \mathcal{O}_{\Delta_1}$. Thus we can find positive constants B_i such that, for every $z \in \Delta_R$, we have $B_1 \leq \|e\|(z) \leq B_2$.

Let $s \in H_B^0(M, L^d)$ and let $w_0 \in W_0$ a point such that $\|s\|_W = \|\alpha^*(s)\|(w_0)$. We can choose a automorphism ι of Δ_1 , such that $\iota(0) = w_0$. Since the hyperbolic metric is invariant by the disk automorphisms, we can still find $r_1 < r$ such that $\iota^{-1}(B) \subset \Delta_{r_1}$.

Consequently since, for every measurable set $H \subset U$ we have that $\mu(H) \leq \mu((\iota \circ \alpha)^{-1}(H) \cap \iota^{-1}(B))$, we may suppose that $w_0 = 0$ it suffices to give an upper estimate for the measure of the set of elements $z \in \Delta_r$ such that

$$(6.21) \quad \ln \|\alpha^*(s)\|(z) \leq -C_1 \left(\ln \left(\frac{1}{\eta} \right) + 1 \right) (\log^+ \|s\|_M + d) + 3 \ln \|\alpha^*(s)\|(0).$$

Write $\alpha^*(s) = f \cdot e^d$ with f a holomorphic function on Δ_1 . Consequently, since z and w_0 are elements of Δ_r , we may find constants A_i , not depending on z , w_0 or s , such that $\ln \|\alpha^*(s)\|(z) \geq \ln |f|(z) + A_1 d$ and $\ln \|\alpha^*(s)\|(0) \leq \ln |f|(0) + A_2 d$. Moreover we can find a constant A_3 (independent on z , w_0 or s) such that $\ln^+ \|s\|_M \geq \ln |f|_{3r} + A_3 \cdot d$.

Consequently, choosing C_1 sufficiently big, the Theorem reduces to the estimate of Lemma 6.9 and the conclusion follows in this case.

When the universal covering \widetilde{M} is \mathbb{C} , choose again a compact subset $B \subset \alpha^{-1}(\overline{U})$ such that $\alpha|_B : B \rightarrow \overline{U}$ is surjective. Let h be the diameter of B (for the standard flat metric on \mathbb{C}). Choose a disk Δ of radius bigger than $3h$ centered in a point of B and proceed as before replacing Δ_1 by the chosen disk Δ . \square

6.2. proof of Theorem 6.1. We start with the following easy observation: suppose that $B_1 \subset M$ is a subset such that $\varphi(B_1) \cap S(X_K) \neq \emptyset$, then B_1 is a subset of type S with respect to φ . It suffices then to prove the following:

Claim 6.11. *If U is a relatively compact open set in M containing a subset B_U of type S with respect to φ then $\varphi^{-1}(S(X_K)) \cap U$ is full in U .*

Indeed, by hypothesis, we can choose a relatively compact open U_0 set containing B and $\varphi^{-1}(S(X_K)) \cap U_0$ will be full in U_0 , in particular it will be dense in U_0 . Choose a covering \mathcal{U} of M made by countably many relatively compact open sets. For each open set U belonging to \mathcal{U} and such that $U \cap U_0 \neq \emptyset$, we will have that $\varphi^{-1}(S(X_K)) \cap U$ is full on U . For each open set $U_1 \in \mathcal{U}$ intersecting one of the open sets which intersect U_0 , we will have that $\varphi^{-1}(S(X_K)) \cap U_1$ is full in U_1 etc. Consequently, by connectedness of M , the conclusion follows.

Proof. (Of the Claim). Let $d \geq 1$ and let $s \in H^0(\mathcal{X}, \mathcal{L}^d) \setminus \{0\}$. Since, B is of type S with respect to φ , we can find positive constants C_2 and a such that $\log \|s\|_B \geq -C_2(\log^+ \|s\| + d)^a$.

Choosing the constants C_i suitably and applying Proposition 6.4 (and the fact that B is of type S with respect to φ) we find that the area $V(s)$ of the set

$$(6.22) \quad \{z \in U \mid \log \|\varphi^*(s)\|(z) \leq -C_3(\log^+ \|s\| + d)^{\sup\{a, N+1\}}\}$$

is smaller than $\frac{C_4}{C_5^{d^{N+1}} \cdot (\|s\|_+)^{C_6 \cdot d^N}}$.

Consequently we find the following estimate:

$$\begin{aligned} & \sum_{d=1}^{\infty} \sum_{s \in H^0(\mathcal{X}; \mathcal{L}^d)} V(s) \\ & \leq \sum_{d=0}^{\infty} \sum_{s \in H^0(\mathcal{X}; \mathcal{L}^d)} \frac{C_4}{(\|s\|_+)^{C_6 d^n} C_5^{d^{n+1}}} \quad \text{by the estimate above} \\ & = \sum_{d=1}^{\infty} \sum_{N=1}^{\infty} \sum_{\substack{s \in H^0(\mathcal{X}; \mathcal{L}^d) \\ N \leq \|s\|_+ < N+1}} \frac{C_4}{(\|s\|_+)^{C_6 d^n} C_5^{d^{n+1}}} \\ & \leq \sum_{d=1}^{\infty} \sum_{N=1}^{\infty} \frac{C_4 C_3^{d^{n+1}} N^{d^n}}{N^{C_6 d^n} C_5^{d^{n+1}}} \quad \text{by formula 2.6.} \end{aligned}$$

The last series converges as soon as we take C_5 and C_6 sufficiently big. Thus we may apply the Borel–Cantelli Lemma 2.4 and the conclusion of the Theorem follows. \square

7. SOME CONCLUDING REMARKS

It is now evident that there is a strict relationship between rational and S points contained in a Riemann surface inside a projective variety. Essentially this relationship may be resumed in the slogan: "Many rational points imply no S points and a single S point implies many S points and few rational points". Moreover, even when we know that the growth of rational points of height less than T is just sub exponential (and not less than that), we know that "for many values of T " the growth is just polynomial.

From our point of view, a consequence of the results found in this paper is that the distribution of rational points on analytic sub varieties of a projective variety is still very mysterious and many natural questions arise. For instance, a natural list of question could be:

a) Suppose that $A_U(T)$ grows not less than sub exponentially, can we estimate how big is the set of T 's for which $A_U(T)$ is "big"?

Theorem 4.8 proved that, even if $A_U(T)$ grows not less than sub exponentially, we can find intervals as big as we want for which $A_U(T)$ is bounded by a fixed polynomial in T . Given a

polynomial $P(t) = C \cdot t^\delta + \dots$, how wide can be an interval for which $A_U(T) \geq P(T)$? It is possible that, for high values of δ , these intervals are quite small.

b) Can we prove a higher dimensional analogue of Theorem 4.8?

As quoted in the introduction, Pila and Wilkie proved an estimate for the analogue of the function $A_U(T)$ for definable sets in a σ -minimal structure. Roughly speaking, given a definable set X , if we remove from it an obvious "algebraic part" (over which there are, most likely, too many rational points), the set of rational points of the remaining part - called the "transcendental part of X - satisfy an estimate similar to the Bombieri - Pila's. One may expect that, for many values (as in Theorem 4.8 for instance) of T , the number of rational points in the transcendental part of X is just polynomial in T .

c) Can we characterize a priori analytically and arithmetically Riemann surfaces which contain few (a lot) of rational points? and same questions for those which intersect the set $S(X_K)$?

Riemann surfaces which are almost all contained in $S(X_K)$ are generic, in many senses. They contain few rational points. One expect that Riemann surfaces which contain many rational points are in some way special so it is possible that one can detect them by some general property which can be computed. Known examples of these surfaces are very artificial, Theorems 5.9 and 6.3 tell us that other examples cannot be constructed as solutions of differential equations, thus at the moment we have a lack of methods to construct them.

d) Can we detect the structure of S points on a variety from its geometry?

The classical conjecture by Lang asserts that varieties of general type should contain very few rational points (the set of rational points shouldn't be Zariski dense). Is it possible that the set of S points on varieties of general type has some special, and peculiar to varieties of these varieties, properties? This would be in the spirit of the slogan quoted at the beginning of this section. We would like to observe that the introduction of relatively compact, non dense open sets in sub section 6.1 seems to go in this direction: the actual proof of Theorem 6.1 requires open sets of this kind but, at the moment, we do not know if, in the hyperbolic case, this hypothesis is necessary.

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